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INFINITE TYPE HOMEOMORPHISMS OF THE CIRCLE AND CONVERGENCE OF FOURIER SERIES

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Abstract. We consider the problem of convergence of Fourier series when we make a change of variable. Under a certain reasonable hypothesis, we give a necessary and sufficient condition for a homeomorphism of the circle to transform absolutely convergent Fourier series into uniformly convergent Fourier series.

1. Introduction

Let $A(T)$ be the space of all continuous functions of the circle T with absolutely convergent Fourier series, and let $U(T)$ be the space of all continuous functions on the circle T that have uniformly convergent Fourier series. If φ is a homeomorphism of the circle T, we say that φ transports $A(T)$ to $U(T)$ if $f \circ \varphi \in U(T)$ for all $f \in A(T)$.

A great deal of attention has been given to the following question: which homeomorphisms of the circle T transport $A(T)$ to $U(T)$? We say that a homeomorphism φ of the circle is of finite type if there is an integer v with $v \geq 3$ such that: φ is of class C^v and $\mid \varphi''(t) \mid + \cdots + \mid \varphi^{(v)}(t) \mid \neq 0$ for all $t \in \mathbb{R}$. In 1974, R. Kaufman showed that if φ is of finite type, then it transports $A(T)$ to $U(T)$. If φ is an analytic homeromorphism of the circle, it is easy to see that either it is of finite type or $\varphi''(t) = 0$ for all t. In the first case it transports $A(T)$ to $U(T)$ by the result of Kaufman; in the second case φ transports $A(T)$ to $A(T)$, see [1]. But, not every C^∞ homeomorphism of the circle transports $A(T)$ to $U(T)$, see [1] for the counterexample. We will be interested, therefore, in the case when φ is of class C^∞, but it may have a flat point, i.e. a point $t \in \mathbb{R}$ such that $\varphi^{(k)}(t) = 0$ for all $k = 2, 3, ...$ (note that a C^∞ homeomorphism with no flat point is of finite type). In a certain class of homeomorphisms of the circle, we will give a necessary and sufficient condition to transport $A(T)$ to $U(T)$.

2. Statement of the results

The theorem of R. Kaufman that was mentioned above states that a homeomorphism φ of the circle transports $A(T)$ to $U(T)$ if it is of finite type. The proof of this fact can be found in [1] and it is based on Lemma 2 below, but we suggest an alternative proof based on a result due to Stein and Wainger. See [2]. We state that result here as a lemma:

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Lemma 1 (Stein-Wainger)**.** *Let $p(t)$ be a real polynomial of degree d. Then,*

$$\left|\int_{-r}^{r} e^{ip(t)}\frac{1}{t}dt\right| \leq 6(2^{d+1}) - 2d - 10 \text{ for all } r > 0.$$

They proved that lemma in a more general form in 1965 and the proof was published five years later [2].

The second lemma we state below was proved by R. Kaufman in 1974, see [1].

Lemma 2 (R. Kaufman)**.** *Let f be a function of class C^k on the interval $[-r,r]$ with $k \geq 2$. Suppose $1 \leq \mid f^{(k)}(t) \mid \leq b$ for all $t \in [-r,r]$. Then,*

$$\left|\int_{-r}^{r} e^{if(t)}\frac{1}{t}dt\right| \leq C(k,b)$$

where $C(k,b)$ is a constant that depends only on k and b.

The fact is that Lemma 2 can be proved from Lemma 1 in a quite simple way. The proof of the second lemma given in [1] does not use Lemma 1 at all. Also, it is not difficult to see that Lemma 1 follows from Lemma 2 if we consider $d \geq 2$. So, they are indeed equivalent results.

The primary tool in dealing with oscillatory integrals as those in the lemmas is the Van der Corput lemma, see [3].

Our purpose is to deal with homeomorphisms of the circle not necessarily of finite type. To be precise, let φ be a homeomorphism of a circle of class C^{υ}, with $\upsilon \geq 3$, such that

$$(1) \qquad \mid \varphi''(t) \mid + \mid \varphi'''(t) \mid + \cdots + \mid \varphi^{(\upsilon)}(t) \mid \neq 0 \text{ for all } t \neq 0, t \in [-\pi, \pi].$$

Suppose that there is a neighbourhood of zero, say $(-r,r)$ with $r < \pi$, such that:

$$(2) \qquad \varphi \text{ is an odd function on } (-r,r).$$

$$(3) \qquad \varphi'(0) = 0 \text{ and } \varphi''(t) > 0 \text{ for all } t \in (0,r].$$

Also, assume that there is a constant θ , with $0 < \theta < 1$ so that

$$(4) \qquad \varphi'((1-\theta)a + \theta b) \leq \frac{1}{2}[\varphi'(a) + \varphi'(b)]$$
$$\text{for all } a, b \in [0,r) \text{ with } a \leq b.$$

We describe (4) by saying that φ' has uniform bounded doubling time. In particular φ' has what is called bounded doubling time, for if we put $a = 0$ above, we have that $\varphi'(\theta t) \leq \frac{1}{2}\varphi'(t)$ for all $t \in [0,r)$. Also, condition (4) with $\theta = \frac{1}{2}$ means that φ' is a convex function on $[0,r)$; but it does not imply convexity of φ' (see [5] for an example of a homeomorphism φ of the circle that satisfies all the four conditions and φ' is not convex in any interval of the form $(0,r)$, $r > 0$).

We shall prove the following result:

Theorem 1. *Let φ be a homeomorphism of the circle of class $C^{\upsilon}, \upsilon \geq 3$. Suppose that φ satisfies (1), (2), (3) and (4). Then, φ transports $A(T)$ to $U(T)$ if and only if there are constants ζ, M and λ with $0 < \zeta < 1$, $M > 0$ and $\lambda > 1$ such that*

$$(5) \qquad \frac{\varphi'(b) - \varphi'((1-\zeta)a + \zeta b)}{\varphi'((1-\zeta)a + \zeta b) - \varphi'(a)} \leq M$$

whenever $\varphi'((1-\zeta)a + \zeta b) \leq \lambda\varphi'(a)$, *for all* $0 < a < b < r$.

An easy way to verify condition (5) is that: suppose $\varphi'(0) = 0$ and $\varphi''(t) > 0$ for all $t \in (0, r)$; then all three conditions below imply condition (5). Moreover, they are related in the following way: *(i)*⇒ *(ii)*⟺ *(iii)* , where

(i) $\log \circ \varphi'(t)$ is a concave function on $(0, r)$.

(ii) There is ζ, with $0 < \zeta < 1$, such that for all $a, b \in (0, r)$ with $a \leq b$,

$$\varphi'((1-\zeta)a + \zeta b) \geq \sqrt{\varphi'(a)}\sqrt{\varphi'(b)}.$$

(iii) There are ζ and σ with $0 < \zeta < 1, 0 < \sigma < 1$ such that for all $a, b \in (0, r)$ with $a \leq b$ we have

$$\varphi'((1-\zeta)a + \zeta b) \geq [\varphi'(a)]^{1-\sigma}[\varphi'(b)]^{\sigma}.$$

To see that condition *(iii)* implies (5) we use the mean value theorem to get

$$\frac{\varphi'(b) - \varphi'((1-\zeta)a + \zeta b)}{\varphi'((1-\zeta)a + \zeta b) - \varphi'(a)}$$

$$\leq \frac{\varphi'((1-\zeta)a + \zeta b)[\varphi'((1-\zeta)a + \zeta b)^{\frac{1-\sigma}{\sigma}} - \varphi'(a)^{\frac{1-\sigma}{\sigma}}]}{\varphi'(a)^{\frac{1-\sigma}{\sigma}}[\varphi'((1-\zeta)a + \zeta b) - \varphi'(a)]}$$

$$\leq \frac{1-\sigma}{\sigma}\left[\frac{\varphi'((1-\zeta)a + \zeta b)}{\varphi'(a)}\right]^{\frac{1-\sigma}{\sigma}} \quad \text{for} \quad \sigma < \frac{1}{2}.$$

Also, assuming *(iii)*, we can prove by induction that

$$\varphi'((1-\zeta)^k a + [1-(1-\zeta)^k]b) \geq [\varphi'(a)]^{(1-\sigma)^k}[\varphi'(b)]^{1-(1-\sigma)^k} \quad \text{for all} \quad k \in \mathbb{N}.$$

So, *(iii)* implies *(ii)*. Condition *(ii)* does not imply *(i)*, see [5].

Using the theorem we can see that homeomorphisms of the circle as $\varphi'(t) = e^{-1/t}$ or $\varphi'(t) = e^{-1/t^2}$ for $t \in (0, 1/4)$, transport $A(T)$ to $U(T)$.

In [5] there is an example of a C^∞ homeomorphism φ of the circle that satisfies conditions (1), (2), (3) and (4) but does not satisfy condition (5).

3. Proof of the theorem

The space $A(T)$ is a Banach space with the norm

$$\| f \|_{A(T)} = \sum_{n \in \mathbb{Z}} | \hat{f}_n |, \text{ where } \hat{f}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt, n \in \mathbb{Z},$$

are the Fourier coefficients of f. Also, $U(T)$ is a Banach space with the norm

$$\| f \|_{U(T)} = \sup \{| S_m(f, x) |; x \in [-\pi, \pi], m = 0, 1, 2, ...\},$$

where $S_m(f, x) = \sum_{-m}^{m} \hat{f}_n e^{inx}$ are the partial sums of the Fourier series of f.

A homeomorphism of the circle φ of class $C^k, k \geq 1$, transports $A(T)$ to $U(T)$ if and only if there is a constant C, that does not depend on m, n or x, such that

$$\left|\int_{-\pi}^{\pi} e^{in\varphi(t+x)} D_m(t)dt\right| \leq C \tag{6}$$

for all $m = 0, 1, 2, ..., n \in \mathbb{Z}$ and $x \in [-\pi, \pi]$, where $D_m(t)$ is the Dirichlet Kernel, i.e.

$$D_m(t) = \frac{\sin(m+\frac{1}{2})t}{\sin(\frac{t}{2})} = \frac{2\sin mt}{t} + \mathcal{O}(1)$$

on any compact subset of $(-2\pi, 2\pi)$.

Let $(-\tau, \tau)$ be a small neighbourhood of zero and suppose that $\tau \leq |\, x \,| \leq \pi$. By (1), there is $\delta > 0$ such that: if $\tau \leq |\, x \,| \leq \pi$, then, for some k, depending on x, with $2 \leq k \leq v$ we have $|\, \varphi^{(k)}(t+x) \,| \geq \delta$ for all t with $|\, t \,| \leq \delta$. Therefore, as in reference [1], the integral in (6) is bounded for $\tau \leq |\, x \,| \leq \pi$. So, φ transports $A(T)$ to $U(T)$ if and only if

$$\text{(7)} \qquad \left| \int_{-\pi}^{\pi} e^{in\varphi(t)} \frac{\sin m(t-x)}{t-x} dt \right| \leq C$$
$$\text{for all } n \in \mathbb{Z}, m = 0, 1, 2, ... \text{ and } x \in (-\tau, \tau),$$

where τ is any positive number less than or equal to $\frac{\pi}{2}$ and C is a constant that does not depend on n,m and x. (The number τ will be chosen conveniently and it will depend on the constant θ.) Let τ be a number such that $\tau < \frac{r}{2} < \frac{\pi}{2}$. So,

$$\left| \int_{-\pi}^{\pi} e^{in\varphi(t)} \frac{\sin m(t-x)}{t-x} dt \right| \leq \left| \int_{-r}^{r} e^{in\varphi(t)} \frac{\sin m(t-x)}{t-x} dt \right| + \frac{4\pi}{r}$$

for all $n \in \mathbb{Z}, m = 0, 1, 2, \ldots$ and for all $x \in (-\tau, \tau)$.

Also, since φ is odd on $(-r, r)$, then, if we prove that the integral

$$\text{(8)} \qquad \int_{0}^{r} e^{in\varphi(t)} \frac{\sin m(t+x)}{t+x} dt$$

is bounded for all $x \in [0, \tau)$, $n > 0$ and $m > 0$, we conclude that (7) holds if and only if the integral

$$\text{(9)} \qquad \int_{0}^{r} e^{in\varphi(t)} \frac{\sin m(t-x)}{t-x} dt$$

is bounded for all $x \in (0, \tau), n > 0$ and $m > 0$.

The boundedness of the integral (8) was proved by Nagel, Vance, Wainger and Weinberg in the second paragraph of [4]. The proof does not use the full force of condition (4); it is based on the bounded doubling time only.

The main part of the theorem is the question of boundedness of the integral (9) for $x \in (0, \tau)$ and $n > 0, m > 0$.

To get boundedness of the integral (9) we have to deal with two major problems: the first one is that the derivative of $mt - n\varphi(t)$ may vanish somewhere between zero and r; and in that case we do not have enough oscillation around the vanishing point. The second one is that we have a bad singularity at the point x, and around that point, oscillation will not be enough to bound the integral. But, condition (5) will tell us whether there is cancellation around the point x or not. So, to estimate the integral (9) we deal simultaneously with oscillation and cancellation.

Put $f(t) = mt - n\varphi(t)$ and suppose that $x \leq \frac{1}{m}$. Assume first that $f'(r) = m - n\varphi'(r) \geq 0$. Then, we have two possibilities: the first one is $\theta r \leq \frac{2}{m}$. If so, (9) is bounded by $\frac{2}{\theta}$. The second one is $\theta r > \frac{2}{m}$, and in that case we divide the integral in three parts: over $[0, \frac{2}{m}]$ it is bounded by 2, over $[\theta r, r]$ by $\frac{2}{\theta}$ and over $[\frac{2}{m}, \theta r]$ we have

$$\left|\int_{2/m}^{\theta r} e^{in\varphi(t)} \frac{\sin\ m(t-x)}{t-x} dt\right|$$

$$\leq \left|\int_{2/m}^{\theta r} e^{i(mt+n\varphi(t))} \frac{1}{t-x} dt\right| + \left|\int_{2/m}^{\theta r} e^{if(t)} \frac{1}{t-x} dt\right|$$

$$\leq \frac{4}{m(\frac{2}{m}-x)} + \frac{4}{[m-n\varphi'(\theta r)](\frac{2}{m}-x)} \leq 4 + \frac{4m}{[m-n\frac{1}{2}\varphi'(r)]} \leq 12.$$

We used the Van der Corput lemma, the bounded doubling time, the monotonicity of φ' and the assumption $m - n\varphi'(r) \geq 0$ to obtain the above estimates. We have not used the full force of condition (4). Now, let's assume that $f'(r) = m - n\varphi'(r) < 0$. Then, there is a number ξ with $0 < \xi < r$ such that $f'(\xi) = 0$. This means that $\varphi'(\xi) = \frac{m}{n}$. As before, we have two possibilites: $\theta\xi \leq \frac{2}{m}$ or $\frac{2}{m} < \theta\xi$. If $\theta\xi \leq \frac{2}{m}$, then $\xi < \frac{1}{\theta}\xi \leq \frac{1}{\theta^2}\frac{2}{m}$. We can assume that $\frac{1}{\theta^2}\frac{2}{m} < r$, because otherwise, (9) is bounded by $mr \leq \frac{2}{\theta^2}$. So, assuming $\frac{2}{\theta^2 m} < r$ we divide the integral in two parts: over $[0, \frac{2}{\theta^2 m}]$ it is bounded by $\frac{2}{\theta^2}$, and over $[\frac{2}{\theta^2 m}, r]$ we have

$$\left|\int_{2/\theta^2 m}^{r} e^{in\varphi(t)} \frac{\sin\ m(t-x)}{t-x} dt\right|$$

$$\leq \left|\int_{2/\theta^2 m}^{r} e^{i(mt+n\varphi(t))} \frac{1}{t-x} dt\right| + \left|\int_{2/\theta^2 m}^{r} e^{if(t)} \frac{1}{t-x} dt\right|$$

$$\leq \frac{4}{m(\frac{2}{\theta^2 m}-x)} + \frac{4}{[n\varphi'(\frac{2}{\theta^2 m})-m](\frac{2}{\theta^2 m}-x)}$$

$$\leq 4 + \frac{4m}{[n\varphi'(\frac{\xi}{\theta})-m]} \leq 4 + \frac{4m}{2n\varphi'(\xi)-m} = 8.$$

If $\frac{2}{m} < \theta\xi$, then we do the following: the integral is bounded by 2 over the interval $[0, \frac{2}{m}]$ and

$$\left|\int_{2/m}^{\theta\xi} e^{in\varphi(t)} \frac{\sin\ m(t-x)}{t-x} dt\right| \leq 4 + \left|\int_{2/m}^{\theta\xi} e^{if(t)} \frac{1}{t-x} dt\right|$$

$$\leq 4 + \frac{4}{[m-n\varphi'(\theta\xi)](\frac{2}{m}-x)} \leq 4 + \frac{4m}{[m-\frac{1}{2}n\varphi'(\xi)]} = 12.$$

Now, if $\frac{\xi}{\theta} \geq r$, then

$$\left|\int_{\theta\xi}^{r} e^{in\varphi(t)} \frac{\sin\ m(t-x)}{t-x} dt\right| \leq \frac{r}{\theta\xi - x} \leq \frac{\frac{\xi}{\theta}}{\frac{1}{2}\theta\xi} = \frac{2}{\theta^2}.$$

So, we may assume $\frac{\xi}{\theta} < r$. In that case we have

$$\left|\int_{\theta\xi}^{\xi/\theta} e^{in\varphi(t)} \frac{\sin\ m(t-x)}{t-x} dt\right| \leq \frac{\frac{\xi}{\theta}}{\theta\xi - x} \leq \frac{2}{\theta^2}$$

and

$$\left|\int_{\xi/\theta}^{r} e^{in\varphi(t)} \frac{\sin\; m(t-x)}{t-x}dt\right| \leq 4 + \frac{4}{[n\varphi'(\frac{\xi}{\theta}) - m](\frac{\xi}{\theta} - x)}$$

$$\leq 4 + \frac{4m}{2n\varphi'(\xi) - m} = 8.$$

Again we used the Van der Corput lemma and the bounded doubling time to obtain the above estimates. Hence, the integral (9) is bounded if $x \leq \frac{1}{m}$.

Let's consider now the case when $x > \frac{1}{m}$. We have that (7) holds if and only if

$$\left|\int_{0}^{x-\frac{1}{m}} e^{if(t)} \frac{1}{t-x}dt + \int_{x+\frac{1}{m}}^{r} e^{if(t)} \frac{1}{t-x}dt\right| \leq C \tag{10}$$

for all $x \in (0, \tau), m, n \in \mathbb{N}$ with $\frac{1}{m} < x$. (C is of course a constant that does not depend on x, m and n.) So, we will prove that (10) holds if and only if the homeomorphism φ satisfies condition (5).

Let's prove first that condition (5) is necessary. For this, let $(\zeta_k)_{k\in\mathbb{N}}$ and $(\lambda_k)_{k\in\mathbb{N}}$ be sequences such that,

$$0 < \zeta_k < 1, \lambda_k > 1 \text{ for all } k \in \mathbb{N} \text{ and } \zeta_k \to 1, \lambda_k \to 1.$$

Suppose that φ does not satisfy condition (5). Then, for each $k \in \mathbb{N}$ there exist numbers a_k and b_k, with $0 < a_k < b_k < \frac{1}{k}r$, so that

$$\frac{\varphi'(b_k) - \varphi'(x_k)}{\varphi'(x_k) - \varphi'(a_k)} > \frac{\zeta_k}{1-\zeta_k} \tag{11}$$

and

$$\varphi'(x_k) \leq \lambda_k \varphi'(a_k) \tag{12}$$

where $x_k = (1-\zeta_k)a_k + \zeta_k b_k$. Put

$$\begin{aligned} \epsilon_k &= \zeta_k(b_k - a_k), \\ \epsilon'_k &= (1-\zeta_k)(b_k - a_k), \\ \nu_k &= \frac{1}{\epsilon_k[\varphi'(x_k) - \varphi'(a_k)]}, \\ \mu_k &= \varphi'(x_k)\nu_k. \end{aligned}$$

Let m_k be the smallest integer greater than μ_k and n_k be the greatest integer less than ν_k. So, by (11), for k sufficiently large we have

$$\frac{1}{2} \leq \epsilon_k[m_k - n_k\varphi'(x_k - \epsilon_k)] \leq \frac{3}{2} \tag{13}$$

and

$$\epsilon'_k[n_k\varphi'(x_k + \epsilon'_k) - m_k] > \frac{1}{2}. \tag{14}$$

Inequality (14) implies that there is $\bar{\epsilon_k}$ with $0 < \bar{\epsilon_k} < \epsilon'_k$ such that

$$\bar{\epsilon_k}[n_k\varphi'(x_k + \bar{\epsilon_k}) - m_k] = \frac{1}{2}. \tag{15}$$

Also, by (12), $\epsilon_k m_k \to +\infty$ and since $x_k > \epsilon_k$, then $x_k > \frac{1}{m_k}$. Since $x_k \to 0$, then $0 < x_k < \tau$ for k large enough.

Suppose that $\bar{\epsilon_k} \leq \frac{1}{m_k}$ for an infinite number of indices k.

Let $f_k(t) = m_k\ t - n_k\varphi(t)$. Then, by (15) and the Van der Corput lemma we have that

$$\left|\int_{x_k+\frac{1}{m_k}}^{r} e^{if_k(t)}\frac{1}{t-x_k}dt\right| \leq \frac{4m_k}{\mid f_k'(x_k+\bar{\varepsilon}_k)\mid} \leq \frac{4}{\bar{\varepsilon}_k \mid f_k'(x_k+\bar{\varepsilon}_k)\mid} = 8,$$

for an infinite number of indices k. By (13) and the Van der Corput lemma,

$$\left|\int_{0}^{x_k-\varepsilon_k} e^{if_k(t)}\frac{1}{t-x_k}dt\right| \leq \frac{4}{\varepsilon_k f_k'(x_k-\varepsilon_k)} \leq 8.$$

Now, using (13),

$$\left|\int_{x_k-\varepsilon_k}^{x_k-\frac{1}{m_k}} [e^{if_k(t)} - e^{if_k(x_k)}]\frac{1}{t-x_k}dt\right| \quad \leq \quad \varepsilon_k f_k'(x_k-\varepsilon_k) \leq \frac{3}{2}$$

and

$$\left|\int_{x_k-\varepsilon_k}^{x_k-\frac{1}{m_k}} e^{if_k(x_k)}\frac{1}{t-x_k}dt\right| \quad = \quad \int_{1/m_k}^{\varepsilon_k} \frac{1}{t}dt = \log(\varepsilon_k m_k).$$

Hence, we conclude that (10) does not hold. Suppose now that $\bar{\varepsilon}_k > \frac{1}{m_k}$ for all k large enough. Then, by (13) and (15),

$$\left|\int_{x_k+\bar{\varepsilon}_k}^{r} e^{if_k(t)}\frac{1}{t-x_k}dt\right| \quad \leq \quad \frac{4}{\bar{\varepsilon}_k \mid f_k'(x_k+\bar{\varepsilon}_k)\mid} = 8$$

and

$$\left|\int_{0}^{x_k-\varepsilon_k} e^{if_k(t)}\frac{1}{t-x_k}dt\right| \quad \leq \quad \frac{4}{\varepsilon_k f_k'(x_k-\varepsilon_k)} \leq 8,$$

for all k large enough. Finally, by (13) and (15),

$$\left|\int_{x_k-\varepsilon_k}^{x_k-\frac{1}{m_k}} (e^{if_k(t)} - e^{if_k(x_k)})\frac{1}{t-x_k}dt\right| \quad \leq \quad \varepsilon_k f_k'(x_k-\varepsilon_k) \leq \frac{3}{2}$$

and

$$\left|\int_{x_k+\frac{1}{m_k}}^{x_k+\bar{\varepsilon}_k} (e^{if_k(t)} - e^{if_k(x_k)})\frac{1}{t-x_k}dt\right| \quad \leq \quad \varepsilon_k f_k'(x_k-\varepsilon_k) \leq \frac{3}{2}.$$

Hence, since

$$\left|\int_{x_k-\varepsilon_k}^{x_k-\frac{1}{m_k}} e^{if_k(x_k)}\frac{1}{t-x_k}dt + \int_{x_k+\frac{1}{m_k}}^{x_k+\bar{\varepsilon}_k} e^{if_k(x_k)}\frac{1}{t-x_k}\right|$$

$$= \left|\int_{-\varepsilon_k}^{-\frac{1}{m_k}} \frac{1}{t}dt + \int_{1/m_k}^{\bar{\varepsilon}_k} \frac{1}{t}dt\right| = \log\left(\frac{\varepsilon_k}{\bar{\varepsilon}_k}\right) \geq \log\left(\frac{\varepsilon_k}{\varepsilon_k'}\right) = \log\left(\frac{\zeta_k}{1-\zeta_k}\right),$$

we conclude that (10) does not hold. Therefore, condition (5) is necessary.

Now, let's prove sufficiency. So, assuming that φ satisfies condition (5), we have to show that (10) is true. For this, let $x \in (0, \tau), n, m \in \mathbb{N}$ with $x > \frac{1}{m}, m > 0, n > 0$. We said before that the number τ would be chosen conveniently. So, let $\tau < \frac{1}{2}\theta r$. (The reason for that choice will be clear in the proof that follows. It is not quite necessary, but it simplifies the proof a lot.) Put $f(t) = mt - n\varphi(t)$ as before and suppose that $f'(r) = m - n\varphi'(r) \geq 0$. Since $x + \frac{1}{m} < 2x < 2\tau < \theta r$, then using the bounded doubling time, the monotonicity of φ', and the Van der Corput lemma we have that

$$\left|\int_0^{x-\frac{1}{m}} e^{if(t)} \frac{1}{t-x} dt\right| \leq \frac{4m}{f'(x-\frac{1}{m})} \leq \frac{4m}{f'(\theta r)} \leq \frac{4m}{m - n\frac{1}{2}\varphi'(r)} \leq 8,$$

$$\left|\int_{x+\frac{1}{m}}^{\theta r} e^{if(t)} \frac{1}{t-x} dt\right| \leq \frac{4m}{f'(\theta r)} \leq 8$$

and

$$\left|\int_{\theta r}^{r} e^{if(t)} \frac{1}{t-x} dt\right| \leq \frac{r}{\theta r - x} \leq \frac{2}{\theta}.$$

Hence, (10) holds if $f'(r) = m - n\varphi'(r) \geq 0$ and $\frac{1}{m} < x < \tau$.

Suppose now that $f'(r) = m - n\varphi'(r) < 0$. So, there exists a number ξ with $0 < \xi < r$ such that $f'(\xi) = m - n\varphi'(\xi) = 0$. Since $mx > 1$, then there exists a number ε with $0 < \varepsilon < x$ such that $\varepsilon(m - n\varphi'(x - \varepsilon)) = 1 - \frac{1}{\lambda}$. We have two cases to consider: one is when $x \leq \xi$ and the other is when $\xi < x$. We will treat each one of them separately. From now on we have to deal simultaneously with oscillation and cancellation. Roughly, the problem is that: when we approach near x, the singularity is so bad that even if there is oscillation at x, this is not enough to guarantee boundedness. So, we have to use the cancellation we have around x. In order to have an effective cancellation we need to use condition (5). On the other hand, when we approach ξ, we don't have oscillation since $f'(\xi) = 0$. So, to deal with the integral around ξ we have to use condition (4). An important factor will be how far (with respect to ε) is ξ from x.

Assume that $x \leq \xi$. Call $\alpha = \xi - x$. The arguments we will use depend on the relation between ε, α and m. Suppose we have the following situation: $0 < \frac{1}{m} < \varepsilon < \frac{1}{2}\theta\alpha$. Let's choose a number A such that $\alpha + \varepsilon = \theta(A + \alpha + \varepsilon)$. We see that $A = \frac{1-\theta}{\theta}(\alpha + \varepsilon)$ is that number. We are going to divide the integral in several parts and prove boundedness of each one separately:

$$\left|\int_0^{x-\varepsilon} e^{if(t)} \frac{1}{t-x} dt\right| \leq \frac{4}{\varepsilon f'(x-\varepsilon)} = \frac{4\lambda}{\lambda - 1},$$

$$\left|\int_{x-\varepsilon}^{x-\frac{1}{m}} e^{if(t)} \frac{1}{t-x} dt + \int_{x+\frac{1}{m}}^{x+\varepsilon} e^{if(t)} \frac{1}{t-x} dt\right|$$

$$= \left|\int_{x-\varepsilon}^{x-\frac{1}{m}} (e^{if(t)} - e^{if(x)}) \frac{1}{t-x} dt + \int_{x+\frac{1}{m}}^{x+\varepsilon} (e^{if(t)} - e^{if(x)}) \frac{1}{t-x} dt\right|$$

$$\leq 2\varepsilon f'(x-\varepsilon) = \frac{2(\lambda - 1)}{\lambda}.$$

To estimate the integral from $x+\varepsilon$ to $x+\frac{1}{2}\theta\alpha$ we use condition (4). This condition assures enough oscillation in that interval. We claim that

$$m - n\varphi'(x+\frac{1}{2}\theta\alpha) \geq \frac{1}{2}[m - n\varphi'(x-\varepsilon)]. \tag{16}$$

To prove (16) we use condition (4): since $\varepsilon < \frac{1}{2}\theta\alpha$, then

$$x + \frac{1}{2}\theta\alpha \leq (1-\theta)(x-\varepsilon) + \theta\xi;$$

this implies that

$$\varphi'(x+\frac{1}{2}\theta\alpha) \leq \varphi'((1-\theta)(x-\varepsilon)+\theta\xi) \leq \frac{1}{2}(\varphi'(x-\varepsilon)+\varphi'(\xi))$$

and (16) follows. Therefore, by (16)

$$\left|\int_{x+\varepsilon}^{x+\frac{1}{2}\theta\alpha} e^{if(t)}\frac{1}{t-x}dt\right| \leq \frac{4}{\varepsilon f'(x+\frac{1}{2}\theta\alpha)} \leq \frac{8}{\varepsilon f'(x-\varepsilon)} = \frac{8\lambda}{\lambda-1}.$$

Now, we have to pass through the point ξ where we don't have oscillation at all. But since we are away from x, things become much simpler:

$$\left|\int_{x+\frac{1}{2}\theta\alpha}^{\xi+A} e^{if(t)}\frac{1}{t-x}dt\right| \leq \log\left(\frac{\xi+A-x}{\frac{1}{2}\theta\alpha}\right) \leq \log\left(\frac{2+\theta(1-\theta)}{\theta^2}\right).$$

Note that we can assume that $\xi + A < r$, because otherwise,

$$\left|\int_{x+\frac{1}{2}\theta\alpha}^{r} e^{if(t)}\frac{1}{t-x}dt\right| \leq \log\left(\frac{r-x}{\frac{1}{2}\theta\alpha}\right) \leq \log\left(\frac{\xi+A-x}{\frac{1}{2}\theta\alpha}\right)$$

and we are done. Finally, to evaluate the integral from $\xi + A$ to r we use again condition (4). This condition, as before, assures enough ocillation in that interval. We claim that

$$n\varphi'(\xi+A) - m \geq m - n\varphi'(x-\varepsilon). \tag{17}$$

By the choice of A, we have that $\xi = (1-\theta)(x-\varepsilon)+\theta(\xi+A)$. So,

$$\varphi'(\xi) \leq \frac{1}{2}(\varphi'(x-\varepsilon)+\varphi'(\xi+A))$$

and (17) follows. Using (17) now we have

$$\begin{aligned}\left|\int_{\xi+A}^{r} e^{if(t)}\frac{1}{t-x}dt\right| &\leq \frac{4}{(\xi+A-x)\mid f'(\xi+A)\mid} = \frac{4}{(\alpha+A)\mid f'(\xi+A)\mid} \\ &\leq \frac{4}{\varepsilon f'(x-\varepsilon)} = \frac{4\lambda}{\lambda-1}\end{aligned}$$

since $\alpha + A > \alpha > \varepsilon$.

Let's see now another possible situation: suppose that $0 < \varepsilon \leq \frac{1}{m} < \frac{1}{2}\theta\alpha$. Then,

$$\begin{aligned}\left|\int_{0}^{x-\frac{1}{m}} e^{if(t)}\frac{1}{t-x}dt\right| &\leq \frac{4m}{f'(x-\frac{1}{m})} \leq \frac{4}{\varepsilon f'(x-\varepsilon)} = \frac{4\lambda}{\lambda-1}, \\ \left|\int_{x+\frac{1}{m}}^{x+\frac{1}{2}\theta\alpha} e^{if(t)}\frac{1}{t-x}dt\right| &\leq \frac{4m}{f'(x+\frac{1}{2}\theta\alpha)} \leq \frac{8}{\varepsilon f'(x-\varepsilon)} = \frac{8\lambda}{\lambda-1},\end{aligned}$$

by (16).

From $x+\frac{1}{2}\theta\alpha$ to $\xi + A$ and from $\xi + A$ to r there is no change.

If $0 < \varepsilon < \frac{1}{2}\theta\alpha \le \frac{1}{m}$, then

$$\left|\int_0^{x-\frac{1}{m}} e^{if(t)}\frac{1}{t-x}dt\right| \le \frac{4m}{f'(x-\frac{1}{m})} \le \frac{4}{\varepsilon f'(x-\varepsilon)} = \frac{4\lambda}{\lambda-1}.$$

If $x+\frac{1}{m} < \xi + A$, then

$$\begin{aligned}\left|\int_{x+\frac{1}{m}}^{\xi+A} e^{if(t)}\frac{1}{t-x}dt\right| &\le \log\left(\frac{\xi+A-x}{\frac{1}{m}}\right) \le \log\left(\frac{\alpha+A}{\frac{1}{2}\theta\alpha}\right)\\ &\le \log\left(\frac{2+\theta(1-\theta)}{\theta^2}\right).\end{aligned}$$

Again we can assume $\xi+A<r$. Note that $x+\frac{1}{m}<r$ because $\frac{1}{m}<x<\tau<\frac{r}{2}$.

$$\left|\int_{\xi+A}^{r} e^{if(t)}\frac{1}{t-x}dt\right| \le \frac{4\lambda}{\lambda-1} \quad \text{as before.}$$

If $\xi+A \le x+\frac{1}{m}$, then

$$\begin{aligned}\left|\int_{x+\frac{1}{m}}^{r} e^{if(t)}\frac{1}{t-x}dt\right| &\le \frac{4m}{\mid f'(x+\frac{1}{m})\mid} \le \frac{4}{\varepsilon\mid f'(\xi+A)\mid}\\ &\le \frac{4}{\varepsilon f'(x-\varepsilon)} = \frac{4\lambda}{\lambda-1},\end{aligned}$$

by (17).

The next situation is $\frac{1}{2}\theta\alpha \le \varepsilon$ and $\frac{1}{m} \ge \varepsilon$. Assuming this we have

$$\left|\int_0^{x-\frac{1}{m}} e^{if(t)}\frac{1}{t-x}dt\right| \le \frac{4m}{f'(x-\frac{1}{m})} \le \frac{4}{\varepsilon f'(x-\varepsilon)} = \frac{4\lambda}{\lambda-1}.$$

If $x+\frac{1}{m} < \xi+A$, then

$$\left|\int_{x+\frac{1}{m}}^{\xi+A} e^{if(t)}\frac{1}{t-x}dt\right| \le \log\left(\frac{\alpha+A}{\frac{1}{m}}\right) \le \log\left(\frac{2+\theta(1-\theta)}{\theta^2}\right).$$

As before, we can assume $\xi+A<r$, so

$$\left|\int_{\xi+A}^{r} e^{if(t)}\frac{1}{t-x}dt\right| \le \frac{4}{(\alpha+A)\mid f'(\xi+A)\mid} \le \frac{4}{\varepsilon f'(x-\varepsilon)} = \frac{4\lambda}{\lambda-1},$$

by (17).

If $\xi+A \le x+\frac{1}{m}$, then, by (17), the integral from $x+\frac{1}{m}$ to r is bounded by $\frac{4\lambda}{\lambda-1}$ as in the preceding situation.

Now suppose that $\frac{1}{2}\theta\alpha \le \varepsilon$ and $\frac{1}{m} < \varepsilon$. Let B be a number such that $\alpha+\varepsilon = \zeta(B+\alpha+\varepsilon)$. We see that $B = \frac{1-\zeta}{\zeta}(\alpha+\varepsilon)$ is that number. Since we can assume that $\theta<\zeta$, then $B<A$. Indeed we will assume $\theta<\frac{1}{2}$ and $\zeta>\frac{1}{2}$. So, we have that $x+\frac{1}{m}<\xi+A$, because $A=\frac{1-\theta}{\theta}(\alpha+\varepsilon)>\alpha+\varepsilon\ge\varepsilon>\frac{1}{m}$. But, it may happen that $\xi+B\le x+\frac{1}{m}$, because α can be even zero. Let's see first this special case, i.e. if $\frac{1}{2}\theta\alpha \le \varepsilon$ and $\alpha + B \le \frac{1}{m} < \varepsilon$. We do the following:

$$\begin{aligned}
\left|\int_0^{x-\varepsilon} e^{if(t)}\frac{1}{t-x}dt\right| &\le \frac{4}{\varepsilon f'(x-\varepsilon)} = \frac{4\lambda}{\lambda-1},\\
\left|\int_{x-\varepsilon}^{x-\frac{1}{m}} e^{if(t)}\frac{1}{t-x}dt\right| &\le m\varepsilon \le \frac{\varepsilon}{\alpha+B} \le \frac{\zeta}{1-\zeta},\\
\left|\int_{x+\frac{1}{m}}^{\xi+A} e^{if(t)}\frac{1}{t-x}dt\right| &\le \log\left(\frac{\alpha+A}{\frac{1}{m}}\right) \le \log\left(\frac{\alpha+A}{\alpha+B}\right)\\
&\le \log\left[\left(\frac{1-\theta}{\theta}\right)\left(\frac{\zeta}{1-\zeta}\right)\right].
\end{aligned}$$

(As before, we can assume $\xi + A < r$. Indeed, since $\frac{1}{2}\theta\alpha \le \varepsilon$, then $\xi + A < r$ if x is sufficiently small.)

$$\left|\int_{\xi+A}^{r} e^{if(t)}\frac{1}{t-x}dt\right| \le \frac{4}{(\alpha+A) \mid f'(\xi+A) \mid} \le \frac{4}{\varepsilon f'(x-\varepsilon)} = \frac{4\lambda}{\lambda-1},$$

by (17).

Let's see now the case that was left over, i.e. we assume $\frac{1}{2}\theta\alpha \le \varepsilon, \frac{1}{m} < \varepsilon$ and $\frac{1}{m} < \alpha + B$. We claim that

$$n\varphi'(\xi+B) - m \le M[m - n\varphi'(x-\varepsilon)]. \tag{18}$$

To prove (18) we use condition (5). By the choice of B we have that $\xi = (1-\zeta)(x-\varepsilon) + \zeta(\xi+B)$. Also, since $\varepsilon f'(x-\varepsilon) = \frac{\lambda-1}{\lambda}$ and $\varepsilon m > 1$, then $\varphi'(\xi) < \lambda\varphi'(x-\varepsilon)$. Hence, (18) follows from (5). Now, let's see how to get boundedness of the integral in that last situation. From zero to $x-\varepsilon$ is quite easy and we have $\left|\int_o^{x-\varepsilon} e^{if(t)}\frac{1}{t-x}dt\right| \le \frac{4}{\varepsilon f'(x-\varepsilon)} = \frac{4\lambda}{\lambda-1}$. (We recall again here that we are free to choose the number τ. So, let's choose τ sufficiently small in order to have $\xi+B < r$. This procedure is not quite necessary, but it simplifies a little bit that part of the proof. Note that it is possible to choose such τ, for in that situation we have

$$\begin{aligned}
\xi + B = x + \alpha + \frac{1-\zeta}{\zeta}(\alpha+\varepsilon) &\le x + \frac{2}{\theta}\varepsilon + \frac{1-\zeta}{\zeta}(\frac{2}{\theta}\varepsilon+\varepsilon)\\
&\le x + \frac{2}{\theta}x + \frac{1-\zeta}{\zeta}(\frac{2}{\theta}x + x)\\
&\le (\frac{4}{\theta}+2)x.
\end{aligned}$$

So, $\tau < \frac{1}{6}\theta r$ will do the job.)

$$\left|\int_{\xi+B}^{\xi+A} e^{if(t)}\frac{1}{t-x}dt\right| \le \log\left(\frac{\alpha+A}{\alpha+B}\right) \le \log\left[\left(\frac{1-\theta}{\theta}\right)\left(\frac{\zeta}{1-\zeta}\right)\right].$$

We can also assume that $\xi + A < r$, because if not, we get boundedness of the integral over the interval $[\xi + B, r]$ as we did above. So,

$$\left|\int_{\xi+A}^{r} e^{if(t)}\frac{1}{t-x}dt\right| = \frac{4}{(\alpha+A) \mid f'(\xi+A) \mid} \le \frac{4}{\varepsilon f'(x-\varepsilon)} = \frac{4\lambda}{\lambda-1},$$

by (17).

It remains to prove boundedness over $[x-\varepsilon, x-\frac{1}{m}]\cup[x+\frac{1}{m}, \xi+B]$. We will use cancellation around x to do this. The cancellation depends on the inequality (18) as well as on the assumption that $\varepsilon \geq \frac{1}{2}\theta\alpha$. We have

$$\left|\int_{x-\varepsilon}^{x-\frac{1}{m}} e^{if(t)}\frac{1}{t-x}dt + \int_{x+\frac{1}{m}}^{\xi+B} e^{if(t)}\frac{1}{t-x}dt\right|$$

$$= \Big| \int_{x-\varepsilon}^{x-\frac{1}{m}} (e^{if(t)} - e^{if(x)})\frac{1}{t-x}dt + \int_{x+\frac{1}{m}}^{\xi+B} (e^{if(t)} - e^{if(x)})\frac{1}{t-x}dt$$

$$+ \int_{x-\varepsilon}^{x-\frac{1}{m}} e^{if(x)}\frac{1}{t-x}dt + \int_{x+\frac{1}{m}}^{\xi+B} e^{if(x)}\frac{1}{t-x}dt\Big|$$

$$\leq \varepsilon f'(x-\varepsilon) + Mf'(x-\varepsilon)(\alpha+B) + \left|\int_{1/m}^{\alpha+B}\frac{1}{t}dt - \int_{1/m}^{\varepsilon}\frac{1}{t}dt\right|$$

$$\leq \varepsilon f'(x-\varepsilon) + M\varepsilon f'(x-\varepsilon)\left[\frac{2}{\theta} + \frac{1-\zeta}{\zeta}\left(\frac{2}{\theta}+1\right)\right] + \left|\log\left(\frac{\alpha+B}{\varepsilon}\right)\right|$$

$$= \frac{\lambda-1}{\lambda} + M\left(\frac{\lambda-1}{\lambda}\right)\left[\frac{2}{\theta} + \frac{1-\zeta}{\zeta}\left(\frac{2}{\theta}+1\right)\right] + \left|\log\left(\frac{\alpha+B}{\varepsilon}\right)\right|,$$

by (18).

Since $\frac{\alpha+B}{\varepsilon} \geq \frac{B}{\varepsilon} = \frac{(\frac{1-\zeta}{\zeta})(\alpha+\varepsilon)}{\varepsilon} \geq \frac{1-\zeta}{\zeta}$ and $\frac{\alpha+B}{\varepsilon} < \frac{\frac{2}{\theta}\varepsilon+(\frac{1-\zeta}{\zeta})(\frac{2}{\theta}\varepsilon+\varepsilon)}{\varepsilon} \leq \frac{4+\theta}{\theta}$, then $\left|\log(\frac{\alpha+B}{\varepsilon})\right| \leq \max\left\{\left|\log(\frac{1-\zeta}{\zeta})\right|, \log(\frac{4+\theta}{\theta})\right\}$. This concludes the case $x \leq \xi$.

Assume now that $\xi < x$. Call $\beta = x - \xi$. This case is quite different from the previous one, and it requires different arguments. We will introduce a new number, say δ, and that number controls the amount of oscillation and the amount of cancellation that we have. The control is done in the following sense: when δ becomes small, this means that we lose cancellation but we gain oscillation; and when δ becomes large this means that we lose oscillation, but we gain cancellation.

Let δ be a positive number such that $\delta[n\varphi'(\xi+\frac{1+\zeta}{2}\beta) - m] = \lambda - 1$. Since $f'(x-\varepsilon) > 0$, then $x-\varepsilon < \xi$. So, $\varepsilon > \beta$. Let's consider first the following situation. Suppose that $0 < \frac{1}{m} < \delta < \frac{1-\zeta}{2}\beta$ and $\varepsilon \leq \frac{1+\zeta}{1-\zeta}\beta$. We claim that

$$n\varphi'(\xi+\frac{1+\zeta}{2\zeta}\beta) - m \leq (M+1)[n\varphi'(\xi+\frac{1+\zeta}{2}\beta) - m]. \tag{19}$$

To prove (19) we use condition (5). (Note that we can choose τ sufficiently small in order to have $\xi+\frac{1+\zeta}{2\zeta}\beta < r$.) We have that $\xi+\frac{1+\zeta}{2}\beta = (1-\zeta)\xi + \zeta(\xi+\frac{1+\zeta}{2\zeta}\beta)$. Since $\delta \mid f'(\xi+\frac{1+\zeta}{2}\beta) \mid= \lambda-1$ and $\delta m > 1$, then $\varphi'(\xi+\frac{1+\zeta}{2}\beta) < \lambda\varphi'(\xi)$. Hence, (19) follows from (5). Let's work with the integral now. We divide it in several parts.

$$\left|\int_0^{x-\varepsilon} e^{if(t)}\frac{1}{t-x}dt\right| \leq \frac{4}{\varepsilon f'(x-\varepsilon)} = \frac{4\lambda}{\lambda-1},$$

$$\left|\int_{x-\varepsilon}^{\xi+\frac{1+\zeta}{2}\beta} e^{if(t)}\frac{1}{t-x}dt\right| \leq \log\left(\frac{\varepsilon}{\frac{1-\zeta}{2}\beta}\right) \leq \log\left(\frac{2(1+\zeta)}{(1-\zeta)^2}\right),$$

$$\left|\int_{\xi+\frac{1+\zeta}{2}\beta}^{x-\delta} e^{if(t)}\frac{1}{t-x}dt\right| \leq \frac{4}{\delta \mid f'(\xi+\frac{1+\zeta}{2}\beta) \mid} = \frac{4}{\lambda-1},$$

and

$$\left|\int_{x+\delta}^{r} e^{if(t)}\frac{1}{t-x}dt\right| \leq \frac{4}{\delta \mid f'(x+\delta) \mid} \leq \frac{4}{\delta \mid f'(\xi+\frac{1+\zeta}{2}\beta) \mid} = \frac{4}{\lambda-1}.$$

To prove boundedness around x we use the small amount of cancellation that we have there:

$$\left|\int_{x-\delta}^{x-\frac{1}{m}} e^{if(t)}\frac{1}{t-x}dt + \int_{x-\frac{1}{m}}^{x+\delta} e^{if(t)}\frac{1}{t-x}dt\right|$$

$$= \left|\int_{x-\delta}^{x-\frac{1}{m}} (e^{if(t)} - e^{if(x)})\frac{1}{t-x}dt + \int_{x+\frac{1}{m}}^{x+\delta} (e^{if(t)} - e^{if(x)})\frac{1}{t-x}dt\right|$$

$$\leq 2\delta[n\varphi'(\xi + \frac{1+\zeta}{2\zeta}\beta) - m]$$

$$\leq 2\delta(M+1)[n\varphi'(\xi + \frac{1+\zeta}{2}\beta) - m] = 2(M+1)(\lambda-1),$$

by (19).

Another possible situation is that: $0 < \delta \leq \frac{1}{m} < \frac{1-\zeta}{2}\beta$ and $\varepsilon \leq \frac{1+\zeta}{1-\zeta}\beta$. This case is better than the previous one because the small δ here means that we have a large amount of oscillation, and $\frac{1}{m} \geq \delta$ means that $\frac{1}{m}$ is large enough to put us away from the bad singularity at x. Hence, as we shall see, we don't need conditions (4) and (5) in this case. We have:

$$\begin{aligned}
\left|\int_{0}^{x-\varepsilon} e^{if(t)}\frac{1}{t-x}dt\right| &\leq \frac{4}{\varepsilon f'(x-\varepsilon)} = \frac{4\lambda}{\lambda-1},\\
\left|\int_{x-\varepsilon}^{\xi+\frac{1+\zeta}{2}\beta} e^{if(t)}\frac{1}{t-x}dt\right| &\leq \log\left(\frac{\varepsilon}{\frac{1-\zeta}{2}\beta}\right) \leq \log\left(\frac{2(1+\zeta)}{(1-\zeta)^2}\right),\\
\left|\int_{\xi+\frac{1+\zeta}{2}\beta}^{x-\frac{1}{m}} e^{if(t)}\frac{1}{t-x}dt\right| &\leq \frac{4m}{\mid f'(\xi+\frac{1+\zeta}{2}\beta) \mid}\\
&\leq \frac{4}{\delta \mid f'(\xi+\frac{1+\zeta}{2}\beta) \mid} = \frac{4}{\lambda-1},\\
\left|\int_{x+\frac{1}{m}}^{r} e^{if(t)}\frac{1}{t-x}dt\right| &\leq \frac{4m}{\mid f'(x+\frac{1}{m}) \mid}\\
&\leq \frac{4}{\delta \mid f'(\xi+\frac{1+\zeta}{2}\beta) \mid} = \frac{4}{\lambda-1}.
\end{aligned}$$

The next situation is: $0 < \delta < \frac{1-\zeta}{2}\beta \leq \frac{1}{m}$ and $\varepsilon \leq \frac{1+\zeta}{1-\zeta}\beta$. In that case we do the following: if $x - \frac{1}{m} \leq x - \varepsilon$, then

$$\begin{aligned}\left|\int_0^{x-\frac{1}{m}} e^{if(t)}\frac{1}{t-x}dt\right| &\leq \frac{4m}{f'(x-\frac{1}{m})} \leq \frac{4}{\varepsilon f'(x-\varepsilon)} = \frac{4\lambda}{\lambda-1},\\ \left|\int_{x+\frac{1}{m}}^{r} e^{if(t)}\frac{1}{t-x}dt\right| &\leq \frac{4m}{\mid f'(x+\frac{1}{m})\mid} \leq \frac{4}{\delta \mid f'(\xi+\frac{1+\zeta}{2}\beta)\mid} = \frac{4}{\lambda-1}.\end{aligned}$$

If $x - \varepsilon < x - \frac{1}{m}$, then

$$\begin{aligned}\left|\int_0^{x-\varepsilon} e^{if(t)}\frac{1}{t-x}dt\right| &\leq \frac{4}{\varepsilon f'(x-\varepsilon)} = \frac{4\lambda}{\lambda-1},\\ \left|\int_{x-\varepsilon}^{x-\frac{1}{m}} e^{if(t)}\frac{1}{t-x}dt\right| &\leq \log(\varepsilon m) \leq \log\left(\frac{2(1+\zeta)}{(1-\zeta)^2}\right),\\ \left|\int_{x+\frac{1}{m}}^{r} e^{if(t)}\frac{1}{t-x}dt\right| &\leq \frac{4m}{\mid f'(x+\frac{1}{m})\mid}\\ &\leq \frac{4}{\delta \mid f'(\xi+\frac{1+\zeta}{2}\beta)\mid} = \frac{4}{\lambda-1}.\end{aligned}$$

Suppose now that $\frac{1}{m} < \frac{1-\zeta}{2}\beta \leq \delta$ and $\varepsilon \leq \frac{1+\zeta}{1-\zeta}\beta$. In that situation we have to use conditions (4) and (5) because the large δ means that we don't have enough oscillation, and the small $\frac{1}{m}$ means that we are too close to the singularity at x. But, on the other hand, the large δ means that we have much cancellation around x, and we will use that fact to compensate the lack of oscillation in this case.

$$\begin{aligned}\left|\int_0^{x-\varepsilon} e^{if(t)}\frac{1}{t-x}dt\right| &\leq \frac{4}{\varepsilon f'(x-\varepsilon)} = \frac{4\lambda}{\lambda-1},\\ \left|\int_{x-\varepsilon}^{\xi+\frac{1+\zeta}{2}\beta} e^{if(t)}\frac{1}{t-x}dt\right| &\leq \log\left(\frac{\varepsilon}{\frac{1-\zeta}{2}\beta}\right)\\ &\leq \log\left(\frac{2(1+\zeta)}{(1-\zeta)^2}\right),\end{aligned}$$

$$\left|\int_{\xi+\frac{1+\zeta}{2}\beta}^{x-\frac{1}{m}} e^{if(t)}\frac{1}{t-x}dt + \int_{x+\frac{1}{m}}^{x+\frac{1-\zeta}{2}\beta} e^{if(t)}\frac{1}{t-x}dt\right|$$

$$= \left|\int_{\xi+\frac{1+\zeta}{2}\beta}^{x-\frac{1}{m}} (e^{if(t)}-e^{if(x)})\frac{1}{t-x}dt + \int_{x+\frac{1}{m}}^{x+\frac{1-\zeta}{2}\beta} (e^{if(t)}-e^{if(x)})\frac{1}{t-x}dt\right|$$

$$\leq 2\delta \mid f'(\xi + \frac{1+\zeta}{2\zeta}\beta)\mid$$

$$\leq 2\delta(M+1)\mid f'(\xi+\frac{1+\zeta}{2}\beta)\mid = 2(M+1)(\lambda-1),$$

by (19).

We still have to prove boundedness of the integral over the interval $[x+\frac{1-\zeta}{2}\beta, r]$. Since δ is large, then we may not have enough oscillation at the point $x+\frac{1-\zeta}{2}\beta$.

So, we cannot just integrate from $x + \frac{1-\zeta}{2}\beta$ to r using the Van der Corput lemma. We have to use condition (4).

Let A' be a number such that $\varepsilon - \beta = \theta(A' + \varepsilon - \beta)$. We can see that $A' = \frac{1-\theta}{\theta}(\varepsilon - \beta)$ is that number. We claim that

$$
\begin{aligned}
n\varphi'(x + \frac{1-\zeta}{2}\beta + A') - m &\geq n\varphi'(\xi + A') - m, \\
n\varphi'(\xi + A') - m &\geq m - n\varphi'(x - \varepsilon).
\end{aligned} \tag{20}
$$

To prove (20) we use (4). By the choice of A' we have that $\xi = (1-\theta)(x-\varepsilon) + \theta(\xi + A')$. Since $\xi + A' < x + \frac{1-\zeta}{2}\beta + A'$, (20) follows from (4).

We will use inequality (20) to show boundedness over the interval $[x + \frac{1-\zeta}{2}\beta, r]$. We divide the interval from $x + \frac{1-\zeta}{2}\beta$ to $x + \frac{1-\zeta}{2}\beta + A'$ and from $x + \frac{1-\zeta}{2}\beta + A'$ to r. (Note that we can choose τ sufficiently small in order to have $x + \frac{1-\zeta}{2}\beta + A' < r$, but again this is not quite necessary because if not we just integrate from $x + \frac{1-\zeta}{2}\beta$ to r as we have done before.) We have

$$
\left| \int_{x+\frac{1-\zeta}{2}\beta}^{x+\frac{1-\zeta}{2}\beta + A'} e^{if(t)} \frac{1}{t-x} dt \right| \leq \log \left(\frac{\frac{1-\zeta}{2}\beta + A'}{\frac{1-\zeta}{2}\beta} \right) \leq \log \left(\frac{\frac{1-\zeta}{2} + \frac{1-\theta}{\theta}\left(\frac{1+\zeta}{1-\zeta}\right)}{\frac{1-\zeta}{2}} \right)
$$

and

$$
\begin{aligned}
\left| \int_{x+\frac{1-\zeta}{2}\beta + A'}^{r} e^{if(t)} \frac{1}{t-x} dt \right| &\leq \frac{4}{\mid f'\left(x + \frac{1-\zeta}{2}\beta + A'\right) \mid \left(\frac{1-\zeta}{2}\beta + A'\right)} \\
&\leq \frac{4}{\left(\frac{1-\zeta}{2}\beta\right) f'(x-\varepsilon)},
\end{aligned}
$$

by (20).

But now, since $\varepsilon \leq \frac{1+\zeta}{1-\zeta}\beta$, then $\frac{1-\zeta}{2}\beta \geq \left(\frac{1-\zeta}{2}\right)\left(\frac{1-\zeta}{1+\zeta}\right)\varepsilon$. Hence,

$$
\begin{aligned}
\left(\frac{1-\zeta}{2}\beta\right) f'(x-\varepsilon) &\geq \left(\frac{1-\zeta}{2}\right)\left(\frac{1-\zeta}{1+\zeta}\right)\varepsilon f'(x-\varepsilon) \\
&= \left(\frac{\lambda-1}{\lambda}\right)\left(\frac{1-\zeta}{2}\right)\left(\frac{1-\zeta}{1+\zeta}\right)
\end{aligned}
$$

and we are done. (Note that in order to bound the above integral over $[x + \frac{1-\zeta}{2}\beta, r]$ we could have worked with the point $x + \varepsilon + A'$ instead of $x + \frac{1-\zeta}{2}\beta + A'$.)

The next situation is the following. Assume that $\frac{1-\zeta}{2}\beta \leq \frac{1}{m}$, $\frac{1-\zeta}{2}\beta \leq \delta$ and $\varepsilon \leq \frac{1+\zeta}{1-\zeta}\beta$. This case is a little better than the previous one because the large $\frac{1}{m}$ means that we are far away from the singularity at x. But since δ can be still large, we might not have enough oscillation. So we still have to use condition (4). If $x - \frac{1}{m} \leq x - \varepsilon$, then

$$
\left| \int_0^{x-\frac{1}{m}} e^{if(t)} \frac{1}{t-x} dt \right| \leq \frac{4m}{f'(x-\varepsilon)} \leq \frac{4}{\varepsilon f'(x-\varepsilon)} = \frac{4\lambda}{\lambda - 1}.
$$

If $x-\frac{1}{m}>x-\varepsilon$, then

$$\left|\int_o^{x-\varepsilon} e^{if(t)}\frac{1}{t-x}dt\right| \le \frac{4}{\varepsilon f'(x-\varepsilon)} = \frac{4\lambda}{\lambda-1}$$

and

$$\left|\int_{x-\varepsilon}^{x-\frac{1}{m}} e^{if(t)}\frac{1}{t-x}dt\right| \le \log(\varepsilon m) \le \log\left(\frac{2(1+\zeta)}{(1-\zeta)^2}\right).$$

This proves boundedness over the interval $[0, x-\frac{1}{m}]$. To work out the integral over $[x+\frac{1}{m}, r]$ we do the following:

$$\left|\int_{x+\frac{1}{m}}^{x+\frac{1}{m}+A'} e^{if(t)}\frac{1}{t-x}dt\right| \le \log\left(\frac{\frac{1}{m}+A'}{\frac{1}{m}}\right) \le \left(1+\frac{(\frac{1-\theta}{\theta})(\frac{1+\zeta}{1-\zeta})}{(\frac{1-\zeta}{2})}\right)$$

and

$$\begin{aligned}\left|\int_{x+\frac{1}{m}+A'}^{r} e^{if(t)}\frac{1}{t-x}dt\right| &\le \frac{4m}{\mid f'(x+\frac{1}{m}+A')\mid} \\ &\le \frac{4}{\frac{1-\zeta}{2}\beta \mid f'(x+\frac{1-\zeta}{2}\beta+A')\mid} \\ &\le \frac{4}{(\frac{1-\zeta}{2})(\frac{1-\zeta}{1+\zeta})\varepsilon f'(x-\varepsilon)} = \frac{4}{(1-\frac{1}{\lambda})(\frac{1-\zeta}{2})(\frac{1-\zeta}{1+\zeta})},\end{aligned}$$

by (20).

So far we studied all possible situations with $\varepsilon \le \frac{1+\zeta}{1-\zeta}\beta$. We still have to analyse the problem when $\varepsilon > \frac{1+\zeta}{1-\zeta}\beta$. For this, let B' be a number such that $\varepsilon-\beta = \zeta(B'+\varepsilon-\beta)$. We see that $B' = \frac{1-\zeta}{\zeta}(\varepsilon-\beta)$ is that number.

Suppose we have the following situation : $\frac{1}{m} < B'-\beta$ and $\varepsilon > \frac{1+\zeta}{1-\zeta}\beta$. (Note that $B' > 2\beta$ since $\varepsilon > \frac{1+\zeta}{1-\zeta}\beta$.) Since $\frac{1}{m} < B'-\beta$, then $x+\frac{1}{m} < \xi + B'$. Also, $B'-\beta \le B' + \frac{1-\zeta}{\zeta}(\varepsilon-\beta) \le \varepsilon-\beta < \varepsilon$. So, $\frac{1}{m} < B'-\beta$ implies that $\frac{1}{m} < \varepsilon$. We claim that

$$n\varphi'(\xi+B') - m \le M[m - n\varphi'(x-\varepsilon)]. \tag{21}$$

To prove (21) we use condition (5). By the choice of B' we have that $\xi = (1-\zeta)(x-\varepsilon)+\zeta(\xi+B')$. Since $\varepsilon f'(x-\varepsilon) = \frac{\lambda-1}{\lambda}$ and $\varepsilon m > 1$, $\varphi'(\xi) \le \lambda\varphi'(x-\varepsilon)$. Hence, (21) follows from (5).

Now, let's estimate the integral. We have

$$\left|\int_0^{x-\varepsilon} e^{if(t)}\frac{1}{t-x}dt\right| \le \frac{4}{\varepsilon f'(x-\varepsilon)} = \frac{4\lambda}{\lambda-1},$$

$$\left|\int_{x-\varepsilon}^{x-\frac{1}{m}} e^{if(t)}\frac{1}{t-x}dt + \int_{x+\frac{1}{m}}^{\xi+B'} e^{if(t)}\frac{1}{t-x}dt\right|$$

$$\le M\varepsilon f'(x-\varepsilon) + M(B'-\beta)f'(x-\varepsilon) + \left|\int_{1/m}^{B'-\beta}\frac{1}{t}dt - \int_{1/m}^{\varepsilon}\frac{1}{t}dt\right|$$

$$\leq 2M\varepsilon f'(x-\varepsilon) + \log\left(\frac{\varepsilon}{B'-\beta}\right) \leq 2M\left(\frac{\lambda-1}{\lambda}\right) + \log\left(\frac{1+\zeta}{1-\zeta}\right),$$

by (21).

$$\begin{aligned}\left|\int_{\xi+B'}^{\xi+A'} e^{if(t)}\frac{1}{t-x}dt\right| &\leq \log\left(\frac{A'-\beta}{B'-\beta}\right)\\ &\leq \log\left(\frac{2A'}{B'}\right) = \log\left[\left(\frac{1-\theta}{\theta}\right)\left(\frac{2\zeta}{1-\zeta}\right)\right].\\ \left|\int_{\xi+A'}^{r} e^{if(t)}\frac{1}{t-x}dt\right| &\leq \frac{4}{(A'-\beta)\mid f'(\xi+A')\mid} \leq \frac{4}{(B'-\beta)f'(x-\varepsilon)}\\ &\leq \frac{8}{B'f'(x-\varepsilon)} \leq \frac{4(1+\zeta)\lambda}{(\lambda-1)(1-\zeta)},\end{aligned}$$

by (20).

Finally, the last situation is that: $B'-\beta \leq \frac{1}{m}$ and $\varepsilon > \frac{1+\zeta}{1-\zeta}\beta$. If $x-\frac{1}{m} \leq x-\varepsilon$, then

$$\left|\int_{0}^{x-\frac{1}{m}} e^{if(t)}\frac{1}{t-x}dt\right| \leq \frac{4m}{f'(x-\frac{1}{m})} \leq \frac{4}{\varepsilon f'(x-\varepsilon)} = \frac{4\lambda}{\lambda-1}.$$

If $x-\frac{1}{m} > x-\varepsilon$, then

$$\left|\int_{0}^{x-\varepsilon} e^{if(t)}\frac{1}{t-x}dt\right| \leq \frac{4}{\varepsilon f'(x-\varepsilon)} = \frac{4\lambda}{\lambda-1},$$

and

$$\begin{aligned}\left|\int_{x-\varepsilon}^{x-\frac{1}{m}} e^{if(t)}\frac{1}{t-x}dt\right| &\leq \log(\varepsilon m) \leq \log\left(\frac{\varepsilon}{B'-\beta}\right)\\ &\leq \log\left(\frac{1+\zeta}{1-\zeta}\right).\end{aligned}$$

This proves boundedness of the integral over $[0, x-\frac{1}{m}]$. To estimate the integral on $[x+\frac{1}{m}, r]$ we do as follows: if $x+\frac{1}{m} \geq \xi+A'$, then

$$\begin{aligned}\left|\int_{x+\frac{1}{m}}^{r} e^{if(t)}\frac{1}{t-x}dt\right| &\leq \frac{4m}{\mid f'(x+\frac{1}{m})\mid} \leq \frac{4}{(B'-\beta)\mid f'(\xi+A')\mid}\\ &\leq \frac{4}{\left(\frac{1-\zeta}{1+\zeta}\right)\varepsilon f'(x-\varepsilon)} = \frac{4\lambda(1+\zeta)}{(\lambda-1)(1-\zeta)},\end{aligned}$$

by (20).

If $x+\frac{1}{m}<\xi+A'$, then

$$\begin{aligned}\left|\int_{x+\frac{1}{m}}^{\xi+A'} e^{if(t)}\frac{1}{t-x}dt\right| &\leq \log\left(\frac{A'-\beta}{\frac{1}{m}}\right)\leq \log\left(\frac{A'}{B'-\beta}\right)\\ &\leq \log\left(\frac{2A'}{B'}\right)=\log\left(\frac{2\zeta(1-\theta)}{\theta(1-\zeta)}\right),\end{aligned}$$

and

$$\begin{aligned}\left|\int_{\xi+A'}^{r} e^{if(t)}\frac{1}{t-x}dt\right| &\leq \frac{4}{(A'-\beta)\mid f'(\xi+A')\mid}\leq \frac{8}{B'f'(x-\varepsilon)}\\ &\leq \frac{4}{\left(\frac{1-\zeta}{1+\zeta}\right)\varepsilon f'(x-\varepsilon)}=\frac{4\lambda(1+\zeta)}{(\lambda-1)(1-\zeta)},\end{aligned}$$

by (20).

This concludes the proof of the theorem.

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