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A GENERALIZED DEDEKIND–MERTENS LEMMA AND ITS CONVERSE

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To Wolmer V. Vasconcelos on the occasion of his sixtieth birthday

Abstract. We study content ideals of polynomials and their behavior under multiplication. We give a generalization of the Lemma of Dedekind–Mertens and prove the converse under suitable dimensionality restrictions.

1. Introduction

Let R be a commutative ring and let t be an indeterminate over R. For a polynomial $f \in R[t]$, the *content ideal* $c(f)$ of f is the ideal of R generated by the coefficients of f.

If f and g are two polynomials in $R[t]$, one clearly has that $c(fg) \subseteq c(f)c(g)$ and the classical Lemma of Gauss, in one of its forms, says that equality holds if R is a principal ideal domain. More generally, $c(f)c(g)$ is integral over $c(fg)$. Since ideals in principal ideal domains are integrally closed, Gauss's Lemma follows from this statement. An even more precise statement is given by the Lemma of Dedekind–Mertens. The lemma asserts that if f and g are two polynomials and n is the degree of g, then

$$c(fg)c(f)^n = c(f)^{n+1}c(g). \tag{1}$$

Interchanging the roles of f and g, there is obviously an analogous formula involving the degree of the polynomial f and powers of $c(g)$.

There has recently been renewed interest in this lemma for a variety of reasons. See [AG], [AK], [BG], [CVV], [GGP], [GV], [HH1], [HH2], and [N].

Easy examples show that the degrees of the polynomials may be too crude a measure of the relation between the ideals $c(f)c(g)$ and $c(fg)$. In order to obtain a sharper form of (1) as well as a finer measure of comparison between $c(f)c(g)$ and $c(fg)$, the last two authors of the present paper introduced the *Dedekind–Mertens number* of a polynomial $g \in R[t]$ (see [HH2]). This number $\mu_R(g)$ is defined as the smallest positive integer k such that

$$c(fg)c(f)^{k-1} = c(f)^k c(g) \tag{2}$$

for every polynomial $f \in R[t]$.

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The relation between the minimal number of generators $\mu_R(c(g))$ of $c(g)$ and $\mu_R(g)$ is addressed in [HH2], and the main result of that paper states that

$$\mu_R(g) \leq \mu_R(c(g)) \tag{3}$$

(see [HH2, Theorem 2.1]). Since $\mu_R(c(g)) \leq \deg(g) + 1$, this statement implies the usual Dedekind–Mertens Lemma. In the same paper, the following question is raised (see [HH2, Question 1.3]): Let $(R, \mathfrak{m})$ be an excellent local domain, and let $g \in R[t]$. Is $\mu(g) = \mu(c(g))$? In the case $\mu(g) = 1$, the veracity of the above equality reduces to a question posed in the early sixties in the Ph.D. thesis of I. Kaplansky's student H.T. Tsang. An affirmative answer to the question of Tsang in a broad variety of cases (including all Noetherian domains) is given in two recent works on Gaussian polynomials: see [GV] and [HH1].

The main theorem in this paper proves the converse under one extra assumption on the dimension of the ring. Theorem 4.2 includes all of Theorem 1.1 and a bit more.

Theorem 1.1. *Let $(R, \mathfrak{m})$ be a universally catenary, analytically unramified Noetherian local ring. Suppose $g \in R[t]$ has Dedekind–Mertens number $\mu_R(g) = k$. Assume that $\dim(R/\mathfrak{p}) \geq k$ for all minimal primes $\mathfrak{p}$ of R. Then $\mu(c(g)) \leq k$. Therefore, $\mu_R(c(g)) = \mu_R(g)$.*

In Section 5 we prove that at least in certain examples, the assumption of Theorem 1.1 concerning the dimension is necessary. We give in Example 5.1 a polynomial over a one-dimensional complete local Gorenstein domain whose content ideal requires three generators, but whose Dedekind–Mertens number is two.

In order to prove Theorem 1.1, we introduce another version of the Dedekind–Mertens number which we call the polarized Dedekind–Mertens number. We define the *polarized Dedekind–Mertens number* $\widetilde{\mu}_R(g)$ of a polynomial $g \in R[t]$ to be the smallest positive integer k such that

$$\sum_{i=1}^{k} c(f_i g) c(f_1) \cdots \widehat{c(f_i)} \cdots c(f_k) = c(f_1) \cdots c(f_k) c(g) \tag{4}$$

for all polynomials $f_1, \ldots, f_k \in R[t]$, where $\widehat{c(f_i)}$ indicates the deletion of $c(f_i)$. We refer to (4) as the *polarized Dedekind–Mertens formula.* Clearly, (2) follows from (4) by choosing $f = f_1 = \ldots = f_k$; thus we have:

Remark 1.2. For every polynomial $g \in R[t]$, the Dedekind–Mertens number $\mu_R(g)$ is less than or equal to $\widetilde{\mu}_R(g)$, the polarized Dedekind–Mertens number of g.

It turns out that we rely heavily on the *a priori* stronger version of Dedekind–Mertens provided by this polarized form. We show in Theorem 2.5 that the polarized Dedekind–Mertens number $\widetilde{\mu}_R(g)$ is related to the minimal number of generators of $c(g)$ in the same way as the Dedekind–Mertens number $\mu_R(g)$, namely

$$\widetilde{\mu}_R(g) \leq \mu_R(c(g)).$$

This raises the issue of whether $\mu_R(g) = \widetilde{\mu}_R(g)$. In Theorem 2.8 we establish this equality under certain dimensionality restrictions.

The strategy for the proof of our main theorem is as follows: we first study the 0-dimensional local Gorenstein case in detail. To study the polarized form of the Dedekind–Mertens Lemma in such a ring, it is convenient to study its dual form, which effectively converts information about equality of ideals into linear equations. In particular the following theorem is an important step in this translation.

Theorem 3.1. *Let $(R, \mathfrak{m})$ be a local Artinian ring, let $\mathbf{s} = (0\colon \mathfrak{m})$ denote the socle of R, and let s be the dimension of $\mathbf{s}$ as a vector space over $R/\mathfrak{m}$. Suppose $g(t) \in R[t]$ is a polynomial of degree n and let $I = (0 :_R c(g))$. For any ideal J such that $I \subseteq J \subseteq (I\colon \mathfrak{m})$ and $\dim(J/I) = r > s$ and for any m such that $(m+1)r > (n+m+1)s$ there exists a polynomial $f(t)$ of degree m with the following properties:*

(*a*) $c(f) \subseteq J$;
(*b*) $fg = 0$;
(*c*) $c(f)c(g) \neq 0$.

After our study of 0-dimensional Gorenstein rings, we are ready to give the proof of the main result. We reduce to the 0-dimensional Gorenstein case by using the fact that the rings we are dealing with are *approximately Gorenstein*. This means that one of the following two equivalent conditions holds (see [Ho]):

(i) For every integer $n > 0$ there is an ideal $I \subseteq \mathfrak{m}^n$ such that R/I is Gorenstein.
(ii) For every integer $n > 0$ there is an $\mathfrak{m}$-primary irreducible ideal $I \subseteq \mathfrak{m}^n$.

The statement of Theorem 4.2 is the same as the one of Theorem 1.1 with the additional conclusion that under these hypotheses we have

$$\mu_R(c(g)) = \widetilde{\mu}_R(g) = \mu_R(g).$$

The restriction on the dimension of $R/\mathfrak{p}$ for minimal primes $\mathfrak{p}$ of R comes into play from the translation into linear equations which occurs in the 0-dimensional case. At this point, we need to ensure that we have more variables than equations, and after winding back to our original situation, we need to know there exist ideals $I_1, \ldots, I_k$ such that the number of minimal generators of the product of these ideals is sufficiently larger than the minimal number of generators of $\sum_i I_1 \cdots \widehat{I_i} \cdots I_k$. To prove this, we need to assume the dimension is sufficiently large. For example, over a 1-dimensional local ring, the number of generators of an arbitrary ideal is bounded by the multiplicity of the ring.

In Section 5 several classes of examples are developed over one-dimensional domains which show that additional assumptions are necessary in general to have the formula $\mu_R(g) = \mu(c(g))$.

2. Bounding the polarized Dedekind–Mertens number

Throughout the paper we use the integral closure of an ideal. Recall the following definition.

Definition 2.1. Given an ideal I of a ring R, an element $x \in R$ is in the *integral closure* $\overline{I}$ of I if x satisfies an equation of the form $x^k + a_1x^{k-1} + \cdots + a_k = 0$ where $a_i \in I^i$.

Remark 2.2. As mentioned above, it is well known that $c(f)c(g) \subseteq \overline{c(fg)}$, so the ideals $c(f)c(g)$ and $c(fg)$ have the same integral closure. For example, see [E].

Another observation we use is:

Remark 2.3. For every ideal I of a ring R and polynomial $g \in R[t]$ the Dedekind–Mertens number and polarized Dedekind–Mertens number of the image of g in $(R/I)[t]$ are less than or equal to the corresponding numbers associated to g.

Let R be a commutative ring and let S be a polynomial extension of R. A polynomial g over R can also be viewed as a polynomial over S. Clearly, one has that $\mu_R(g) \leq \mu_S(g)$. Lemma 2.4 relates $\widetilde{\mu}_R(g)$ to $\mu_S(g)$, where S is a suitable polynomial extension of R.

Lemma 2.4. *Let R be a commutative ring. If $g \in R[t]$ is such that for all polynomials $F \in S[t]$, with $S = R[x_1, \ldots, x_{k-1}]$ and $x_1, \ldots, x_{k-1}$ indeterminates, the equality*

$$c(Fg)c(F)^{k-1} = c(F)^k c(g) \tag{5}$$

holds, then $\widetilde{\mu}_R(g) \leq k$. In particular, if $\mu_S(g) = k$, then (5) holds, so $\widetilde{\mu}_R(g) \leq \mu_S(g)$.

Proof. For any $f_1, \ldots, f_k \in R[t]$ consider the polynomial

$$F = f_1 + \sum_{i=2}^{k} f_i x_{i-1} t^{N_{i-1}} \in R[x_1, \ldots, x_{k-1}][t]$$

where the N_i's are chosen recursively so that $N_1 > \max\{\deg_t(f_1), \deg_t(f_1 g)\}$ and

$$N_i > \max\{\deg_t(f_i), \deg_t(f_i g)\} + N_{i-1}, \quad \text{for} \quad i > 1.$$

By assumption the equality

$$c\left(\left(f_1 + \sum_{i=2}^{k} f_i x_{i-1} t^{N_{i-1}}\right) g\right) \left(c\left(f_1 + \sum_{i=2}^{k} f_i x_{i-1} t^{N_{i-1}}\right)\right)^{k-1} = \left(c\left(f_1 + \sum_{i=2}^{k} f_i x_{i-1} t^{N_{i-1}}\right)\right)^{k} c(g)$$

holds, and from our choice of the N_i's it follows that this is equivalent to

$$\left(c(f_1 g) + \sum_{i=2}^{k} c(f_i g) x_{i-1}\right) \left(c(f_1) + \sum_{i=2}^{k} c(f_i) x_{i-1}\right)^{k-1} = \left(c(f_1) + \sum_{i=2}^{k} c(f_i) x_{i-1}\right)^{k} c(g).$$

Finally, the comparison of the coefficient of the monomial $x_1 \cdots x_{k-1}$ in both terms of the previous identity gives the formula

$$\sum_{i=1}^{k} c(f_i g) c(f_1) \cdots \widehat{c(f_i)} \cdots c(f_k) = c(f_1) \cdots c(f_k) c(g)$$

and thus shows that $\widetilde{\mu}_R(g) \leq k$. □

It is shown in [HH2, Theorem 2.1] that the Dedekind–Mertens number $\mu(g)$ is bounded above by the number of generators needed to generate locally the content ideal of g. Applying this result to a polynomial extension ring of R yields the following consequence of Lemma 2.4.

Theorem 2.5. *For a polynomial $g \in R[t]$, the polarized Dedekind–Mertens number $\widetilde{\mu}_R(g)$ is bounded above by the number of local generators of the content ideal $c(g)$ of g, i.e., we have $\widetilde{\mu}_R(g) \leq \mu_R(c(g))$.*

Remark 2.6. Suppose R is a subring of a ring S. If S is faithfully flat over R and $g \in R[t] \subseteq S[t]$ is a polynomial, it is easily seen that $\mu_R(g) \leq \mu_S(g)$. Sometimes, however, this inequality is strict. For example, if $(R, \mathfrak{m})$ is a local Artinian ring that is not Gorenstein, then as noted in [HH1, Remark 1.6], there exists a polynomial $f \in R[t]$ such that $\mu_R(f) = 1$, while $\mu(c(f)) > 1$. It follows [HH1, Remark 1.7] that if S is a polynomial ring extension in two variables over R, then $\mu_S(f) > 1$.

Remark 2.7. We use in several places the following result of Rees [Re2, Theorem 2.1]: Suppose $(R, \mathfrak{m})$ is a formally equidimensional local ring of dimension d and $I = (a_1, \ldots, a_r)$ is an ideal of the principal class, i.e., $\dim(R/I) = d - r$. If $F(X_1, \ldots, X_r) \in R[X_1, \ldots, X_r]$ is a homogeneous polynomial such that $F(a_1, \ldots, a_r)$ is nilpotent, then the coefficients of F belong to the integral closure $\overline{I}$ of I.

We show in Theorem 2.8 the equality of the Dedekind–Mertens number and the polarized Dedekind–Mertens number under certain dimensionality restrictions.

Theorem 2.8. *Let $(R, \mathfrak{m})$ be a universally catenary, analytically unramified Noetherian local ring. Suppose $g \in R[t]$ has Dedekind–Mertens number $\mu_R(g) = k$. If $\dim(R/\mathfrak{p}) \geq k$ for each minimal prime $\mathfrak{p}$ of R, then $\widetilde{\mu}_R(g) \leq k$. Therefore $\widetilde{\mu}_R(g) = \mu_R(g)$.*

Proof. Let $f_1, \ldots, f_k$ be arbitrary polynomials in $R[t]$. For simplicity, we let $I = c(f_1) \cdots c(f_k)c(g)$ and $J = \sum_{i=1}^{k} c(f_i g)c(f_1) \cdots \widehat{c(f_i)} \cdots c(f_k)$.

By the Artin–Rees lemma, there exists an integer t so that $\mathfrak{m}^t \cap I \subseteq \mathfrak{m}I$. Also, by [Re1, Theorem 1.4] there exists an integer n with the property that $\overline{\mathfrak{m}^n} \subseteq \mathfrak{m}^t$, where $\overline{\mathfrak{m}^n}$ denotes the integral closure of $\mathfrak{m}^n$. Choose now $a_1, \ldots, a_k \in \mathfrak{m}^n$ so that for all minimal primes $\mathfrak{p}$ of R the height of $((a_1, \ldots, a_k)R + \mathfrak{p})/\mathfrak{p}$ is exactly k.

Consider the polynomial

$$f = f_1 a_1 + \sum_{i=2}^{k} f_i a_i t^{N_{i-1}} \in R[t]$$

where the N_i's are chosen recursively so that $N_1 > \max\{\deg_t(f_1), \deg_t(f_1 g)\}$ and

$$N_i > \max\{\deg_t(f_i), \deg_t(f_i g)\} + N_{i-1}, \quad \text{for} \quad i > 1.$$

By assumption we have $c(fg)c(f)^{k-1} = c(f)^k c(g)$, so the equality

$$c\left(\left(f_1 a_1 + \sum_{i=2}^{k} f_i a_i t^{N_{i-1}}\right) g\right) \left(c\left(f_1 a_1 + \sum_{i=2}^{k} f_i a_i t^{N_{i-1}}\right)\right)^{k-1} = \left(c\left(f_1 a_1 + \sum_{i=2}^{k} f_i a_i t^{N_{i-1}}\right)\right)^{k} c(g)$$

holds. From our choice of the N_i's, this is equivalent to

$$\left(c(f_1 g)a_1 + \sum_{i=2}^{k} c(f_i g)a_i\right) \left(c(f_1)a_1 + \sum_{i=2}^{k} c(f_i)a_i\right)^{k-1} = \left(c(f_1)a_1 + \sum_{i=2}^{k} c(f_i)a_i\right)^{k} c(g).$$

This last equality implies that if K is the ideal of R generated by the elements of the form $a_1^{e_1}\cdots a_k^{e_k}$, where the e_i are nonnegative integers such that $\sum_{i=1}^k e_i = k$ and not all $e_i = 1$, then

$$Ia_1\cdots a_k \subseteq (K, Ja_1\cdots a_k). \tag{6}$$

Equation (6) implies that for every $b \in I$ there exists $c \in J$ such that

$$a_1\cdots a_k(b-c) \in K. \tag{7}$$

Since R is universally catenary, $R/\mathfrak{p}$ is formally equidimensional for each minimal prime $\mathfrak{p}$ of R [Mat, Theorem 31.7]. Equation (7) implies the existence of a homogeneous polynomial $F(X_1,\dots,X_k) \in (R/\mathfrak{p})[X_1,\dots,X_k]$ of degree k having the image of $(b-c)$ in $R/\mathfrak{p}$ as the coefficient of $X_1\cdots X_k$ and having the property that $F(a_1,\dots,a_k) = 0$. Hence by the result of Rees in Remark 2.7, the image of $b-c$ in $R/\mathfrak{p}$ is integral over the image of $(a_1,\dots,a_k)$ in $R/\mathfrak{p}$ for every minimal prime $\mathfrak{p}$. Thus

$$b-c \in \overline{(a_1,\dots,a_k)} \cap I \subseteq \overline{\mathfrak{m}^n} \cap I \subseteq \mathfrak{m}^t \cap I \subseteq \mathfrak{m}I,$$

so that $I \subseteq J + \mathfrak{m}I$; hence by Nakayama's lemma, $I = J$. □

3. Zero-dimensional results

Theorem 3.1. *Let $(R,\mathfrak{m})$ be a local Artinian ring, let $\mathbf{s} = (0\colon \mathfrak{m})$ denote the socle of R, and let s be the dimension of $\mathbf{s}$ as a vector space over $R/\mathfrak{m}$. Suppose $g(t) \in R[t]$ is a polynomial of degree n and let $I = (0:_R c(g))$. For any ideal J such that $I \subseteq J \subseteq (I\colon \mathfrak{m})$ and $\dim(J/I) = r > s$ and for any m such that $(m+1)r > (n+m+1)s$, there exists a polynomial $f(t)$ of degree m with the following properties:*

(*a*) $c(f) \subseteq J$;
(*b*) $fg = 0$;
(*c*) $c(f)c(g) \neq 0$.

Proof. Let $g(t) = a_nt^n + a_{n-1}t^{n-1} + \ldots + a_1t + a_0 \in R[t]$. The condition for another polynomial $f(t) = b_mt^m + \ldots + b_1t + b_0$ that the product $fg = 0$ is clearly equivalent to a system of $n+m+1$ linear equations in the $m+1$ 'variables' $b_m,\dots,b_1,b_0$. More precisely, for each d with $0 \le d \le m+n$, we have

$$\sum_{i+j=d} a_ib_j = 0. \tag{8}$$

Let A denote the matrix of coefficients of the system of $n+m+1$ equations in $m+1$ variables defined by (8). Let $x_1,\dots x_r$ be the preimages in J of a basis for J/I over $R/\mathfrak{m}$, and let $y_1,\dots,y_s$ be a basis of $\mathbf{s}$. As $a_ix_j\mathfrak{m} = 0$ for any $0 \le i \le n$ and $1 \le j \le r$, there exist $c_{ijk} \in R$ such that

$$a_ix_j = \sum_{k=1}^{s} c_{ijk}y_k. \tag{9}$$

For indeterminates $\{z_{ij}\}_{0\le i\le m,\,1\le j\le r}$ over R, let $h(t) = w_mt^m + \cdots + w_1t + w_0$, where

$$w_i = z_{i1}x_1 + \cdots + z_{ir}x_r \qquad \text{for} \quad 0 \le i \le m. \tag{10}$$

The product $gh = 0$ is equivalent to the system of linear equations $A\vec{w} = \vec{0}$, where $\vec{w}$ is the transpose of $(w_m,\dots,w_0)$.

In general, for each d with $0 \le d \le m+n$, $\sum_{i+j=d} a_i w_j = 0$ implies, by (10),

$$\sum_{i+j=d} a_i(z_{j1}x_1 + \cdots + z_{jr}x_r) = 0,$$

which, by (9), implies

$$\left(\sum_{i+j=d} (c_{i1k}z_{j1} + \cdots + c_{irk}z_{jr})\right) y_k = 0, \quad 0 \le d \le m+n,\ 1 \le k \le s. \tag{11}$$

Since $y_1, \ldots, y_s$ form a basis for the socle $\mathbf{s}$ of R, to solve the system of equations (11) over R in the $(m+1)r$ variables $\{z_{ij}\}$ is equivalent to solving over the residue field of R the system

$$\sum_{i+j=d} (c'_{i1k}z_{j1} + \cdots + c'_{irk}z_{jr}) = 0, \qquad 0 \le d \le m+n, \quad 1 \le k \le s, \tag{12}$$

where c'_{ijk} denotes the image in $R/\mathfrak{m}$ of $c_{ijk} \in R$.

In (12) we have a system of $(m+n+1)s$ homogeneous linear equations over the field $R/\mathfrak{m}$ in the $(m+1)r$ variables $\{z_{ij}\}$. For any m such that $(m+1)r > (m+n+1)s$ the system (12) has a nontrivial solution $z_{ij} = e'_{ij}$. Let $e_{ij} \in R$ be a preimage of e'_{ij} for $0 \le i \le m, 1 \le j \le r$. Then for $f(t) = b_n t^n + \cdots + b_1 t + b_0$, where $b_i = e_{i1}x_1 + \cdots + e_{ir}x_r$, we have $fg = 0$ in $R[t]$. The linear independence of the images of the x_i in J/I implies that $c(f)$ is not contained in I. Hence $c(f)c(g) \ne 0$. □

Given $g(t) \in R[t]$ under what conditions do there exists polynomials f and h in $R[t]$ such that $c(fg)c(h) = 0 = c(hg)c(f)$ but $c(f)c(g)c(h) \ne 0$? Theorem 3.3 below answers this question in a special case.

Remark 3.2. Let R be a zero-dimensional local Gorenstein ring. An immediate consequence of duality is that for each ideal I of R the dimension of the socle of $R/(0\colon I)$ equals the minimal number of generators of I (see [BH, Proposition 3.2.12]).

Theorem 3.3. *Suppose $(R, \mathfrak{m})$ is a zero-dimensional local Gorenstein ring and let $g, f_1, \ldots, f_n \in R[t]$. Set*

$$A = c(f_1)\cdots c(f_n) \qquad \text{and} \qquad B = \sum_{i=1}^{n} c(f_i g)c(f_1)\cdots \widehat{c(f_i)} \cdots c(f_n)$$

where $\widehat{c(f_i)}$ indicates that $c(f_i)$ has been omitted. If

$$\dim(Ac(g)/(\mathfrak{m}Ac(g) + B)) \ge \mu(A) + 1, \tag{13}$$

then there exists an $h \in R[t]$ such that $Ac(h)c(g) \ne 0$ but $Bc(h) = 0 = c(hg)A$.

Proof. Let $J = 0\colon (\mathfrak{m}Ac(g) + B)$. Using duality and the inclusions

$$0\colon Ac(g) \subseteq J \subseteq 0\colon \mathfrak{m}Ac(g),$$

one concludes that

$$\dim(J/(0\colon Ac(g))) > \dim(\operatorname{soc}(R/(0\colon A))) = \mu(A).$$

By Theorem 3.1 applied to the ring $R/(0\colon A)$ there exists $h(t) \in R[t]$ such that $c(h) \subseteq J$ and $Ac(h)c(g) \ne 0$ but $c(hg)A = 0$. □

4. Minimal generators of the content ideal

We observe in Proposition 4.1 good behavior of the (polarized) Dedekind–Mertens number under passage from a Noetherian local ring to its completion.

Proposition 4.1. *Let $(R, \mathfrak{m})$ be a local Noetherian ring and let $(\widehat{R}, \widehat{\mathfrak{m}})$ denote the $\mathfrak{m}$-adic completion of R. For every polynomial $g(t) \in R[t]$ we have*

$$\mu_R(g) = \mu_{\widehat{R}}(g) \quad \text{and} \quad \widetilde{\mu}_R(g) = \widetilde{\mu}_{\widehat{R}}(g).$$

In other words, the (polarized) Dedekind–Mertens number of g over R equals the (polarized) Dedekind–Mertens number of g over $\widehat{R}$.

Proof. Let g^* denote the image of g in $(R/\mathfrak{m}^s)[t]$. Since all ideals of R are closed in the $\mathfrak{m}$-adic topology on R, $\mu_R(g) = \mu_{R/\mathfrak{m}^s}(g^*)$ and $\widetilde{\mu}_R(g) = \widetilde{\mu}_{R/\mathfrak{m}^s}(g^*)$ for sufficiently large s. Moreover, $R/\mathfrak{m}^s \cong \widehat{R}/\widehat{\mathfrak{m}}^s$ for each s, so we have

$$\mu_{R/\mathfrak{m}^s}(g^*) = \mu_{\widehat{R}/\widehat{\mathfrak{m}}^s}(g^*) \quad \text{and} \quad \widetilde{\mu}_{R/\mathfrak{m}^s}(g^*) = \widetilde{\mu}_{\widehat{R}/\widehat{\mathfrak{m}}^s}(g^*),$$

from which the assertions follow. □

Theorem 4.2. *Let $(R, \mathfrak{m})$ be a universally catenary, analytically unramified Noetherian local ring. Suppose $g \in R[t]$ has Dedekind–Mertens number $\mu_R(g) = k$. Assume that $\dim(R/\mathfrak{p}) \geq k$ for all minimal primes $\mathfrak{p}$ of R. Then $\mu(c(g)) \leq k$. Therefore, $\mu_R(c(g)) = \widetilde{\mu}_R(g) = \mu_R(g)$.*

Proof. By Theorem 2.8, the polarized Dedekind–Mertens number $\widetilde{\mu}_R(g) = k$, i.e., for all polynomials $f_1, \ldots, f_k \in R[t]$, we have

$$\sum_{i=1}^{k} c(f_i g) c(f_1) \cdots \widehat{c(f_i)} \cdots c(f_k) = c(f_1) \cdots c(f_k) c(g).$$

In view of Proposition 4.1, we may assume that R is complete. Suppose the ideal $c(g)$ is minimally generated by $z_1, \ldots, z_m$ with $m \geq k+1$.

By the Artin–Rees lemma, there exists an integer t so that $c(g) \cap \mathfrak{m}^t \subseteq \mathfrak{m}c(g)$. Also, by [Re1, Theorem 1.4] there exists an integer n with the property that $\overline{\mathfrak{m}^n} \subseteq \mathfrak{m}^t$.

Choose now $a, b_1, \ldots b_{k-1} \in \mathfrak{m}^n$ so that for all minimal primes $\mathfrak{p}$ of R the height of $(a, b_1, \ldots, b_{k-1})R + \mathfrak{p}/\mathfrak{p}$ is exactly k and consider the ideals

$$\prod_{i=1}^{k-1} (a, b_i)^{r_i}, \qquad c(g) \prod_{i=1}^{k-1} (a, b_i)^{r_i},$$

where the r_i's are nonnegative integers.

For simplicity we use $\mathbf{b}^{\mathbf{v}}$ for $b_1^{v_1} \cdots b_{k-1}^{v_{k-1}}$, $|\mathbf{v}|$ to denote $\sum_{j=1}^{k-1} v_j$, the length of $\mathbf{v}$, and P to denote $\sum_{j=1}^{k-1} r_j$.

We observe that the minimal number of generators of $\prod_{i=1}^{k-1}(a, b_i)^{r_i}$ is

$$\mu\left(\prod_{i=1}^{k-1} (a, b_i)^{r_i}\right) = \prod_{i=1}^{k-1} (r_i + 1).$$

Indeed, the ideal $\prod_{i=1}^{k-1}(a, b_i)^{r_i}$ is generated by elements of the form $a^u \mathbf{b}^{\mathbf{v}}$, where $u + |\mathbf{v}| = P$ and $0 \leq v_i \leq r_i$ for all $1 \leq i \leq k-1$. For every minimal prime $\mathfrak{p}$ of R, Remark 2.7 implies the images of these elements in $R/\mathfrak{p}$ form an irredundant generating set. Hence, these elements are irredundant generators in R. Moreover, these elements are in one-to-one correspondence with the k-tuples $(u, v_1, \dots, v_{k-1})$ of integers satisfying the above restrictions; an easy calculation shows that the number of such k-tuples is exactly $\prod_{i=1}^{k-1}(r_i+1)$.

We claim that the minimal number of generators of $c(g)\prod_{i=1}^{k-1}(a,b_i)^{r_i}$ is exactly

$$\mu\left(c(g)\prod_{i=1}^{k-1}(a,b_i)^{r_i}\right) = m\prod_{i=1}^{k-1}(r_i+1).$$

If not, there exists an element $z_i a^u \mathbf{b}^{\mathbf{v}}$, with $u+|\mathbf{v}|=P$, which can be written as a combination of the remaining ones, namely

$$z_i a^u \mathbf{b}^{\mathbf{v}} = z_i\left(\sum_{\substack{u'+|\mathbf{v}'|=P\\ \mathbf{v}\neq\mathbf{v}'}} q_{(i,u',\mathbf{v}')} a^{u'}\mathbf{b}^{\mathbf{v}'}\right) + \sum_{\substack{j=0\\ j\neq i}}^{m} z_j\left(\sum_{u''+|\mathbf{v}''|=P} q_{(j,u'',\mathbf{v}'')} a^{u''}\mathbf{b}^{\mathbf{v}''}\right). \tag{14}$$

An easy rearrangement of the terms in (14) yields

$$a^u\mathbf{b}^{\mathbf{v}}\left(z_i - \sum_{\substack{j=0\\ j\neq i}}^{m} q_{(j,u,\mathbf{v})} z_j\right) \in \left(a^l\mathbf{b}^{\mathbf{w}} \ :\ l+|\mathbf{w}|=P,\ l\neq u\right),$$

which, *a fortiori*, implies, by Remark 2.7, that

$$z_i - \sum_{\substack{j=0\\ j\neq i}}^{m} q_{(j,u,\mathbf{v})} z_j \in \overline{(a, b_1, \dots, b_{k-1})} \cap c(g). \tag{15}$$

However by our choice of n and t we have the inclusions

$$\overline{(a, b_1, \dots, b_{k-1})} \cap c(g) \subseteq \overline{\mathfrak{m}^n} \cap c(g) \subseteq \mathfrak{m}^t \cap c(g) \subseteq \mathfrak{m}c(g),$$

so that (15) contradicts the minimality of the generators of $c(g)$.

If we pick $r_1, \dots, r_{k-1}$ so that

$$\left(\sum_{i=1}^{k-1}\frac{1}{r_i+1}\right)^{-1} > N,$$

where N is the degree of g, then the integer

$$m\prod_{i=1}^{k-1}(r_i+1) - \sum_{i=1}^{k-1}(N+r_i+1)(r_1+1)\cdots\widehat{(r_i+1)}\cdots(r_{k-1}+1)$$

is strictly greater than $\prod_{i=1}^{k-1}(r_i+1)$.

To complete the proof of Theorem 4.2, we use that R is approximately Gorenstein; hence there exists an irreducible $\mathfrak{m}$-primary ideal Q of R such that the image of the ideal $c(g)\prod_{i=1}^{k-1}(a,b_i)^{r_i}$ in R/Q requires $m\prod_{i=1}^{k-1}(r_i+1)$ generators.

Let g^*, respectively f_i^*, denote the image of g, respectively of the polynomial

$$f_i = a^{r_i}t^{r_i} + a^{r_i-1}b_i t^{r_i-1} + \dots + ab_i^{r_i-1}t + b^{r_i},$$

in R/Q. Then $c(f_i^*)$ is the image of $(a, b_i)^{r_i}$ in R/Q and the inequality in (13) is satisfied with the n of Theorem 3.3 equal to $k-1$. Hence there exists a polynomial f_k^* in $(R/Q)[t]$ (or polynomial h in the notation of Theorem 3.3) such that

$$c(f_1^*)\cdots c(f_{k-1}^*)c(f_k^*)c(g^*) \neq 0$$

while

$$\left(\sum_{i=1}^{k-1} c(f_i^* g^*)c(f_1^*)\cdots \widehat{c(f_i^*)}\cdots c(f_{k-1}^*)\right) c(f_k^*) + c(f_k^* g^*)c(f_1^*)\cdots c(f_{k-1}^*) = 0.$$

This contradicts the fact that the polarized Dedekind–Mertens number of g^* is at most k (see Remark 2.3). Hence $\mu(c(g)) = m \leq k$.

By Remark 1.2 and Theorem 2.5, we have $\mu_R(c(g)) = \widetilde{\mu}_R(g) = \mu_R(g)$. □

Remark 4.3. It would be interesting to know if Theorem 4.2 holds more generally without the hypothesis that the ring is reduced. In the next section we present examples to show that an assumption on the dimension of the ring is necessary in Theorem 4.2.

5. One-dimensional examples

In the presentation of the examples of this section we use Remark 2.2 that if f and g are polynomials in $R[t]$, then the ideal $c(f)c(g)$ of R is integral over $c(fg)$.

Example 5.1. Let F be a field and let R be the Gorenstein subring $F[\![s^3, s^4]\!]$ of the power series ring $\overline{R} = F[\![s]\!]$. Consider the polynomial

$$g = s^7 + s^6 t + s^8 t^2 \in R[t].$$

We claim that $\mu_R(g) = \widetilde{\mu}_R(g) = 2$ while $\mu_R(c(g)) = 3$.

Proof. We use the following facts about the ring R.

(1) The integral closure $\overline{I}$ of an ideal I of R is $I\overline{R} \cap R$.
(2) If I is a nonzero ideal of R, then $I\overline{R} = s^n\overline{R}$ for some nonnegative integer n. We have $\mu_R(I) \leq 3$ and $\mu_R(I) = 3$ if and only if I also contains power series in s of order $n+1$ and $n+2$. In this case, I is integrally closed and $I = I\overline{R} = s^n\overline{R}$, i.e., I is also an ideal of $\overline{R}$, and $n \geq 6$.
(3) If $\mu_R(I) = 3$ and J is a nonzero ideal of R, then $\mu_R(IJ) = 3$, so IJ is integrally closed.

Statement (1) follows for example from [ZS, Theorem 1, page 350]. Since every nonzero ideal of $\overline{R}$ is of the form $s^n\overline{R}$, the first sentence of Statement (2) is clear. Since R is a free module of rank 3 over the principal ideal domain $F[\![s^3]\!]$, every nonzero ideal I of R is a free $F[\![s^3]\!]$-module of rank 3. Therefore $\mu_R(I) \leq 3$. Suppose $\mu_R(I) = 3$ and $I = (h_1, h_2, h_3)R$. If $I\overline{R} = s^n\overline{R}$, then at least one of the h_i has order n as a power series in s. We may assume h_1 has order n. There exist $a, b \in F \subset R$ such that $h_2' = h_2 - ah_1$ and $h_3' = h_3 - bh_1$ have order greater than n. We may assume that h_2' has order less than or equal to that of h_3', and by subtracting from h_3' a scalar multiple of h_2', we obtain $I = (h_1, h_2', h_3'')R$, where the order of h_2' is greater than n and the order of h_3'' is greater than the order of h_2'. Moreover, if the order of h_2' or h_3'' is $n+3$ or $n+4$, we can subtract a scalar multiple of s^3h_1 or s^4h_1 to get new generators of higher order. Thus we may assume h_2' and h_3'' have order different from $n+3$ and $n+4$.

Since R contains all power series in $\overline{R}$ of order greater than or equal to 6, the ideal h_1R contains all power series of order at least $n+6$. If the order of h'_2 and h''_3 are not $n+1$ and $n+2$, then the order of h''_3 is at least $n+5$. However, if the order of h''_3 is $n+5$, then by subtracting from h''_3 a suitable scalar multiple of $s^3h'_2$ or $s^4h'_2$, we obtain a new minimal generator of order at least $n+6$. This contradicts the fact that $\mu_R(I) = 3$. Therefore the order of h'_2 and h''_3 must be $n+1$ and $n+2$. Since R contains no power series of order 1, 2, or 5, if $\mu_R(I) = 3$, then I has order at least 6.

Conversely, if I has order n and contains power series of order $n+1$ and $n+2$, then I is minimally generated by any elements in I with these orders, and in view of the fact that I contains all power series of order at least $n+6$, we see that I contains all power series of order greater than or equal to n and hence $I = s^n\overline{R}$.

Statement (3) follows from the characterization of 3-generated ideals given in part (2).

To establish Example 5.1, we first show that $\mu_R(g) = 2$. Let f be an arbitrary polynomial in $R[t]$. If $c(f)$ requires three generators, then the equality $c(f)c(fg) = c(f)^2c(g)$ holds, as $c(f)c(fg)$ is integrally closed by (2) and $c(f)^2c(g)$ is integral over $c(f)c(fg)$. On the other hand, if $c(f)$ is principal, then the equality $c(f)c(fg) = c(f)^2c(g)$ also holds, as f is a Gaussian polynomial in the terminology of [GV], [HH1].

In order to complete the proof it remains to consider the case in which $\mu_R(c(f)) = 2$. We show also in this case that the product $c(f)c(fg)$ requires three generators and hence is integrally closed. Suppose first that $c(f)$ has order n and contains a power series of order $n+1$. Since $c(f)c(g)$ is integral over $c(fg)$, the ideal $c(fg)$ contains a power series of order $n+6$. Suppose $f = \sum a_it^i$ and let j be minimal such that a_j as a power series in s has order n or $n+1$. If a_j has order n, then $c(fg)$ contains a power series of order $n+7$. On the other hand, if a_j has order $n+1$, then $c(fg)$ contains a power series of order $n+8$. In either case, since $c(f)$ contains power series of order n and $n+1$ it follows that $c(f)c(fg)$ requires three generators and hence is integrally closed. The remaining possibility is that $c(f)$ has order n, contains a power series of order $n+2$, and does not contain a power series of order $n+1$. In this case[1], $c(fg)$ contains power series of order $n+6$ and $n+7$. It follows that $c(f)c(fg)$ contains power series of order $2n+6, 2n+7$, and $2n+8$ and therefore is integrally closed.

We show that the polarized Dedekind–Mertens number $\widetilde{\mu}_R(g) = 2$ as well. Let f and h be two arbitrary polynomials in $R[t]$. If $c(f)$ is principal, then $c(fg) = c(f)c(g)$, so $c(fg)c(h) = c(f)c(g)c(h)$. Similarly, if $c(h)$ is principal, we have $c(hg)c(f) = c(h)c(g)c(f)$. On the other hand, if $\mu_R(c(f)) = 3$, then $c(hg)c(f)$ is integrally closed and hence equal to $c(h)c(g)c(f)$. Similarly, if $\mu_R(c(h)) = 3$, we have $c(fg)c(h) = c(f)c(g)c(h)$. It remains to consider the case where $\mu_R(c(f)) = 2 = \mu_R(c(h))$. In this case both $c(fg)c(h)$ and $c(hg)c(f)$ are integrally closed, the argument being the same as that given in the paragraph above. □

Remark 5.2. We remark that the order of the coefficients of g in Example 5.1 is important. The polynomial $g' = s^6 + s^7t + s^8t^2$ has $c(g') = c(g)$, but $\mu_R(g') = 3$. To see this, consider the polynomial $f = s^6 - s^7t$. We have $fg' = s^{12} - s^{15}t^3$ so

[1] In this case the arrangement of the coefficients of g does not matter.

that $c(fg') = s^{12}R$. A direct calculation shows that

$$c(fg')c(f) = (s^{18}, s^{19})R \subsetneq (s^{18}, s^{19}, s^{20})R = c(f)^2c(g').$$

Setup 5.3. To generalize Example 5.1, let $(R, \mathfrak{m})$ be a one-dimensional local Noetherian domain such that the integral closure $(\overline{R}, \overline{\mathfrak{m}})$ of R is again local and is a finitely generated R-module. Assume that the canonical injection $R/\mathfrak{m} \hookrightarrow \overline{R}/\overline{\mathfrak{m}}$ is also surjective, and let $s \in \overline{\mathfrak{m}}$ be a generator for the maximal ideal of the DVR $\overline{R}$. Then $\mathfrak{m}\overline{R} = s^e\overline{R}$, where e is the multiplicity of R. Since we are assuming $\overline{R}$ to be a finitely generated R-module, the conductor of $\overline{R}$ to R is a nonzero ideal of $\overline{R}$, so it is of the form $s^c\overline{R}$ for some nonnegative integer c.

If I is a nonzero ideal of R, then $I\overline{R} = s^n\overline{R}$ for some nonnegative integer n. To better measure and compare ideals of R, we associate with I a subset $\gamma(I)$ of the positive integers less than e, defined as follows:

$$\gamma(I) = \{i \in \mathbb{N} \mid 1 \leq i < e \text{ and } \exists a \in I \text{ with } a\overline{R} = s^{n+i}\overline{R}\}.$$

In analogy with the observations made at the beginning of the proof of Example 5.1, we have the following facts about R.

(1) The integral closure $\overline{I}$ of an ideal I of R is $I\overline{R} \cap R$. A nonzero ideal I of R has $\gamma(I) = \{1, 2, \ldots, e-1\}$ if and only if $I = I\overline{R}$; in this case, I is integrally closed and is contained in the conductor of $\overline{R}$ to R.

(2) For every ideal I of R we have $\mu_R(I) \leq e$ and $\mu_R(I) \geq 1 + |\gamma(I)|$, where $|\gamma(I)|$ denotes the cardinality of $\gamma(I)$. A nonzero integrally closed ideal I of R contained in $s^c\overline{R}$ has $\gamma(I) = \{1, 2, \ldots, e-1\}$. In particular, every nonzero integrally closed ideal I of R that is contained in the conductor of $\overline{R}$ to R has[2] precisely $\mu_R(I) = e$.

(3) If I and J are nonzero ideals of R then $\gamma(IJ)$ contains $\gamma(I)$ and $\gamma(J)$ as well as all $i + j \leq e-1$ where $i \in \gamma(I)$ and $j \in \gamma(J)$. In particular, if $|\gamma(I)| = e-1$ and J is a nonzero ideal of R, then $|\gamma(IJ)| = e-1$, so IJ is integrally closed.

(4) Let $f, g \in R[t]$ be nonzero polynomials and let m be a positive integer less than e. If $\{1, \ldots, m\} \cap \gamma(c(f)) = \varnothing$ and $\{1, \ldots, m\} \subseteq \gamma(c(g))$, then $\{1, \ldots, m\} \subseteq \gamma(c(fg))$.

In view of (2), Theorem 2.5 implies that $\widetilde{\mu}_R(g) \leq e$ for every polynomial $g \in R[t]$.

General Example 5.4. With notation as in Setup 5.3, assume that R has multiplicity $e(R) = e \geq 3$, and let $s^c\overline{R}$ be the conductor of $\overline{R}$ to R. Specific examples for R are, for instance, the subrings of the formal power series ring $\overline{R} = F[\![s]\!]$ of the form $R = F[\![s^e, s^{e+1}]\!]$ or $R = F[\![s^e, s^{e+1}, \ldots, s^{2e-1}]\!]$. In the first case R is a complete intersection (therefore Gorenstein) and $c = (e-1)e$, while in the second case $c = e$ and R fails to be Gorenstein.

In analogy with Remark 5.2, consider the polynomial

$$g' = s^c + s^{c+1}t + s^{c+2}t^2 + \ldots + s^{c+e-1}t^{e-1} \in R[t].$$

Let $f = s^c - s^{c+1}t$. Then $fg' = s^{2c} - s^{2c+e}t^e$, and hence $c(fg') = s^{2c}R$. By Setup 5.3(1), we have $c(g') = s^c\overline{R}$ and by Setup 5.3(3) we have that $c(g')c(f)^k$ is

[2]In this more general setting, as contrasted with Example 5.1, there exist rings R with multiplicity e and nonintegrally closed ideals I of R with $\mu_R(I) = e$. For example, consider the subring $R = F[\![s^3, s^5]\!]$ of the formal power series ring $\overline{R} = F[\![s]\!]$. Then $I = (s^3, s^5)^3 = (s^9, s^{11}, s^{13})$ has $\mu_R(I) = 3$, but I is not integrally closed. Note that $\gamma(I) = \{2\}$.

integrally closed for each positive integer k. Therefore the smallest positive integer k for which

$$c(fg')c(f)^{k-1} = c(f)^k c(g')$$

is the smallest integer k such that $c(fg')c(f)^{k-1}$ is integrally closed. Since $c(fg') = s^{2c}R$ is principal in R, $c(fg')c(f)^{k-1}$ is integrally closed precisely when $c(f)^{k-1}$ is integrally closed. As $c(f)^{k-1}$ is generated by k elements, it is, in view of Setup 5.3(2), integrally closed only for $k \geq e$. Therefore $\mu_R(g') = e = \mu_R(c(g'))$.

On the other hand, the polynomial

$$g = s^{c+1} + s^c t + s^{c+2}t^2 + \ldots + s^{c+e-1}t^{e-1}$$

has $c(g) = c(g')$, but $\mu_R(g) \leq e-1 < e = \mu_R(c(g))$. To justify this assertion, let $f \in R[t]$ be a nonzero polynomial; the following two cases are possible: (a) $1 \notin \gamma(c(f))$; (b) $1 \in \gamma(c(f))$.

(a) Choose $1 \leq m \leq e-1$ maximal with the property that $\{1,\ldots,m\} \cap \gamma(c(f)) = \varnothing$. As $\gamma(c(g)) = \{1,\ldots,e-1\}$, by Setup 5.3(4) we have $\{1,\ldots,m\} \subseteq \gamma(c(fg))$. By Setup 5.3(3), $\gamma(c(fg)c(f)^{k-1})$ contains each of the integers $1,\ldots,km+k-1$ that is less than e. Hence, for k a positive integer such that $km+k-1 \geq e-1$, the ideal $c(fg)c(f)^{k-1}$ is integrally closed and therefore equal to $c(f)^k c(g)$. Choosing $k = \lceil \frac{e}{m+1} \rceil$ guarantees that $c(fg)c(f)^{k-1} = c(f)^k c(g)$. Finally, notice that

$$k = \left\lceil \frac{e}{m+1} \right\rceil < e.$$

If not, $\lceil \frac{e}{m+1} \rceil \geq e$ or, equivalently, $\frac{e}{m+1} > e-1$. But this last inequality yields the contradicting conclusion $1 > m(e-1)$.

(b) If $\gamma(c(f)) = \{1,\ldots,e-1\}$, then by Setup 5.3(4), $c(fg)c(f)$ is integrally closed. Hence $c(fg)c(f) = c(f)^2 c(g)$ in this case. In the case that remains, there exists a positive integer m less than $e-1$ such that $\{1,\ldots,m\} \subseteq \gamma(c(f))$ while $m+1 \notin \gamma(c(f))$.

The following fact that depends on the specific arrangement of the coefficients of g implies that $\{1,\ldots,m+1\} \subseteq \gamma(c(fg)c(f))$.

(b.1) Let $f \in R[t]$ be a nonzero polynomial and let m be a positive integer less than $e-1$. If $\{1,\ldots,m\} \subseteq \gamma(c(f))$ and $m+1 \notin \gamma(c(f))$, then $\{1,\ldots,m+1\} \cap \gamma(c(fg)) \neq \varnothing$.

By Setup 5.3(3), for an integer $k \geq 2$, $\gamma(c(fg)c(g)^{k-1})$ contains each of the integers $1,\ldots,(k-1)m+1$ that is less than e. Hence, for k a positive integer such that $(k-1)m+1 \geq e-1$, the ideal $c(fg)c(f)^{k-1}$ is integrally closed and therefore equal to $c(f)^k c(g)$. Choosing $k = \lceil \frac{e+m-2}{m} \rceil$ guarantees that $c(fg)c(f)^{k-1} = c(f)^k c(g)$. Finally, notice that

$$k = \left\lceil \frac{e+m-2}{m} \right\rceil < e.$$

If not, $\lceil \frac{e+m-2}{m} \rceil \geq e$ or, equivalently, $e+m-2 > (e-1)m$. But this last inequality yields the contradicting conclusion $2(m-1) > e(m-1)$.

Putting (a) and (b) together we conclude that $\mu_R(g) \leq e-1$, as claimed.

To show the polarized Dedekind–Mertens number $\widetilde{\mu}_R(g)$ is also at most $e-1$, we need to show for polynomials $f_1, \dots, f_{e-1} \in R[t]$ that we have

$$\sum_{i=1}^{e-1} c(f_i g)c(f_1)\cdots\widehat{c(f_i)}\cdots c(f_{e-1}) = c(f_1)\cdots c(f_{e-1})c(g). \tag{16}$$

Since $c(f_1)\cdots c(f_{e-1})c(g)$ is integral over $c(f_i g)c(f_1)\cdots\widehat{c(f_i)}\cdots c(f_{e-1})$ for each i, it suffices to show that one of these ideals is integrally closed in order to establish (16). To do this, we consider the following cases.

(1) Suppose $\gamma(c(f_i)) = \{1, \dots, e-1\}$ for some i. Then Setup 5.3(3) implies that the ideal $c(f_j g)c(f_1)\cdots\widehat{c(f_j)}\cdots c(f_{e-1})$ is integrally closed for every $j \neq i$, so (16) holds in this case. Hence we assume $|\gamma(c(f_i))| \leq e-2$ for each i. Applying either Setup 5.3(4) or (b.1), it follows that $\gamma(c(f_i g)) \neq \varnothing$ for each i.

(2) Suppose $\gamma(c(f_i)) = \varnothing$ for some i. Then Setup 5.3(4) implies that $|\gamma(c(f_i g))| = e-1$. By Setup 5.3(3), $c(f_i g)c(f_1)\cdots\widehat{c(f_i)}\cdots c(f_{e-1})$ is integrally closed, so (16) holds in this case. Hence we assume each $\gamma(c(f_i)) \neq \varnothing$.

(3) Suppose $1 \in \gamma(c(f_i))$ for every i, $1 \leq i \leq e-1$. In view of (1), Setup 5.3 implies that the ideal $c(f_1 g)c(f_2)\cdots c(f_{e-1}) = I$ has $|\gamma(I)| = e-1$ and hence is integrally closed. Therefore (16) holds in this case. Hence we assume $1 \notin \gamma(c(f_i))$ for some i.

(4) Let m be the positive integer maximal with the property that for every i, $1 \leq i \leq e-1$, there exists a positive integer $k_i \in \gamma(c(f_i))$ with $k_i \leq e-m$. In view of (2), there exists such an integer m and in view of (3), $m \leq e-2$. By the maximality of m, we have $\gamma(c(f_i)) \subseteq \{e-m, e-m+1, \cdots, e-1\}$ for some i. Setup 5.3(4) implies that $\gamma(c(f_i g))$ contains $\{1, \dots, e-m-1\}$. Since for each $j \neq i$, there exists a positive integer $k_j \in \gamma(c(f_j))$ with $k_j \leq e-m$, it follows from Setup 5.3 that the ideal $I = c(f_i g)c(f_1)\cdots\widehat{c(f_i)}\cdots c(f_{e-1})$ has $|\gamma(I)| = e-1$ and hence is integrally closed. Therefore (16) holds in general.

We conclude that $\widetilde{\mu}_R(g) \leq e-1$.

Remark 5.5. (1) For the polynomial g in General Example 5.4 and $f = s^c - s^{c+1}t$, a simple computation shows that $c(fg) = (s^{2c}, s^{2c+1}, s^{2c+2})R$. Since $\gamma(c(f)) = \{1\}$ and $\gamma(c(fg)) = \{1, 2\}$, for k a positive integer the ideal $c(fg)c(f)^{k-1}$ is integrally closed (and hence equal to $c(f)^k c(g)$) if and only if $k \geq e-2$. Hence the Dedekind–Mertens number $\mu_R(g)$, as well as the polarized Dedekind–Mertens number $\widetilde{\mu}_R(g)$, is either $e-1$ or $e-2$.

(2) If $e = 3$, then $\mu_R(g) = e-1 = 2 = \widetilde{\mu}_R(g)$; for otherwise g would be Gaussian, and it is known [HH1, Theorem 1.5] that over a Noetherian local domain a Gaussian polynomial has principal content ideal.

(3) To obtain an example where $\mu_R(g) < \mu_R(c(g))-1$, we modify the g of General Example 5.4 as follows: With R as in General Example 5.4 and $e = 5$, let

$$g = s^{c+1} + s^c t + s^{c+2}t2 + s^{c+4}t^3 + s^{c+3}t^4.$$

Then $\mu_R(g) \leq 3$, while $\mu_R(c(g)) = 5$. To see that $\mu_R(g) \leq 3$, we examine cases similar to what is done above. The 'new' case is where $\gamma(c(f)) = \{1\}$. In this case with our modified g, the set $\gamma(c(fg))$ contains either 1 or 2 and also contains either 3 or 4.

Remark 5.6. If R is a Noetherian domain of dimension at least 2, it would be interesting to know whether, in situations where the hypotheses of Theorem 4.2 are not satisfied, there exist polynomials g over R with $\mu(g) < \mu(c(g))$. For example, if R is the polynomial ring $k[x, y]$ with k a field and $g \in R[t]$ is a polynomial such that $(c(g)) = (x, y)^3$, could it happen that $\mu(g) \leq 3$?

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