MULTI–SEPARATION, CENTRIFUGALITY AND CENTRIPETALITY IMPLY CHAOS

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ABSTRACT. Let I be an interval. I need not be compact or bounded. Let $f : I \to I$ be a continuous map, and $(x_0, x_1, \cdots, x_n)$ be a trajectory of f with $x_n \leq x_0 < x_1$ or $x_1 < x_0 \leq x_n$. Then there is a point $v \in I$ such that $\min\{x_0, \cdots, x_n\} < v = f(v) < \max\{x_0, \cdots, x_n\}$. A point $y \in I$ is called a centripetal point of f relative to v if $y < f(y) < v$ or $v < f(y) < y$, and y is centrifugal if $f(y) < y < v$ or $v < y < f(y)$. In this paper we prove that if there exist k centripetal points of f in $\{x_0, \cdots, x_{n-1}\}, k \geq 1$, then f has periodic points of some odd ($\neq 1$) period $p \leq (n-2)/k+2$. In addition, we also prove that if $(x_0, x_1, \cdots, x_{n-1})$ is multi-separated by $\mathrm{Fix}(f)$, or there exists a centrifugal point of f in $\{x_0, \cdots, x_{n-1}\}$, then f is turbulent and hence has periodic points of all periods.

1. INTRODUCTION

Let I be a real interval, which need not be compact or bounded, and let $f : I \to I$ be a continuous map. A point $x \in I$ is called a *periodic point* of f *with period n* or simply an *n-periodic point* if $f^n(x) = x$ and $f^k(x) \neq x$ for $1 \leq k < n$. For some given positive integer n, does f have n-periodic points? This is an interesting problem. One expects to find some succinct conditions to decide whether f has n-periodic points. A well known result is the following theorem, due to A.N. Sarkovskii [8].

Theorem A. *Suppose f has m-periodic points. If $n \prec m$ in the sequence*

$$1 \prec 2 \prec 4 \prec 8 \prec \cdots \prec \cdots \prec 2^2 \cdot 7 \prec 2^2 \cdot 5 \prec 2^2 \cdot 3 \prec \cdots$$

$$\prec 2 \cdot 7 \prec 2 \cdot 5 \prec 2 \cdot 3 \prec \cdots \prec 7 \prec 5 \prec 3,$$

then f has n-periodic points.

T.Y. Li and J.A. Yorke in [7] proved that if f has 3-periodic points, then f has n-periodic points for every positive integer n, and f is chaotic. It is well known (see [4] or [5]) that f is also chaotic if it has a periodic point of a period which is not a power of 2.

Definition 1.1. A sequence $(x_0, x_1, \cdots, x_n)$ of points in I is called a *trajectory* of f if $x_{i+1} = f(x_i)$ for $i = 0, 1, \cdots, n-1$. A trajectory $(x_0, x_1, \cdots, x_n)$ is said to be *return* if $x_n \leq x_0 < x_1$ or $x_1 < x_0 \leq x_n$.

T.Y. Li et al. considered return trajectories and obtained the following two theorems (see Proposition 2.2 and Remark after it, and Proposition 3.4 in [6]).

Theorem B. *If f has a return trajectory $(x_0, x_1, \cdots, x_n)$, and $n \geq 3$ is odd, then f has n-periodic points.*

Theorem C. *If f has a return trajectory $(x_0, x_1, \cdots, x_n), n \geq 4$ is even, and there is no division for $(x_0, x_1, \cdots, x_n)$, then f has p-periodic points, where $p = n/2$ if $n/2$ is odd, or $p = n/2 + 1$ if $n/2$ is even.*

In this paper we will further discuss return trajectories. Our main result is the following theorem.

Theorem D. *Let $(x_0, x_1, \cdots, x_n)$ be a return trajectory of $f : I \to I$ with $n \geq 3$, and v be a fixed point of f in the interval $(\min\{x_0, \cdots, x_n\}, \max\{x_0, \cdots, x_n\})$.*

(i) *If $(x_0, x_1, \cdots, x_{n-1})$ is multi-separated by $\mathrm{Fix}(f)$, or there exists a centrifugal point of f relative to v in $\{x_0, \cdots, x_{n-1}\}$, then f is turbulent, and hence f has periodic points of all periods.*

(ii) *If there exist k centripetal points of f relative to v in $\{x_0, \cdots, x_{n-1}\}, k \geq 1$, then f has periodic points of some odd $(\neq 1)$ period $p \leq (n-2)/k + 2$.*

Remark 1.2. Taking $k = 1$ and $k = 2$ in Theorem D, we can obtain Theorem B and Theorem C respectively. Thus these two theorems are two particular situations of our conclusion.

2. Multi-separation implies chaos

In the following we still assume that $f : I \to I$ is a continuous map. Denote by $\mathrm{Fix}(f)$ the set of fixed points of f. For any real numbers $x < y$, put $[x; y] = [y; x] = [x, y], (x; y) = (y; x) = (x, y)$, and $[x; x] = [x, x] = \{x\}$.

Definition 2.1. Let $(x_0, x_1, \cdots, x_n)$ be a return trajectory of f. Take $m, M \in \{1, \cdots, n\}$ such that

$$x_m = \min\{x_0, x_1, \cdots, x_n\}, \quad x_M = \max\{x_0, x_1, \cdots, x_n\}.$$

Write $V = [x_m, x_M] \cap \mathrm{Fix}(f)$. Then $V \neq \emptyset$. Let $u = \min V, u' = \max V$. The trajectory $(x_0, x_1, \cdots, x_n)$ is said to be *multi-separated by* $\mathrm{Fix}(f)$ if $[u, u'] \cap \{x_0, x_1, \cdots, x_n\} \neq \emptyset$.

Recall that f is called *turbulent* if there exist $w, y, z \in I$ with $w < y < z$ such that $f([w, y]) \cap f([y, z]) \supset [w, z]$. The following proposition is well known (see [1] or [3]).

Proposition 2.2. *If f is turbulent, then f has periodic points of all periods.*

Theorem 2.3. *Let $(x_0, x_1, \cdots, x_n)$ be a return trajectory of f with $n \geq 2$. If $(x_0, x_1, \cdots, x_n)$ is multi-separated by $\mathrm{Fix}(f)$, then f is turbulent.*

Proof. We may discuss only the case $x_n \leq x_0 < x_1$. Let x_m, x_M and u, u' be as in Definition 2.1. Then $M < n$. Write

$$W = [x_m, x_{M-1}] \cap \mathrm{Fix}(f).$$

If $W \neq \emptyset$, then $x_{M-1} > x_m$, and there is $w \in W$ such that $w < x_{M-1}$ and $(w, x_{M-1}] \cap \mathrm{Fix}(f) = \emptyset$. Evidently, there is an $i \in \{0, \cdots, n-1\}$ such that $x_i > x_{M-1}$ and $x_{i+1} (= f(x_i)) \leq w$. Hence we have

$$f([w, x_{M-1}]) \cap f([x_{M-1}, x_i]) \supset [w, x_M] \supset [w, x_i].$$

This implies that f is turbulent.

Now we assume $W = \emptyset$. Then $[u, u'] \subset (x_{M-1}, x_M)$.

If $\{x_0, x_1, \cdots, x_M\} \cap [u, u'] \neq \emptyset$, then there is some $i \in \{0, 1, \cdots, M - 2\}$ such that $x_i \in (u, u')$ and $\{x_{i+1}, \cdots, x_M\} \cap [u, u'] = \emptyset$. If $x_{i+1} > u'$, then $i < M - 2$, and $\min(f(u', x_{i+1})) < u$, (otherwise we will get $x_{i+1} > x_{i+2} > \cdots > x_{M-2} > x_{M-1} > u'$, which will yield a contradiction). Thus we have

$$(2.1) \qquad f([u, x_i]) \cap f([x_i, x_{i+1}]) \supset [u, x_{i+1}], \quad \text{if} \ \ x_{i+1} > u'.$$

Similarly, we have

$$(2.2) \qquad f([x_{i+1}, x_i]) \cap f([x_i, u']) \supset [x_{i+1}, u'], \quad \text{if} \ \ x_{i+1} < u.$$

From (2.1) and (2.2) we see that f is turbulent.

If $\{x_0, x_1, \cdots, x_M\} \cap [u, u'] = \emptyset$, then $x_n \leq x_0 < u$ since $(x_0, x_M) \cap \text{Fix}(f) \neq \emptyset$, and $\{x_{M+1}, \cdots, x_{n-1}\} \cap (u, u') = \{x_{M+1}, \cdots, x_{n-1}\} \cap [u, u'] \neq \emptyset$ since $(x_0, x_1, \cdots, x_n)$ is multi-separated by $\text{Fix}(f)$. Take $i \in \{M+1, \cdots, n-1\}$ such that $x_i \in (u, u')$ and $\{x_{i+1}, \cdots, x_n\} \cap [u, u'] = \emptyset$. Then it is easy to check that (2.1) or (2.2) is still true, and hence f is also turbulent. Theorem 2.3 is proven.   □

By Definition 2.1, a return trajectory $(x_0, x_1, \cdots, x_n)$ must be multi-separated by $\text{Fix}(f)$ if $\{x_0, x_1, \cdots, x_n\} \cap \text{Fix}(f) \neq \emptyset$. Thus, from Theorem 2.3 we obtain the following corollary.

Corollary 2.4. *Let $(x_0, x_1, \cdots, x_n)$ be a return trajectory of $f, n \geq 2$. If $\{x_0, x_1, \cdots, x_n\} \cap \text{Fix}(f) \neq \emptyset$, then f is turbulent.*   □

3. Centripetal–centrifugal structure

In this section we will raise the concepts of centripetal point and centrifugal point. We will show that, under some conditions, a return trajectory can be "homotopically" changed to a periodic trajectory which has the same centripetal-centrifugal structure.

Definition 3.1. Let v be a fixed point of $f : I \to I$. A point $y \in I$ is called a *centripetal point of f relative to v* (or simply a centripetal point if there is no confusion) if $y < f(y) < v$ or $v < f(y) < y$. y is called a *centrifugal point of f (relative to v)* if $f(y) < y < v$ or $v < y < f(y)$. And y is called a *striding point* of f if $y < v < f(y)$ or $f(y) < v < y$.

Definition 3.2. Let $(x_0, x_1, \cdots, x_n)$ and $(y_0, y_1, \cdots, y_n)$ be two trajectories of $f : I \to I$. Suppose v is a given fixed point of f. We say that *$(x_0, x_1, \cdots, x_n)$ and $(y_0, y_1, \cdots, y_n)$ have the same centripetal-centrifugal structure (relative to v)*, and write

$$(x_0, x_1, \cdots, x_n) \sim (y_0, y_1, \cdots, y_n), \quad (\text{rel} \ \ v)$$

if the following two conditions hold:

(i)   For $i = 0, 1, \cdots, n$, $\ y_i < v$ if and only if $x_i < v$, $\ $ and $y_i > v$ if and only if $x_i > v$.

(ii)   For $i = 0, 1, \cdots, n - 1$, $\ y_i < y_{i+1}$ if and only if $x_i < x_{i+1}$, $\ $ and $y_i > y_{i+1}$ if and only if $x_i > x_{i+1}$.

Remark 3.3. We define the number of centripetal (resp. centrifugal, resp. striding) points of trajectory $(x_0, x_1, \cdots, x_n)$ as the cardinal number of the set $\{i : x_i$ is a centripetal (resp. centrifugal, resp. striding) point, and $0 \leq i \leq n - 1\}$. Obviously,

if $(x_0, x_1, \cdots, x_n)$ and $(y_0, y_1, \cdots, y_n)$ have the same centripetal-centrifugal structure, then the number of centripetal (resp. centrifugal, resp. striding) points of $(x_0, x_1, \cdots, x_n)$ is equal to that of $(y_0, y_1, \cdots, y_n)$.

Theorem 3.4. *Let $(x_0, x_1, \cdots, x_n)$ be a return trajectory of f, $n \geq 2$, and $V = [x_m, x_M] \cap \mathrm{Fix}(f)$ be the same as in Definition 2.1. Let $v \in V$ be given. If f is not turbulent, and $(x_0, x_1, \cdots, x_n)$ contains at least a centripetal point (relative to v), then there exists a return trajectory $(y_0, y_1, \cdots, y_n)$ of f which has the same centripetal-centrifugal structure as $(x_0, x_1, \cdots, x_n)$ and satisfies $y_n = y_0$.*

Proof. We may consider only the case $x_n < x_0 < x_1$. Since f is not turbulent, by Theorem 2.3, $(x_0, x_1, \cdots, x_n)$ is not multi-separated by $\mathrm{Fix}(f)$. Thus $V \subset (x_0, x_M)$. Put $u = \min V$. Because $f(t) > t$ for all $t \in [x_m, u)$, there is a point $z \in [x_0, u)$ such that $f(z) > u$, (otherwise, we will have $x_0 < x_1 \leq x_2 \leq \cdots \leq x_M \leq \cdots \leq x_n \leq u$, which would yield a contradiction). Write $t_i = f^i(t)$ for any $t \in I$ and any $i \geq 0$. Then t_i continuously depends on t. Since $f^n(x_0) = x_n < x_0$ and $f^n(u) = u$, there exists a point $y = y_0 \in (x_0, u]$ such that $y_n = y_0$ and

$$(3.1) \qquad t_n < t_0 = t < y \leq u \leq v, \quad \text{for any } t \in [x_0, y).$$

From (3.1) and Corollary 2.4 it follows that

$$(3.2) \qquad \{t_0, t_1, \cdots, t_n\} \cap \mathrm{Fix}(f) = \emptyset, \quad \text{for any } t \in [x_0, y).$$

By (3.2) it is easy to show

$$(3.3) \qquad (t_0, t_1, \cdots, t_n) \sim (x_0, x_1, \cdots, x_n), \ (\mathrm{rel} \ v), \ \text{for any } t \in [x_0, y).$$

We now claim $y < u$. In fact, if $x_{n-1} < x_n$, then from the continuity of $f^{n-1}|[x_0, u]$ we see that there exists $w \in (x_0, u)$ such that $w_{n-1} = z$ and $w_n = f(z) > u$, and by (3.1) we get $y < w < u$. If $x_{n-1} \geq x_n$, then there is an $i \in \{0, 1, \cdots, n-2\}$ such that x_i is a centripetal point, i.e. $x_{i+1} \in (v; x_i)$. Since $(x_0, x_1, \cdots, x_n)$ is not multi-separated by $\mathrm{Fix}(f)$, we have $\{x_i, x_{i+1}\} \cap [u, v] = \emptyset$, and hence $x_{i+1} \in (u; x_i)$. Since $f^i(u) = u$ and $f^i(x_0) = x_i$, there exists $r \in (x_0, u)$ such that $r_i(= f^i(r)) = x_{i+1}$, and hence

$$r_j = x_{j+1} \quad \text{for } j = i, i+1, \cdots, n-1.$$

If $r < y$, then by (3.3) we will obtain

$$(r_0, \cdots, r_n) = (r_0, \cdots, r_{i-1}, x_{i+1}, x_{i+2}, \cdots, x_n, r_n)$$

$$(3.4) \qquad\qquad \sim (x_0, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_{n-1}, x_n), \quad (\mathrm{rel} \ v).$$

According to (3.4), x_i being centripetal implies that $x_{i+1}, x_{i+2}, \cdots, x_{n-1}$ are all centripetal. This contradicts the assumption $x_{n-1} \geq x_n$. Thus we still have $y \leq r < u$.

From $y_n = y_0 = y < u$ we know that $(y_0, y_1, \cdots, y_n)$ is also a return trajectory. By Corollary 2.4 we obtain

$$(3.5) \qquad\qquad \{y_0, y_1, \cdots, y_n\} \cap \mathrm{Fix}(f) = \emptyset.$$

Analogous to (3.3), from (3.2) and (3.5) we get

$$(3.6) \qquad (y_0, y_1, \cdots, y_n) \sim (x_0, x_1, \cdots, x_n), \quad (\mathrm{rel} \ v).$$

Theorem 3.4 is proven.   □

Remark 3.5. Suppose $(x_0, x_1, \cdots, x_n), (y_0, y_1, \cdots, y_n)$ and $t_i = f^i(t)$ are the same as in Theorem 3.4 and its proof. Let

$$S = \{(t_0, t_1, \cdots, t_n) : t \in [x_0, y_0]\}$$

Then S is a set of trajectories satisfying (3.3) and (3.6). We can regard S as a "homotopy" from $(x_0, x_1, \cdots, x_n)$ to $(y_0, y_1, \cdots, y_n)$ preserving the centripetal-centrifugal structure.

4. Centrifugality and centripetality imply chaos

Let $(x_0, x_1, \cdots, x_n)$ be a return trajectory of continuous map $f : I \to I$. If there is no division for $(x_0, x_1, \cdots, x_n)$, then it is proved in [6] that f is chaotic, see Theorems B and C stated above. However, if we further consider the number of centripetal and centrifugal points, we can obtain stronger results.

We first consider centrifugal points. The following proposition is slightly stronger than Corollary 3.2 in [6]. Using our Theorem 2.3, we can give a short proof of this proposition.

Proposition 4.1. *Let $(x_0, x_1, \cdots, x_n)$ be a return trajectory of f, and $V = [x_m, x_M]$ $\cap \operatorname{Fix}(f)$ be as in Definition 2.1. Let $v \in V$ be given. If there exists a centrifugal point of f relative to v in $\{x_0, x_1, \cdots, x_{n-1}\}$, then f is turbulent.*

Proof. We may assume $x_n \leq x_0 < x_1$. Let x_i be a centrifugal point of f, where $i \in \{0, 1, \cdots, n-1\}$.

If $x_i > v$, then there is a fixed point w of f in (x_i, x_M), and $x_i \in (v, w)$.

If $x_0 > v$, then there is a fixed point w of f in (x_0, x_M), and $x_0 \in (v, w)$.

If $x_i < v$ and $x_0 < v$, then there is a fixed point w of f in $(x_0; x_i)$, and $\{x_0, x_i\} \cap (w, v) \neq \emptyset$.

Thus, in any case, the trajectory $(x_0, x_1, \cdots, x_n)$ is always multi-separated by $\operatorname{Fix}(f)$. By Theorem 2.3, f is turbulent. $\square$

Theorem 4.2. *Let $(x_0, x_1, \cdots, x_n)$ be a return trajectory of f, and let $V = [x_m, x_M]$ $\cap \operatorname{Fix}(f)$ be as in Definition 2.1. Let $v \in V$ be given. If the trajectory $(x_0, x_1, \cdots, x_n)$ contains k centripetal points relative to v, (i.e., the cardinal number of the set $\{j : x_j$ is a centripetal point, and $0 \leq j \leq n-1\}$ is k), $k \geq 1$, then f has periodic points of some odd $(\neq 1)$ period $p \leq (n-2)/k + 2$.*

Proof. By Proposition 4.1 and Theorem 2.3, we may assume that there is no centrifugal point of f in $\{x_0, x_1, \cdots, x_{n-1}\}$, and the trajectory $(x_0, x_1, \cdots, x_n)$ is not multi-separated by $\operatorname{Fix}(f)$. By Theorem 3.4, we may assume $x_n = x_0$. Then the number of striding points of the trajectory $(x_0, x_1, \cdots, x_n)$ is $n - k, n - k \geq 2$ is even, and

$$(4.1) \qquad f(x_i) > x_i, \quad \text{if } x_i < v, \ i \in \{0, 1, \cdots, n\};$$

$$(4.2) \qquad f(x_i) < x_i, \quad \text{if } x_i > v, \ i \in \{0, 1, \cdots, n\}.$$

If $k = 1$, then n is odd, and Theorem 4.2 is true. Now we assume $k_0 \geq 2$ is a given integer, and Theorem 4.2 holds for $1 \leq k < k_0$. We need only to prove that the theorem still holds for $k = k_0$.

Suppose the period of $x_0(= x_n)$ under f is n'. Then n' is a factor of n, and $n' \geq 3$. Suppose there are k' centripetal points in the trajectory $(x_0, x_1, \cdots, x_{n'})$. Then $k' = kn'/n$. If $n' < n$, then by the inductive hypothesis f has periodic points

of some odd ($\neq 1$) period $p \leq (n' - 2)/k' + 2$. Since $(n' - 2)/k' < (n - 2)/k$, we have $p < (n - 2)/k + 2$, and hence Theorem 4.2 holds.

Now we assume $n' = n$, i.e., the period of x_0 under f is n. Let the k centripetal points in $\{x_0, x_1, \cdots, x_{n-1}\}$ be $x_{a(1)}, x_{a(2)}, \cdots, x_{a(k)}$ with

$$x_{a(1)} < x_{a(2)} < \cdots < x_{a(k)}.$$

Write $O = \{x_0, x_1, \cdots, x_{n-1}\}$. If $x_{a(1)} < v < x_{a(k)}$, noting that any nonempty proper subset of O is not an invariant set of f, from (4.1) and (4.2) we see that at least one of the following two inequalities is true:

(4.3) $\max(f(O \cap [x_{a(1)}, v])) \geq x_{a(k)},$

or

(4.4) $\min(f(O \cap [v, x_{a(k)}])) \leq x_{a(1)}.$

By symmetry, we may assume that (4.3) is true. It follows from (4.3) that

(4.5) $f([x_{a(1)}, v]) \supset [v, x_{a(k)}].$

If $x_{a(1)} < v < x_{a(k)}$ does not hold, then $x_{a(1)} < x_{a(k)} < v$ or $v < x_{a(1)} < x_{a(k)}$. By symmetry, we may assume $x_{a(1)} < v$.

In addition, for convenience, we may assume $a(1) = 0$, i.e. $x_{a(1)} = x_0$. Then $x_0 < x_1 < v$.

For any integer j and any $i \in \{0, 1, \cdots, n - 1\}$, write $x_{jn+i} = x_i$. Since $x_0 = \min\{x_{a(1)}, \cdots, x_{a(k)}\} < v$, we see that x_{-1} is a striding point, and $x_{-1} > v$. For any real number r, let $\lfloor r \rfloor$ denote the greatest integer not greater than r. Put

$$b = \lfloor (n - 2)/k \rfloor.$$

There are three cases to consider.

Case 1. There exists $q \in \{2, 3, \cdots, b\}$ such that x_{-q} is centripetal,   and $x_{-1}, x_{-2}, \cdots, x_{-q+1}$ are all striding.

Subcase 1.1. If $x_{-q} < v$, then $x_0 < x_{-q} < x_{-q+1} < v$. Since the trajectory $(x_{-q}, x_{-q+1}, \cdots, x_{-1}, x_0\}$ is return and has a centripetal point x_{-q}, by the inductive hypothesis, f has periodic points of some odd ($\neq 1$) period $p \leq q \leq b < (n-2)/k+2$.

Subcase 1.2. If $x_{-q} > v$, then $x_{a(1)} = x_0 < v < x_{-q+1} < x_{-q} \leq x_{a(k)}$. By (4.5), we can take $y_{-q-1} \in [x_0, v) \cap f^{-1}(x_{-q})$ and obtain a return trajectory $(y_{-q-1}, x_{-q}, x_{-q+1}, \cdots, x_{-1}, x_0\}$, which contains a centripetal point x_{-q}. By the inductive hypothesis, f has periodic points of some odd ($\neq 1$) period $p \leq q + 1 \leq b + 1 < (n - 2)/k + 2$.

Case 2. $x_{-1}, x_{-2}, \cdots, x_{-b}$ are all striding points, and $x_{-b} < v$.

Subcase 2.1. If $x_1 < x_{-b} < v$, then the trajectory $(x_{-b}, x_{-b+1}, \cdots, x_0, x_1)$ is return, which contains a centripetal point x_0. By the inductive hypothesis, f has periodic points of some odd ($\neq 1$) period $p \leq b + 1 < (n - 2)/k + 2$.

Subcase 2.2. If $x_{-b} < x_1 < v$, then the trajectory $(x_1, x_2, \cdots, x_{n-b})$ is return, which contains $k-1$ centripetal points $x_{a(2)}, \cdots, x_{a(k)}$. By the inductive hypothesis, f has periodic points of some odd ($\neq 1$) period $p \leq (n - b - 3)/(k - 1) + 2$. Since $b = \lfloor (n - 2)/k \rfloor > (n - 2)/k - 1$ and $k \geq 2$, we have

$$n - b - 3 < n - \frac{n - 2}{k} - 2 = \frac{k - 1}{k} \cdot (n - 2),$$

which yields

$$\frac{n-b-3}{k-1} < \frac{n-2}{k}.$$

Thus $p < (n-2)/k + 2$.

Case 3. $x_{-1}, x_{-2}, \cdots, x_{-b}$ are all striding points, and $x_{-b} > v$.
   Similar to (4.3) and (4.4), in this case we have

$$\max(f(O \cap [x_1, v])) \geq x_{-b}$$

or

$$\min(f(O \cap [v, x_{-b}])) \leq x_1,$$

which yields

(4.6) $$f([x_1, v]) \supset [v, x_{-b}]$$

or

(4.7) $$f([v, x_{-b}]) \supset [x_1, v].$$

Subcase 3.1. If (4.6) holds, then we can take $y_{-b-1} \in [x_1, v) \cap f^{-1}(x_{-b})$ and obtain a return trajectory $(y_{-b-1}, x_{-b}, x_{-b+1}, \cdots, x_0, x_1)$, which contains a centripetal point x_0. By the inductive hypothesis, f has periodic points of some odd ($\neq 1$) period $p \leq b + 2$.

Subcase 3.2. If (4.7) holds, then we can take $y_0 \in (v, x_{-b}] \cap f^{-1}(x_1)$ and obtain a return trajectory $(y_0, x_1, x_2, \cdots, x_{n-b})$, which contains $k - 1$ centripetal points $x_{a(2)}, \cdots, x_{a(k)}$. By the inductive hypothesis, f has periodic points of some odd ($\neq 1$) period $p \leq (n - b - 2)/(k - 1) + 2$. Suppose $b = (n - 2 - j)/k$, where $j \in \{0, 1, \cdots, k - 1\}$. Then

$$\frac{n-b-2}{k-1} = b + \frac{j}{k-1} \leq b + 1.$$

   If b is odd, then the odd number $p \leq (n - b - 2)/(k - 1) + 2 \leq b + 3$ must satisfy $p \leq b + 2$.

   If b is even, then $n - j = kb + 2$ is even, and $k - j = (n - j) - (n - k)$ is also even. Thus $j \leq k - 2$, and $\lfloor (n - b - 2)/(k - 1) \rfloor = \lfloor b + j/(k - 1) \rfloor = b$. This also implies $p \leq b + 2$.

   Theorem 4.2 is proven.   □

   Combining Theorem 2.3, Proposition 4.1 and Theorem 4.2, we obtain Theorem D given in Section 1.

Remark 4.3. Let $(x_0, x_1, \cdots, x_n)$ be a return trajectory of f, let x_m and x_M be the same as in Definition 2.1, and let $v \in [x_m, x_M] \cap \mathrm{Fix}(f)$ be given. By Corollary 2.4, we only consider the case $\{x_0, x_1, \cdots, x_n\} \cap \mathrm{Fix}(f) = \emptyset$. Suppose the number of centripetal points with centrifugal points of $(x_0, x_1, \cdots, x_n)$ relative to v is k. Then $n - k$ is even, and hence we have
   (i) $k \geq 1$ is odd if n is odd;
   (ii) $k \geq 2$ is even if n is even and there is no division for $(x_0, x_1, \cdots, x_n)$.
Therefore, Theorems B and C stated in Section 1 are two special situations of Theorem D.

   As a corollary of Proposition 4.1 and Theorem 4.2, we have

Theorem 4.4. *Let $(x_0, x_1, \cdots, x_n)$ be a trajectory of $f, n \geq 3$. Suppose there is some $k \in \{2, 3, \cdots, n-1\}$ such that*

$$x_n \leq x_0 < x_1 < x_2 < \cdots < x_k \quad or \quad x_k < \cdots < x_2 < x_1 < x_0 \leq x_n.$$

Then f has periodic points of some odd $(\neq 1)$ period p, which satisfies

(i) $p \leq (n-2)/k + 2$ *if $n - k$ is even;*
(ii) $p \leq (n-2)/(k-1) + 2$ *if $n - k$ is odd.*

Proof. We may consider only the case $x_n \leq x_0 < x_1 < x_2 < \cdots < x_k$. Let $V = [x_m, x_M] \cap \text{Fix}(f)$ be as in Definition 2.1. Take $v \in V$. If the trajectory $(x_0, x_1, \cdots, x_n)$ contains a centrifugal point of f relative to v, then, by Proposition 4.1, f is turbulent. If $(x_0, x_1, \cdots, x_n)$ contains no centrifugal point, then $v \in (x_{k-1}, x_M)$, and the trajectory $(x_0, x_1, \cdots, x_n)$ contains at least k (resp. $k-1$) centripetal points if $n - k$ is even (resp. odd). Thus, by Theorem 4.2, f has periodic points of some odd (> 1) period p satisfying the condition (i) (resp. (ii)). $\square$

Now we give two examples to show that under the assumptions of Theorem 4.2 and 4.4 we cannot obtain stronger results.

Examples. Let $a \geq 1$ and $k \geq c \geq 1$ be integers, $v = (a+1)k + c$, and $n = (2a+1)k + 2c$. Then $(n-2)/k + 2 < 2a + 5$. Write $I = [0, n]$. Take continuous maps $f : I \to I$ and $g : I \to I$ such that $f|[j-1, j]$ and $g|[j-1, j]$ are monotone for $j = 1, 2, \cdots, n$, and

$$\begin{cases} f(v-i) = v+i, & \text{for} \quad i = 0, 1, \cdots, ak+c; \\ f(v+i) = v-1-i, & \text{for} \quad i = 1, 2, \cdots, c-1; \\ f(v+i) = v-k-i, & \text{for} \quad i = c, \cdots, ak+c; \\ f(i) = ak+i+1, & \text{for} \quad i = 0, 1, \cdots, k-2; \\ f(k-1) = v-1, & \end{cases}$$

and

$$\begin{cases} g(n) = 0, \\ g(j(a+1)) = (j+1)(a+1), & \text{for} \quad j = 0, 1, \cdots, k-1; \\ g(k(a+1)) = v+1; \\ g(v+i) = v-i, & \text{for} \quad i = 0, 1, \cdots, c-1; \\ g(v-i) = v+i+1, & \text{for} \quad i = 1, 2, \cdots, c-1; \end{cases}$$

$$\begin{cases} g(v+c+ja+i-1) = v-c-j(a+1)-i, \\ \qquad\qquad \text{for} \quad j = 0, \cdots, k-1 \quad \text{and} \quad i = 1, \cdots, a; \\ g(v-c-j(a+1)-i) = v+c+ja+i, \\ \qquad\qquad \text{for} \quad j = 0, \cdots, k-1 \quad \text{and} \quad i = 1, \cdots, a. \end{cases}$$

Put $O = \{0, 1, \cdots, n\} - \{v\}$. Then O is an n-periodic orbit both of f and of g. Since the return trajectory $(0, f(0), \cdots, f^n(0))$ has k centripetal points, which are $0, 1, \cdots, k-1$, and since $g^n(0) = 0 < g(0) < g^2(0) < \cdots < g^{k-1}(0) < g^k(0)$, by Theorems 4.2 and 4.4, we see that both f and g have $(2a+3)$-periodic points. On the other hand, by using the method of [2], it is easy to show that neither f nor g has $(2a+1)$-periodic points.

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