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SCRAMBLED SETS OF CONTINUOUS MAPS OF 1-DIMENSIONAL POLYHEDRA

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Abstract. Let K be a 1-dimensional simplicial complex in R^3 without isolated vertexes, $X = |K|$ be the polyhedron of K with the metric d_K induced by K, and $f : X \to X$ be a continuous map. In this paper we prove that if K is finite, then the interior of every scrambled set of f in X is empty. We also show that if K is an infinite complex, then there exist continuous maps from X to itself having scrambled sets with nonempty interiors, and if $X = R$ or R_+, then there exist C^∞ maps of X with the whole space X being a scrambled set.

1. Introduction

Chaotic behavior is a manifestation of the complexity of nonlinear dynamical systems. There are some distinct definitions given by different authors. The following definition of chaos mainly stems from Li and Yorke [11].

Definition 1.1. Let (X, d) be a metric space, and $f : X \to X$ be a continuous map. A subset S of X containing at least two points is called a *scrambled set* of f if for any $x, y \in S$ with $x \neq y$,

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0, \tag{1.1}$$

and

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0. \tag{1.2}$$

f is said to be *chaotic* (*in the sense of Li and Yorke*) if f has an uncountable scrambled set.

Remark 1.2. Let $P(f)$ denote the set of all periodic points of f. In Definition 1.1, we do not insist that

$$\limsup_{n\to\infty} d(f^n(x), f^n(p)) > 0 \tag{1.3}$$

holds for any $x \in S$ and any $p \in P(f)$ because condition (1.3) is not important. In fact, if (1.1) and (1.2) hold for any $x, y \in S$ with $x \neq y$, then the set

$$\{x \in S : \limsup_{n\to\infty} d(f^n(x), f^n(p)) = 0 \ \text{ for some } \ p \in P(f)\} \tag{1.4}$$

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contains at most one point. Also we do not insist that $S \cap P(f) = \emptyset$ because if (1.2) holds, then $S \cap P(f)$ also contains at most one point (see (ii) of Lemma 2.1 below).

Let I be a compact interval. For the case $X = I$, Li and Yorke in [11] first showed that if $f : I \to I$ has a periodic point of period 3, then it is chaotic, i.e. "period three implies chaos". Kuchta and Smital in [9] indicated that if $f : I \to I$ has a two point scrambled set, then it has an uncountable scrambled set. In [5]–[8], [12], [13] and [15], scrambled sets of some maps were further discussed from the point of view of measure.

In this paper we will consider the case of X being a 1-dimensional polyhedron. The tree is a particular kind of 1-dimensional polyhedron. In [1], [2], [3] and [10], the sets of periods of periodic orbits and the topological entropies of tree maps were discussed. Now we will study scrambled sets of continuous maps for general 1-dimensional polyhedra from the point of view of topology. Our main result is the following theorem.

Theorem A. *Let K be a finite 1-dimensional simplicial complex in R^3 without isolated vertexes, and let $X = |K|$ be the polyhedron of K. Suppose $f : X \to X$ is a continuous map. Then the interior of any scrambled set of f in X is empty.*

In addition, we will show that if K is an infinite 1-dimensional complex, then Theorem A is not true. Particularly, if K is a triangulation of R or R_+, i.e., if X is the real line R or the real half-line R_+, then there exist C^∞ maps from X to itself with the whole space X being a scrambled set.

2. Some elementary properties of scrambled sets

Let (X, d) be a metric space, and $f : X \to X$ be a continuous map. A point $x \in X$ is called an *eventually periodic point* of f if there are integers $n > m \geq 0$ such that $f^n(x) = f^m(x)$. If $m = 0$, i.e. $f^n(x) = x$, then x is called a *periodic point.*

The following lemma will be useful, of which the proof is easy and is omitted.

Lemma 2.1. *Let S be a scrambled set of $f : X \to X$. Then*

(i) *$f|S$ is an injection.*
(ii) *There is at most one eventually periodic point of f in S.*
(iii) *For any integer $n \geq 0$, $f^n(S)$ is also a scrambled set of f.*
(iv) *Let $S' \subset X$ contain at least two points. If $f(S') \subset S$ and $f|S'$ is injective, then S' is also a scrambled set of f.*
(v) *If f is uniformly continuous, then S is also a scrambled set of f^n for any integer $n > 0$.*

Definition 2.2. Let (X, d) and (X', d') be two metric spaces, and $h : X \to X'$ be a homeomorphism. h is called a *uniform homeomorphism* if both h and h^{-1} are uniformly continuous. X and X' are said to be *uniformly homeomorphic* if there exists a uniform homeomorphism $h : X \to X'$.

Obviously, every homeomorphism between two compact metric spaces must be a uniform homeomorphism, and we have

Lemma 2.3. *Let $h : X \to Y$ be a uniform homeomorphism, and $S \subset X$, $T = h(S)$. Suppose $f : X \to X$ is a continuous map, and $g = h \circ f \circ h^{-1}$. Then T is a scrambled set of g if and only if S is a scrambled set of f.*

3. Scrambled sets of continuous maps of 1-dimensional polyhedra

Let K be a 1-dimensional simplicial complex in R^3. Every 0-dimensional simplex of K is called a *vertex*, and every 1-dimensional simplex of K is called an *edge*. Denote by K_0 the set of all vertexes of K. Let $X = |K|$ be the polyhedron of K (see [4]). Define the metric d_K on X as follows:

(i) If points x and y lie on the same edge E, the two vertexes of E are u and v, and $x = ru+(1-r)v, y = su+(1-s)v$ for some $r, s \in [0, 1]$, then $d_K(x, y) = |r-s|$.

(ii) If there is a connected subcomplex K' of K such that $\{x, y\} \subset |K'|$, then

$$d_K(x,y) = \min\Big\{ \sum_{i=1}^{n} d_K(x_{i-1}, x_i) : (x_0, x_1, ..., x_n) \text{ is a sequence of points in } X \text{ with } x_0 = x,\ x_n = y, \text{ and } x_{i-1} \text{ and } x_i \text{ lying on the same edge of } K \text{ for } i = 1, ..., n\Big\}.$$

(iii) If there is no connected subcomplex K' of K such that $\{x, y\} \subset |K'|$, then $d_K(x, y) = \infty$.

Remark 3.1. In order to avoid that the case $d_K(x, y) = \infty$ arises, we can give another metric d'_K on X by

$$d'_K(x,y) = \begin{cases} \operatorname{arctg} d_K(x,y), & \text{if } d_K(x,y) < \infty; \\ 1, & \text{if } d_K(x,y) = \infty. \end{cases}$$

However, it is easy to see that the identical map $id : (X, d_K) \to (X, d'_K)$ is a uniform homeomorphism. Thus, for convenience, we use d_K rather than d'_K.

Remark 3.2. If for any bounded subset B of R^3, the number of the simplexes of K intersecting B is finite, then the topology on X induced by d_K coincides with that as subspace of the Euclidean space R^3.

Recall that an *arc* is a space homeomorphic to the unit interval [0,1]. Let $A \subset X$ be an arc. Denote by ∂A the two endpoints of A, and write $\overset{\circ}{A} = A - \partial A$. Let x and y be two points on arc A. Denote by $A[x, y]$ the subarc of A from x to y. If $A[x, y]$ is a straight line segment, then it is simply written as $[x, y]$, and put $(x, y] = [x, y] - \{x\}, (x, y) = (x, y] - \{y\}$. Let u and v be the two endpoints of A. We denote by $(A; u, v)$ the directed arc A from u to v. In addition, we denote by $l(A)$ the length of arc A under metric d_K.

Lemma 3.3. *Let $(A; u, v)$ be a directed arc on $X = |K|$. Suppose $\{u, v\} \cap K_0 = \emptyset$. Then there is a unique sequence $(w_0, w_1, \cdots, w_n)$ of vertexes of K with $n \geq 1$ satisfying the following four conditions:*

(i) *For $i = 1, 2, \cdots, n$, $[w_{i-1}, w_i]$ is an edge of K.*

(ii) *$u \in [w_0, w_1)$, $v \in (w_{n-1}, w_n]$.*

(iii) *If $n = 1$, then $u \in [w_0, v]$ and $A = [u, v]$. If $n > 1$, then*

$$A = [u, w_1] \cup (\bigcup_{i=2}^{n-1} [w_{i-1}, w_i]) \cup [w_{n-1}, v].$$

(iv) *$w_i \neq w_j$ for $1 \leq i < j \leq n-1$.*

Lemma 3.3 is evident. The sequence $(w_0, w_1, \cdots, w_n)$ in Lemma 3.3 will be called the *carrier sequence* of the directed arc $(A; u, v)$ and we write $\mathrm{CS}(A; u, v) = (w_0, w_1, \cdots, w_n)$.

Theorem A. *Let K be a finite 1-dimensional simplicial complex in R^3 without isolated vertexes, and let $X = |K|$ be the polyhedron of K. Suppose $f : X \to X$ is a continuous map. Then the interior of any scrambled set of f in X is empty.*

Proof. If not, there is a scrambled set S of f having a nonempty interior in X. Then S contains an arc L. By (ii) of Lemma 2.1, we may assume that L contains no eventually periodic points of f. By (i) of Lemma 2.1, $f^k(L)$ is also an arc in $X(k = 1, 2, \cdots)$. Let the two endpoints of L be x' and y'. By Definition 1.1, we have $\limsup_{k\to\infty} d_k(f^k(x'), f^k(y')) > 0$. Thus $\sum_{k=0}^{\infty} l(f^k(L)) = \infty$. This implies that $f^\mu(L) \cap f^m(L) \neq \emptyset$ for some integers $\mu > m \geq 0$. Write $A' = f^m(L)$. Take $u, v \in A'$ such that $v = f^{\mu-m}(u)$ and $l(A'[u, v])$ achieves the minimum. Let $A = A'[u, v]$, and $g = f^{\mu-m}$. By (iii) and (v) of Lemma 2.1 we know that $A \subset A'$ is a scrambled set of g. Write $A_k = g^k(A)$ and $u_k = g^k(u)$ for $k = 0, 1, \cdots$. Then $v = g(u) = u_1$. It follows from (i) of Lemma 2.1 that A_k is an arc and the two endpoints of A_k are u_k and u_{k+1}. Since $A \cap P(g) = \emptyset$, we have

$$A_i \not\subset A_j, \quad \text{for any nonnegative integers } i \neq j. \tag{3.1}$$

From (3.1) we get the following

Claim 1. Let $k \geq 0$. If there exist edges E and E' of K such that $A_k \subset E$ and $A_{k+1} \subset E'$, then $A_k \cap A_{k+1} = \{u_{k+1}\}$, and $E = E'$ if $u_{k+1} \notin K_0$.

Since K_0 is a finite set and A contains no eventually periodic points of g, there is a $k_0 \geq 0$ such that $u_k \notin K_0$ for all $k \geq k_0$. Noting $\limsup_{k\to\infty} l(A_k) > 0$, by (3.1) and Claim 1 we have

Claim 2. Write $Z_0 = \{k : k \geq k_0 + 2, \text{ and } A_k \cap K_0 \neq \emptyset\}$. Then Z_0 is an infinite set.

By (iv) of Lemma 3.3, the number of carrier sequences of all directed arcs in X is finite. Hence there exist integers a and $b \in Z_0$ with $|a - b| \geq 3$ such that

$$\mathrm{CS}(A_a; u_a, u_{a+1}) = \mathrm{CS}(A_b; u_b, u_{b+1}), \tag{3.2}$$

$$\mathrm{CS}(A_{a+1}; u_{a+1}, u_{a+2}) = \mathrm{CS}(A_{b+1}; u_{b+1}, u_{b+2}). \tag{3.3}$$

Suppose the carrier sequence $\mathrm{CS}(A_a; u_a, u_{a+1})$ is $(v_0, v_1, \cdots, v_n)$. Then $n \geq 2$. By (3.2), we have $u_a \in (v_0, u_b)$ or $u_b \in (v_0, u_a)$. By the symmetry, we may assume that

$$u_a \in (v_0, u_b). \tag{3.4}$$

It follows from (3.4), (3.2) and (3.1) that $u_b \in (u_a, v_1)$, and $u_{a+1} \in (v_{n-1}, u_{b+1})$, $u_{b+1} \in (u_{a+1}, v_n)$. We now claim

$$g([u_a, u_b]) = [u_{a+1}, u_{b+1}]. \tag{3.5}$$

In fact, if (3.5) does not hold, then $g([u_a, u_b])$ is an arc in X with endpoints u_{a+1} and u_{b+1} which does not intersect (u_{a+1}, u_{b+1}). Noting $A_{a+1} = g(A_a) = g([u_a, u_b]) \cup g(A_a[u_b, u_{a+1}])$ is an arc and $g|A_a$ is injective, we have

$$g(A_a[u_b, u_{a+1}]) \subset [u_{b+1}, u_{a+1}), \quad \text{and} \quad u_{a+2} \in (u_{b+1}, u_{a+1}). \tag{3.6}$$

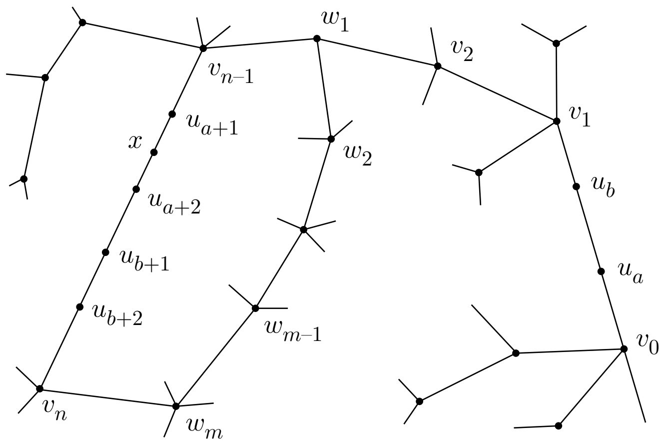


FIGURE 3.1

By (3.6) and (3.3) we get

$$\begin{aligned}\mathrm{CS}(g([u_a,u_b]);u_{a+1},u_{b+1}) &= \mathrm{CS}(A_{a+1};u_{a+1},u_{a+2})\\ &= \mathrm{CS}(A_{b+1};u_{b+1},u_{b+2}) = \mathrm{CS}(g([u_{a+1},u_{b+1}]);u_{a+2},u_{b+2})\\ &= (v_n,v_{n-1},w_1,\cdots,w_m,v_n,v_{n-1}), \qquad \text{for some } \{w_1,\cdots,w_m\}\subset K_0.\end{aligned}$$

(see Fig. 3.1). This implies that there is a point $x \in (u_{a+1}, u_{b+1})$ such that $g([u_{a+1}, x]) = [u_{a+2}, v_{n-1}]$ and $g(x) = v_{n-1}$, and hence there is a fixed point p of g in (u_{a+1}, x). However, $(u_{a+1}, x) \subset [u_{a+1}, u_{b+1}] \subset A_{b+1}$, which contains no fixed points of g. This leads to a contradiction. Thus (3.5) must hold.

From (3.5) and (i) of Lemma 2.1 it is easy to see that, for sufficiently small $\varepsilon > 0$, $g(u_b + \varepsilon(v_1 - u_b)) \in (u_{b+1}, v_n) - A_b$. Thus we have $u_b + \varepsilon(v_1 - u_b) \notin A_{b-1}$ and hence

$$[u_a, u_b] \subset A_{b-1}. \tag{3.7}$$

There are two cases to consider:

Case 1. $b > a$. In this case, let $Q = \bigcup_{k=a}^{b} A_k$. By (3.5) and (3.7) we can easily verify that $g(Q) = Q$. Let $S^1 = \{e^{2\pi it} : t \in R\}$ be the unit circle in the complex plane, and d be the usual metric on S^1. Take a sequence $t_a < t_{a+1} < \cdots < t_b$ of real numbers such that $t_{b-1} < t_a + 1 < t_b < t_{a+1} + 1$. Put $z_k = e^{2\pi it_k}$ for $k = a, a+1, \cdots, b$. Let $C_k = \{e^{2\pi it} : t_k \le t \le t_{k+1}\}$ for $k = a+1, \cdots, b-2, b-1$, and let $C_b = \{e^{2\pi it} : t_b \le t \le t_{a+1} + 1\}$ (see Fig. 3.2). Then C_j is an arc on S^1 $(j = a+1, \cdots, b)$. For $k = a+1, \cdots, b-1$, choose a homeomorphism $h_k : C_k \to A_k$ such that $h_k(z_k) = u_k$, $h_k(z_{k+1}) = u_{k+1}$, and $h_{b-1}(z_a) = u_a$. Choose again a homeomorphism $h_b : C_b \to A_b[u_b, u_{a+1}]$ such that $h_b(z_b) = u_b$, $h_b(z_{a+1}) = u_{a+1}$. Define a projection $h : S^1 \to Q$ by $h|C_k = h_k$ for $k = a+1, \cdots, b$. Then h is continuous, and $h(S^1) = Q$. Define $\varphi : S^1 \to S^1$ by

$$\begin{aligned}&\varphi|C_k = h_{k+1}^{-1} \circ g \circ h_k, \qquad \text{for } k = a+1, \cdots, b-2;\\ &\varphi|C_{b-1}[z_{b-1}, z_a] = h_b^{-1} \circ g \circ h_{b-1}|C_{b-1}[z_{b-1}, z_a];\\ &\varphi|C_{b-1}[z_a, z_b] = h_{a+1}^{-1} \circ g \circ h_{b-1}|C_{b-1}[z_a, z_b];\\ &\varphi|C_b = h_{a+1}^{-1} \circ g \circ h_b.\end{aligned}$$

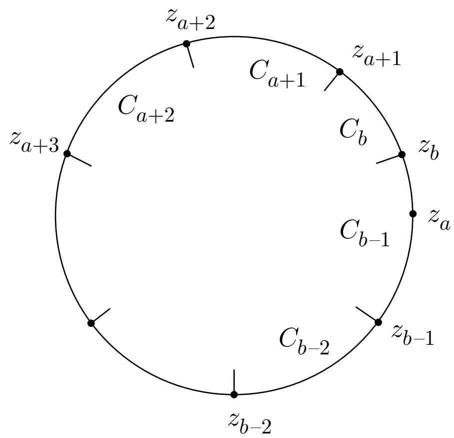


FIGURE 3.2

Then φ is also continuous, and $h \circ \varphi = g \circ h$. We say that φ is the lift of $g|Q$ relative to the projection h, or relative to the sequence $(h_{a+1}, h_{a+2}, \cdots, h_b)$ of homeomorphisms. Note that $\varphi : S^1 \to S^1$ is both injective and surjective. Thus φ is a homeomorphism.

φ has no periodic points because $g|Q$ has no periodic points and $h \circ \varphi = g \circ h$. This implies that the rotation number of φ is irrational. If φ has wandering points, then we know (for example, see [14, Chap.1]) that the wandering set $W(\varphi)$ of φ is an open set dense in S^1. Let z and w be two different points on the same connected component of $W(\varphi) \cap C_{a+1}$. Then $\lim_{k\to\infty} d(\varphi^k(z), \varphi^k(w)) = 0$ (see [14]). Since S^1 is compact, $h : S^1 \to Q$ is uniformly continuous. Thus

$$\lim_{k\to\infty} d_K(g^k(h(z)), g^k(h(w))) = \lim_{k\to\infty} d_K(h(\varphi^k(z)), h(\varphi^k(w))) = 0.$$

This implies that the points $h(z)$ and $h(w)(\neq h(z))$ of A_{a+1} can not lie in the same scrambled set of g. However, as indicated above, A is a scrambled set of g, and hence $A_{a+1} = g^{a+1}(A)$ is also a scrambled set. This reduces to a contradiction.

Therefore, φ has no wandering points. Thus φ is topologically conjugate to an irrational rotation of S^1, i.e. there exist an irrational number c and an orientation preserving homeomorphism $\eta : S^1 \to S^1$ such that

$$\eta^{-1}\varphi\eta(e^{2\pi i t}) = e^{2\pi i(t+c)}, \quad \text{for any } \ t \in R. \tag{3.8}$$

Let $\psi = \eta^{-1}\varphi\eta : S^1 \to S^1$, and $\xi = h\eta : S^1 \to Q$. Then ξ is a continuous surjection, and

$$\xi \circ \psi = g \circ \xi. \tag{3.9}$$

For any given positive number r, if there exists $s \in R$ such that $\xi(e^{2\pi i(s+r)}) = \xi(e^{2\pi i s})$, then it follows from (3.8) and (3.9) that

$$\begin{aligned} \xi(e^{2\pi i(s+kc+r)}) &= \xi\psi^k(e^{2\pi i(s+r)}) = g^k\xi(e^{2\pi i(s+r)}) \\ &= g^k\xi(e^{2\pi i s}) = \xi\psi^k(e^{2\pi i s}) = \xi(e^{2\pi i(s+kc)}) \end{aligned} \tag{3.10}$$

holds for all $k = 0, 1, \cdots$. Since the point set $\{e^{2\pi i(s+kc)} : k = 0, 1, \cdots\}$ is dense in S^1, by (3.10) and the continuity of ξ we have

$$\xi(e^{2\pi i(t+r)}) = \xi(e^{2\pi it}), \qquad \text{for any} \quad t \in R. \tag{3.11}$$

Let $T = \{r : r \in (0, 1]$, and there exists $s = s(r) \in R$ such that $\xi(e^{2\pi i(s+r)}) = \xi(e^{2\pi is})\}$. Then T is evidently a nonempty closed set in $(0, 1]$. Let $r_0 = \inf T$. Then $r_0 > 0$ since $\xi|\eta^{-1}(C_{a+1})$ is a homeomorphism from $\eta^{-1}(C_{a+1})$ to $A_{a+1} \subset Q$. It is easy to see that there is an integer $q \geq 1$ such that $r_0 = 1/q$. By (3.11) and the definition of r_0 we know that, for any $t, t' \in R$, $\xi(e^{2\pi it}) = \xi(e^{2\pi it'})$ if and only if $q(t' - t)$ is an integer. Thus we can define $\zeta : S^1 \to Q$ by

$$\zeta(e^{2\pi it}) = \xi(e^{2\pi it/q}), \qquad \text{for any} \quad t \in R.$$

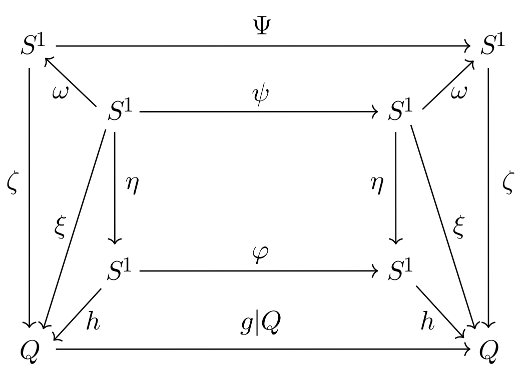
Obviously, this ζ is injective, surjective and continuous. Hence ζ is a homeomorphism. Define $\omega : S^1 \to S^1$ by

$$\omega(e^{2\pi it}) = e^{2\pi iqt}, \qquad \text{for any} \quad t \in R.$$

Then $\zeta \circ \omega = \xi$. Define $\Psi : S^1 \to S^1$ by

$$\Psi(e^{2\pi it}) = e^{2\pi i(t+qc)}, \qquad \text{for any} \quad t \in R.$$

Then $\Psi \circ \omega = \omega \circ \psi$. Therefore, we have the following commutative diagram.



Thus $\zeta\Psi\omega = \zeta\omega\psi = \xi\psi = g\xi = g\zeta\omega$, and hence $\zeta \circ \Psi = g \circ \zeta$ because ω is a surjection. This implies that $g|Q$ and Ψ are topologically conjugate. Since the irrational rotation Ψ has no scrambled set, by Lemma 2.3, $g|Q$ also has no scrambled set. However, as indicated above, $A_{a+1} (\subset Q)$ is a scrambled set of g. This is still a contradiction.

Case 2. $a > b$. Analogous to Case 1, Case 2 also leads to a contradiction.

Thus, the interior of any scrambled set of f in X must be empty. Theorem A is proven. □

If complex K is not finite, then Theorem A is not true. In fact, we have

Theorem 3.1. *Let K be an infinite 1-dimensional simplicial complex in R^3, and let $X = |K|$ be the polyhedron of K with the metric d_K. Then there exists a continuous map $f : X \to X$ which has a scrambled set containing a nonempty interior in X.*

Proof. If K has isolated vertexes, take an edge E of K, and let $f_0 : E \to E$ be a continuous map which has an uncountable scrambled set S_0. Suppose the set of isolated vertexes of K is V. Choose a continuous map $g : X \to E$ such that $g(x) = x$ for all $x \in E, g(V) \subset S_0$, and $g|V$ is injective. Let $f = f_0 \circ g$, and let $S = S_0 \cup V - g(V)$. Then S is an uncountable scrambled set of f, the interior of S in X contains V, and is nonempty.

Now we assume that K has no isolated vertexes. Then K has infinitely many edges. Take countably infinitely many edges $E_0, E_1, E_2, \cdots$ of K with $E_i \neq E_j$ for $i \neq j$ such that one of the following two conditions holds:

(C.1) If K has infinitely many connected components, then for any $0 \leq i < j < \infty$, E_i and E_j belong to different components of K.

(C.2) If K has only finitely many connected components, then all of $E_0, E_1, E_2, \cdots$ belong to the same component of K.

For $n = 0, 1, 2, \cdots$, suppose $\partial E_n = \{v_n, w_n\}$. Let $x_n = (2v_n + w_n)/3, y_n = (v_n + 2w_n)/3$, and $A_n = [x_n, y_n]$. Denote by J the open interval $(0, 1)$. For any rational number $r \in J$, write $u_n(r) = (1 - r)x_n + ry_n$. Put

$$W = \{(a, b, r, s) : a, b, r, s \text{ are all rational numbers, and } 0 < a < b < 1,\ 0 < r < s < 1\}.$$

Then W is a countable set in $J^4 (\subset R^4)$. Arrange all points in W to be an infinite sequence. Assume the sequence is

$$W = \{(a_n, b_n, r_n, s_n) : n = 0, 1, 2, \cdots\}.$$

For $n = 1, 2, 3, \cdots$, choose a homeomorphism $g_n : A_0 \to A_n$ such that $g_n(x_0) = x_n, g_n(y_0) = y_n, g_n(u_0(a_n)) = u_n(r_n)$, and $g_n(u_0(b_n)) = u_n(s_n)$, and then define the homeomorphism $h_n : A_n \to A_{n+1}$ by $h_n = g_{n+1} \circ g_n^{-1}$. Let $h_0 = g_1 : A_0 \to A_1$. Put $X_0 = \bigcup_{n=0}^{\infty} A_n$. Define $f_0 : X_0 \to X_0$ by $f_0|A_n = h_n$ for $n = 0, 1, 2, \cdots$. Then it is easy to see that $\overset{\circ}{A}_0$ is a scrambled set of f_0.

If condition (C.1) holds, suppose the connected component of K containing E_n is $K^{(n)}$, and $Y_n = |K^{(n)}|, (n = 0, 1, 2, \cdots)$. Obviously, we can construct a continuous map $f : X \to X$ such that $f|X_0 = f_0, f(Y_n) = E_{n+1}$ for $n = 0, 1, 2, \cdots$, and $f(z) = z$ for any $z \in X - \bigcup_{n=0}^{\infty} Y_n$. Clearly, $\overset{\circ}{A}_0$ is still a scrambled set of f.

If condition (C.2) holds, put $X_1 = X - \bigcup_{n=0}^{\infty} \overset{\circ}{E}_n$, and $X_2 = X_0 \cup X_1$. Define $f_2 : X_2 \to X_2$ by $f_2|X_0 = f_0$, and $f_2(z) = z$ for any $z \in X_1$. Then f_2 is continuous. Evidently, f_2 can be extented to be a continuous map $f : X \to X$, and $\overset{\circ}{A}_0$ is also a scrambled set of this f.

Note that $\overset{\circ}{A}_0$ is a nonempty open set in X. Theorem 3.1 is proven. □

4. Totally chaotic continuous maps of the real line

In the proof of Theorem 3.1 we construct a continuous map $f : X \to X$, which has a scrambled set with a nonempty interior in X. But this f is not a totally chaotic map defined as follows.

Definition 4.1. Let (X, d) be a metric space. A continuous map $f : X \to X$ is called *totally chaotic* if the whole space X is a scrambled set of f.

What metric space X can admit a totally chaotic map? This is an interesting problem. In this section we will consider the real line R and the real half-line $R_+ = [0, \infty)$, which can be regarded as the polyhadra of the infinite 1-dimensional simplicial complexes $K(R)$ and $K(R_+)$, respectively, where

$$K(R) = \{n, [n, n+1] : n \text{ is an integer}\},$$

$$K(R_+) = \{n, [n, n+1] : n \text{ is a nonnegative integer}\}.$$

We have the following theorem.

Theorem 4.2. *There exists a totally chaotic continuous map $f : R \to R$, which satisfies that $f(R_+) \subset R_+$, and $f|R_+$ is also a totally chaotic map.*

Proof. Let $c_0, c_1, c_2, \cdots$ be a given infinite sequence of positive numbers with $c_k \le 1/2$ for every even number $k \ge 0$. We first choose a C^∞ map $f_0 : R \to R$ such that

(0.1) $f_0(0) = 1$, $f_0(x) > x$ for any $x \le 0$, and $\lim_{x\to-\infty} f_0(x) = -\infty$;

(0.2) $f_0'(x) \equiv 1 - c_0$ for any $x \ge 1$ and $f_0'(x) > 0$ for all $x \in R$, where $f_0'(x)$ is the derivative of $f_0(x)$;

(0.3) $1 - c_0 < f_0'(x) < 1$ for $0 < x < 1$.

The equation $f_0(x) = x$ has a unique root. Suppose this root is a_0. Then $a_0 > 2$. Obviously, there is an integer $n_0 \ge 2$ such that $f_0^{n_0}([-1, 1]) \subset [a_0 - 1, a_0)$. Let $\varepsilon_0 = a_0 - f_0^{n_0}(1)$. Then $0 < \varepsilon_0 < 1$.

Next, we take a C^∞ map $f_1 : R \to R$ such that

(1.1) $f_1(x) = f_0(x)$ for any $x \le a_0 - \varepsilon_0$, and $f_1(x) > x$ for all $x \in R$;

(1.2) the derivative $f_1'(x) \equiv 1 + c_1$ for any $x \ge a_0$;

(1.3) $f_0'(x) < f_1'(x) < 1 + c_1$ for $a_0 - \varepsilon_0 < x < a_0$.

Evidently, there is an integer $n_1 > n_0$ such that $f_1^{n_1}(-a_0) > 2a_0$, and the derivative $d(f_1^{n_1}(x))/dx > 2$ for all $x \in [-a_0, a_0]$. Let $a_1 = f_1^{n_1}(a_0)$. Then $a_1 > 2a_0$. Put $\varepsilon_1 = \varepsilon_0$.

Now we choose again a C^∞ map $f_2 : R \to R$ such that

(2.1) $f_2(x) = f_1(x)$ for any $x \le a_1 + 1$;

(2.2) $f_2'(x) \equiv 1 - c_2$ for any $x \ge a_1 + 2$;

(2.3) $1 - c_2 < f_2'(x) < 1 + c_1$ for $a_1 + 1 < x < a_1 + 2$.

It is easy to see that the equation $f_2(x) = x$ has a unique root. Suppose this root is a_2. Then $a_2 > a_1 + 1$. Clearly, there is an integer $n_2 > n_1$ such that $f_2^{n_2}([-a_1, a_1]) \subset [a_2 - 2^{-2}, a_2)$. Let $\varepsilon_2 = a_2 - f_2^{n_2}(a_1)$. Then $0 < \varepsilon_2 < 2^{-2}$.

Continuing this process, for every positive integer $k = 1, 2, 3, \cdots$, we can choose a C^∞ map $f_k : R \to R$ and take a positive integer n_k and two positive numbers a_k, ε_k satisfying the following conditions:

(a) If $k \ge 1$ is odd, then

(k.1.a) $f_k(x) = f_{k-1}(x)$ for all $x \le a_{k-1} - \varepsilon_{k-1}$, and $f_{k-1}(x) > x$ for all $x \in R$;

(k.2.a) the derivative $f_k'(x) \equiv 1 + c_k$ for all $x \ge a_{k-1}$;

(k.3.a) $f_{k-1}'(x) < f_k'(x) < 1 + c_k$ for $a_{k-1} - \varepsilon_{k-1} < x < a_{k-1}$;

(k.4.a) $n_k > n_{k-1}$, $f_k^{n_k}(-a_{k-1}) > 2a_{k-1}$, and the derivative $d(f_k^{n_k}(x))/dx > 2^k$ for all $x \in [-a_{k-1}, a_{k-1}]$;

(k.5.a) $a_k = f_k^{n_k}(a_{k-1}) > 2a_{k-1}$, and $\varepsilon_k = \varepsilon_{k-1}$.

(b) If $k \ge 2$ is even, then

(k.1.b) $f_k(x) = f_{k-1}(x)$ for all $x \le a_{k-1} + 1$;

(k.2.b) $f_k'(x) \equiv 1 - c_k$ for all $x \ge a_{k-1} + 2$;

(k.3.b) $1 - c_k < f_k'(x) < 1 + c_{k-1}$ for $a_{k-1} + 1 < x < a_{k-1} + 2$;

(k.4.b) the equation $f_k(x) = x$ has a unique root, and a_k is this root, $a_k > a_{k-1} + 1$;

(k.5.b) $n_k > n_{k-1}$, and $f_k^{n_k}([-a_{k-1}, a_{k-1}]) \subset [a_k - 2^{-k}, a_k)$;

(k.6.b) $\varepsilon_k = a_k - f_k^{n_k}(a_{k-1}) \in (0, 2^{-k}]$.

From these conditions we know that there exists a limit function $f = \lim_{k\to\infty} f_k$ with $f(x) = f_k(x)$ for $x \le a_k - \varepsilon_k$. Thus $f : R \to R$ is a C^∞ map. For any given $u, v \in R$ with $u \ne v$, take a positive integer $j \ge 1$ such that $\{u, v\} \subset [-a_j, a_j]$. Then by (k.4.a) we have $|f^{n_k}(u) - f^{n_k}(v)| > 2^k|u - v|$ for every odd $k > j$, and by (k.5.b) we have $|f^{n_k}(u) - f^{n_k}(v)| < 2^{-k}$ for every even $k > j$. This implies that

$$\liminf_{k\to\infty} |f^k(u) - f^k(v)| = 0, \qquad \text{and} \quad \limsup_{k\to\infty} |f^k(u) - f^k(v)| = \infty.$$

Hence f is totally chaotic.

Noting that $f(x) > x$ for any $x \in R$, we have $f(R_+) \subset R_+$. Therefore, $f|R_+$ is also totally chaotic. Theorem 4.2 is proven. □

Remark 4.3. Let $f : R \to R$ be as in the proof of Theorem 4.2. Then f is a C^∞ diffeomorphism. For any $n \ge 2$, define $F_n : R^n \to R^n$ by

$$F_n(x_1, x_2, \cdots, x_n) = (f(x_1), f(x_2), \cdots, f(x_n)), \quad \text{for any} \quad (x_1, x_2, \cdots, x_n) \in R^n.$$

It is easy to see that F_n is also a C^∞ diffeomorphism, and F_n is totally chaotic.

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