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ERGODIC SEQUENCES IN THE FOURIER-STIELTJES ALGEBRA AND MEASURE ALGEBRA OF A LOCALLY COMPACT GROUP

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ABSTRACT. Let G be a locally compact group. Blum and Eisenberg proved that if G is abelian, then a sequence of probability measures on G is strongly ergodic if and only if the sequence converges weakly to the Haar measure on the Bohr compactification of G. In this paper, we shall prove an extension of Blum and Eisenberg's Theorem for ergodic sequences in the Fourier-Stieltjes algebra of G. We shall also give an improvement to Milnes and Paterson's more recent generalization of Blum and Eisenberg's result to general locally compact groups, and we answer a question of theirs on the existence of strongly (or weakly) ergodic sequences of measures on G.

0. INTRODUCTION

Let G be a locally compact group and π be a continuous unitary representation of G on a Hilbert space H. Let H_f denote the fixed point set of π in H, i.e.

$$H_f = \{\xi \in H;\ \pi(x)\xi = \xi \quad \text{for all} \quad x \in G\}.$$

A sequence $\{\mu_n\}$ of probability measures on G is called a *strongly* (resp. *weakly*) *ergodic* sequence if for every representation π of G on a Hilbert space H and for every $\xi \in H$, $\{\pi(\mu_n)\xi\}$ converges in norm (resp. weakly) to a member of H_f. When G is abelian or compact, or G is a [Moore]-group (i.e. every irreducible representation of G is finite dimensional), then every weakly ergodic sequence is strongly ergodic. However, this is not true in general (see [8, Proposition 1 and Proposition 5]).

In [1], Blum and Eisenberg proved that if G is a locally compact abelian group, and $\{\mu_n\}$ is a sequence of probability measures on G, then the following are equivalent:

(i) $\{\mu_n\}$ is strongly ergodic.
(ii) $\widehat{\mu}_n(\gamma) \to 0$ for all $\gamma \in \widehat{G}\backslash\{1\}$.
(iii) $\{\mu_n\}$ converges weakly to the Haar measure on the Bohr compactification of G.

More recently Milnes and Paterson [8] obtained the following generalization of Blum and Eisenberg's result to general locally compact groups:

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Theorem A (Milnes and Paterson [8]). *Let G be a second countable locally compact group. Then the following statements about a sequence $\{\mu_n\}$ of probability measures in $M(G)$ are equivalent:*

(i) *$\{\mu_n\}$ is a weakly ergodic sequence.*
(ii) *$\pi(\mu_n) \to 0$ in the weak operator topology for every $\pi \in \widehat{G}\backslash\{1\}$.*
(iii) *$\widehat{\mu}_n$ converges to the unique invariant mean on $B_I(G)$, the closure in $C(G)$ of the linear span of the set of coefficient functions of the irreducible representations of G.*

(Here $\widehat{G}$ denotes the set of irreducible continuous representations of G which is the same as the dual group of G when G is abelian.)

Let $P_1(G)$ denote the continuous positive definite functions ϕ on G such that $\phi(e) = 1$ (where e is the identity of G). When G is abelian, $P_1(G)$ corresponds to the set of probability measures on the dual group $\widehat{G}$ of G (by Bochner's Theorem). In this paper, we shall prove an extension of Blum and Eisenberg's Theorem for ergodic sequences in $P_1(G)$ (Theorems 3.1 and 3.3). We shall give an improvement to condition (iii) of Theorem A by replacing "$B_I(G)$" by the Fourier-Stieltjes algebra "$B(G)$" for any G (Theorems 4.1 and 4.4) and remove the condition of second countability (= separability in [8]) in Theorem A. We shall also show that (Theorem 4.6) G is σ-compact if and only if it has a strongly (or weakly) ergodic sequence of measures. This completely answers a question in [8, p. 693].

A "strongly ergodic sequence" is called a "general summing sequence" by Blum and Eisenberg in [1]. It was also introduced by Rindler under the name "unitarily distributed sequences" in Def. 4 of [13] for point sequences and their Cesàro averages and by Maxones and Rindler in [9] for sequences of measures.

1. Some preliminaries

Throughout this paper, G denotes a locally compact group with a fixed left Haar measure μ. Integration with respect to μ will be given by $\int \cdots dx$. Let $C(G)$ denote the Banach space of bounded continuous functions on G with the supremum norm. Then G is *amenable* if there exists a positive linear functional ϕ on $C(G)$ of norm one such that $\phi(\ell_a f) = \phi(f)$ for each $a \in G$ and $f \in C(G)$ where $(\ell_a f)(x) = f(ax), \quad x \in G$. Amenable groups include all solvable groups and all compact groups. However, the free group on two generators is not amenable (see [11] or [12] for more details).

Let $C^*(G)$ denote the completion of $L^1(G)$ with respect to the norm $\|f\|_c = \sup\{\|T_f\|\}$, where the supremum is taken over all $*$-representations T of $L^1(G)$ as an algebra of bounded operators on a Hilbert space. Let $P(G)$ denote the subset of $C(G)$ consisting of all continuous positive definite functions on G, and let $B(G)$ be its linear span. Then $B(G)$ (the *Fourier-Stieltjes algebra* of G) can be identified with the dual of $C^*(G)$, and $P(G)$ is precisely the set of positive linear functionals on $C^*(G)$.

Let $\mathcal{B}(L^2(G))$ be the algebra of bounded linear operators from $L^2(G)$ into $L^2(G)$ and let $VN(G)$ denote the weak operator topology closure of the linear span of $\{\rho(a) : \ a \in G\}$, where $\rho(a)f(x) = f(a^{-1}x), \ x \in G, \ f \in L^2(G)$, in $\mathcal{B}(L^2(G))$. Let $A(G)$ denote the subalgebra of $C_0(G)$ (continuous complex-valued functions vanishing at infinity), consisting of all functions of the form $h * \widetilde{k}$ where $h, \ k \in L^2(G)$ and $\widetilde{k}(x) = \overline{k(x^{-1})}$, $x \in G$. Then each $\phi = h * \widetilde{k}$ in $A(G)$ can be regarded as an ultraweakly continuous functional on $VN(G)$ defined by

$$\phi(T) = \langle Th, k \rangle \quad \text{for each} \quad T \in VN(G).$$

Furthermore, as shown by Eymard in [3, pp. 210, Theorem 3.10], each ultraweakly continuous functional on $VN(G)$ is of this form. Also $A(G)$ with pointwise multiplication and the norm $\|\phi\| = \sup\{|\phi(T)|\}$, where the supremum runs through all $T \in VN(G)$ with $\|T\| \leq 1$, is a semisimple commutative Banach algebra with spectrum G; $A(G)$ is called the *Fourier algebra* of G and it is an ideal of $B(G)$.

There is a natural action of $A(G)$ on $VN(G)$ given by $\langle \phi \cdot T, \gamma \rangle = \langle T, \phi \cdot \gamma \rangle$ for each ϕ, $\gamma \in A(G)$ and each $T \in VN(G)$. A linear functional m on $VN(G)$ is called a *topological invariant mean* if

(i) $T \geq 0$ implies $\langle m, T \rangle \geq 0$,
(ii) $\langle m, I \rangle = 1$ where $I = \rho(e)$ denotes the identity operator, and
(iii) $\langle m, \phi \cdot T \rangle = \phi(e) \langle m, T \rangle$ for $\phi \in A(G)$.

As known, $VN(G)$ always has a topological invariant mean. However $VN(G)$ has a unique topological invariant mean if and only if G is discrete (see [14, Theorem 1] and [6, Corollary 4.11]).

Let $C^*_\delta(G)$ denote the norm closure of the linear span of $\{\rho(a);\ a \in G\}$. Let $B_\delta(G)$ denote the linear span of $P_\delta(G)$, where $P_\delta(G)$ is the pointwise closure of $A(G) \cap P(G)$. Then $B_\delta(G)$ can be identified with $C^*_\delta(G)^*$ by the map $\pi(\phi)(f) = \sum\{\phi(t)f(t),\ t \in G\}$ for each $f \in \ell_1(G)$ and $\phi \in B_\delta(G)$ (see [3, Proposition 1.21]). Furthermore $B_\delta(G)$ with pointwise multiplication and dual norm is a commutative Banach algebra. If m is topological invariant mean on $VN(G)$, then $m' =$ restriction of m to $C^*_\delta(G)$, is also a topological invariant mean on $C^*_\delta(G)$. Furthermore, if m'' is another topological invariant mean on $C^*_\delta(G)$, then $m' = m''$, by commutativity of $B_\delta(G)$. If G is amenable, then $B(G) \subseteq B_\delta(G)$. In particular, each $\phi \in B(G)$ corresponds to a continuous linear functional on $C^*_\delta(G)$ defined by $\langle \phi, \rho(a) \rangle = \phi(a),\ a \in G$. Also if G is abelian, then $C^*_\delta(G) \cong AP(\widehat{G})$, the space of continuous almost periodic functions on $\widehat{G}$ (see [5]).

2. Some lemmas

Let G be a locally compact group, and $M^+(G)$ be the positive finite regular Borel measures on G.

Lemma 2.1. *Let $\mu \in M^+(G)$. For each $\phi \in A(G)$, define S_ϕ an operator on $L_2(G, \mu)$ by*

$$S_\phi h = \phi h, \quad h \in L_2(G, \mu).$$

Then the mapping $\phi \to S_\phi$ is a cyclic $*$-representation of $A(G)$ as bounded operators on $L_2(G, \mu)$.

Proof. It is easy to see that $\phi \to S_\phi$ is a $*$-representation as bounded operators on $L_2(G, \mu)$. Also the element $1 \in L_2(G, \mu)$ is a cyclic vector for S, since we have $\{S_\phi 1;\ \phi \in A(G)\} = \{\phi;\ \phi \in A(G)\}$. Let $f \in C_{00}(G)$ (continuous function with compact support); then there exists $\{\phi_n\} \subseteq A(G)$ such that $\|\phi_n - f\|_\infty \to 0$. In particular, $\phi_n \to f$ in the L_2-norm of $L_2(G, \mu)$. The result now follows by density of $C_{00}(G)$ in $L_2(G, \mu)$, and $\mu(G) < \infty$. □

Lemma 2.2. *Let $\{T, H\}$ be a cyclic $*$-respresentation of $A(G)$. There exists a measure $\mu \in M^+(G)$ such that T is unitarily equivalent to a representation S defined by μ as in Lemma 2.1.*

Proof. Indeed, for any $\phi \in A(G)$,

$$\|T(\phi)\|_{\rm sp} \leq \|\phi\|_{\rm sp}$$

($\|\cdot\|_{\rm sp}$ denotes the spectral-radius). Since $A(G)$ is commutative, $\|T(\phi)\|_{\rm sp}$ = operator norm in $\mathcal{B}(H)$ and $\|\phi\|_{\rm sp} = \sup \{|\phi(x)| : x \in G\}$ (by semi-simplicity of $A(G)$, and the fact that the spectrum of $A(G)$ is G). Hence

$$\|T(\phi)\| \leq \|\phi\|_\infty.$$

In particular T extends to a $*$-representation of the C^*-algebra $C_0(G)$ (by density of $A(G)$ in $C_0(G)$). Let $\eta \in H$ be a cyclic vector of $\{T, H\}$. Then

$$f \to \langle T(f)\eta, \eta\rangle, \quad f \in C_0(G),$$

defines a positive linear functional on the C^*-algebra $C_0(G)$. Let $\mu \in M^+(G)$ which represents this functional and S be the cyclic representation of $A(G)$ as defined in Lemma 2.1. Then T and S are unitarily equivalent. Indeed, define a map $W: \{T(\phi)\eta;\ \phi \in A(G)\} \to \{\phi\cdot 1;\ \phi \in A(G)\} \subseteq L_2(G, \mu)$ by $W\big(T(\phi)\eta\big) = \phi\cdot 1$.

Then $\langle T(\phi)\eta, \eta\rangle = \int \phi d\mu = 0$ whenever $\phi = 0$ μ-a.e. Hence W is well-defined. Also

$$\begin{aligned}\langle T(\phi)\eta, T(\phi)\eta\rangle &= \langle T^*(\phi)T(\phi)\eta, \eta\rangle = \langle T(\overline{\phi}\phi)\eta, \eta\rangle \\ &= \int \overline{\phi}\phi(x)d\mu(x) = \langle \phi, \phi\rangle.\end{aligned}$$

Consequently W extends to a linear isometry from H onto $L^2(G, \mu)$. Finally, if $\psi, \phi \in A(G)$, then

$$S(\phi)W\big(T(\psi)\eta\big) = S(\phi)(\psi \cdot 1) = \phi\psi \cdot 1$$

and

$$WT(\phi)\big(T(\psi)\eta\big) = WT(\phi\psi)\eta = \phi\psi \cdot 1.$$

Hence $\{S, L_2(G, \mu)\}$ and $\{T, H\}$ are unitarily equivalent. □

Assume that G is amenable. Then it is well known that $A(G)$ has an approximate identity bounded by 1 . Let $\{T, H\}$ be a $*$-representation of $A(G)$ which is non-degenerate. Next, we will show that for each $\psi \in B(G)$, there is a unique bounded linear operator $\widetilde{T}(\psi)$ on H such that

(i) $\widetilde{T}(\psi)T(\phi) = T(\psi\phi)$ for all $\phi \in A(G)$.

Uniqueness is clear from the fact that vectors of the form $\{T(\phi)\xi,\ \phi \in A(G),\ \xi \in H\}$ span H. For existence, consider first the case when T is cyclic, and let $\xi_0 \in H$ be such that $[T(A(G))\xi_0] = H$. We claim that

(ii) $\|T(\psi\phi)\xi_0\| \leq \|\psi\|\ \|T(\phi)\xi_0\|$ for each $\phi \in A(G)$.

Indeed, let $\phi \in A(G)$ be fixed. Choose a bounded approximate identity $\{\psi_n\}$ in $A(G)$ such that $\|\psi_n\| \leq 1$. Then

$$\begin{aligned}\|T(\psi\phi)\xi_0\| &= \lim \ \|T(\psi\psi_n\phi)\xi_0\| \\ &= \lim_n \ \|T(\psi\psi_n)T(\phi)\xi_0\| \\ &\leq \|\psi\psi_n\| \ \|T(\phi)\xi_0\| \\ &\leq \|\psi\| \ \|T(\phi)\xi_0\|\end{aligned}$$

(since any $*$-homomorphism from an involutive Banach algebra into a C^*-algebra is norm decreasing) as asserted.

Now it follows that the map $T(\phi)\xi_0 \to T(\psi\phi)\xi_0$ (where $\phi \in A(G)$) extends uniquely to an operator $\widetilde{T}(\psi)$ on $[T(A(G))\xi_0] = H$ having norm at most $\|\psi\|$. The relation $\widetilde{T}(\psi)T(\phi) = T(\psi\phi)$ holds on all vectors of the form $T(\theta)\xi_0, \quad \theta \in A(G)$, so it holds on H.

For a general non-degenerate $*$-representation T of $A(G)$, we simply write $T = \sum \bigoplus T_\alpha$, each T_α cyclic, and define $\widetilde{T}(\psi) = \sum \bigoplus \widetilde{T}_\alpha(\psi), \quad \psi \in B(G)$.

3. Ergodic sequences in $B(G)$

A sequence $\{\phi_n\}$ in $A(G) \cap P_1(G)$ is called *strongly* (respectively *weakly*) *ergodic* if whenever $\{T, H\}$ is a $*$-representation of $A(G)$, $\xi \in H$, the sequence $T(\phi_n)\xi$ converges in the norm (resp. weak) topology to a member of the fixed point set:

$$H_f = \{\xi \in H;\ T(\phi)\xi = \xi \quad \text{for all} \quad \phi \in A(G) \cap P_1(G)\}.$$

Theorem 3.1. *Let G be a locally compact group. The following are equivalent for a sequence $\{\phi_n\}$ in $A(G) \cap P_1(G)$:*

(i) *$\{\phi_n\}$ is strongly ergodic.*
(ii) *$\{\phi_n\}$ is weakly ergodic.*
(iii) *For each $g \in G,\ g \neq e,\ \phi_n(g) \to 0$.*
(iv) *For each $T \in C^*_\delta(G)$, $\langle \phi_n, T\rangle \to \langle m, T\rangle$, where m is the unique topological invariant mean on $C^*_\delta(G)$.*

Proof. We first observe that $\langle m, \rho(g)\rangle = 0$ for all $g \in G\backslash\{e\}$, and $\langle m, \rho(e)\rangle = 1$. Indeed, if $\phi \in A(G) \cap P_1(G)$, then $\langle m, \rho(g)\rangle = \langle m, \phi\cdot\rho(g)\rangle = \langle m, \phi(g)\rho(g)\rangle = \phi(g)\langle m, \rho(g)\rangle$ (note $\langle \phi\cdot\rho(g), \psi\rangle = \langle \rho(g), \phi\psi\rangle = \phi(g)\psi(g) = \phi(g)\langle\rho(g), \psi\rangle$; hence $\phi\cdot\rho(g) = \phi(g)\rho(g)$). Now if $g \neq e$, then there exists $\phi \in A(G) \cap P_1(G)$ such that $\phi(g) \neq 1$ so $\langle m, \rho(g)\rangle = 0$. Consequently, (iii) and (iv) are equivalent.

(ii) $\Longrightarrow$ (iii). Consider, for $g \in G$ (fixed), the representation $\{T, H\}$, where $H = \mathbb{C}, \quad T(\phi)\lambda = \phi(g)\lambda$. If $g \neq e$, then $H_f = \{\lambda;\ T(\phi)\lambda = \lambda,\ \phi \in A(G) \cap P_1(G)\} = \{\lambda;\ \phi(g)\lambda = \lambda,\ \phi \in A(G) \cap P_1(G)\} = \{0\}$. Hence if $g \neq e$, then

$$\phi_n(g) = T(\phi_n) = T(\phi_n)1 = \phi_n(g)\cdot 1 \to 0$$

by ergodicity of the sequence $\{\phi_n\}$.

(iii) $\Longrightarrow$ (i). We first assume that $\{T, H\}$ is a cyclic $*$-representation of $A(G)$. By Lemma 2.2 there exists a measure $\mu \in M^+(G)$ such that T is unitarily equivalent to a representation S on $L_2(G, \mu)$ as in Lemma 2.1. Hence we may assume that $T = S$, and $H = L_2(G, \mu)$.

Let $h \in L_2(G,\mu)$. Then for each n, m,

$$\|T(\phi_n)h - T(\phi_m)h\|^2 = \int (\phi_n - \phi_m)(x)h(x)\,\overline{(\phi_n - \phi_m)(x)h(x)}\,d\mu(x).$$

The integrand converges pointwise to "0" as $n, m \to \infty$, and it is dominated by the integrable function $4|h|^2$. Hence by the dominated convergence theorem

$$\|T(\phi_n)h - T(\phi_m)h\|^2 \to 0 \quad \text{as } n, m \to \infty,$$

i.e. $\{T(\phi_n)h\}$ is Cauchy. Let f be the limit of $T(\phi_n)h$ in $L_2(G,\mu)$. Now if $\phi \in A(G) \cap P_1(G)$, $\quad h \in L_2(G,\mu)$, then

$$\|T(\phi)\,(T(\phi_n)h) - T(\phi_n)h\|^2 = \int (\phi\cdot\phi_n - \phi_n)h \;\cdot\; \overline{(\phi\cdot\phi_n - \phi_n)\cdot h}\,d\mu$$

which again converges to zero as $n \to \infty$ by the dominated convergence theorem. So $T(\phi)f = f$, i.e. f is a fixed point of $\{T, H\}$.

If $\{T, H\}$ is any $*$-representation of $A(G)$, then $T = \{T_0, H_0\} \oplus \sum_{\alpha\in\Gamma}\{T_\alpha, H_\alpha\}$ where $\{T_0, H_0\}$ is the degenerate part of $\{T, H\}$ and $\{T_\alpha, H_\alpha\}$ is cyclic. The result follows by applying the cyclic case to each $\{T_\alpha, H_\alpha\}$ to obtain a fixed point $f_\alpha \in H_\alpha$ of $\{T_\alpha, H_\alpha\}$. Then $f = (f_\alpha)$ is the limit of the sequence $\{T(\phi_n)h\}$ in H, and $T(\phi)f = f$ for all $\phi \in A(G) \cap P(G)$. □

Corollary 3.2. *A locally compact group G is first countable if and only if $A(G)$ contains an ergodic sequence.*

Proof. Let $\{U_n\}$ be a sequence of compact symmetric neighborhoods of the identity of G, such that

(i) $U_n \downarrow \{e\}$,
(ii) $U_n \cdot U_n \subseteq U_{n-1}$.

For each n, let $\phi_n = \frac{1}{\lambda(U_n)}\left(1_{U_n} * 1_{U_n}\right)$. Then $\phi_n \in A(G) \cap P_1(G)$, and $\phi_n(g) \to 0$ for each $g \in G$ ($g \neq e$). Hence $\{\phi_n\}$ is ergodic by Theorem 3.1. Conversely if $\{\phi_n\}$ is an ergodic sequence on $A(G)$, then the topology on G defined by the sequence of pseudometrics $\{d_n\}$, where $d_n(x,y) = |\phi_n(x) - \phi_n(y)|$ is Hausdorff (by Theorem 3.1(iii)) and hence must agree on any compact neighbourhood of x, $x \in G$. Consequently G is first countable. □

For G amenable, a sequence $\{\phi_n\}$ in $P_1(G)$ is called *strongly* (resp. *weakly*) *ergodic* if whenever $\{T, H\}$ is a non-degenerate $*$-representation of $A(G)$, the sequence $\widetilde{T}(\phi_n)\xi$ converges in norm (resp. weakly) to a member of the fixed point set:

$$H_f = \{\xi \in H;\; T(\phi)\xi = \xi \quad \text{for all} \quad \phi \in A(G) \cap P_1(G)\}$$

where $\widetilde{T}$ is the unique extension of T to $B(G)$ as defined earlier in Section 2.

Theorem 3.3. *Let G be an* amenable *locally compact group. The following are equivalent for a sequence $\{\phi_n\}$ in $P_1(G)$:*

(i) *$\{\phi_n\}$ is strongly ergodic.*
(ii) *$\{\phi_n\}$ is weakly ergodic.*
(iii) *For each $g \in G$, $g \neq e$, $\phi_n(g) \to 0$.*
(iv) *For each $T \in C^*_\delta(G)$, $\langle \phi_n, T\rangle \to \langle m, T\rangle$, where m is the unique topological invariant mean on $C^*_\delta(G)$.*

Proof. Note that if m is a topological invariant mean on $C^*_\delta(G)$ (i.e. $\langle m, \phi\cdot T\rangle = \langle m,T\rangle$ for any $\phi \in P_1(G)\cap A(G)$, $\quad T\in C^*_\delta(G)$), then $\langle m,\psi\cdot T\rangle = \langle m,T\rangle$, for $\psi\in P_1(G), T\in C^*_\delta(G)$, where $\langle \psi\cdot T,\ \phi\rangle = \langle T,\psi\phi\rangle$, for $\phi\in A(G):\quad$ indeed, let $\psi_n \subseteq P(G)\cap A(G)$ be a bounded approximate identity for $A(G)$. Then $\|\psi_n\cdot T - T\|\to 0$ for all $T\in UC(\widehat{G}) = A(G)\cdot VN(G)\supseteq C^*_\delta(G)$. Hence $\langle m,\psi\cdot T\rangle = \lim_n \langle m,\psi\cdot\psi_n\cdot T\rangle = \langle m,T\rangle$. So (iii) $\iff$ (iv) as in the proof of Theorem 3.1.

(i) $\iff$ (ii) $\iff$ (iii): same as Theorem 3.1 (*Note*: the representation $T(\phi)\lambda = \phi(g)\lambda$, where $\phi\in A(G)$ has a unique extension $\widetilde{T}$ to $B(G)$, $\quad \widetilde{T}(\phi)\lambda = \phi(g)\lambda$, for $\phi\in B(G)$; similarly, the unique extension of S from $A(G)$ to $B(G)$ is $S_\phi h = \phi h, \quad h\in L^2(G,\mu)$.) □

4. Ergodic sequences of measures

Let $M(G)$ denote the space of finite regular Borel measures on G. We put $\langle \mu, f\rangle = \int_G f(t)d\mu(t)$, $\quad$ for $\mu\in M(G)$, $f\in C(G)$ (in [8] this is denoted by $\widehat{\mu}(f)$). If π is a continuous unitary representation of G, let P_f denote the orthogonal projection from H^π onto the closed subspace H^π_f of fixed points.

Theorem 4.1. *Let G be a locally compact group. Then the following statements about a sequence $\{\mu_n\}$ of probability measures on G are equivalent:*

(i) *$\{\mu_n\}$ is a weakly ergodic sequence.*
(ii) *$\pi(\mu_n)\to 0$ in the weak operator topology for every $\pi\in\widehat{G}\backslash\{1\}$.*
(iii) *$\mu_n \to m$ in the weak*-topology ($\sigma(B(G)^*, B(G))$), where m is the unique translation-invariant mean on $B(G)$.*

Proof. (i)$\Longrightarrow$(ii). Let $\pi\in\widehat{G}$. Then $\pi(\mu_n)\to P_f$. But $P_f = 0$ or I by irreducibility of π. Hence if $\pi\neq I$, $\quad\pi(\mu_n)\to 0$ in the weak operator topology.

(ii)$\Longrightarrow$(iii). Let $\pi\in\widehat{G}$, $\quad \xi,\eta\in H^\pi$ and $\phi^\pi_{\xi,\eta}(x) = \langle \pi(x)\xi,\eta\rangle$, $\quad x\in G$. Then

$$\begin{aligned}\langle \mu_n, \phi^\pi_{\xi,\eta}\rangle &= \int \phi^\pi_{\xi,\eta}(x)d\mu_n(x)\\ &= \int \langle \pi(x)\xi,\eta\rangle d\mu_n(x)\\ &= \langle \pi(\mu_n)\xi,\eta\rangle \to 0 \quad \text{if} \quad \pi\neq I.\end{aligned}$$

Let $\mathcal{E}(G)$ denote the extreme points of $P_1(G)$. The above implies that $\langle \mu_n, \ell_y\phi\rangle\to 0$ for any $y\in G$, $\phi\in\mathcal{E}(G)$, $\quad \phi\neq 1$ where 1 denotes the constant one function on G. We will show that $\langle\mu_n,\phi\rangle\to\langle m,\phi\rangle$ for all $\phi\in P_1(G)$.

Note that if E is a locally convex space, and C a compact subset of E, and f_n a sequence of continuous linear functionals on E which are uniformly bounded on C and converge to 0 on C, then convergence to 0 holds on the closed convex hull of C (see [15] or [10] for an elementary proof). This applies easily if G is discrete. In the general case, slight complications arise: the set $P_1(G)$ is not weak*-compact, and measures are in general not weak*-continuous on $B(G)$. Nevertheless the method of proof generalizes to this case:

If G is second countable, then the weak*-topology on the unit ball of $B(G)$ is metrizable. Then $P_0(G)$ $\quad$ (= intersection with the cone of positive definite functions) is compact and convex; the extreme points of $P_0(G)$ are 0 and the extreme points of $P_1(G)$. Let $\phi\in P_1(G)$. By Choquet's theorem, there is a probability measure Φ concentrated on ext $(P_0(G))$ representing ϕ, i.e., for $T\in C^*(G)$, we have

$$\langle T, \phi \rangle = \int_{P_0(G)} \langle T, \gamma \rangle \, d\Phi(\gamma) \quad \text{for all} \quad x \in G.$$

Using a bounded approximate unit (v_n) in $L^1(G) \subseteq C^*(G)$, it follows that the map $(x, \gamma) \to \gamma(x) = \lim \langle v_n, \ell_x \gamma \rangle$ is Borel measurable on $G \times P_0(G)$ and by dominated convergence that

$$\phi(x) = \int_{P_0(G)} \gamma(x) \, d\Phi(\gamma) \quad \text{for all} \quad x \in G,$$

in particular that 0 has weight zero (take $x = e$). Thus, Φ is concentrated on $\mathcal{E}(G)$. Hence if $\mu \in M(G)$, one gets by Fubini's theorem

$$\langle \phi, \mu \rangle = \int_{\mathcal{E}(G)} \langle \gamma, \mu \rangle \, d\Phi(\gamma).$$

Hence if $\{\mu_n\}$ is a sequence of probability measures on G, satisfying (ii), it follows from the Lebesgue dominated convergence theorem that $\langle \phi, \mu_n \rangle \to \Phi(\{1\})$, and similarly $\langle \phi, \ell_y^* \mu_n \rangle \to \Phi(\{1\})$ for $\phi \in P_1(G)$, $y \in G$. Consequently, μ_n and $\ell_y^* \mu_n$ have the same limit on $P_1(G)$; hence μ_n converges to the unique invariant mean m on $B(G)$.

For general G, if there is a weakly ergodic sequence of measures in $M(G)$ (resp. (ii) holds), then G has to be σ-compact (see Theorem 4.6 and Remark 4.3).

If G is σ-compact, and π is a cyclic representation of G on a Hilbert space H, then H is separable, and hence the strong operator topology on $\mathcal{B}(H)$ is metrizable on bounded sets. Consequently, the quotient group $G/\operatorname{Ker} \pi$ is second countable, and the above argument applies.

(iii)$\Longrightarrow$(i). Let π be a continuous unitary representation of G. Then, by (iii), $\{\langle \pi(\mu_n)\xi, \eta \rangle\}$ converges for all $\xi, \eta \in H^\pi$, and hence $\pi(\mu_n) \to T$ in the weak operator topology for some $T \in \mathcal{B}(H^\pi)$. Clearly, $\langle T\xi, \eta \rangle = \langle m, \phi^\pi_{\xi,\eta} \rangle$, and since m is translation-invariant, we have $\pi(y)T = T = T\pi(y)$ for all $y \in G$. So, $T = P_f$ i.e. $\pi(\mu_n) \to P_f$ in the weak operator topology for all π. Hence (iii) holds. □

Lemma 4.2. *If H is an open subgroup of G with G/H infinite, (μ_n) a weakly ergodic sequence of measures, then $\mu_n(H) \to 0$.*

Proof. Let π be the regular representation on $\ell^2(G/H)$, $\xi = 1_H$. Then $\langle \pi(\mu)\xi, \xi \rangle = \mu(H)$. If G/H is infinite, it follows easily that $\ell^2(G/H)_f = (0)$; hence $\mu_n(H) \to 0$. □

Remark 4.3. By a similar argument one shows that if the sequence (μ_n) satisfies (ii) of Theorem 4.1, then the measures μ_n cannot be concentrated on a subgroup H as above: We have $1_H \in P_1(G)$ and the set of $\phi \in P_1(G)$ for which $\phi(x) = 1$ for $x \in H$ is easily seen to be weak*-compact in $B(G)$. Hence it has an extreme point $\phi \neq 1$ and this is also an extreme point of $P_1(G)$. Thus we get $\pi \in \widehat{G} \backslash \{1\}$, $\xi \in H^\pi \backslash \{0\}$ with $\pi(x)\xi = \xi$ for $x \in H$. If all μ_n would be concentrated on H, we would get $\pi(\mu_n)\xi = \xi$ for all n, contradicting (ii).

Theorem 4.4. *Let G be a locally compact group. Then the following statements about a sequence (μ_n) of probability measures on G are equivalent:*

(i) *(μ_n) is a strongly ergodic sequence.*
(ii) *$\pi(\mu_n) \to 0$ in the strong operator topology for every $\pi \in \widehat{G} \backslash \{1\}$.*

Proof. (i)$\Longrightarrow$(ii) follows as in Theorem 4.1.

(ii)$\Longrightarrow$(i): Let (π, H) be a (continuous, unitary) representation of G, P_f denotes the orthogonal projection onto H_f. As in the proof of Theorem 4.1, (ii)$\Longrightarrow$(iii), we may assume that H is separable, G second countable. Then $C^*(G)$ is separable. By [16, Theorem IV.8.32] there exists a disintegration $(\pi, H) = \int_\Gamma^\oplus (\pi_\gamma, H(\gamma))d\nu(\gamma)$ of the representation (π, H) of $C^*(G)$ such that π_γ is an irreducible representation of $C^*(G)$ for almost all γ. Each π_γ defines a representation of G and, putting $\Gamma_f = \{\gamma : \pi_\gamma = 1\}$, we have clearly $H_f = \int_{\Gamma_f}^\oplus H(\gamma)d\nu(\gamma)$. For $\xi = \int_\Gamma^\oplus \xi(\gamma)d\nu(\gamma) \in H$, we get $P_f\xi = \int_{\Gamma_f}^\oplus \xi(\gamma)d\nu(\gamma)$. If μ is a bounded measure on G, it follows as in [2, 18.7.4] that $\pi(\mu) = \int_\Gamma^\oplus \pi_\gamma(\mu)d\nu(\gamma)$; hence $\pi(\mu)\xi = \int_\Gamma^\oplus \pi_\gamma(\mu)\xi(\gamma)d\nu(\gamma)$. Since $\pi_\gamma(\mu_n)\xi(\gamma) \to 0$ for almost all $\gamma \notin \Gamma_f$, it follows as in the proof of Theorem 3.1, (iii)$\Longrightarrow$(i), from Lebesgue's dominated convergence theorem that $\pi(\mu_n)\xi \to P_f\xi$. □

Remark 4.5. The question of the existence of weakly ergodic sequences of measures was stated as a problem in [8]. In fact, the case of separable groups G had already been settled before in [7], Theorem 3: for the sequences (x_n) constructed there, $\mu_n = \frac{1}{n} \sum_{j=1}^n \delta_{x_j}$ has the property that $\pi(\mu_n)$ converges to P_f in the strong operator topology for any continuous representation of G on a Banach space B for which all orbits $\{\pi(x)b : x \in G\}$ are relatively weakly compact. In particular, (μ_n) is even a strongly ergodic sequence. More generally, the following result holds:

Theorem 4.6. *The following statements about a locally compact group G are equivalent:*

(i) *There exists a strongly ergodic sequence of measures.*
(ii) *There exists a weakly ergodic sequence of measures.*
(iii) *G is σ-compact.*

Proof. (i)$\Longrightarrow$(ii) is trivial.

(ii)$\Longrightarrow$(iii): See Lemma 4.2 (any sequence of finite measures is supported by a countable union of compact sets, hence by an open σ-compact subgroup).

(iii)$\Longrightarrow$(i): By the Kakutani-Kodaira theorem, G has a compact normal subgroup N such that G/N is metrizable. In particular, G/N is separable. Let λ be the normalized Haar measure on N and let M be a closed separable subgroup of G such that $G = M \cdot N$. Let (x_n) be a sequence in M satisfying the properties of [7], Theorem 3, mentioned above. Put $\mu_n = \frac{1}{n} \sum_{j=1}^n \delta_{x_j} * \lambda$. We claim that (μ_n) is strongly ergodic. Let (π, H) be a unitary representation of G. Put $H_{f,N} = \{\xi \in H : \pi(x)\xi = \xi \text{ for all } x \in N\}$, similarly for $H_{f,M}$, and denote the orthogonal projections on these spaces by $P_{f,N}$ resp. $P_{f,M}$. Clearly, $P_{f,N} = \pi(\lambda)$. Since N is normal, $H_{f,N}$ is a π-invariant subspace; hence the same is true for $H_{f,N}^\perp$. This entails that $P_{f,N}$ and $P_{f,M}$ commute; hence $P_f = P_{f,M} \circ P_{f,N}$. By assumption, $(\frac{1}{n} \sum_{j=1}^n \pi(x_n))$ converges strongly to $P_{f,M}$; hence $(\pi(\mu_n))$ converges to $P_{f,M} \circ P_{f,N} = P_f$. □

Examples. a) Let H be the *Heisenberg* group. If (μ_n) is a sequence of probability measures, we claim that the following statements are equivalent:

(i) (μ_n) is strongly ergodic.
(ii) (μ_n) is weakly ergodic.
(iii) $\widehat{\mu}_n(\gamma) \to 0$ for all $\gamma \in \widehat{H} \backslash \{1\}$.

Here $\widehat{H}$ denotes the set of *abelian* continuous characters of H, i.e. in this example strong (or weak) ergodicity is uniquely determined by the projections of μ_n to H/Z, where $Z = [H, H]$ is the center of H.

Proof. We use the notation of [8], Proposition 6. Condition (iii) is clearly necessary, since $\widehat{G}$ describes the one-dimensional representations of G. Hence it is sufficient to show that (iii) implies (i). We write $H = \mathbb{R}^3$ (as a set). Then the infinite dimensional irreducible representations of H act on $H^\pi = L^2(\mathbb{R})$ by

$$\big(\pi(x)f\big)(t) = e^{2\pi i(x_1 - x_2 t)a} f(t - x_3)$$

($x = (x_1, x_2, x_3)$, $a \in \mathbb{R}\backslash\{0\}$ is a fixed parameter). It is clearly enough to show that $\pi(\mu_n)f \to 0$ for f with bounded support, i.e. supp $f \subseteq [-K, K]$ for some $K > 0$. Then $\langle \pi(x)f, f\rangle = 0$ if $|x_3| > 2K$.

Put $A = \{(x, y) \in H \times H : |x_3 - y_3| \leq 2K\}$. Then it follows that $\|\pi(\mu)f\|^2 \leq \|f\|^2 \mu \otimes \mu(A)$. Hence it is sufficient to show that

$$\mu_n \otimes \mu_n(A) \to 0$$

for every sequence (μ_n) satisfying (iii).

Put $A_j = \mathbb{R}^2 \times]2K(j-1), 2K(j+1)]$, $\quad \alpha_{nj} = \mu_n(A_j)$. Then $A \subseteq \bigcup_{j\in\mathbb{Z}} A_j \times A_j$; hence

$$\mu_n \otimes \mu_n(A) \leq \sum_j \alpha_{nj}^2.$$

Furthermore, $\sum_j \alpha_{nj} \leq 2$ (observe that $A_j \cap A_k = \phi$ for $|j - k| \geq 2$).

Put $\overline{\mu}_n(M) = \mu_n(\mathbb{R}^2 \times M)$. Then $(\overline{\mu}_n)$ is a sequence of probability measures on $\mathbb{R}$. By assumption (iii), the sequence $(\overline{\mu}_n)$ converges to the Bohr-von Neumann mean m on $AP(\mathbb{R})$ (we have $Z = \{(x_1, 0, 0)\}$).

Fix $t \in \mathbb{N}$ with $t \geq 6$. Let f be a continuous function on $\mathbb{R}$ with period tK, satisfying $0 \leq f \leq 1$ and

$$f(x) = \begin{cases} 1 & \text{for} \quad |x| \leq 2K, \\ 0 & \text{for} \quad 3K \leq x \leq (t-3)K. \end{cases}$$

Then

$$m(f) = \frac{1}{tK} \int_0^{tK} f(x)dx < \frac{6}{t} \ .$$

Hence there exists n_0 such that $\langle f, \overline{\mu}_n\rangle < 6/t$ for $n \geq n_0$. Then it follows that $\alpha_{nj} < 6/t$ for $n \geq n_0$, $j = 0, \pm t, \pm 2t, \ldots$. Considering appropriate translates of f, we get the same estimate for the other residue classes mod t, i.e.

$$\alpha_{nj} < \frac{6}{t} \quad \text{for} \quad n \geq n_1, \quad j \in \mathbb{Z}.$$

This implies $\sum_j \alpha_{nj}^2 < 2 \cdot \dfrac{6}{t}$ for $n \geq n_1$, and for $t \to \infty$ our claim follows.

Further results of this type (in the setting of uniform distribution) have been shown in [17].

b) A similar description holds for the '$ax+b$'-group (compare [8], Proposition 7). In particular, a) and b) provide examples of non-Moore groups for which strong and weak ergodicity are equivalent.

c) For $G = \mathbb{C} \underset{\sigma}{\times} \mathbb{T}$, the *euclidean motion* group of the plane, the situation is different. For measures μ_n on G, let as before $\overline{\mu}_n$ be the projections to $\mathbb{T}$, m denotes normalized Haar measure on $\mathbb{T}$. Then we have

(i) (μ_n) is weakly ergodic if and only if $\overline{\mu}_n \to m$ (w^*) and $\mu_n \to 0$ (with respect to $C_0(G)$).

(ii) (μ_n) is strongly ergodic if and only if $\overline{\mu}_n \to m$ (w^*) and $\delta_{x_n} * \mu_n \to 0$ (with respect to $C_0(G)$) for arbitrary sequences $(x_n) \subseteq G$, i.e. the convergence $\mu_n(xK) \to 0$ holds uniformly for the translates of a given compact set K.

(δ_x denotes the Dirac measure concentrated at x.) For example, $\mu_n = \delta_{x_n} * m$, where x_n is a sequence in G tending to infinity, establishes a sequence of measures that is weakly but not strongly ergodic. □

Proof. (i) follows immediately from [8], Proposition 8.

(ii) results from the following lemma. (Necessity of the condition is obvious since (μ_n) strongly ergodic implies $(\delta_{x_n} * \mu_n)$ strongly ergodic.) □

Lemma 4.7. *Let G be a locally compact group, π a unitary representation of G whose coefficients $\phi^\pi_{\xi,\eta}$ belong to $C_0(G)$ and let (μ_n) be a sequence of probability measures on G such that $\mu_n(xK) \to 0$ uniformly for $x \in G$ (where K is a fixed compact subset of G with non-empty interior). Then $\pi(\mu_n) \to 0$ in the strong operator topology.*

Proof. Assume $\|\xi\| \leq 1$. We have

$$\|\pi(\mu_n)\xi\|^2 = \int_G \int_G \phi^\pi_{\xi,\xi}(y^{-1}x) d\mu_n(x) d\mu_n(y).$$

For $\varepsilon > 0$ choose K such that $|\phi^\pi_{\xi,\xi}(z)| < \varepsilon$ for $z \notin K$ (the condition for (μ_n) does not depend on the choice of K). Then $\mu_n(yK) < \varepsilon$ for $n \geq n_0,\ y \in G$. Since $y^{-1}x \in K$ is equivalent to $x \in yK$, and $|\phi^\pi_{\xi,\xi}(z)| \leq 1$ for all z, this gives combined

$$\Big| \int_G \phi^\pi_{\xi,\xi}(y^{-1}x) d\mu_n(x) \Big| < 2\,\varepsilon \quad \text{for} \quad n \geq n_0,\ y \in G.$$

Hence $\|\pi(\mu_n)\xi\|^2 < 2\varepsilon$ for $n \geq n_0$. □

d) A similar description (as in c) but taking into account that there are no non-trivial finite dimensional unitary representations) holds for the case of non-compact, connected, simple Lie groups with finite center (compare [8], Proposition 5).

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