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ON MINIMAL PARABOLIC FUNCTIONS AND TIME-HOMOGENEOUS PARABOLIC h-TRANSFORMS

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Abstract. Does a minimal harmonic function h remain minimal when it is viewed as a parabolic function? The question is answered for a class of long thin semi-infinite tubes $D \subset \mathbb{R}^d$ of variable width and minimal harmonic functions h corresponding to the boundary point of D "at infinity." Suppose $f(u)$ is the width of the tube u units away from its endpoint and f is a Lipschitz function. The answer to the question is affirmative if and only if $\int^\infty f^3(u)du = \infty$. If the test fails, there exist parabolic h-transforms of space-time Brownian motion in D with infinite lifetime which are not time-homogenous.

1. Introduction and main results

We want to compare the parabolic Martin boundary of a domain in $\mathbb{R}^d$ with its Martin boundary, both topologically and probabilistically. In many cases, the two boundaries are related in a very simple way. This provides a complete description of the parabolic Martin boundary in those cases (quite many) when the Martin boundary is known. We plan to present a detailed discussion of this general problem in a separate publication. This paper is devoted to a narrower aspect of the relationship between the two boundaries. We will start with a very informal discussion of a special case which motivated our study. The concepts of the usual and parabolic Martin boundary will be reviewed in a rigorous way later in the introduction. The basic ideas of classical potential theory and Brownian motion may be found in Doob's book ([Db], 1984), to which we will frequently refer.

Consider a strip $D = \{(x^1, x^2) \in \mathbb{R}^2 : |x^2| < 1\}$. Let X_t be a Brownian motion starting from $(0, 0)$. Then $\dot{X}_t = (X_t, -t)$ is a space-time Brownian motion starting from $(0, 0, 0)$. First fix some $s > 0$, a point $z \in \partial D$ and a sequence of points $\{z_k\}$ in D converging to z as $k \to \infty$. Condition $\dot{X}$ to be at $(z_k, -s)$ at time s and to not leave $D \times \mathbb{R}$ before time s. Then let k go to infinity. The conditioned processes converge in distribution to a process whose first coordinate is a Brownian motion conditioned to exit D through z at time s. The lifetime of this process is finite. This conditioned space-time Brownian motion is not time-homogeneous, i.e., its

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transition probabilities $P(\dot{X}_u \in (dy, -du) \mid \dot{X}_t \in (dx, -dt))$ depend not only on $u - t$, but on the values of t and u as well.

Next suppose that $c > 0$ is a constant and consider $\dot{X}$ conditioned to be at $(ck, 0, -k)$ at time k and to not leave $D \times \mathbb{R}$ before time k. In the limit, as $k \to \infty$, we obtain a process whose spatial component escapes to "$+\infty$" within D at rate c. The first coordinate of the space process is a one-dimensional Brownian motion with drift c. This conditioned space-time Brownian motion is time-homogeneous and its lifetime is infinite.

The domain in our example, a strip, seems to be typical and we would expect that many domains have the property stated in the following problem.

(1.1) Problem. *Find necessary and sufficient conditions of a geometric nature in D such that for every minimal parabolic function h in $\dot{D}$, the corresponding h-transform of the space-time Brownian motion is time-homogeneous if and only if its lifetime is a.s. infinite.*

Another source of motivation may be explained in purely analytic language. Recall the domain of our first example, $D = \{(x^1, x^2) \in \mathbb{R}^2 : |x^2| < 1\}$. Consider a minimal positive harmonic function $h(x)$, $x \in D$. Let $g(x,t) = h(x)$ for all $x \in D$ and $t \in \mathbb{R}$. Evidently, g is a parabolic function, and we may therefore identify every harmonic function with a parabolic function. Since h is minimal harmonic, it corresponds to a minimal Martin boundary point y of D. Suppose that y is also a Euclidean boundary point, say, $y = (1,1)$. Then g is not minimal as a parabolic function, i.e., it is a mixture of different parabolic functions. An easy probabilistic justification can be based on the fact that Brownian motion conditioned by h has a random lifetime. Thus the space-time Brownian motion conditioned by g is a mixture of processes conditioned to exit D through y at different times s, i.e., a mixture of g_s-transforms for different parabolic functions g_s. However, if y is the point at "$+\infty$", then g is minimal in the space of parabolic functions. While not completely obvious, this is simple to show directly, and also follows from our main result, Theorem 1.3 below. Our informal discussion suggests that in many domains, a minimal harmonic function is also minimal in the space of parabolic functions if and only if it corresponds to a "point at infinity." We propose the following problem.

(1.2) Problem. *Determine which minimal harmonic functions are minimal in the space of parabolic functions.*

We are not able to give a complete answer to either of the two problems but we hope that our main result, Theorem 1.3 below, will shed light on both.

We proceed with a rigorous presentation of our results. We start with a review of basic definitions and facts concerning Martin boundaries and conditioned Brownian motion. Let D be a Euclidean domain, that is, an open connected subset of $\mathbb{R}^d$ for some $d \geq 2$. We will consider the domain $\dot{D} \stackrel{\mathrm{df}}{=} D \times (-\infty, 0) \subset \mathbb{R}^{d+1}$. Let $G(x,y) = G_D(x,y)$ and $\dot{G}(u,v) = \dot{G}_{\dot{D}}(u,v)$ be the Green functions for $(1/2)\Delta$ on D and for the heat operator $(1/2)\Delta - \partial/\partial t$ on $\dot{D}$ where Δ is the Laplace operator (see 1.VII.1 and 1.XVII.4 of [Db]). Thus $G : D \times D \to (0, \infty]$ and $\dot{G} : \dot{D} \times \dot{D} \to [0, \infty]$. For $u = (x,s) \in \dot{D}$ and $v = (y, s-t) \in \dot{D}$ we have that

$$\dot{G}(u,v) = \begin{cases} p_t(x,y), & \text{for } t > 0, \\ 0, & \text{for } s < t \leq 0, \end{cases}$$

where $p_t = p_t^D$ is the heat kernel on D (that is, the transition function for Brownian motion killed upon leaving D). Note that this formula can also be used to define $\dot{G}((x,s),v)$ when $s=0$. A function $h : D \to [0,\infty)$ is *harmonic* if $\Delta h = 0$ on D. A function $g : \dot{D} \to [0,\infty)$ is *parabolic* if it solves the heat equation

$$\frac{\partial g}{\partial t} = \frac{1}{2}\Delta_x g$$

in $\dot{D}$. In this case, it is *superparabolic* as well; that is,

$$g(x,s) \geq \int g(y,s-t)p_t(x,y)dy$$

for every $(x,s) \in \dot{D}$ and $t > 0$. We may extend g by letting

$$g(x,0) \stackrel{\text{df}}{=} \lim_{t \downarrow 0} \int g(y,-t)p_t(x,y)dy$$

(the limit is easily seen to be monotone). Fix some $x_0 \in D$. We say that g is *admissible* if $g(x_0,0) < \infty$.

Now recall the definitions of the Martin boundary in the elliptic and parabolic contexts (1.XII.3 and 1.XIX.3 of [Db]). Let

$$K(x,y) \stackrel{\text{df}}{=} \frac{G(x,y)}{G(x_0,y)}$$

for $x, y \in D$. Then, up to homeomorphism, there is a unique metrizable compactification D^M of D such that

(i) the function $K(\,\cdot\,,\,\cdot\,)$ may be extended continuously to $D \times (D^M \setminus \{x_0\})$;
(ii) $K(\,\cdot\,,x) \equiv K(\,\cdot\,,y)$ if and only if $x = y$.

The set $\partial^M D \stackrel{\text{df}}{=} D^M \setminus D$ is called the *Martin boundary* of D. For $z \in \partial^M D$ and $y_k \in D$, we have $y_k \to z$ if and only if $K(x,y_k) \to K(x,z)$ for every $x \in D$. A harmonic function $h > 0$ is said to be minimal if, whenever $h' > 0$ is harmonic, and $h' \leq h$, it follows that $h' = ch$ for some constant c. A point $z \in \partial^M D$ is said to be minimal if $K(\,\cdot\,,z)$ is minimal. For every $h > 0$ harmonic, there is a unique measure μ, concentrated on the set $\partial_0^M D$ of minimal points of $\partial^M D$, such that

$$h(x) = \int_{\partial_0^M D} K(x,z)\mu(dz),$$

for every $x \in D$ (see 1.XII.9 of [Db]).

Now define $\dot{K}$ on $\dot{D} \times \dot{D}$ by

$$\begin{aligned}\dot{K}((x,s),(y,t)) &\stackrel{\text{df}}{=} \frac{\dot{G}((x,s),(y,t))}{\dot{G}((x_0,0),(y,t))}\\ &= \begin{cases} p_{s-t}(x,y)/p_{-t}(x_0,y), & t < s < 0,\\ 0, & s \leq t < 0.\end{cases}\end{aligned}$$

Then up to homeomorphism, there is a unique metrizable compactification $\dot{D}^M$ of $\dot{D}$ with the following properties:

(i) the function $\dot{K}$ has an extension to $\dot{D} \times \dot{D}^M$ such that for each $(x,s) \in \dot{D}$, the function $\dot{K}((x,s),\,\cdot\,)$ is finite valued and continuous on $\dot{D}^M \setminus \{(x,s)\}$;
(ii) $\dot{K}(\,\cdot\,,u) = \dot{K}(\,\cdot\,,v)$ if and only if $u = v$.

We call u the *pole* of $\dot K(\,\cdot\,,u)$. We write $\partial^M\dot D \stackrel{\text{df}}{=} \dot D^M\setminus\dot D$ and call it the *Martin boundary* of $\dot D$ (or the parabolic Martin boundary of D). Again we have that, for $z\in\partial^M\dot D$ and $(y_k,t_k)\in\dot D$, $(y_k,t_k)\to z$ if and only if $\dot K((x,t),(y_k,t_k))\to \dot K((x,t),z)$ for every $(x,t)\in\dot D$. Every $\dot K(\,\cdot\,,z)$ is admissible (see 1.XIX.3.1 of [Db]).

We denote by $\dot 0$ the unique point of $\partial^M\dot D$ for which $K(\,\cdot\,,\dot 0)\equiv 0$. It is unique by (ii) and exists as the limit of some subsequence of $(x_0,1/n)$. A point $z\in\partial^M\dot D$ is *minimal* if $\dot K(\,\cdot\,,z)$ is minimal as a parabolic function, and $\dot K((x_0,0),z)=1$. The set of minimal points is denoted by $\partial^M_0\dot D$. The integral representation of admissible parabolic functions as

$$g(x,t)=\int_{\partial_0^M\dot D}\dot K((x,t),z)\mu(dz)$$

is entirely analogous to that of the harmonic setting (see 1.XIX.7 of [Db]).

Let $(\Omega,\mathcal F)$ be a measurable space with $X:\Omega\times[0,\infty)\to\mathbb R^d\cup\{\delta\}$ a stochastic process. We use the notation X_t and $X(t)$ interchangeably. P^x is a probability measure under which X is a standard d-dimensional Brownian motion started from x, and killed upon leaving D. We write E^x for the corresponding expectation. In particular, δ is a cemetery point adjoined to D, X is continuous on a random time interval $[0,\zeta)$, and $X_t=\delta$ for $t\ge\zeta$.

Let $\tau_t=\tau_0-t$ be a process measuring absolute time, and write $\dot X_t=(X_t,\tau_t)$. By enlarging Ω if necessary, we may suppose that for each $s\le 0$, there are probability measures $P^{x,s}$ under which X has the same law as under P^x, and $\tau_0=s$. That is, $\{\dot X_t,t\ge 0\}$ is a space-time Brownian motion starting from (x,s).

If $h:D\to(0,\infty]$ is a superharmonic function, then

$$p^h_t(x,y)\stackrel{\text{df}}{=}\frac{h(y)p_t(x,y)}{h(x)}$$

is the transition function of a Markov process X^h, called an h-transform, or conditioned Brownian motion. We write P^x_h and E^x_h for the corresponding probability measure, and its expectations. By convention, h is taken to vanish at δ. If $x\in D^M$, $x\ne x_0$, then we write X^x for $X^{K(\,\cdot\,,x)}$. If $h=\int_{\partial_0^M}K(\,\cdot\,,z)\mu(dz)$ is harmonic, then

$$P^x_h=\frac{1}{h(x)}\int_{\partial_0^M}K(x,z)P^x_z\,\mu(dz).$$

The paths of X^h converge a.s. to points of the minimal Martin boundary, at their lifetimes (see 3.III.1 of [Db], or section 7.2 of Pinsky ([P], 1995)).

Similarly, if $g:\dot D\to[0,\infty]$ is a superparabolic function, then

$$\dot p^g_t((x,s),(y,s-t))\stackrel{\text{df}}{=}\frac{g(y,s-t)p_t(x,y)}{g(x,s)}$$

is the transition function for a Markov process $\dot X^g$ taking values in $\dot D\cup\{\delta\}$ (actually in $\{\delta\}\cup\{u\in\dot D;\,g(u)>0\}$) that we call a conditioned space-time Brownian motion. We will use $P^{x,s}_g$ to denote a probability measure under which $\dot X^g$ has this transition function and starts from (x,s). We write X^g for the spatial component of $\dot X^g$ (with

$X^g_t = \delta$ for $t \geq \zeta$), and note that

$$\dot{X}^g_t = \begin{cases} (X^g_t, \tau_t) \in \dot{D}, & \text{for } t < \zeta, \\ \delta & \text{for } t \geq \zeta. \end{cases}$$

We will also refer to X^g as a g-transform. This abuse should cause no confusion, as it is easy to check that if h is superharmonic and we define a superparabolic function g by $g(x,t) = h(x)$, then $X^h = X^g$. If $u \in \dot{D}^M$, then we write $\dot{X}^u$, X^u, $P^{x,s}_u$ instead of $\dot{X}^{\dot{K}(\,\cdot\,,u)}$, etc. Strictly speaking, the above formulae hold under $P^{x,s}_g$ only for $s < 0$, but by taking $X^g_0 = x$ under $P^{x,0}_g$, we obtain extensions valid for $s = 0$ as well, provided g is admissible. If g is actually parabolic, then each g-process approaches the one-point boundary of $\dot{D}$ at its lifetime ζ (2.X.12 of [Db]), in other words, it eventually leaves every compact subset of $\dot{D}$. In the Martin topology, the paths of $\dot{X}$ converge at their lifetimes, to points of the minimal parabolic Martin boundary, and the measures $P^{x,s}_g$ can be represented in terms of the $P^{x,s}_u$, for $u \in \partial^M_0 \dot{D}$, just as in the harmonic setting.

For $(x^1, x^2, \dots, x^d) \in \mathbb{R}^d$ let $\widetilde{x} = (x^1, x^2, \dots, x^{d-1})$. We will restrict our attention to "tubes" with variable width. For a non-negative function $f : \mathbb{R} \to \mathbb{R}$, let

$$D_f \stackrel{\mathrm{df}}{=} \{x \in \mathbb{R}^d : |\widetilde{x}| < f(x^d)\}.$$

We will always assume that f is strictly positive on (a,b) for some $-\infty \leq a < b \leq \infty$ and equal to 0 on $(-\infty, a] \cup [b, \infty)$. We will focus on domains D_f corresponding to functions f which are Lipschitz on (a,b) (the function may have a jump at a or b). If f is Lipschitz and $b = \infty$, then each sequence x_k of points in D_f such that $x^d_k \to \infty$ converges in the Martin topology to a point (the same for all such sequences) which we will denote as ∞. The proof of this claim is easy — it may be based on the boundary Harnack principle. The same result should be true for all functions f (not necessarily Lipschitz) but we do not see an obvious argument. An analogous remark applies to $-\infty$. Any positive harmonic function h corresponding to $\infty \in \partial^M D_f$ vanishes on $\{x \in \partial D_f : x^d < b\}$ and, moreover, $h(x) \to 0$ when $x^d \to -\infty$.

Let $\Lambda_s = \{x \in D_f : x^d = s\}$. The stopping time $\inf\{t > 0 : X_t \in A\}$ will be denoted $T(A)$. We write $\tau(A)$ for the absolute time $\tau_{T(A)} = \tau_0 - T(A)$.

Recall that a harmonic function h is identified with a parabolic function by letting $h(x,t) = h(x)$.

(1.3) Theorem. *Suppose that $b = \infty$ and f is a function which is Lipschitz on (a,b) and such that*

$$\limsup_{v \to \infty} f(v) < \infty$$

and

$$\int_u^\infty f(v)dv = \infty \tag{1.4}$$

for all $u < \infty$. Let h be the minimal harmonic function corresponding to $\infty \in \partial^M_0 D_f$. Fix some $x_0 \in D_f$.

(i) *Suppose that either*

(a) $\int_a^\infty f^3(v)dv < \infty$ *or*

(b) *the Lipschitz constant of f is sufficiently small (it will suffice to assume that it is less than the λ in (iv) of Theorem 1.6) and $\int_u^\infty f^3(v)dv < \infty$ for some $u < \infty$.*

Each one of assumptions (a) or (b) implies (A)–(D) below.

(A) *For some function $g : (a,\infty) \to (-\infty, 0]$ with $\lim_{u\to\infty} g(u) = -\infty$, we have the following. For each $s \in \mathbb{R}$ there is a minimal point $z_s \in \partial_0^M \dot{D}_f$, which is the limit of all sequences $(x_k, (g(x_k^d) - s_k) \wedge 0)$ with $x_k^d \to \infty$ and $s_k \to s$.*

(B) *If $s_1 \neq s_2$, then $z_{s_1} \neq z_{s_2}$.*

(C) *Let h_s denote a minimal parabolic function with pole at z_s. Then $h = \int_{\mathbb{R}} h_s \mu(ds)$ for some measure μ which charges all non-degenerate intervals. In particular, h is not minimal in the space of parabolic functions on $\dot{D}_f$.*

(D) *Let $s \in \mathbb{R}$ and $(x,t) \in \dot{D}$. The process $\dot{X}$ is not time-homogeneous under $P_{z_s}^{x,t}$. In fact, $g(u) - \tau(\Lambda_u) \to s$ as $u \to \infty$ $P_{z_s}^{x,t}$-a.s. Hence, $\lim_{u\to\infty}(T(\Lambda_u) + g(u))$ exists P_h^x-a.s.*

(ii) *If $\int_u^\infty f^3(v)dv = \infty$ for all $u < \infty$, then h is minimal in the space of parabolic functions on $\dot{D}_f$.*

(1.5) *Remarks.* The lifetime of Brownian motion conditioned by h is infinite if and only if $\int_u^\infty f(v)dv = \infty$ for all $u < \infty$, according to Theorem 1.6 below. If this condition is not satisfied, the function h is not minimal as a parabolic function (see the discussion preceding Problem 1.2).

The proof of Theorem 1.3 hinges on estimates of the variance of h-path lifetimes. Since the estimates may have some independent interest, we state them as Theorem 1.6 below.

Several authors have addressed the problem of when, given a domain $D \subset \mathbb{R}^d$, there is a constant $c = c(D) < \infty$ such that for any $x \in D$ and any positive harmonic function h in D we have $E_h^x \zeta < c$. The pioneering work was done by Cranston and McConnell ([CM], 1983) and Cranston ([C], 1985). The existence of the finite upper bound c is known for a wide class of domains; see, e.g., Bañuelos and Davis ([BD], 1992) or Bass and Burdzy ([BB2], 1992) and references therein. Higher moments of h-path lifetimes have been studied by Davis ([Dv], 1988), Davis and Zhang ([DZ], 1994) and Zhang ([Z], 1996).

Chris Rogers has pointed out to us that a related equivalence, between non-minimality and the variance of hitting times, has been established in the context of one-dimensional diffusions. There, the speed measure and coupling can be used to give a simple proof. See Rogers ([Rg], 1988), which synthesizes earlier work of Fristedt and Orey ([FO], 1978), Küchler and Lunze ([KL], 1980), and Rösler ([Rs], 1979).

Recall that we are concerned with functions f which are strictly positive and Lipschitz on (a,b) and equal to 0 on $(-\infty, a] \cup [b,\infty)$. Our next result holds for all functions f which are Lipschitz on (a,b). However, in order to simplify the notation we will prove it only in the case when f is Lipschitz with the constant equal to 1, i.e., from now on we will assume that $|f(u) - f(v)| \le |u - v|$ for $u, v \in (a,b)$. Fix some $s_0 \in (a,b)$ and define s_k inductively by $s_{k+1} = s_k + f(s_k)/2$ for $k \ge 0$ and $s_{k-1} = s_k - f(s_k)/2$ for $k \le 0$. If $s_k \ge b$ for some k, then we redefine s_j for $j \ge k$ and we let $s_j = b$ for all $j \ge k$. A similar remark applies to the case when $s_k \le a$. Note that it may happen that $s_k < b$ for all $k > 0$ and/or $s_k > a$ for all $k < 0$. However, we always have $\lim_{k\to\infty} s_k = b$ and $\lim_{k\to-\infty} s_k = a$. Let

$k_f = \inf\{k : s_k = b\}$ and recall that $\Lambda_{s_k} = \{x \in D_f : x^d = s_k\}$. Let D_j be the component of $D_f \setminus \Lambda_{s_j}$ which contains points x with $x^d < s_j$.

(1.6) Theorem. *Let h be a positive harmonic function in D_f which vanishes on $\{x \in \partial D : x^d < b\}$. If $b = \infty$, then h corresponds to $\infty \in \partial_0^M D_f$. In the following statements, x ranges over the elements of D_f with $x^d < b - f(b-)$ (here $\infty - \infty = \infty$).*

(i) *For some $c_1, c_2 \in (0, \infty)$,*

$$c_1 \int_{x^d}^b f(v)dv \le E_h^x \zeta \le c_2 \int_{x^d}^b f(v)dv. \tag{1.7}$$

(ii) *If $\int_{x^d}^b f(v)dv = \infty$, then $\zeta = \infty$ P_h^x-a.s.*

(iii) *If $\zeta < \infty$ P_h^x-a.s., then for some $c_3, c_4 \in (0, \infty)$,*

$$c_3 \int_{x^d}^b f^3(v)dv \le \operatorname{Var}_h^x \zeta \le c_4 \int_a^b f^3(v)dv. \tag{1.8}$$

(iv) *There exists $\lambda > 0$ such that if the Lipschitz constant of f is less than λ, then*

$$\operatorname{Var}_h^x \zeta \le c_5 \int_{x^d}^b f^3(v)dv. \tag{1.9}$$

(v) *If $\int_{x^d}^b f^3(v)dv = \infty$, then for each $c_6 < \infty$ and $c_7 > 0$ there is a $k_0 < \infty$ such that for all $k > k_0$ and $u \in \mathbb{R}$,*

$$P_h^x(T(\Lambda_{s_k}) \in (u, u + c_6)) < c_7.$$

(1.10) *Remarks.* (i) The constants c_j in Theorem 1.6 depend only on the dimension d and the Lipschitz constant of f. However, the proof will be given only in the case when the Lipschitz constant of f is equal to 1 so all the constants in Section 2 will depend only on dimension d.

(ii) The bound (1.9) holds for $d \ge 4$ without any assumptions on the value of the Lipschitz constant of f but it does not hold without such an assumption for $d < 4$. We are not going to prove the latter. It essentially follows from a theorem of Davis and Zhang ([DZ], 1994).

(iii) We can give a meaning to (1.8) and (1.9) even if $\zeta = \infty$ P_h^x-a.s. Note that in such a case we necessarily have $b = \infty$ (see (1.7)). For all $k < \infty$ and $x \in D_f$ such that $x^d < s_k$,

$$\operatorname{Var}_h^x T(\Lambda_{s_k}) < c_4 \int_a^b f^3(v)dv$$

with the same constant c_4 as in (1.8). This and the analogous modification of (1.9) can be proved by applying the theorem to the function $\widetilde{f}(v) \stackrel{\mathrm{df}}{=} f(v)\mathbf{1}_{(-\infty, s_k)}(v)$.

(iv) In the two-dimensional case, part (i) of Theorem 1.6 is due to Xu ([X], 1990). This was generalized by Bañuelos and Davis ([BD], 1992).

(v) Suppose that $d = 2$, the Lipschitz constant of f is small and let ρ be the supremum of areas of discs contained in D_f. Then (1.7) and (1.9) imply that $\operatorname{Var}_h^x \zeta \le c_1 \rho E_h^x \zeta$. Davis ([Dv], 1988) discovered this inequality and proved that it holds for all simply connected planar domains D provided h is a minimal positive harmonic function or a Green function.

We would like to thank Rodrigo Bañuelos, Rich Bass and Burgess Davis for some very useful discussions of h-path lifetimes.

2. Moments of h-transform lifetimes

This section contains the proof of Theorem 1.6. We start with a short review of some useful facts about h-processes. The proofs may be found in [Db] and in a paper of Meyer, Smythe and Walsh ([MSW], 1972).

Let $D \subset \mathbb{R}^d$ be a Greenian domain and h be a positive superharmonic function in D. Suppose that M is a closed subset of D and let $L = \sup\{t < \zeta : X_t \in M\}$ be the last exit time from M. Let

$$\begin{aligned} &Y_1(t) = X(t), \quad t \in (0, T(M)),\\ &Y_2(t) = X(T(M)+t), \quad t \in (0, \zeta - T(M)),\\ &Y_3(t) = X(t), \quad t \in (0, L),\\ &Y_4(t) = X(L+t), \quad t \in (0, \zeta - L),\\ &Y_5(t) = X(\zeta - t), \quad t \in (0, \zeta). \end{aligned}$$

Under P_h^x, each process Y_k is an h_k-transform in a domain D_k, where $D_1 = D_4 = D \backslash M$ and $D_2 = D_3 = D_5 = D$. Moreover, $h_1 = h_2 = h$. The function h_3 is a potential supported by ∂M. The function h_4 is harmonic and has the boundary values 0 on ∂M and the same boundary values as h on $\partial D \backslash \partial M$. The function h_5 is the Green function $G_D(x, \cdot\,)$ if $x \in D$, or a harmonic function with a pole at x if $x \in \partial D$.

If $\mu(dy)$ is the P^x-distribution of $X(T(M))$, then the P_h^x-distribution of this random variable is $\mu(dy)h(y)/h(x)$.

(2.1) Lemma (Brownian scaling)**.** *Suppose h is a positive superharmonic function in a domain $D \subset \mathbb{R}^d$ and $x \in D^M$. For a fixed $a \in (0, \infty)$ let*

$$\begin{aligned} D_a &\overset{\mathrm{df}}{=} \{y \in \mathbb{R}^d : y/a \in D\},\\ h_a(y) &\overset{\mathrm{df}}{=} h(y/a) \quad \textit{for} \;\; y \in D_a,\\ x_a &\overset{\mathrm{df}}{=} ax,\\ X_t^a &\overset{\mathrm{df}}{=} aX_{t/a^2} \quad \textit{for} \;\; t \geq 0. \end{aligned}$$

If X has the distribution P_h^x, then X^a has the distribution $P_{h_a}^{x_a}$.

Proof. The lemma follows immediately from the scaling properties of Brownian motion and superharmonic functions. □

A domain $D \subset \mathbb{R}^d$, $d \geq 2$, is called a *Lipschitz domain* if for every $x \in \partial D$ there is a neighborhood U_x of x, an orthonormal coordinate system CS_x and a Lipschitz function $f_x : \mathbb{R}^{d-1} \to \mathbb{R}$ with constant λ (independent of x) such that $\partial D \cap U_x$ is a part of the graph of f_x in CS_x. Note also that the index on any constant $c_1, c_2, \ldots$ is local in nature. That is, new results or sections of proofs will start numbering their constants with c_1 as well.

(2.2) Lemma (Boundary Harnack principle)**.** *(a) Suppose $f : \mathbb{R}^{d-1} \to \mathbb{R}$ is a Lipschitz function with constant $\lambda > 0$, $|f(x)| \leq 1$ for all $x \in \mathbb{R}^{d-1}$, and let*

$$\begin{aligned} D &= \{x \in \mathbb{R}^d : |\widetilde{x}| < 1, f(\widetilde{x}) < x^d < 2\},\\ D_1 &= \{x \in D : |\widetilde{x}| < 1/2, x^d < 3/2\}. \end{aligned}$$

There exists $c_1 > 0$ which depends on λ but otherwise does not depend on f such that for all $x, y \in D_1$ and all positive harmonic functions g, h in D which vanish continuously on $\{z \in \partial D : z^d = f(\widetilde{z})\}$ we have

$$\frac{g(x)}{g(y)} \geq c_1 \frac{h(x)}{h(y)}.$$

(b) Suppose D is a Lipschitz domain, Q is a compact set and A is an open set such that $Q \cap \overline{D} \subset A$. There exists $c_2 > 0$ such that for all $x, y \in Q \cap D$ and all positive harmonic functions g, h in D which vanish continuously on $\partial D \cap A$ we have

$$\frac{g(x)}{g(y)} \geq c_2 \frac{h(x)}{h(y)}.$$

For the first proofs of the boundary Harnack principle, see Ancona ([An], 1978), Dahlberg ([Dg], 1977) and Wu ([W], 1978). Stronger versions of the result may be found in Bass and Burdzy ([BB1], 1991) or Bañuelos, Bass and Burdzy ([BBB], 1991).

Part (a) of Lemma 2.2 holds (with the same c_1) in domains which may be obtained from D by scaling.

When applying the boundary Harnack principle we will sometimes leave it to the reader to find the right choice of D and D_1 or D, A and Q.

(2.3) Lemma. *Suppose D is a domain, D_1 is a Lipschitz subdomain of D, Q is a compact set, A is an open set such that $Q \cap \overline{D} \subset A$, $A \cap D \subset D_1$, and M is a Borel subset of $D \setminus A$. Assume that h is a positive superharmonic function in D which vanishes on $\partial D \cap A$ and is harmonic in D_1. Then*

$$P_h^x(T(M) < \infty) \leq c_1 P_h^y(T(M) < \infty)$$

for all $x, y \in Q \cap D$. The constant c_1 depends only on D_1, Q and A.

Proof. The function

$$x \to E^x[T(M) < T(\partial D), h(X(T(M)))]$$

is positive and harmonic in $A \cap D$ and the same is true for $x \to h(x)$. Let D_2 be a Lipschitz subdomain of $A \cap D$ which contains Q. By the boundary Harnack principle, Lemma 2.2(b), applied in D_2,

$$\begin{aligned} P_h^x(T(M) < \infty) &= \frac{1}{h(x)} E^x[T(M) < T(\partial D), h(X(T(M)))] \\ &\leq c_2 \frac{1}{h(y)} E^y[T(M) < T(\partial D), h(X(T(M)))] \\ &= c_2 P_h^y(T(M) < \infty). \quad \square \end{aligned}$$

(2.4) Lemma. *Suppose D is a domain and for each $k = 1, 2$,*

(i) *D_k is a subdomain of D,*
(ii) *$A_k \stackrel{\text{df}}{=} \partial D_k \cap D$,*
(iii) *V_k is an open set and Q_k is a compact set such that $Q_k \cap \overline{D} \subset V_k$ and $\overline{V}_k \cap D \subset D_k$,*
(iv) *$(D_1 \cup V_1) \cap (D_2 \cup V_2) = \emptyset$,*

(v) *there is a $c_k > 0$ such that for all $x, y \in Q_k \cap D$ and all positive harmonic functions f, g in D_k which vanish on $V_k \cap \partial D$ we have*

$$\frac{f(x)}{f(y)} \geq c_k \frac{g(x)}{g(y)}. \tag{2.5}$$

Assume that $x_1, x_2 \in \overline{Q_1 \cap D}$ and h_1, h_2 are positive superharmonic functions in D which vanish continuously on $\partial D \setminus V_2$ and are harmonic in $D \setminus Q_2$. Let $T_1 \stackrel{\text{df}}{=} T(A_1)$ and let T_2 be the last exit time from A_2. The distributions of $\{X_t, t \in [T_1, T_2]\}$ under $P^{x_1}_{h_1}$ and $P^{x_2}_{h_2}$ are mutually absolutely continuous and their Radon-Nikodym derivative is bounded below by $c_1 c_2$.

Proof. We will consider only the case when $x_k \in Q_1 \cap D$ and $h_k(\,\cdot\,) = G_D(\,\cdot\,, y_k)$ for some $y_k \in Q_2 \cap D$. Other points x_k and functions h_k may be treated analogously.

Under $P^{x_k}_{y_k}$, the process $\{X_t, t \in [T_1, \zeta]\}$ is a $G_D(\,\cdot\,, y_k)$-process with the initial distribution

$$\mu_k(\,\cdot\,) \stackrel{\text{df}}{=} P^{x_k}_{y_k}(X(T_1) \in \,\cdot\,) = P^{x_k}(T_1 < T(D^c), X(T_1) \in \,\cdot\,) G_D(\,\cdot\,, y_k)/G_D(x_k, y_k),$$

supported on A_1. For a fixed $z \in A_1$, the process $Y_t \stackrel{\text{df}}{=} X_{\zeta - t}$ under $P^z_{y_k}$ has the distribution $P^{y_k}_z$. If $T_3 = \inf\{t : Y_t \in A_2\}$, then $T_3 = \zeta - T_2$. The process $\{Y_t, t \in [T_3, \zeta)\}$ under $P^z_{y_k}$ is a $G_D(\,\cdot\,, z)$-process with the initial distribution

$$\nu_k(\,\cdot\,) \stackrel{\text{df}}{=} P^{y_k}(T(A_2) < T(D^c), X(T(A_2)) \in \,\cdot\,) G_D(\,\cdot\,, z)/G_D(y_k, z).$$

For a fixed $v \in A_2$, the function $y \to P^y(T(A_2) < T(D^c), X(T(A_2)) \in dv)$ is positive and harmonic in D_2 and vanishes on $V_2 \cap \partial D$ and the same is true for $z \to G_D(v, z)$. By (2.5),

$$\frac{d\nu_k}{d\nu_{3-k}}(v) = \frac{P^{y_k}(T(A_2) < T(D^c), X(T(A_2)) \in dv) G_D(v, z) G_D(y_{3-k}, z)}{G_D(y_k, z) P^{y_{3-k}}(T(A_2) < T(D^c), X(T(A_2)) \in dv) G_D(v, z)} \geq c_2.$$

After reversing time again, we see that the distributions of $X(T_2)$ under $P^z_{y_1}$ and $P^z_{y_2}$ have Radon-Nikodym derivative bounded below by c_2. The process $\{X_t, t \in [T_1, T_2]\}$ under $P^z_{y_1}$ is a mixture of h-transforms converging to w with the mixing measure $P^z_{y_1}(X(T_2) \in dw)$ and the same remark applies to $P^z_{y_2}$. Hence, the distributions of $\{X_t, t \in [T_1, T_2]\}$ under $P^z_{y_1}$ and $P^z_{y_2}$ have a Radon-Nikodym derivative bounded below by c_2.

We can prove in a similar way that $d\mu_k(\,\cdot\,)/d\mu_{3-k}(\,\cdot\,) \geq c_1$. The distributions of $\{X_t, t \in [T_1, T_2]\}$ under $P^{x_1}_{y_1}$ and $P^{x_2}_{y_2}$ have the Radon-Nikodym derivative bounded below by $c_1 c_2$ because $P^{x_k}_{y_k}$ is a mixture of the measures $P^z_{y_k}$ with the mixing measure μ_k. □

(2.6) Lemma. *Suppose that $f : \mathbb{R}^{d-1} \to \mathbb{R}$ is Lipschitz with constant λ and assume that $|f(x)| \leq 1$ for all x. Let*

$$D = \{x \in \mathbb{R}^d : |\widetilde{x}| < 1, f(\widetilde{x}) < x^d < 2\}.$$

There exists $c < \infty$ (which may depend on λ but does not otherwise depend on f) such that for every $x \in \overline{D}$ and every positive harmonic function h in D

$$E^x_h \zeta < c. \tag{2.7}$$

Proof. The result is essentially due to Cranston ([C], 1985) but we refer the reader to the paper by Bass and Burdzy ([BB2], 1992). Our domain D is a special case of a "twisted Hölder domain" and (2.7) follows from Theorem 1.1 (i), (a), (C) of [BB2]. A direct inspection of its proof shows that c depends only on the volume and diameter of D (under the assumption that f is Lipschitz with constant λ) and these quantities may be bounded independently of the particular form of f. $\square$

(2.8) *Remark.* It is not necessary to assume in Lemma 2.6 that f is Lipschitz. It is enough to suppose that f is upper semicontinuous and $f(x)$ is bounded in the L^p-norm for a suitable $p = p(d)$. This version of the result uses Theorem 1.1 (i), (a), (A) of [BB2] which has a considerably more complicated proof than Theorem 1.1 (i), (a), (C). We feel it would not be fair to ask the reader to go through the former proof in order to check that the constants may be chosen independently of f.

(2.9) Lemma. *Suppose that $D \subset \mathbb{R}^d$ is a domain, $x, y \in \overline{D}$, and for each $v = x, y$ there exist an orthonormal coordinate system CS_v, a point $z_v \in D$, a Lipschitz function f_v with constant λ and a constant $c_v > 0$ such that $|f_v| \le c_v$,*

$$\begin{aligned} D_v &\overset{\mathrm{df}}{=} \{z \in D : |\widetilde{z}| < c_v, -c_v < z^d < 2c_v \quad in\ CS_v\} \\ &= \{z \in \mathbb{R}^d : |\widetilde{z}| < c_v, f_v(\widetilde{z}) < z^d < 2c_v \quad in\ CS_v\}, \end{aligned}$$

$$\begin{gathered} z_v = (0, 0, \dots, 0, 3c_v/2) \quad in\ CS_v, \\ |\widetilde{v}| \le c_v/2 \quad and \quad v^d \le 3c_v/2 \quad in\ CS_v, \\ D_x \cap D_y = \emptyset. \end{gathered}$$

If $E^{z_x}_{z_y}\zeta = c_1$, then

$$E^x_y \zeta \le c_2 c_1 + c_3(c_x^2 + c_y^2)$$

where c_2 and c_3 depend only on the dimension d and the Lipschitz constant λ.

Proof. For $v = x, y$ let

$$\begin{aligned} D_v^1 &= \{z \in D_v : |\widetilde{z}| < 3c_v/4, z^d < 7c_v/4 \ \text{ in } CS_v\}, \\ A_v &= \partial D_v^1 \cap D, \\ Q_v &= \{z \in \overline{D}_v : |\widetilde{z}| \le c_v/2, z^d \le 3c_v/2 \ \text{ in } CS_v\}, \\ V_v &= \{z \in \mathbb{R}^d : |z - Q_v| < c_v/8\}. \end{aligned}$$

By the boundary Harnack principle, Lemma 2.2(a), applied in D_v, assumption (2.5) of Lemma 2.4 holds. Let T_1 be the first hitting time of A_x and let T_2 be the last exit time from A_y. By Lemma 2.4,

$$E^x_y(T_2 - T_1) \le c_4 E^{z_x}_{z_y}(T_2 - T_1) \le c_4 E^{z_x}_{z_y}\zeta. \tag{2.10}$$

Lemma 2.6 and Brownian scaling (2.1) imply that

$$E^x_y T_1 \le c_5 c_x^2. \tag{2.11}$$

The same lemma and time-reversal show that

$$E^x_y(\zeta - T_2) \le c_5 c_y^2. \tag{2.12}$$

The lemma follows from (2.10)–(2.12). $\square$

We now return to the specific domains, hypotheses, and notation of Theorem 1.6.

(2.13) Lemma. *Assume that $a < s_{j-1} < s_j < b$. There exists $c_1 > 0$ such that for every positive harmonic function h in D_j which vanishes on $\partial D_j \setminus \Lambda_{s_j}$ and every $x \in \Lambda_{s_{j-1}}$,*

$$E_h^x \zeta \geq c_1 f^2(s_{j-1}).$$

Moreover, there is a non-negative, non-constant and bounded random variable Y such that for every j and $x \in \Lambda_{s_{j-1}}$, the distribution of ζ under P_h^x is stochastically larger than that of $f^2(s_{j-1})Y$.

Proof. Let $B(y,r)$ denote the ball with center y and radius r. Let c_2 be the expected lifetime of conditioned Brownian motion in $B(0,1)$ starting from 0 and converging to $x \in \partial B(0,1)$. The constant c_2 is strictly positive and does not depend on x by symmetry. For any harmonic function g in $B(0,1)$, the g-process starting from 0 is a mixture of processes conditioned to go to some point of $\partial B(0,1)$ so its expected lifetime is also equal to c_2. By scaling, the expected lifetime of any Brownian motion conditioned by a harmonic function in $B(y,r)$ and starting from y is equal to $c_2 r^2$.

Let

$$B_0 = B((0,\dots,0,s_{j-1}+f(s_{j-1})/4), f(s_{j-1})/8),$$
$$T_1 = \inf\{t > T(B_0) : |X_t - X(T(B_0))| = f(s_{j-1})/16\}.$$

Note that $B_0 \subset D_j$. By the strong Markov property applied at $T(B_0)$,

$$E_h^x \zeta \geq E_h^x[(T_1 - T(B_0))\mathbf{1}_{\{T(B_0)<\infty\}}] = c_2(f(s_{j-1})/16)^2 P_h^x(T(B_0) < \infty). \tag{2.14}$$

Let $x_0 = (0,\dots,0,s_{j-1})$. By Lemma 2.3, for all $x \in \Lambda_{s_{j-1}}$,

$$P_h^x(T(B_0) < \infty) \geq c_3 P_h^{x_0}(T(B_0) < \infty). \tag{2.15}$$

It is not hard to see that the constant c_3 may be chosen independently of the particular form of f. The probability $P_h^{x_0}(T(B_0) < \infty)$ is not less than

$$P^{x_0}(T(B_0) < T(\partial D_j)) \inf_{y \in B_0} h(y)/h(x_0).$$

It is elementary to see that $P^{x_0}(T(B_0) < T(\partial D_j))$ is bounded below and the usual Harnack principle shows that the same is true for $\inf_{y\in B_0} h(y)/h(x_0)$. Hence, $P_h^{x_0}(T(B_0) < \infty)$ is bounded below by $c_4 > 0$ which together with (2.14) and (2.15) implies

$$E_h^x \zeta \geq c_2(f(s_{j-1})/16)^2 c_3 c_4.$$

It is clear from our proof that Y can be chosen as follows. Let $\widetilde{\zeta}$ be the hitting time of $\partial B(0,1/16)$ by a Brownian motion starting from 0 and let W be an independent random variable with $P(W=1) = 1 - P(W=0) = c_3 c_4$. Then let $Y = WY'$, where $Y' = c_2 \min(\widetilde{\zeta}, 1)$. □

(2.16) Lemma. *Suppose that $s_j < s_n$. Let $T_j^1 = T(\Lambda_{s_j})$ and*

$$S_j^k = \inf\{t > T_j^k : X_t \in \Lambda_{s_{j-1}} \cup \Lambda_{s_{j+1}}\}, \quad k \geq 1,$$
$$T_j^k = \inf\{t > S_j^{k-1} : X_t \in \Lambda_{s_j}\}, \quad k > 1.$$

There exist $c_1 < \infty$ and $p < 1$ such that for all k and for every positive harmonic function h in D_n which vanishes on $\partial D_n \setminus \Lambda_{s_n}$ and every $x \in D_n$

$$P_h^x(T_j^k < \infty) < c_1 p^k.$$

Moreover, if $i \geq 0$, $j + i < n$ and $x \in \Lambda_{s_{j+i}}$, then

$$P_h^x(T_j^k < \infty) < c_1 p^{k+i}.$$

Proof. Suppose $s_k < s_{k+1} \leq s_n$. We have

$$h(x) = \int_{\Lambda_{s_{k+1}}} h(y) P^x(X(T(\Lambda_{s_{k+1}})) \in dy) \tag{2.17}$$

for $x \in \Lambda_{s_k}$. The boundary Harnack principle implies that

$$\frac{P^{x_1}(X(T(\Lambda_{s_{k+1}})) \in dy)}{P^{x_2}(X(T(\Lambda_{s_{k+1}})) \in dy)} \cdot \frac{P^{x_2}(T(\Lambda_{s_{k+1}}) < \infty)}{P^{x_1}(T(\Lambda_{s_{k+1}}) < \infty)} < c_3 < \infty \tag{2.18}$$

for $x_1, x_2 \in \Lambda_{s_k}$. Let $z_k = (0, \dots, 0, s_k)$. It is easy to see that there is $c_4 > 0$ such that for all $x \in \Lambda_{s_k}$ with $|\widetilde{x}| > (1 - c_4) f(s_k)$, we have

$$P^x(T(\Lambda_{s_{k+1}}) < \infty) < (c_3^{-1}/2) P^{z_k}(T(\Lambda_{s_{k+1}}) < \infty).$$

This, (2.17) and (2.18) imply that $h(x) \leq h(z_k)/2$ for $x \in \Lambda_{s_k}$ with $|\widetilde{x}| > (1 - c_4) f(s_k)$. It follows that the maximum of h on Λ_{s_k} is attained at a point in the set

$$A_k \stackrel{\text{df}}{=} \{x \in \Lambda_{s_k} : |\widetilde{x}| \leq (1 - c_4) f(s_k)\}.$$

Let a_k be the maximum of h over Λ_{s_k}. Since

$$P^x(T(\Lambda_{s_{k+1}}) \leq T(\partial D_n)) < c_5 < 1$$

for $x \in \Lambda_{s_k}$, we have $a_k < c_5 a_{k+1}$ assuming $a < s_k < s_{k+1} < b$. It follows that $a_k < c_5^j a_{k+j}$. By the Harnack principle, $h(x) > c_6 a_k$ for some $c_6 > 0$ and all $x \in A_k$. Let m be so large that $c_6 c_5^{-m} > 2$. Then $a_k < h(x)/2$ for all $x \in A_{k+m}$ provided $a < s_k < s_{k+m} < b$. We obtain

$$P_h^x(T(\Lambda_{s_j}) < \infty) = \int_{\Lambda_{s_j}} \frac{h(y)}{h(x)} P^x(X(T(\Lambda_{s_j})) \in dy) \leq 1/2 \tag{2.19}$$

for $x \in A_{j+m}$. Here and later in the proof we assume that $a < s_j < s_{j+m} < b$. This assumption could be easily disposed of. We have

$$P^{z_k}(T(A_{k+1}) < T(\partial D_n \cup \Lambda_{s_{k-1}})) > c_7 > 0$$

and an application of the Harnack principle shows that

$$P_h^{z_k}(T(A_{k+1}) < T(\Lambda_{s_{k-1}})) > c_8 > 0.$$

By Lemma 2.3,

$$P_h^x(T(A_{k+1}) < T(\Lambda_{s_{k-1}})) > c_9 > 0 \tag{2.20}$$

for all $x \in \Lambda_{s_k}$. By the strong Markov property applied at the hitting times of A_i,

$$P_h^x(T(A_{j+m}) < T(\Lambda_{s_j})) > c_9^{m-1} \tag{2.21}$$

for all $x \in \Lambda_{s_{j+1}}$. Let

$$U_1 = \inf\{t > T(\Lambda_{s_{j+1}}) : X_t \in A_{j+m}\},$$
$$U_2 = \inf\{t > T(\Lambda_{s_{j+1}}) : X_t \in \Lambda_{s_j}\},$$
$$U_3 = \inf\{t > U_1 : X_t \in \Lambda_{s_j}\}.$$

Then (2.19)–(2.21) imply that for $x \in \Lambda_{s_j}$

$$P_h^x(T_j^2 = \infty) \geq P_h^x(T(\Lambda_{s_{j+1}}) < T(\Lambda_{s_{j-1}}), U_1 < U_2, U_3 = \infty) > c_9^m/2 > 0$$

for $x \in \Lambda_{s_j}$. Both conclusions of the lemma now follow by the repeated application of the strong Markov property at the stopping times T_j^k. □

(2.22) Lemma. *For all $x_1 \in D_f$ such that $s_{k+1} \leq x_1^d \leq s_{k+2}$ and $x_2 \in \Lambda_{s_k}$ we have $E_{x_2}^{x_1}\zeta < c_1 f^2(s_k)$ where $E_{x_2}^{x_1}$ refers to the conditioned Brownian motion in D_f.*

Proof. We will suppose that $x_1 \in \Lambda_{s_{k+1}}$. The modifications needed for the general case are obvious.

By Brownian scaling (2.1), we may assume that $f(s_k) = 1$ and prove that $E_{x_2}^{x_1}\zeta < c_1$. Note that then $|x_1^d - x_2^d| = 1/2$.

We have

$$E_{x_2}^{x_1}\zeta = c_2 \int_{D_f} \frac{G_{D_f}(x_1, z)G_{D_f}(z, x_2)}{G_{D_f}(x_1, x_2)} dz.$$

In view of Lemma 2.9 it will suffice to prove the lemma for $x_1 \in \Lambda_{s_{k+1}}$, $|\widetilde{x}_1| < c_3 f(s_{k+1})$, and $x_2 \in \Lambda_{s_k}$, $|\widetilde{x}_2| < c_3$ for some $c_3 < 1$. Under this additional assumption, x_1 and x_2 may be connected in D_f by a Harnack chain of balls of bounded length and this implies that $G_{D_f}(x_1, x_2) > c_4 > 0$. Hence,

$$E_{x_2}^{x_1}\zeta < c_5 \int_{D_f} G_{D_f}(x_1, z)G_{D_f}(z, x_2) dz. \tag{2.23}$$

Let

$$A_j = \{z \in D_f : |z - x_j| < 5, |z - x_{3-j}| > |x_1 - x_2|/2\}, \quad j = 1, 2,$$
$$A_3 = \{z \in D_f : |z - x_1| \geq 5, z^d < s_k\},$$
$$A_4 = \{z \in D_f : |z - x_1| \geq 5, z^d > s_{k+1}\}.$$

Assume for now that $d \geq 3$, and recall that $G(x, y) \stackrel{\text{df}}{=} G_{\mathbb{R}^d}(x, y) = c_6|x - y|^{2-d}$. For $j = 1, 2$ we obtain

$$\begin{aligned}\int_{A_j} G_{D_f}(x_1, z)G_{D_f}(z, x_2) dz &\leq \int_{A_j} G(x_1, z)G(z, x_2) dz \\ &\leq c_7 \int_{A_j} (|x_1 - x_2|/2)^{2-d}|z - x_j|^{2-d} dz \\ &\leq c_7(|x_1 - x_2|/2)^{2-d} \int_0^5 r^{2-d} r^{d-1} dr < c_8 < \infty.\end{aligned} \tag{2.24}$$

Let $x_0 = (0, \ldots, 0, s_k)$,

$$\widetilde{D} = \{x \in \mathbb{R}^d : x^d < s_k\},$$
$$D_* = D_f \cup \{x \in \mathbb{R}^d : x^d \in (-\infty, s_k) \cup (s_{k+1}, \infty)\},$$
$$M = \{x \in \widetilde{D} : |x - x_1| = 4\}.$$

The Poisson kernel $K(x)$ in $\widetilde{D}$ with the pole at x_0 has the form $c_9|x^d - s_k|/|x-x_0|^d$ (1.VIII.9 of [Db]). By the boundary Harnack principle,

$$G_{D_*}(x_1, x) \leq c_{10} K(x)$$

for $x \in M$ and, therefore, for all $x \in \widetilde{D}$ such that $|x - x_1| \geq 4$, in particular, for $x \in A_3$. Hence, for $x \in A_3$,

$$G_{D_*}(x_1, x) \leq c_{11}|x^d - s_k|/|x - x_0|^d \leq c_{11}|x - x_0|^{1-d}$$

and the same estimate holds for $G_{D_*}(x_2, x)$. It follows that

$$\tag{2.25} \begin{aligned} \int_{A_3} G_{D_f}(x_1, z) G_{D_f}(z, x_2) dz &\leq \int_{A_3} G_{D_*}(x_1, z) G_{D_*}(z, x_2) dz \\ &\leq \int_{A_3} (c_{11}|z - x_0|^{1-d})^2 dz \\ &\leq c_{12} \int_2^\infty r^{2(1-d)} r^{d-1} dr < c_{13} < \infty \end{aligned}$$

and a similar estimate holds for A_4. Since $D_f \subset A_1 \cup A_2 \cup A_3 \cup A_4$, the lemma follows from (2.23)–(2.25).

If $d = 2$, an argument similar to the above could be given. In this case, $\widetilde{D}$ should be replaced by a suitable wedge with angle $\alpha < \pi$. The Green function in such a wedge decays like $r^{-\pi/\alpha}$, and this is sufficient to make the bounding integrals finite. □

(2.26) Lemma. *For $x \in D_f$ and $y \in \Lambda_{s_k}$, let*

$$g_x^k(y) dy \stackrel{\text{df}}{=} P_h^x(X(T(\Lambda_{s_k})) \in dy).$$

Then there exist $c_1 < \infty$ and $c_2 < 1$ such that

$$\tag{2.27} \frac{g_{x_1}^n(y_1)}{g_{x_1}^n(y_2)} \geq a_i \frac{g_{x_2}^n(y_1)}{g_{x_2}^n(y_2)}$$

and

$$a_i \geq 1 - c_1 c_2^i$$

for all $i > 0$, all n, where $x_1, x_2 \in D_{n-i}$ and $y_1, y_2 \in \Lambda_{s_n}$.

Proof. A standard application of the boundary Harnack principle in the spirit of Lemma 2.3 shows that (2.27) holds for $i = 1$ with some $a_1 > 0$.

Assume that (2.27) holds for all n and for some i; we will show that it holds for $i+1$ as well. Let $j = n - i$. By the strong Markov property applied at $T(\Lambda_{s_{n-1}})$,

$$g_x^n(y) = \int_{\Lambda_{s_{n-1}}} g_x^{n-1}(v) g_v^n(y) dv$$

for $y \in D_{j-1}$. Now apply Lemma 6.1 of Burdzy, Toby and Williams ([BTW], 1989). Set in that lemma $V = W = \Lambda_{s_{n-1}}$ and $U = \emptyset$, set f_1 and f_2 equal to our $g_{x_1}^{n-1}$ and $g_{x_2}^{n-1}$, set $g_z(v)$ equal to our $g_v^n(z)$, and take $c = a_i$, $d = a_1$, and $b = 1$. The aforementioned lemma implies that

$$\frac{g_{x_1}^n(y_1)}{g_{x_1}^n(y_2)} \geq a_{i+1} \frac{g_{x_2}^n(y_1)}{g_{x_2}^n(y_2)}$$

for all $y_1, y_2 \in D_{j-1}$, where

$$a_{i+1} = a_i + a_1^2(1 - a_i).$$

Hence

$$1 - a_{i+1} = 1 - a_i - a_1^2(1 - a_i) = (1 - a_i)(1 - a_1^2)$$

and, by induction,

$$1 - a_{i+1} \le c_1 c_2^i,$$

with $c_2 \stackrel{\mathrm{df}}{=} 1 - a_1^2 < 1$. □

(2.28) Corollary. *With the notation of Lemma 2.26,*

$$a_{n-j}^{-1} \ge \frac{g_{x_1}^n(y)}{g_{x_2}^n(y)} \ge a_{n-j}$$

for every $j < n$, $x_1, x_2 \in D_j$ and $y \in \Lambda_{s_n}$.

Proof. Let M and m be the supremum and infimum of $g_{x_1}^n(y)/g_{x_2}^n(y)$ over $y \in \Lambda_{s_n}$. By Lemma 2.26, $m \ge a_{n-j}M$, and

$$M g_{x_2}^n(y) \ge g_{x_1}^n(y) \ge m g_{x_2}^n(y).$$

Integrating with respect to y shows that $M \ge 1 \ge m$, from which the desired conclusion follows. □

Proof of Theorem 1.6. (i) We will first prove the lower bound in (1.7).

Suppose that $s_{j_0} \le x^d < s_{j_0+1} < s_{j_0+2} < b$. The other cases are left to the reader. Let $T_j = T(\Lambda_{s_j})$. For each $j > j_0 + 2$ the process $\{X_t, t \in [T_{j-1}, T_j)\}$ under P_h^x is a conditioned Brownian motion in D_j starting from a (random) point in $\Lambda_{s_{j-1}}$ and converging to Λ_{s_j} at its lifetime. By Lemma 2.13, for $j \in [j_0 + 2, k_f - 1]$,

$$E_h^x(T_j - T_{j-1}) \ge c_1 f^2(s_{j-1})$$

and, therefore,

$$E_h^x \zeta \ge \sum_{j=j_0+2}^{k_f-1} E_h^x(T_j - T_{j-1}) \ge \sum_{j=j_0+2}^{k_f-1} c_1 f^2(s_{j-1}). \tag{2.29}$$

Since

$$c_2 f^2(s_{j-1}) < \int_{s_{j-1}}^{s_j} f(v)dv < c_3 f^2(s_{j-1}),$$

the sum on the right hand side of (2.29) is bounded below by $c_4 \int_{s_{j_0+1}}^{k_f-2} f(v)dv$. Note that

$$\int_{x^d}^{s_{j_0+1}} f(v)dv < c_5 \int_{s_{j_0+1}}^{s_{j_0+2}} f(v)dv$$

and

$$\int_{k_f-2}^{b} f(v)dv < c_5 \int_{k_f-3}^{k_f-2} f(v)dv.$$

Hence

$$\int_{x^d}^{b} f(v)dv < c_6 \int_{s_{j_0+1}}^{k_f-2} f(v)dv$$

and, therefore,

$$E_h^x \zeta \geq c_7 \int_{x^d}^{b} f(v)dv.$$

(ii) Next we will prove (ii) of Theorem 1.6.

First note that $k_f = \infty$. Recall the definitions of j_0 and the T_j's from part (i) of the proof. By Lemma 2.13 and the strong Markov property applied at T_j's, there exist non-negative (not necessarily independent) random variables Z_j and i.i.d. non-negative random variables Y_j such that

$$\sum_{j=j_0+2}^{\infty} (T_j - T_{j-1}) \tag{2.30}$$

has the same distribution as

$$\sum_{j=j_0+2}^{\infty} (Z_j + f^2(s_{j-1})Y_j). \tag{2.31}$$

For later use, note that, as in the proof of Lemma 2.13, we can write $Y_j = W_j Y_j'$, where the Y_j' are independent of the Z's and W's, with some common mean μ and variance σ^2. Each W_j takes values 0 or 1, and $W_j = 1$ with some common probability p, even if conditioned on the preceding W's and on $\{X_t, t \in [0, T_{j-1}]\}$. Thus the W_j are i.i.d., though they may not be independent of the Z_j.

It is elementary to check that $\sum_{j=j_0+2}^{\infty} f^2(s_{j-1}) = \infty$ because $\int_{x^d}^{b} f(v)dv = \infty$. Hence,

$$\sum_{j=j_0+2}^{\infty} E(f^2(s_{j-1})Y_j) = \infty.$$

Recalling that each Y_j is non-negative, non-constant and bounded, the three-series theorem now easily implies a.s. that

$$\sum_{j=j_0+2}^{\infty} f^2(s_{j-1})Y_j = \infty.$$

It follows that the sums in (2.31), and therefore in (2.30), must be infinite a.s.

(iii) We are going to prove the lower bound in (1.8).

Let j_0, the Y_j's, etc. be as in part (ii) of the proof. By adjusting the first and last Z, if necessary, we can guarantee that

$$\begin{aligned} \zeta &= \sum_{j=j_0+2}^{k_f-1} (Z_j + f^2(s_{j-1})Y_j) \\ &= \sum_{j=j_0+2}^{k_f-1} (Z_j + f^2(s_{j-1})\mu W_j) + \sum_{j=j_0+2}^{k_f-1} (f^2(s_{j-1})W_j(Y_j' - \mu)). \end{aligned} \tag{2.32}$$

Therefore by independence,

$$\begin{aligned}\operatorname{Var}_h^x \zeta &= \operatorname{Var}_h^x \left(\sum_{j=j_0+2}^{k_f-1} (Z_j + f^2(s_{j-1})\mu W_j) \right) + \sum_{j=j_0+2}^{k_f-1} E_h^x((f^2(s_{j-1})W_j(Y_j' - \mu))^2) \\ &\geq \sum_{j=j_0+2}^{k_f-1} E_h^x((f^2(s_{j-1})W_j(Y_j' - \mu))^2) \\ &\geq \sum_{j=j_0+2}^{k_f-1} f^4(s_{j-1})p\sigma^2 \geq c_3 \int_{x^d}^b f^3(v)dv.\end{aligned}$$

(iv) We will now prove part (v) of Theorem 1.6.

We will again invoke the Y_j's and Z_j's of part (ii) of the proof. Suppose that $\int_{x^d}^b f^3(v)dv = \infty$. Then necessarily $b = \infty$. Let us assume that

$$\limsup_{v\to\infty} f(v) < \infty. \tag{2.33}$$

In order to simplify the notation, suppose that $x^d = s_{j_0}$.

First, let $w_1, w_2, \ldots$ be any sequence of 0's and 1's, such that

$$\sum_{j>j_0} f^4(s_{j-1})w_j = \infty.$$

Consider

$$\widetilde{Y}_k = \sum_{j=j_0+1}^{k} f^2(s_{j-1})w_j(Y_j' - \mu) \quad \text{and} \quad \widehat{Y}_k = \widetilde{Y}_k / (\operatorname{Var} \widetilde{Y}_k)^{1/2}.$$

Since the Y_j's are uniformly bounded, the Lindeberg-Feller condition can be easily verified using (2.33) and it follows that the distributions of $\widehat{Y}_k$ converge to the standard normal distribution as $k \to \infty$. In fact it is simple to show, using (2.33) and the Berry-Eseen theorem, that for every $c_1 < \infty$ and $c_2 > 0$ there exists a $c_3 < \infty$ such that

$$P(\widetilde{Y}_k \in (u, u + c_1)) < c_2/2 \text{ for every } u \in \mathbb{R}, \quad \text{if } \operatorname{Var} \widetilde{Y}_k > c_3.$$

Since $\sum_{j>j_0} f^4(s_{j-1})W_j = \infty$ almost surely, we can choose a $k_0 < \infty$ such that

$$P_h^x \left(\sum_{j=j_0+1}^{k} f^4(s_{j-1})W_j > c_3 \right) > 1 - c_2/2$$

for every $k \geq k_0$. Also, as in (2.32) we have that

$$T(\Lambda_{s_k}) = \sum_{j=j_0+2}^{k} (Z_j + f^2(s_{j-1})\mu W_j) + \sum_{j=j_0+2}^{k} (f^2(s_{j-1})W_j(Y_j' - \mu)).$$

Therefore, conditioning on the values of $W_j, j > j_0$ yields that

$$P_h^x(T(\Lambda_{s_k}) \in (u, u + c_1)) < c_2$$

for every $u \in \mathbb{R}$.

The case when (2.33) fails is not hard and is left to the reader.

(v) Next we prove the upper bound in (1.7).

Suppose that $s_{n+1} \le x^d \le s_{n+2}$. Let L be the last exit time from Λ_{s_n}. Under P_h^x, the process $\{X_t, t \in [0, L]\}$ is a conditioned Brownian motion in D_f starting from x and converging to a (random) point of Λ_{s_n}. Lemma 2.22 implies that $E_h^x L < c_1 f^2(s_n)$ and this in turn implies that

$$E_h^x L < c_2 \int_{x^d}^{s_{n+3}} f(v)dv. \tag{2.34}$$

For every $\varepsilon > 0$, the process $\{X_{t+L+\varepsilon}, t \ge 0\}$ under P_h^x is an h-process in the domain D_g where $g(s) = f(s)\mathbf{1}_{(s_n,\infty)}(s)$. This and (2.34) show that (1.7) will follow once we prove that

$$E_h^x \zeta < c_3 \int_a^b f(v)dv.$$

Let $M_k = \{y \in D_f : s_{k-1} < y^d < s_{k+1}\}$ and consider an h_0-process in M_k for some positive harmonic function h_0 in M_k. A variation of Lemma 2.6 shows that

$$E_{h_0}^y \zeta < c_4 \tag{2.35}$$

for all $y \in M_k$, provided $f(s_k) = 1$. By scaling,

$$E_{h_0}^y \zeta < c_4 f^2(s_k) \tag{2.36}$$

for any value of $f(s_k)$.

Recall the stopping times S_j^k and T_j^k from Lemma 2.16 and let $F_j^k \stackrel{\mathrm{df}}{=} \{T_j^k < \infty\}$. Let T_0 be the hitting time of $\bigcup_k \Lambda_{s_k}$. We have

$$\zeta = T_0 + \sum_{j,k} (S_j^k - T_j^k)\mathbf{1}_{F_j^k}. \tag{2.37}$$

Given $T_j^k < \infty$, the process $\{X_t, t \in [T_j^k, S_j^k]\}$ is a conditioned Brownian motion in M_k and, therefore,

$$E_h^x[(S_j^k - T_j^k) \mid F_j^k] < c_4 f^2(s_j).$$

By Lemma 2.16,

$$\sum_k E_h^x (S_j^k - T_j^k)\mathbf{1}_{F_j^k} < c_5 f^2(s_j). \tag{2.38}$$

Recall that $s_{n+1} \le x^d \le s_{n+2}$. Hence $E_h^x T_0 < c_4 f^2(s_n)$. This and (2.37)–(2.38) yield

$$E_h^x \zeta \le c_6 \sum_j f^2(s_j).$$

It is easy to check that the last quantity is bounded by $c_7 \int_a^b f(v)dv$.

(vi) We will now prove the upper bound for the variance in (1.8). Recall M_k and the use of an h_0-process in M_k from part (v) of the proof. The Chebyshev inequality and (2.35) show that $P_{h_0}^x(\zeta > c_1) < c_2$ for some $c_1 < \infty$, $c_2 < 1$ and all $x \in M_k$ provided $f(s_k) = 1$. By the Markov property applied repeatedly at the multiples of c_1, $P_{h_0}^x(\zeta > jc_1) < c_2^j$. Hence $E_{h_0}^x \zeta^2 < c_3$ in the case $f(s_k) = 1$ and, by scaling,

$$E_{h_0}^x \zeta^2 < c_3 f^4(s_k) \tag{2.39}$$

for any value of $f(s_k)$, all $x \in M_k$ and all harmonic functions h_0 in M_k.

Let S_j^k and T_j^k be as in Lemma 2.16. Let $F_j^k \stackrel{\mathrm{df}}{=} \{T_j^k < \infty\}$. Given F_j^k, the process $\{X_t, t \in [T_j^k, S_j^k]\}$ is a conditioned Brownian motion in M_k and this implies in view of (2.36) and (2.39), that

$$\begin{aligned} &E_h^x[(S_j^k - T_j^k) \mid F_j^k] < c_4 f^2(s_j) \quad \text{and} \\ &E_h^x[(S_j^k - T_j^k)^2 \mid F_j^k] < c_3 f^4(s_j). \end{aligned} \tag{2.40}$$

Let $\Theta_j^k \stackrel{\mathrm{df}}{=} (S_j^k - T_j^k)\mathbf{1}_{F_j^k}$. Define q by the condition that $s_{q-1} < x^d \le s_q$, and recall from Lemma 2.16 that

$$P_h^x(F_j^k) \le \begin{cases} c_5 c_6^{k+q-j}, & j < q, \\ c_5 c_6^k, & j \ge q, \end{cases} \tag{2.41}$$

where $c_6 < 1$. This and (2.40) imply that

$$E_h^x[\Theta_j^k] \le \begin{cases} c_4 c_5 c_6^{k+q-j} f^2(s_j), & j < q, \\ c_4 c_5 c_6^k f^2(s_j), & j \ge q, \end{cases} \tag{2.42}$$

and

$$E_h^x[(\Theta_j^k)^2] \le \begin{cases} c_3 c_5 c_6^{k+q-j} f^4(s_j), & j < q, \\ c_3 c_5 c_6^k f^4(s_j), & j \ge q, \end{cases} \tag{2.43}$$

Now assume that $j < n$, and let

$$A = \{T_j^k < T_n^1\}, \quad B = \{T_n^1 < T_j^k\}, \quad B_i = \{T_j^{i-1} < T_n^1 < T_j^i\}$$

where T_j^0 is taken to be 0. Then

$$\begin{aligned} \mathrm{Cov}_h^x(\Theta_j^k, \Theta_n^m) &= E_h^x((\Theta_j^k - E_h^x\Theta_j^k)(\Theta_n^m - E_h^x\Theta_n^m)) \\ &= E_h^x((\Theta_j^k - E_h^x\Theta_j^k)(\Theta_n^m - E_h^x\Theta_n^m)\mathbf{1}_A) \\ &\quad + E_h^x((\Theta_j^k - E_h^x\Theta_j^k)(\Theta_n^m - E_h^x\Theta_n^m)\mathbf{1}_B) \\ &\stackrel{\mathrm{df}}{=} I + II. \end{aligned} \tag{2.44}$$

Consider term I of (2.44). If $q > n$, then $I = 0$ automatically. So suppose that $q \le j$. By Corollary 2.28 and the strong Markov property at T_n^1,

$$|E_h^y\Theta_n^m - E_h^x\Theta_n^m| \le c_7 c_8^{n-j} E_h^x\Theta_n^m$$

for any $y \in D_j$, where $c_8 < 1$. In particular,

$$|E_h^x(\Theta_n^m \mid \mathcal{F}_{S_j^k}) - E_h^x\Theta_n^m| \le c_7 c_8^{n-j} E_h^x(\Theta_n^m)$$

on A. Thus, by (2.42),

$$\begin{aligned} I &= E_h^x\left[(\Theta_j^k - E_h^x\Theta_j^k)\mathbf{1}_A E_h^x(\Theta_n^m - E_h^x(\Theta_n^m) \mid \mathcal{F}_{S_j^k})\right] \\ &\le E_h^x\left[|\Theta_j^k - E_h^x\Theta_j^k| \cdot \mathbf{1}_A \cdot |E_h^x(\Theta_n^m \mid \mathcal{F}_{S_j^k}) - E_h^x\Theta_n^m|\right] \\ &\le 2c_7 c_8^{n-j} E_h^x(\Theta_n^m) E_h^x(\Theta_j^k) \le c_9 c_8^{n-j} c_6^{k+m} f^2(s_j) f^2(s_n). \end{aligned}$$

If, on the other hand, we have $j < q \le n$, then by a similar argument,

$$|E_h^x(\Theta_n^m \mid \mathcal{F}_{S_j^k}) - E_h^x\Theta_n^m| \le c_7 c_8^{n-q} E_h^x(\Theta_n^m)$$

on A, and

$$I \le 2c_7 c_8^{n-q} E_h^x(\Theta_n^m) E_h^x(\Theta_j^k) \le c_9 c_8^{n-q} c_6^{k+m+q-j} f^2(s_j) f^2(s_n).$$

Taking $c_{11} = \max(c_8, c_6)$, it follows that

$$I \le c_9 c_{11}^{n-j} c_6^{k+m} f^2(s_j) f^2(s_n), \tag{2.45}$$

regardless of the value of q.

Consider now the term II of (2.44). By (2.43), and by Lemma 2.16 again,

$$\begin{aligned}
E_h^x((\Theta_j^k)^2 \mathbf{1}_B) &= \sum_{i=1}^k E_h^x((\Theta_j^k)^2 \mathbf{1}_{B_i}) \\
&= \sum_{i=1}^k E_h^x(E_h^x((\Theta_j^k)^2 \mathbf{1}_{B_i} \mid \mathcal{F}_{S_n^1})) \\
&\le \sum_{i=1}^k f^4(s_j) c_0 c_3 c_6^{n-j+k-i+1} P_h^x(B_i) \\
&\le \sum_{i=1}^k f^4(s_j) c_0 c_3 c_6^{n-j+k-i+1} P_h^x(F_j^{i-1}) \\
&\le k c_0^2 c_3 c_6^{n-j+k} f^4(s_j) \le c_{12} c_{13}^{n-j+k} f^4(s_j),
\end{aligned}$$

where $c_{13} < 1$. As a result,

$$\begin{aligned}
II &\le (E_h^x((\Theta_j^k)^2 \mathbf{1}_B))^{1/2} (E_h^x((\Theta_n^m)^2))^{1/2} \\
&\le f^2(s_j) f^2(s_n) (c_{12} c_{13}^{n-j+k} c_0 c_3 c_6^m)^{1/2} \\
&\le c_{14} c_{15}^{k+m+n-j} f^2(s_j) f^2(s_n),
\end{aligned}$$

where $c_{14} < 1$. Combining this with (2.44) and (2.45), it follows that

$$\mathrm{Cov}_h^x(\Theta_j^k, \Theta_n^m) \le c_{16} c_{17}^{k+m+|n-j|} f^2(s_j) f^2(s_n), \tag{2.46}$$

for $j < n$, where $c_{17} < 1$. By symmetry, the same is true for $j > n$, and the inequality is even simpler to prove if $j = n$ ((2.45) is no longer needed). Thus, (2.46) holds for every j, k, m, n.

If $\int_a^b f^3(v)dv = \infty$, then the upper bound in (1.8) is trivial. Assume therefore that $\int_a^b f^3(v)dv < \infty$. Then for each $\varepsilon > 0$ there are only finitely many j such that $f(s_j) > \varepsilon$. Hence we may choose an ordering $\{j_i\}_{i \ge 1}$ of the set $\{k : a < s_k < b\}$ which satisfies $f(s_{j_{i+1}}) \le f(s_{j_i})$ for all i. By (2.46)

$$\begin{aligned}
\mathrm{Var}_h^x \zeta = \mathrm{Var}_h^x \left(\sum_{j,k} \Theta_j^k \right) &= \sum_{j,k,n,m} \mathrm{Cov}_h^x(\Theta_j^k, \Theta_n^m) \\
&\le 2 \sum_i \sum_{n \ge i} \sum_k \sum_m \mathrm{Cov}_h^x(\Theta_{j_i}^k, \Theta_{j_n}^m) \\
&\le 2 \sum_i \sum_{n \ge i} \sum_k \sum_m c_{13} c_{12}^{k+m+|j_n - j_i|} f^2(s_{j_i}) f^2(s_{j_n}) \\
&\le \sum_i \sum_{n \ge i} c_{14} c_{12}^{|j_n - j_i|} f^4(s_{j_i}) \\
&\le \sum_j c_{15} f^4(s_j) \le c_{16} \int_a^b f^3(v)dv.
\end{aligned} \tag{2.47}$$

(vii) Next we will prove part (iv) of Theorem 1.6.

Fix some $x \in D_f$ and suppose for convenience that $x^d = s_q$ for some q. Recall S_j^k, T_j^k, F_j^k and Θ_j^k from part (v) of the proof. With slightly more work, the argument for (2.46) can be seen to yield the following improved estimate:

$$\mathrm{Cov}_h^x(\Theta_j^k, \Theta_n^m) \le \begin{cases} c_1 c_2^{k+m+|n-j|} f^2(s_j) f^2(s_n), & j, n \ge q, \\ c_1 c_2^{k+m+|n-j|} c_3^{q-j} f^2(s_j) f^2(s_n), & j < q \le n, \\ c_1 c_2^{k+m+|n-j|} c_3^{q-j} c_3^{q-n} f^2(s_j) f^2(s_n), & j, n < q, \end{cases}$$

where $c_2, c_3 < 1$.

Now we assume that the Lipschitz constant of f is so small that for each j,

$$\frac{f^2(s_{j-1})}{f^2(s_j)} < \frac{c_3^{-1} + 1}{2}.$$

Therefore

$$\mathrm{Cov}_h^x(\Theta_j^k, \Theta_n^m) \le \begin{cases} c_1 c_2^{k+m+|n-j|} f^2(s_j) f^2(s_n), & j, n \ge q, \\ c_1 c_2^{k+m+|n-j|} c_4^{q-j} f^2(s_q) f^2(s_n), & j < q \le n, \\ c_1 c_2^{k+m+|n-j|} c_4^{q-j} c_4^{q-n} f^4(s_q), & j, n < q, \end{cases} \tag{2.48}$$

for some $c_4 < 1$.

If $\int_{x^d}^b f^3(v)dv = \infty$, then (1.9) obviously holds. Assume that $\int_{x^d}^b f^3(v)dv < \infty$. Then we may choose an ordering $\{j_i\}_{i \ge 1}$ of the set $\{k : x^d \le s_k < b\}$ which satisfies $f(s_{j_{i+1}}) \le f(s_{j_i})$ for all i. Let $j_{i_0} = q$. Then in view of (2.48),

$$\begin{aligned}
\mathrm{Var}_h^x \zeta &= \mathrm{Var}_h^x \left(\sum_{j,k} \Theta_j^k \right) = \sum_{j,k,n,m} \mathrm{Cov}_h^x(\Theta_j^k, \Theta_n^m) \\
&\le 2 \sum_i \sum_{n \ge i} \sum_k \sum_m \mathrm{Cov}_h^x(\Theta_{j_i}^k, \Theta_{j_n}^m) \\
&\quad + 2 \sum_{j \le n < q} \sum_k \sum_m \mathrm{Cov}_h^x(\Theta_j^k, \Theta_n^m) \\
&\quad + 2 \sum_{j<q} \sum_{i \ge i_0} \sum_k \sum_m \mathrm{Cov}_h^x(\Theta_j^k, \Theta_{j_i}^m) \\
&\quad + 2 \sum_{j<q} \sum_{i < i_0} \sum_k \sum_m \mathrm{Cov}_h^x(\Theta_j^k, \Theta_{j_i}^m) \\
&\le 2 \sum_i \sum_{n \ge i} \sum_k \sum_m c_1 c_2^{k+m+|j_n - j_i|} f^2(s_{j_i}) f^2(s_{j_n}) \\
&\quad + 2 \sum_{j \le n < q} \sum_k \sum_m c_1 c_2^{k+m+|n-j|} c_4^{q-j} c_4^{q-n} f^4(s_q) \\
&\quad + 2 \sum_{j<q} \sum_{i \ge i_0} \sum_k \sum_m c_1 c_2^{k+m+|j_i - j|} c_4^{q-j} f^2(s_q) f^2(s_{j_i}) \\
&\quad + 2 \sum_{j<q} \sum_{i < i_0} \sum_k \sum_m c_1 c_2^{k+m+|j_i - j|} c_4^{q-j} f^2(s_q) f^2(s_{j_i}) \\
&\le \sum_i c_5 f^4(s_{j_i}) + c_6 f^4(s_q) + c_7 f^4(s_q) + \sum_i c_8 f^4(s_{j_i}) \\
&\le c_9 \sum_{j \ge q} f^4(s_j) \le c_{10} \int_{x^d}^b f^3(v)dv. \quad \square
\end{aligned}$$

Because they use similar arguments to those just given, we include the following two subsidiary results in this section.

(2.49) Corollary. *Suppose that D_f and h are as in Theorem 1.6. Assume that $\int_a^b f^3(v)dv < \infty$. Then*

$$\lim_{x^d \to \infty} \sup\{\operatorname{Var}_h^x T(\Lambda_u) : u > x^d\} = 0. \tag{2.50}$$

Proof. Recall the notation from the proof of Theorem 1.6. As in the proof of (2.47), for every x and for every $u = s_i$,

$$\operatorname{Var}_h^x T(\Lambda_u) = \sum_{j,k,n,m} \operatorname{Cov}_h^x(\Theta_j^k \mathbf{1}_{\{T_j^k < T(\Lambda_u)\}}, \Theta_n^m \mathbf{1}_{\{T_n^m < T(\Lambda_u)\}}).$$

An examination of the proof of (2.47) shows that the terms of this sum are bounded by the terms of an absolutely convergent series, uniformly in x and in $u = s_i$. With a little more work, it is easy to see that this domination holds for $u \in (a, b)$ as well. For fixed j, k, m and n,

$$\operatorname{Cov}_h^x(\Theta_j^k \mathbf{1}_{\{T_j^k < T(\Lambda_u)\}}, \Theta_n^m \mathbf{1}_{\{T_n^m < T(\Lambda_u)\}}) \to 0$$

as $x^d \to \infty$, uniformly in u, because of (2.41). This easily implies (2.50). $\square$

(2.51) Lemma. *Assume that D_f and h are as in Theorem 1.6. Set*

$$f_*(v) \stackrel{\text{df}}{=} \sup_{u \geq v} f(u).$$

There exists a $c_1 < \infty$ such that for all u and all $x_1, x_2 \in D_f$ with $x_1^d = x_2^d < u$ we have

$$|E_h^{x_1} T(\Lambda_u) - E_h^{x_2} T(\Lambda_u)| \leq c_1 f_*^2(x_1^d).$$

Proof. We will use an argument from part (v) of the proof of Theorem 1.6. For simplicity, assume that $u = s_m$ for some m. Suppose that $s_{n+1} \leq x_1^d \leq s_{n+2}$ and let L be the last exit from Λ_{s_n}. It has been proved that

$$E_h^{x_k} L < c_2 f^2(s_n) \tag{2.52}$$

for $k = 1, 2$ (see the paragraph preceding (2.34)). Recall the definitions of T_0, S_j^k, T_j^k and F_j^k from the same proof, and set

$$G_j^k \stackrel{\text{df}}{=} F_j^k \cap \{T_j^k < T(\Lambda_u)\}.$$

We have

$$E_h^{x_k} T_0 < c_3 f^2(s_{n+1}) \tag{2.53}$$

by an argument analogous to that proving (2.36).

Let $g_{x_i}^j(y)dy = P_h^{x_i}(X(T_j^1) \in dy)$. By Corollary 2.28 it follows that for $j \geq n$,

$$\begin{aligned}
&|E_h^{x_1}[(S_j^k - T_j^k)\mathbf{1}_{G_j^k}] - E_h^{x_2}[(S_j^k - T_j^k)\mathbf{1}_{G_j^k}]| \\
&\qquad = \left| \int_{\Lambda_j} E_h^y[(S_j^k - T_j^k)\mathbf{1}_{G_j^k}](P_h^{x_1}(X(T_j^1) \in dy) - P_h^{x_2}(X(T_j^1) \in dy)) \right| \\
&\qquad \leq \int_{\Lambda_j} E_h^y[(S_j^k - T_j^k)\mathbf{1}_{G_j^k}] \left| 1 - \frac{g_{x_2}^j(y)}{g_{x_1}^j(y)} \right| P_h^{x_1}(X(T_j^1) \in dy) \\
&\qquad \leq c_4 c_5^{|n-j|} E_h^{x_1}[(S_j^k - T_j^k)\mathbf{1}_{G_j^k}],
\end{aligned}$$

where $c_5 < 1$. (Strictly speaking, Corollary 2.28 applies when $j > n+2$, but the other cases follow similarly.) Now (2.38) implies that

$$\begin{aligned} &\left| \sum_k E_h^{x_1}(S_j^k - T_j^k)\mathbf{1}_{G_j^k} - \sum_k E_h^{x_2}(S_j^k - T_j^k)\mathbf{1}_{G_j^k} \right| \\ &\qquad \le c_4 c_5^{|n-j|} \sum_k E_h^{x_1}(S_j^k - T_j^k)\mathbf{1}_{G_j^k} \\ &\qquad \le c_4 c_5^{|n-j|} c_6 f^2(s_j). \end{aligned} \tag{2.54}$$

Since

$$\sum_{j\ge n}\sum_{k\ge 1}(S_j^k - T_j^k)\mathbf{1}_{G_j^k} \le T(\Lambda_u) \le T_0 + L + \sum_{j\ge n}\sum_{k\ge 1}(S_j^k - T_j^k)\mathbf{1}_{G_j^k},$$

we obtain from (2.52)–(2.54) that

$$\begin{aligned} &|E_h^{x_1}T(\Lambda_u) - E_h^{x_2}T(\Lambda_u)| \\ &\qquad \le 2c_2 f^2(s_n) + 2c_3 f^2(s_{n+1}) + \sum_{j\ge n} c_4 c_5^{|n-j|} c_6 f^2(s_j) \\ &\qquad \le 2c_2 f_*^2(s_n) + 2c_3 f_*^2(s_n) + \sum_{j\ge n} c_4 c_5^{|n-j|} c_6 f_*^2(s_n) \le c_7 f_*^2(s_n). \qquad \square \end{aligned}$$

3. Disintegration of harmonic functions

The purpose of this section is to prove Theorem 1.3. Unless otherwise indicated, the notation and general hypotheses of Theorem 1.3 will be assumed throughout this section.

Fix some $x_0 \in D_f$ and let $g(u) \overset{\text{df}}{=} -E_h^{x_0}T(\Lambda_u)$. Recall that $f_*(v) = \sup_{u\ge v} f(u)$. Note that in either case (a) or (b) of Theorem 1.3 (i), we have that $f(v) \to 0$ as $v \to \infty$.

(3.1) Lemma. *Suppose that one of the assumptions (a) or (b) of Theorem 1.3 (i) is satisfied. Then*

$$\lim_{u\to\infty}(T(\Lambda_u) + g(u)) \quad \textit{exists } P_h^{x_0}\textit{-a.s.}$$

Proof. Lemma 2.51 and Corollary 2.49 show that for $k \ge 1$, we can choose u_k such that

$$|E_h^{x_1}T(\Lambda_u) - E_h^{x_2}T(\Lambda_u)| \le c_1 f_*^2(u_k) \le 1/k^2 \tag{3.2}$$

for all $x_1, x_2 \in D_f$ and u with $u_k \le x_1^d = x_2^d < u$. We may also assume that

$$\operatorname{Var}_h^x T(\Lambda_u) \le 1/k^6 \tag{3.3}$$

for $x \in D_f$ and u with $u_k \le x^d < u$.

Suppose $u \in [u_k, u_{k+1})$. Since

$$T(\Lambda_{u_{k+1}}) = (T(\Lambda_{u_{k+1}}) - T(\Lambda_u)) + T(\Lambda_u),$$

we have

$$g(u_{k+1}) = -E_h^{x_0}(T(\Lambda_{u_{k+1}}) - T(\Lambda_u)) + g(u).$$

This, (3.2), and the strong Markov property applied at $T(\Lambda_u)$ imply that

$$|E_h^x T(\Lambda_{u_{k+1}}) + (g(u_{k+1}) - g(u))| \le 1/k^2 \tag{3.4}$$

for all $x \in D_f$ such that $x^d = u$. The Chebyshev inequality and (3.3) yield that

$$P_h^x(|T(\Lambda_{u_{k+1}}) - E_h^x T(\Lambda_{u_{k+1}})| \geq 1/k^2) \leq k^4 \operatorname{Var}_h^x T(\Lambda_{u_{k+1}}) \leq 1/k^2,$$

if $x^d = u$. This and (3.4) give

$$P_h^x(|T(\Lambda_{u_{k+1}}) + (g(u_{k+1}) - g(u))| \geq 2/k^2) \leq 1/k^2,$$

for $x \in D_f$ such that $x^d = u$. By the strong Markov property applied at $T(\Lambda_u)$,

$$P_h^x(|T(\Lambda_{u_{k+1}}) - T(\Lambda_u) + (g(u_{k+1}) - g(u))| \geq 2/k^2) \leq 1/k^2, \tag{3.5}$$

for any $x \in D_f$ with $x^d \leq u$. In particular,

$$P_h^x(|T(\Lambda_{u_{k+1}}) - T(\Lambda_{u_k}) + (g(u_{k+1}) - g(u_k))| \geq 2/k^2) \leq 1/k^2, \tag{3.6}$$

if $x^d \leq u_k$.

Fix some $c_2 > 0$ and find j_0 so large that $\sum_{j \geq j_0} 2/j^2 < c_2$. Suppose that $k > j_0$, $x^d \leq u$, and recall that $u \in [u_k, u_{k+1})$. Then (3.5)–(3.6) imply that with P_h^x-probability larger than $1 - c_2$, the event

$$\begin{aligned}\{|T(\Lambda_{u_{k+1}}) - T(\Lambda_u) + (g(u_{k+1}) - g(u))| \leq 2/k^2\}& \\ \cap \bigcap_{j \geq j_0} \{|T(\Lambda_{u_{j+1}}) - T(\Lambda_{u_j}) + (g(u_{j+1}) - g(u_j))| \leq 2/j^2\}&\end{aligned} \tag{3.7}$$

occurs. Let

$$A_v \stackrel{\text{df}}{=} \{|(T(\Lambda_{u_m}) + g(u_m)) - (T(\Lambda_v) + g(v))| < c_2 \quad \forall u_m \geq v\}.$$

If the event in (3.7) holds, then A_u holds, because in such a case we have

$$\begin{aligned}|(T(\Lambda_{u_m}) &+ g(u_m)) - (T(\Lambda_u) + g(u))| \\ &\leq |(T(\Lambda_{u_{k+1}}) + g(u_{k+1})) - (T(\Lambda_u) + g(u))| \\ &\quad + \sum_{j=k+1}^{m-1} |(T(\Lambda_{u_{j+1}}) - T(\Lambda_{u_j})) + (g(u_{j+1}) - g(u_j))| \\ &\leq 2/k^2 + \sum_{j=k+1}^{m-1} 2/j^2 < c_2.\end{aligned}$$

Hence $P_h^x(A_u) > 1 - c_2$.

Let

$$W = W(u) \stackrel{\text{df}}{=} \inf\{v > u : |(T(\Lambda_v) + g(v)) - (T(\Lambda_u) + g(u))| \geq 2c_2\}.$$

By the strong Markov property applied at $T(\Lambda_W)$ we have $P_h^{x_0}(A_W \mid W < \infty) > 1 - c_2$. Since $A_u \cap \{W < \infty\} \cap A_W = \emptyset$, it follows that $P_h^{x_0}(A_W \cap \{W < \infty\}) < c_2$, and hence $P_h^{x_0}(W < \infty) < c_2/(1 - c_2)$. This proves the lemma, since we may assume that $c_2 > 0$ is arbitrarily small by choosing u sufficiently large. □

We now make some general observations about parabolic Martin boundaries. Let D be a domain. For ϕ a parabolic function on $\dot{D}$, and $v < 0$, define

$$\phi_v(x,t) \stackrel{\text{df}}{=} \phi(x, t+v).$$

Then ϕ_v is also parabolic. Moreover, if ϕ is minimal, then ϕ_v is either minimal or $\phi_v \equiv 0$ (see 1.XV.17 in [Db]).

(3.8) Lemma. *Let D be a domain. Let ϕ be parabolic on $\dot{D}$, and let $v < 0$. Then the laws of X under $P^{x,t}_{\phi_v}$ and $P^{x,t+v}_{\phi}$ are the same.*

Proof. It suffices to show that $P^{x,t}_{\phi_v}(A) = P^{x,t+v}_{\phi}(A)$, for A an event of the form $\{X(t_1) \in A_1, \dots, X(t_n) \in A_n\}$, where $t_1 < t_2 < \cdots < t_n$. But

$$\begin{aligned} P^{x,t}_{\phi_v}(A) &= \frac{1}{\phi_v(x,t)} E^{x,t}[\mathbf{1}_A \phi_v(X_{t_n}, \tau_{t_n})] \\ &= \frac{1}{\phi(x,t+v)} E^{x,t}[\mathbf{1}_A \phi(X_{t_n}, \tau_{t_n} + v)] \\ &= \frac{1}{\phi(x,t+v)} E^{x,t+v}[\mathbf{1}_A \phi(X_{t_n}, \tau_{t_n})] = P^{x,t+v}_{\phi}(A). \quad \square \end{aligned}$$

Now, if $(y_k, t_k) \in \dot{D}$, $(y_k, t_k) \to z \in \partial^M \dot{D}$, and each $t_k < v$, then

$$\begin{aligned} \dot{K}((x,t),(y_k, t_k - v)) &= \frac{p^D_{t-t_k+v}(x, y_k)}{p^D_{-t_k+v}(x_0, y_k)} \\ &= \frac{p^D_{t-t_k+v}(x, y_k)}{p^D_{-t_k}(x_0, y_k)} \cdot \frac{p^D_{-t_k}(x_0, y_k)}{p^D_{-t_k+v}(x_0, y_k)} \\ &\to \frac{\dot{K}((x, t+v), z)}{\dot{K}((x_0, v), z)}. \end{aligned}$$

Thus, provided $\dot{K}((x_0, v), z) > 0$, it follows that $(y_k, t_k - v)$ converges in $\dot{D}^M$ to a point $\Phi_v z \in \partial^M \dot{D}$ with

$$\dot{K}(\,\cdot\,, \Phi_v z) = \frac{\dot{K}_v(\,\cdot\,, z)}{\dot{K}_v((x_0, 0), z)}.$$

Of course, it may happen that $\Phi_v z = z$. Note also that

$$\dot{K}((x_0, 0), \Phi_v z) = 1, \tag{3.9}$$

so that $\Phi_v z$ is a minimal point (according to the definition given in Section 1), if and only if $\dot{K}_v(\,\cdot\,, z)$ is a minimal function.

It would simplify several future arguments, if the map Φ_v could be defined for $v > 0$ as well. A natural way of doing this would be to set

$$\phi_v(x,t) \overset{\text{df}}{=} \begin{cases} \phi(x, t+v), & t + v \le 0, \\ \int p^D_{t+v}(x,y)\phi(y,0)dy, & t+v > 0. \end{cases}$$

The obstacle to this approach is that in general, this integral need not converge.

The following result is well known. See, for example, Theorems C and E of Aronson ([Ar], 1968).

(3.10) Lemma. *Let D be a domain, and let $A \subset \dot{D}$ be compact.*

- (i) *Let $\varepsilon > 0$ and $M < \infty$. There exists a $\delta > 0$ such that if u is parabolic on $\dot{D}$ and $u \le M$, then $|u(z) - u(z')| < \varepsilon$ whenever $z, z' \in A$ and $|z - z'| < \delta$.*
- (ii) *Let $x \in D$. There exists an $M < \infty$ such that if u is parabolic on $\dot{D}$, and $u(x, 0) \le 1$, then $u \le M$ on A.*

(3.11) Lemma. *Let D be a domain. Suppose that $(y_k, t_k) \in \dot{D}$ converge to some $z \in \partial^M \dot{D}$, and that $a_k \to 0$. Let $(x,t) \in \dot{D}$ (so that, in particular, $t < 0$) and suppose that $\dot{K}((x,t), z) > 0$. Then*

$$\frac{p^D_{t-t_k}(x, y_k)}{p^D_{t-t_k-a_k}(x, y_k)} \to 1 \tag{3.12}$$

as $k \to \infty$. Moreover,

$$(y_k, t_k + a_k - t) \to \Phi_t z. \tag{3.13}$$

Proof. If $\dot{K}((x,t), z) > 0$, then the $\dot{K}((x,t),(y_k,t_k))$ are bounded away from 0. Since $\dot{K}((x_0, 0),(y_k, t_k)) = 1$, (ii) of Lemma 3.10 shows that the $\dot{K}(\,\cdot\,,(y_k,t_k))$ are uniformly bounded on a suitable neighbourhood of (x,t). Applying (i) of Lemma 3.10 on this neighbourhood shows that

$$\frac{p^D_{t-t_k}(x, y_k)}{p^D_{t-t_k-a_k}(x, y_k)} = \frac{\dot{K}((x,t),(y_k,t_k))}{\dot{K}((x,t-a_k),(y_k,t_k))} \to 1,$$

as $k \to \infty$, showing (3.12).

To prove (3.13), we must show that

$$\lim_{k\to\infty} \dot{K}((x,s),(y_k, t_k + a_k - t)) = \lim_{k\to\infty} \dot{K}((x,s),(y_k, t_k - t))$$

for every $(x,s) \in \dot{D}$. But as before,

$$\begin{aligned}
\dot{K}((x,s),&(y_k, t_k + a_k - t)) = \frac{p_{s+t-t_k-a_k}(x, y_k)}{p_{t-t_k-a_k}(x_0, y_k)} \\
&= \frac{\dot{K}((x, s+t-a_k),(y_k,t_k))}{\dot{K}((x_0, t-a_k),(y_k,t_k))} \\
&\to \frac{\dot{K}((x,s+t),z)}{\dot{K}((x_0,t),z)} = \dot{K}((x,s),\Phi_t z). \quad \square
\end{aligned}$$

(3.14) Lemma. *Assume that $f(u) \to 0$ as $u \to \infty$. Let $(z_k, t_k) \in \dot{D}_f$ converge to $z \in \partial^M \dot{D}_f$, and suppose that $\dot{K}((x,t), z) > 0$ for every $(x,t) \in \dot{D}_f$. If $y_k \in D_f$ and $z^d_k = y^d_k$ for each k, then for some $c_1 < \infty$ and $c_2 > 0$, and for every $q < 0$ and $(x,t) \in \dot{D}_f$,*

$$\limsup_{k\to\infty} \dot{K}((x,t),(y_k, t_k - q)) \le c_1 \dot{K}((x,t),\Phi_q z), \tag{3.15}$$

$$\liminf_{k\to\infty} \dot{K}((x,t),(y_k, t_k - q)) \ge c_2 \dot{K}((x,t),\Phi_q z). \tag{3.16}$$

Proof. Let $r_0 > 0$ be so small that for each $w \in \partial D_f$, the set $\partial D_f \cap B(w, r_0 f(w^d))$ is the graph of a Lipschitz function F, with Lipschitz constant λ_0 in some orthonormal coordinate system CS_w. Let the coordinates of x in CS_w be $(\hat{x}, x')$, so that

$$D_f \cap B(w, r_0 f(w^d)) = \{(\hat{x}, x') : x' > F(\hat{x})\} \cap B(w, r_0 f(w^d)).$$

Let

$$\begin{aligned}
\Psi_r(w,s) &= \{(x,t) \in \dot{D}_f : |x - w| < r, |s - t| < r^2\}, \\
A_r(w) &= (\hat{w}, w' + r) \quad \text{in } CS_w.
\end{aligned}$$

We fix a suitable $\bar{s} < 0$ and apply Theorem 1.6 of Fabes, Garofalo and Salsa ([FGS], 1986) to some $\Psi_{r/8}(w, \bar{s})$, to see that if $x_1, x_2 \in D_f$, $w \in \partial D_f$, $r < r_0 f(w^d)/2$, $s, s' < \bar{s}$ and $y \in B(w, r/8)$, then

$$\begin{aligned} \frac{p^{D_f}_{-s}(x_2, y)}{p^{D_f}_{-s'}(x_1, y)} &= \frac{\dot{G}_{\dot{D}_f}((y, \bar{s}), (x_2, \bar{s}+s))}{\dot{G}_{\dot{D}_f}((y, \bar{s}), (x_1, \bar{s}+s'))} \\ &\le c_1 \frac{\dot{G}_{\dot{D}_f}((A_r(w), \bar{s}+2r^2), (x_2, \bar{s}+s))}{\dot{G}_{\dot{D}_f}((A_r(w), \bar{s}-2r^2), (x_1, \bar{s}+s'))} \\ &= c_1 \frac{p^{D_f}_{-s+2r^2}(x_2, A_r(w))}{p^{D_f}_{-s'-2r^2}(x_1, A_r(w))}. \end{aligned} \tag{3.17}$$

Note that, although Theorem 1.6 of [FGS] would in principle allow the above constant c_1 to depend on $f(w^d)$, in fact a scaling argument shows that it does not.

Fix $(x,t) \in \dot{D}_f$ and $q < 0$. Let $M = \bigcup_{w \in \partial D_f} B(w, r_0 f(w^d)/32)$. If $y_k \in M$, choose w so that $y_k \in B(w, r/8)$, where $r = r_0 f(w^d)/4$. With this choice of r, set

$$\bar{y}_k = A_r(w), \qquad a_k = 2r^2.$$

If $y_k \notin M$, set

$$\bar{y}_k = y_k, \qquad a_k = 0.$$

The assumption that $\dot{K}((x,t), z) > 0$ for every $(x,t) \in \dot{D}_f$ easily implies that $t_k \to -\infty$. By (3.17),

$$\frac{p^{D_f}_{t+q-t_k}(x, y_k)}{p^{D_f}_{q-t_k}(x_0, y_k)} \le c_1 \frac{p^{D_f}_{t+q-t_k+a_k}(x, \bar{y}_k)}{p^{D_f}_{q-t_k-a_k}(x_0, \bar{y}_k)}, \tag{3.18}$$

for k so large that $t_k - t - q < \bar{s}$.

Let $b_k = f^2(z_k^d)$. A precise version of the parabolic Harnack principle (see Theorem 0.2 of [FGS]) implies that for k large and for every $v \in D_f$ with $|v^d - z_k^d| < f(z_k^d)$ and $v \notin M$, we have

$$\frac{p^{D_f}_{t+q-t_k+a_k}(x, \bar{y}_k)}{p^{D_f}_{q-t_k-a_k}(x_0, \bar{y}_k)} \le c_2 \frac{p^{D_f}_{t+q-t_k+a_k+b_k}(x, v)}{p^{D_f}_{q-t_k-a_k-b_k}(x_0, v)}. \tag{3.19}$$

As above, take $\bar{z}_k$ equal to either z_k (if $z_k \notin M$), or an $A_r(w)$ (if $z_k \in B(w, r/8)$, where $r = r_0 f(w^d)/4$). Take d_k equal to 0 or $2r^2$ respectively. Therefore

$$\frac{p^{D_f}_{t+q-t_k+a_k+b_k}(x, \bar{z}_k)}{p^{D_f}_{q-t_k-a_k-b_k}(x_0, \bar{z}_k)} \le c_1 \frac{p^{D_f}_{t+q-t_k+a_k+b_k+d_k}(x, z_k)}{p^{D_f}_{q-t_k-a_k-b_k-d_k}(x_0, z_k)}, \tag{3.20}$$

for k large, as before. Since $q < 0$, $a_k \to 0$, $b_k \to 0$, and $d_k \to 0$, it follows from (3.12) that

$$\begin{aligned} \lim_{k\to\infty} \frac{p^{D_f}_{t+q-t_k+a_k+b_k+d_k}(x, z_k)}{p^{D_f}_{q-t_k-a_k-b_k-d_k}(x_0, z_k)} &= \lim_{k\to\infty} \frac{p^{D_f}_{t+q-t_k}(x, z_k)}{p^{D_f}_{q-t_k}(x_0, z_k)} \\ &= \lim_{k\to\infty} \dot{K}((x,t), (z_k, t_k - q)) = \dot{K}((x,t), \Phi_q z). \end{aligned}$$

Thus, taking $v = \bar{z}_k$, it follows from this and (3.18)–(3.20) that

$$\limsup_{k\to\infty} \dot{K}((x,t),(y_k,t_k-q)) = \limsup_{k\to\infty} \frac{p^{D_f}_{t+q-t_k}(x,y_k)}{p^{D_f}_{q-t_k}(x_0,y_k)} \le c_3\dot{K}((x,t),\Phi_q z)$$

as well, proving (3.15). The argument for (3.16) is similar. □

We may improve upon the conclusion of Lemma (3.14), by assuming that z is minimal:

(3.21) Lemma. *Assume that $f(u) \to 0$ as $u \to \infty$. Let $(z_k, t_k) \in \dot{D}_f$ converge to a minimal point $z \in \partial_0^M \dot{D}_f$, and suppose that $\dot{K}((x,t),z) > 0$ for every $(x,t) \in \dot{D}_f$. If $y_k \in D_f$ satisfy $z_k^d = y_k^d$ for each k, and $q_k \to q < 0$, then $(y_k, t_k - q_k) \to \Phi_q z$; that is,*

$$\lim_{k\to\infty} \dot{K}((x,t),(y_k,t_k-q_k)) = \dot{K}((x,t),\Phi_q z)$$

for every $(x,t) \in \dot{D}_f$.

Proof. We first consider the limit of $(y_k, t_k - q)$. If w is any limit point of this sequence, then by (3.15) we have that

$$\dot{K}(\,\cdot\,,w) \le c_1\dot{K}(\,\cdot\,,\Phi_q z).$$

By minimality of z (and hence $\Phi_q z$), in fact

$$\dot{K}(\,\cdot\,,w) = c\dot{K}(\,\cdot\,,\Phi_q z)$$

for some $c < \infty$. By (3.16) we must have $c > 0$, so $w \ne \dot{0}$.

Let k_i be a subsequence along which $(y_{k_i}, t_{k_i} - q) \to w$. By passing to a further subsequence, if necessary, we may also ensure that $(y_{k_i}, t_{k_i} - q/2)$ converges to some $w' \ne \dot{0}$. Then $w = \Phi_{q/2}w'$, so by (3.9),

$$\dot{K}((x_0,0),w) = 1 = \dot{K}((x_0,0),\Phi_q z).$$

Thus $c = 1$, and so $w = z$. Since $\Phi_q z$ is the only limit point of $(y_k, t_k - q)$, it follows that the sequence itself converges to $\Phi_q z$.

Similarly, $(y_k, t_k - q/2) \to \Phi_{q/2} z$. Since $\Phi_q z = \Phi_{q/2}(\Phi_{q/2} z)$, we may set $a_k = q - q_k$, and apply (3.13) (with $t = q/2$), to obtain, in addition, that $(y_k, t_k - q_k) \to \Phi_q z$, as required. □

Proof of Theorem 1.3. (i) Assume either (a) or (b) of (i) of the theorem, and recall that this implies that $f(u) \to 0$ as $u \to \infty$.

Let $\mathcal{H}_s$ denote the set of points z of the minimal Martin boundary $\partial_0^M \dot{D}_f$, such that $g(u) - \tau(\Lambda_u) \to s$, $P_z^{x_0,0}$-a.s. Set $\mathcal{H} = \bigcup_{s\in\mathbb{R}} \mathcal{H}_s$. Recall that if ϕ is a minimal parabolic function, then the tail σ-field of every ϕ-transform of space-time Brownian motion is trivial. By Lemma 3.1, the random variable $\lim_{u\to\infty} g(u) - \tau(\Lambda_u)$ is well defined $P_h^{x_0}$-a.s. It is clearly measurable with respect to the tail σ-field of $\dot{X}_t$, and so

$$h(x) = \int_{\mathcal{H}} \dot{K}((x,t),z)\mu(dz),$$

for every $(x,t) \in \dot{D}_f$, where μ is some measure concentrated on $\mathcal{H}$. In particular, it follows that $\mathcal{H}_s$ is nonempty for some $s \in \mathbb{R}$. We will work towards proving that, in fact,

$$\text{every } \mathcal{H}_s \text{ consists of a single point,} \tag{3.22}$$

namely the z_s of (A).

In fact, the conclusion of (B) will follow immediately from (3.22), since $\mathcal{H}_{s_1}$ and $\mathcal{H}_{s_2}$ are disjoint if $s_1 \neq s_2$.

For $z \in \mathcal{H}_s$, we have that $g(u) - \tau(\Lambda_u) \to s$, $P_z^{x_0,0}$-a.s. A standard argument now shows that the same is true, $P_z^{x,t}$-a.s., for every $(x,t) \in \dot{D}_f$. Thus (D) will also follow immediately, once (3.22) is proven.

(ii) It is a routine matter to prove that if

$$P_h^{x_0,0}\left(\lim_{u\to\infty}(g(u) - \tau(\Lambda_u)) \in (s_1, s_2)\right) > 0, \tag{3.23}$$

then for every $s_3 \in \mathbb{R}$,

$$P_h^{x_0,0}\left(\lim_{u\to\infty}(g(u) - \tau(\Lambda_u)) \in (s_1 + s_3, s_2 + s_3)\right) > 0.$$

Hence, (3.23) holds for all $-\infty < s_1 < s_2 < \infty$. Therefore

$$\mu\left(\bigcup_{s\in(s_1,s_2)} \mathcal{H}_s\right) > 0,$$

for every such s_1, s_2. This will establish (C). Moreover, it shows that

$$\exists\ \{s_k\}_{k\geq 1} \text{ such that } \lim_{k\to\infty} s_k = \infty \text{ and for every } k,\ \mathcal{H}_{s_k} \neq \emptyset. \tag{3.24}$$

If $\phi = \dot{K}(\,\cdot\,, z)$, where $z \in \mathcal{H}_s$, and $v < 0$, then by Lemma 3.8,

$$\begin{aligned} 1 = P_\phi^{x,t+v}(g(u) - \tau(\Lambda_u) \to s) &= P_\phi^{x,t+v}(g(u) + T(\Lambda_u) - t - v \to s) \\ = P_{\phi_v}^{x,t}(g(u) + T(\Lambda_u) - t - v \to s) &= P_{\phi_v}^{x,t}(g(u) - \tau(\Lambda_u) \to s + v); \end{aligned}$$

that is, the pole of ϕ_v belongs to $\mathcal{H}_{s+v}$. Thus, Φ_v maps $\mathcal{H}_s$ into $\mathcal{H}_{s+v}$. Appealing to (3.24), we conclude that $\mathcal{H}_s$ is nonempty, for every $s \in \mathbb{R}$.

(iii) Let $s \in \mathbb{R}$, and pick $z \in \mathcal{H}_s$. For any sequence $u_k \to \infty$, we may set $y_k = X(T(\Lambda_{u_k}))$, and $t_k = \tau(\Lambda_{u_k})$. Because $\dot{X}(T(\Lambda_{u_k})) \to z$ in the Martin topology, $P_z^{x,0}$-a.s., it follows that we have constructed a sequence $(y_k, t_k) \to z$ as in part (A), with $s_k \stackrel{\text{df}}{=} g(y_k^d) - t_k \to s$.

Next we will show that $\mathcal{H}_s$ consists of a single point for each s. Suppose to the contrary that $z, \widetilde{z} \in \mathcal{H}_s$ for some s. It is easy to see that we must have $\Phi_{s'-s}z \neq \Phi_{s'-s}\widetilde{z}$ for some $s' < s$. Fix any sequence $u_k \to \infty$. Consider any sequence $(y_k, t_k) \to z$, with $g(y_k^d) - t_k \to s$ and $y_k^d = u_k$, constructed as in the previous paragraph. Let $(\widetilde{y}_k, \widetilde{t}_k)$ be the analogous sequence with $(\widetilde{y}_k, \widetilde{t}_k) \to \widetilde{z}$, $\widetilde{y}_k^d = u_k$ and $g(\widetilde{y}_k^d) - \widetilde{t}_k \to s$. Note that $t_k - \widetilde{t}_k \to 0$ because $y_k^d = \widetilde{y}_k^d = u_k$, $g(y_k^d) - t_k \to s$, and $g(\widetilde{y}_k^d) - \widetilde{t}_k \to s$. Lemma 3.21 implies that $(y_k, t_k + s - s') \to \Phi_{s'-s}z$. But it also implies that $(y_k, t_k + s - s') \to \Phi_{s'-s}\widetilde{z}$, because $y_k^d = \widetilde{y}_k^d$ and $t_k - \widetilde{t}_k \to 0$. This contradicts the fact that $\Phi_{s'-s}z \neq \Phi_{s'-s}\widetilde{z}$ and so it proves our claim, and establishes (3.22).

Now let $r_k \to s$, and consider any sequence x_k such that $x_k^d \to \infty$. Our goal is to show that $(x_k, g(x_k^d) - r_k) \to z$, where z is the only element of $\mathcal{H}_s$. Set $u_k = x_k^d$, and this time choose $s' > s$. Let z' be the element of $\mathcal{H}_{s'}$. Note that $\Phi_{s-s'}z' = z$. By the argument of the first paragraph of (iii), we may choose $(y_k, t_k') \to z'$ with

$y_k^d = u_k = x_k^d$ and $g(y_k^d) - t_k' \to s'$. Since $t_k' - [g(x_k^d) - r_k] \to -s' + s$, we may apply Lemma 3.21 and obtain that

$$(x_k, g(x_k^d) - r_k) \to \Phi_{s-s'} z' = z.$$

This finishes the proof of (A). Thus, part (i) of Theorem 1.3 is proven.

(iv) Turning to part (ii) of Theorem 1.3, suppose that $\int_u^\infty f^3(v)dv = \infty$ for all $u < \infty$. We also assume, since it simplifies the proof, that $f(u) \to 0$ as $u \to \infty$. At the end we will sketch out how to extend the argument to the general case, that $\limsup_{u\to\infty} f(u) < \infty$.

We use a coupling argument. Fix $x_1, x_2 \in D_f$, and $s \le 0$. Let X_1 and X_2 be independent processes, under a probability measure P, with the same distributions as X under $P_h^{x_1,s}$ and $P_h^{x_2,s}$ respectively. Thus, $\dot{X}_1(t) = (X_1(t), \tau_t)$ and $\dot{X}_2(t) = (X_2(t), \tau_t)$ are versions of $\dot{X}$, where $\tau(t) = s - t$. Define

$$W = \inf\{t > 0 : X_1^d(t) = X_2^d(t)\}.$$

We will show that

$$P(W < \infty) = 1. \tag{3.25}$$

Write $T_j(\Lambda_u)$ for the hitting time of Λ_u by X_j. We may assume, without loss of generality, that $x_1^d \le x_2^d$. Set $u_0 = x_2^d + f(x_2^d)$, $Y_j = T_j(\Lambda_{u_0})$ and $Z_j = T_j(\Lambda_u) - T_j(\Lambda_{u_0})$, where the value of u will be chosen later. A standard application of the boundary Harnack principle (Lemma 2.2) shows that the Radon-Nikodym derivative of the hitting distributions of Λ_{u_0} under $P_h^{y_1}$ and $P_h^{y_2}$ is bounded below by $c_1 > 0$ for all $y_1, y_2 \in \Lambda_{x_2^d}$.

Let c_2 be so large that

$$P(Y_1 - Y_2 \ge c_2) < c_1/16. \tag{3.26}$$

Use Theorem 1.6 (v) to find u so large that for every $v \in \mathbb{R}$ we have

$$P(Z_2 \in (v, v + c_2)) < c_1/8. \tag{3.27}$$

Let v_1 be the median of Z_1, in other words,

$$P(Z_1 \le v_1) \ge 1/2, \quad P(Z_1 \ge v_1) \ge 1/2. \tag{3.28}$$

By applying the strong Markov property at $T(\Lambda_{u_0})$, and by our choice of c_1, we have $P(Z_2 \ge v_1) \ge c_1/2$. Now we use (3.27) to obtain that $P(Z_2 \ge v_1 + c_2) \ge 3c_1/8$. This, (3.28) and the independence of Z_1 and Z_2 show that

$$P(Z_2 - Z_1 \ge c_2) \ge P(Z_1 \le v_1, Z_2 \ge v_1 + c_2) \ge 3c_1/16.$$

Inequality (3.26) now implies that

$$\begin{aligned} P(T_1(\Lambda_u) < T_2(\Lambda_u)) &= P(Y_1 + Z_1 < Y_2 + Z_2) \\ &\ge P(Y_1 - Y_2 < c_2 \le Z_2 - Z_1) \\ &\ge P(Z_2 - Z_1 \ge c_2) - P(Y_1 - Y_2 \ge c_2) \ge c_1/8. \end{aligned} \tag{3.29}$$

Let $V_j^0 = x_j$, $\tau^0 = s$, $T^1 = \max(T_1(\Lambda_u), T_2(\Lambda_u))$, $\tau^1 = \tau(T^1)$, $V_j^1 = X(T_j(\Lambda_u))$, $U^1 = u$. Repeat the above argument, starting from (V_j^1, τ^1) in place of (V_j^0, τ^0), and ensuring that U^2 is chosen so large that each $T_j(\Lambda_{U^2}) > T^1$. Then continue this procedure inductively, to obtain sequences of random variables V_j^k, T^k, τ^k, and U^k. By the strong Markov property, (3.29) becomes

$$P(T_1(\Lambda_{U^{k+1}}) < T_2(\Lambda_{U^{k+1}}) \mid \mathcal{F}_{T^k}) \ge c_1/8,$$

where $\mathcal{F}_t$ is the filtration of $(X_1(t), X_2(t))$. It follows that an infinite number of these events will occur, P-a.s. The same is true when the roles of X_1 and X_2 are reversed. Thus (3.25) holds.

(v) According to (3.25), used repeatedly, there are points $(x_{j,k}, t_k)$ on the paths of $\dot{X}_j$ such that $x^d_{1,k} = x^d_{2,k} \to \infty$. Using Lemma 3.21, as in the argument of section (iii) above, we get that $(x_{1,k}, t_k)$ and $(x_{2,k}, t_k)$ have the same limit in $\partial^M_0 \dot{D}_f$. Thus, the limits of $\dot{X}_1(t)$ and $\dot{X}_2(t)$ in $\partial^M_0 \dot{D}_f$, as $t \to \infty$, are the same. Since $\dot{X}_1$ and $\dot{X}_2$ are independent, the measure μ such that $h(x) = \int_{\partial^M_0 \dot{D}_f} \dot{K}((x,0), z)\mu(dz)$ must actually be supported on a singleton. That is, h must be minimal as a parabolic function.

It is the use of Lemma 3.21 that requires the assumption that $f(u) \to 0$. If only $\limsup_{u\to\infty} f(u) < \infty$, we modify the argument as follows. For any $\varepsilon > 0$,

$$P^{x,t}_h(X(f((x^d)^2)) \in B((0,\ldots,0,x+f(x^d)), \varepsilon f(x^d))) \geq c(\varepsilon) > 0, \tag{3.30}$$

for every $(x,t) \in \dot{D}_f$. Let

$$W_\varepsilon \stackrel{\text{df}}{=} \inf\{t > 0 : X^d_1(t), X^d_2(t) \in B((0,\ldots,0,u+f(u)), \varepsilon f(u)) \text{ for some } u\}.$$

Applying (3.30) to $x = X_j(W)$ and using another iterative argument, one can show that $P(W_\varepsilon < \infty) = 1$ for every $\varepsilon > 0$. Taking a sequence $\varepsilon_k \to 0$, now gives sequences $(x_{j,k}, t_k)$ on the paths of $\dot{X}_j$, such that

$$x_{j,k} \in B((0,\ldots,0,u_k + f(u_k)), \varepsilon_k f(u_k)),$$

where $u_k \to \infty$. An argument as in the proof of Lemmas (3.14) and 3.21 now shows that the $(x_{1,k}, t_k)$ and $(x_{2,k}, t_k)$ have the same limit in $\partial^M_0 \dot{D}_f$. As before, this shows that h is parabolically minimal.

□

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