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CLASSES OF SINGULAR INTEGRALS ALONG CURVES AND SURFACES

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ABSTRACT. This paper is concerned with singular convolution operators in $\mathbb{R}^d$, $d \geq 2$, with convolution kernels supported on radial surfaces $y_d = \Gamma(|y'|)$. We show that if $\Gamma(s) = \log s$, then L^p boundedness holds if and only if $p = 2$. This statement can be reduced to a similar statement about the multiplier $m(\tau, \eta) = |\tau|^{-i\eta}$ in $\mathbb{R}^2$. We also construct smooth Γ for which the corresponding operators are bounded for $p_0 < p \leq 2$ but unbounded for $p \leq p_0$, for given $p_0 \in [1, 2)$. Finally we discuss some examples of singular integrals along convex curves in the plane, with odd extensions.

1. INTRODUCTION

This paper is primarily concerned with singular integral operators T in dimensions $d \geq 2$ defined for $f \in C_0^\infty(\mathbb{R}^d)$ by

$$Tf(x', x_d) = \text{p.v.} \int f(x' - y', x_d - \Gamma(|y'|)) \frac{\Omega(y')}{|y'|^{d-1}} dy' \tag{1.1}$$

where $x' \in \mathbb{R}^{d-1}$. We assume that $\Gamma : (0, \infty) \to \mathbb{R}$ is a smooth function, $\Omega \in L^q(S^{d-2})$ for some $q > 1$ and

$$\int_{S^{d-2}} \Omega(\theta) d\sigma(\theta) = 0. \tag{1.2}$$

We include the case $d = 2$ with the interpretation of $S^0 = \{-1, 1\}$ and the surface measure being counting measure.

It is easy to see using (1.2) that the principal value integral (1.1) exists everywhere for $f \in C_0^\infty$. The question is for which $p \in (1, \infty)$ the operator T extends to a bounded operator on $L^p(\mathbb{R}^d)$. If we consider the case of convex Γ it is known that, then L^2 boundedness implies L^p boundedness for $1 < p < \infty$ (see [10], [2] for the case $d = 2$ and [8] for the case $d \geq 3$, at least in the case of smooth Ω). Moreover it was shown in [8] (again assuming that Ω is smooth and Γ is C^1 in $(0, \infty)$) that in dimension $d \geq 3$ the operators T are bounded in $L^2(\mathbb{R}^d)$, without any convexity assumption on Γ. Our primary concern here is whether T extends to a bounded operator on L^p without any further restriction on Γ. Our first theorem shows that this is not the case, in fact in our example Γ is chosen to be *concave.*

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Theorem 1.1. *Suppose that* $\Omega \in L^q(S^{d-2})$ *where* $q > 1$ *and suppose that the cancellation property (1.2) holds. Suppose* $\Gamma(t) = \log t$. *Then* T *extends to a bounded operator on* $L^p(\mathbb{R}^d)$ *if and only if* $p = 2$ *or* $\Omega = 0$ *almost everywhere.*

Remark. The analogous maximal operator M_γ defined as the pointwise supremum of averages over $\{(x+y', \log(|x+y'|) : |y'| \leq h\}$, $h > 0$, is unbounded on all L^p spaces, see the argument in [14, p. 1291]. Moreover the L^2 estimate may fail if the standard homogeneous Calderón-Zygmund kernels $\Omega(y'/|y'|)|y'|^{1-d}$ are replaced by other (standard) singular kernels, such as the kernel for fractional integration of imaginary order, see Remark 2.3 below.

We shall see that the unboundedness of T for $p \neq 2$ follows from a negative result for a Fourier multiplier on $\mathbb{R}^2$. In what follows M^p denotes the class of Fourier multipliers of L^p and $\|m\|_{M^p}$ is the L^p operator norm of the convolution operator with Fourier multiplier m.

Proposition 1.2. *Let* χ *be a bounded function in* $C^1(\mathbb{R})$ *and define*

$$h(\tau, \eta) = \chi(\eta)|\tau|^{-i\eta}. \tag{1.3}$$

Then $h \in M^p(\mathbb{R}^2)$ *if and only if* $p = 2$ *or* $\chi \equiv 0$.

If χ_+ *denotes the characteristic function of* $(0, \infty)$, *then the same statement holds with* $h(\tau, \eta)$ *replaced by* $h_\pm(\tau, \eta) = h(\tau, \eta)\chi_+(\pm\tau)$.

Remark. This result should be compared with the fact that for every η the multiplier $\tau \mapsto |\tau|^{-i\eta}$ is a multiplier in $M^p(\mathbb{R})$ for $1 < p < \infty$ (it is the multiplier corresponding to fractional integration of imaginary order; the L^p boundedness follows from the Marcinkiewicz multiplier theorem).

In our second theorem we exhibit operators T with a prescribed range of L^p boundedness.

Theorem 1.3. *Suppose* $1 < r \leq 2$. *There is a function* Γ *defined on* $[0, \infty)$ *with* $\Gamma(0) = 0$, *such that the symmetric extension* $\Gamma(|x'|)$ *to* $\mathbb{R}^{d-1}$ *is smooth and such that the following holds.*

Let $d \geq 2$ *and* T *be as in (1.1), where* $\Omega \in L^q(S^{d-2})$ *for some* $q > 1$ *and the cancellation property (1.2) is assumed. Then* T *extends to a bounded operator on* $L^p(\mathbb{R}^d)$ *if and only if* $r \leq p \leq r/(r-1)$ *or* $\Omega = 0$ *almost everywhere.*

Remarks. (i) Let $1 \leq r < 2$. A slight modification of our construction yields Γ such that T is bounded on $L^p(\mathbb{R}^d)$ if and only if $r < p < r/(r-1)$ or $\Omega = 0$ a.e.

(ii) Examples where the *maximal* operator associated to the curve is bounded on some L^p spaces but not on others have been constructed by M. Christ [4], see also Vance, Wright and Wainger [15] and unpublished work by Wierdl. Examples of this kind for singular integral operators seem to be new; however in [3] an example of a convex Γ was constructed, so that the Hilbert transform associated to the *odd* extension was bounded only on $L^2(\mathbb{R}^2)$.

(iii) In an appendix (§5) we include some observations related to the examples in [3] and [4], dealing with singular integrals with convolution kernels supported on curves $\{(t, \gamma(t))\}$ in the plane; here γ is the odd extension of a convex function on $(0, \infty)$.

2. L^2-ESTIMATES

We shall now consider the case

$$\Gamma(t) = \log t$$

and show that T is bounded on L^2 (provided that $\Omega \in L^q$, $q > 1$). This is achieved by showing that

$$\begin{aligned} m_R(\xi) &= \int_{|x'|\le R} e^{-i(\langle x',\xi'\rangle + \xi_d \log|x'|)} \frac{\Omega(x'/|x'|)}{|x'|^{d-1}} dx' \\ &= \int_0^R e^{-i\xi_d \log r} \int_{S^{d-2}} e^{-i\langle r\theta,\xi'\rangle} \Omega(\theta) d\sigma(\theta) \frac{dr}{r} \end{aligned} \tag{2.1}$$

is bounded uniformly in ξ and R and converges to a bounded function as $R \to \infty$. By changing variables $r \mapsto r|\xi'|$ and using the cancellation of Ω we see that

$$m_R(\xi) = e^{i\xi_d \log|\xi'|} M_{R|\xi'|}(\xi'/|\xi'|, \xi_d) \tag{2.2}$$

with

$$M_R(\vartheta, \xi_d) = \int_0^R e^{-i\xi_d \log r} \int_{S^{d-2}} (e^{-i\langle r\theta,\vartheta\rangle} - 1) \Omega(\theta) d\sigma(\theta) \frac{dr}{r} \tag{2.3}$$

for $\vartheta \in S^{d-2}$.

We split $M_R = \sum_{i=1}^3 \mathcal{E}_i^R$ where

$$\begin{aligned} \mathcal{E}_1^R(\vartheta,\xi_d) &= \int_0^R e^{-i\xi_d \log r} \int_{\theta: r|\langle\theta,\vartheta\rangle|\le 1} (e^{-i\langle r\theta,\vartheta\rangle} - 1) \Omega(\theta) d\sigma(\theta) \frac{dr}{r}, \\ \mathcal{E}_2^R(\vartheta,\xi_d) &= \int_0^R e^{-i\xi_d \log r} \int_{\theta: r|\langle\theta,\vartheta\rangle|\ge 1} e^{-i\langle r\theta,\vartheta\rangle} \Omega(\theta) d\sigma(\theta) \frac{dr}{r}, \\ \mathcal{E}_3^R(\vartheta,\xi_d) &= -\int_0^R e^{-i\xi_d \log r} \int_{\theta: r|\langle\theta,\vartheta\rangle|\ge 1} \Omega(\theta) d\sigma(\theta) \frac{dr}{r}. \end{aligned} \tag{2.4}$$

First observe that

$$|\mathcal{E}_1^R(\vartheta,\xi_d)| \le \int |\Omega(\theta)| \int_0^{\min\{|\langle\theta,\vartheta\rangle|^{-1}, R\}} |e^{-i\langle r\theta,\vartheta\rangle} - 1| \frac{dr}{r} d\sigma(\theta) \le C.$$

To estimate $\mathcal{E}_2^R$ interchange the order of the integration and observe that after a change of variables $s = r|\langle\theta,\vartheta\rangle|$ in the inner integral we have

$$\begin{aligned} \mathcal{E}_2^R(\vartheta,\xi_d) \ &= \int_{\langle\theta,\vartheta\rangle \ge R^{-1}} \Omega(\theta) e^{i\xi_d \log|\langle\theta,\vartheta\rangle|} u_+(\xi_d, R|\langle\theta,\vartheta\rangle|) d\sigma(\theta) \\ &\quad + \int_{\langle\theta,\vartheta\rangle \le -R^{-1}} \Omega(\theta) e^{i\xi_d \log|\langle\theta,\vartheta\rangle|} u_-(\xi_d, R|\langle\theta,\vartheta\rangle|) d\sigma(\theta) \end{aligned}$$

where

$$u_\pm(\gamma, N) = \int_1^N \exp(-i(\pm s + \gamma \log s)) \frac{ds}{s}. \tag{2.5}$$

We show that u is uniformly bounded in γ and $N \ge 1$.

Assume first that $|\gamma| > 1/2$. Then we split the integral (2.5) into three parts depending on whether $|\gamma| \ge 5s$ or $s < |\gamma|/5$ or $|\gamma|/5 < s < 5|\gamma|$. The integral over $s \in [|\gamma|/5, 5|\gamma|]$ is trivially bounded.

If $N > 5|\gamma|$, then we integrate by parts to get

$$\begin{aligned}\int_{5|\gamma|}^{N} e^{-i(\pm s+\gamma\log s)}\frac{ds}{s} &= \int_{5|\gamma|}^{N}\frac{d(e^{i(\mp s+\gamma\log s)})}{\mp is - i\gamma}\\ &= i\Big(\frac{e^{-i(\pm N+\gamma\log N)}}{\gamma\mp N} - \frac{e^{-i(\pm 5\gamma+\gamma\log 5\gamma)}}{\gamma\mp 5|\gamma|}\Big)\\ &\mp i\int_{5|\gamma|}^{N} e^{-i(\pm s+\gamma\log s)}\frac{ds}{(\gamma\pm s)^2}\end{aligned}$$

and this is bounded (since $|\gamma|\ge 1/2$).

We treat the integral $\int_1^{|\gamma|/5} e^{-i(\pm s+\gamma\log s)}\frac{ds}{s}$ similarly. If $|\gamma|<1/2$ and $N\ge 1$, then

$$\int_1^N e^{-i(\pm s+\gamma\log s)}\frac{ds}{s} = \pm i(e^{\mp iN}N^{-i\gamma-1} - e^{\mp i}) \pm (i\gamma+1)\int_1^N e^{\mp is}s^{-i\gamma-2}ds \tag{2.6}$$

which is bounded. This shows that $|\mathcal{E}_2^R(\vartheta,\xi_d)| = O(1)$, uniformly in R.

Finally to estimate $\mathcal{E}_3^R(\vartheta,\xi_d)$ we observe that

$$\begin{aligned}\mathcal{E}_3^R(\vartheta,\xi_d) &= -\int_{|\langle\theta,\vartheta\rangle|\ge 1/R}\Omega(\theta)\int_{r=|\langle\theta,\vartheta\rangle|^{-1}}^{R} e^{-i\xi_d\log r}\frac{dr}{r}\,d\sigma(\theta)\\ &= -\mathcal{E}_{3,1}^R(\vartheta,\xi_d) + \mathcal{E}_{3,2}^R(\vartheta,\xi_d)\end{aligned}$$

where

$$\begin{aligned}\mathcal{E}_{3,1}^R(\vartheta,\xi_d) &= \int_{S^{d-2}}\Omega(\theta)\int_{r=|\langle\theta,\vartheta\rangle|^{-1}}^{R} e^{-i\xi_d\log r}\frac{dr}{r}\,d\sigma(\theta),\\ \mathcal{E}_{3,2}^R(\vartheta,\xi_d) &= \int_{|\langle\theta,\vartheta\rangle|\le 1/R}\Omega(\theta)\int_{r=|\langle\theta,\vartheta\rangle|^{-1}}^{R} e^{-i\xi_d\log r}\frac{dr}{r}\,d\sigma(\theta).\end{aligned}$$

Now

$$\mathcal{E}_{3,1}^R(\vartheta,\xi_d) = -\int_{S^{d-2}}\Omega(\theta)\frac{R^{-i\xi_d} - |\langle\theta,\vartheta\rangle|^{i\xi_d}}{-i\xi_d}d\sigma(\theta) = -\int_{S^{d-2}}\Omega(\theta)\frac{1-|\langle\theta,\vartheta\rangle|^{i\xi_d}}{-i\xi_d}d\sigma(\theta)$$

where we have used the cancellation of Ω again. We see that

$$\begin{aligned}|\mathcal{E}_{3,1}^R(\vartheta,\xi_d)| &\le \int_{S^{d-2}}|\Omega(\theta)|\frac{|e^{-i\xi_d\log|\langle\theta,\vartheta\rangle|}-1|}{|\xi_d|}d\sigma(\theta)\\ &\le \int_{S^{d-2}}|\Omega(\theta)|\log|\langle\theta,\vartheta\rangle|^{-1}d\sigma(\theta)\end{aligned}$$

and the last integral is bounded uniformly in ϑ because of our assumption $\Omega\in L^q$. Moreover by a straightforward estimate

$$\begin{aligned}\mathcal{E}_{3,2}^R(\vartheta,\xi_d) &\le \int_{|\langle\theta,\vartheta\rangle|\le 1/R}|\Omega(\theta)|\big[\log R + \log|\langle\theta,\vartheta\rangle|^{-1}\big]d\sigma(\theta)\\ &\le 2\int_{S^{d-2}}|\Omega(\theta)|\log|\langle\theta,\vartheta\rangle|^{-1}d\sigma(\theta).\end{aligned}$$

We have shown that M_R is bounded uniformly in (ϑ, ξ_d). An examination of the above argument also shows that if $|\xi_d| \le J$ and $J \ge 1$, then for $J \le R \le R'$

$$|M_R(\vartheta,\xi_d) - M_{R'}(\vartheta,\xi_d)| \le C_J \Big[\int_{|\langle\theta,\vartheta\rangle| \le 10JR^{-1}} |\Omega(\theta)|(1+\log|\langle\theta,\vartheta\rangle|^{-1}) d\sigma(\theta) + \int_{|\langle\theta,\vartheta\rangle| \ge R^{-1}} |\Omega(\theta)|(R|\langle\theta,\vartheta\rangle|)^{-1} d\sigma(\theta) \Big]$$

which is $O(R^{-1+1/q}(1+\log R))$. Therefore $\lim_{R\to\infty} M_{R|\xi'|}(\xi'/|\xi'|, \xi_d)$ exists and the convergence is uniform with respect to (ξ', ξ_d) in compact subsets of $(\mathbb{R}^{d-1}\backslash\{0\})\times\mathbb{R}$. Since each M_R is easily seen to be a smooth function on $S^{d-1}\times\mathbb{R}$ we have proved

Proposition 2.1. *Suppose that $\Gamma(t) = \log t$, $\Omega \in L^q(S^{d-2})$, $q > 1$, and that (1.2) holds. Then T is bounded on $L^2(\mathbb{R}^d)$ and the Fourier transform of its convolution kernel is given by*

$$m(\xi) = e^{i\xi_d \log(|\xi'|)} M(\xi'/|\xi'|, \xi_d) f$$

where M is a bounded continuous function on $S^{d-2}\times\mathbb{R}$.

Remark 2.2. If Ω is odd, then T is L^2 bounded if (1.2) holds and Ω is merely in $L^1(S^{d-2})$. To see this one uses the method of rotations (see [1]). Define

$$H_\theta f(x) = \text{p.v.} \int f(x' - t\theta, x_d - \log|t|) \frac{dt}{t};$$

then one can see by transferring our result in two dimensions to d dimensions that H_θ is bounded on $L^2(\mathbb{R}^d)$ with operator norm independent of θ. If Ω is odd, then $T = c\int_{S^{d-2}} \Omega(\theta) H_\theta d\sigma(\theta)$ and the L^2 boundedness of T follows. For general Ω satisfying (1.2) the assumption $\Omega \in L\log L(S^{d-2})$ yields L^2 boundedness of T.

Remark 2.3. For $\alpha \ne 0$ let $m_\alpha(\tau) = |\tau|^{i\alpha}$ and $k_\alpha = \mathcal{F}^{-1}[m_\alpha]$, then k_α is a standard singular integral kernel on $\mathbb{R}^{d-1}$ (although not homogeneous of degree $1-d$). For $f \in C_0^\infty(\mathbb{R}^d)$ define

$$\mathcal{H}_\alpha f(x) = \int f(x'-t, x_d - \log|t|) k_\alpha(t) dt.$$

Then $\mathcal{H}_\alpha$ is unbounded on $L^2(\mathbb{R}^d)$. To see this observe that the associated multiplier

$$c_\alpha \int_{\mathbb{R}^{d-1}} e^{-i(\langle\xi',x'\rangle) + (\xi_d+\alpha)\log|x'|)} |x'|^{1-d} dx'$$

is unbounded as $\xi_d \to -\alpha$.

For later use we shall now show that for $\xi_d \ne 0$ the function M is actually differentiable as a function of ξ_d; in particular we shall show that

$$\Big|\xi_d \frac{\partial M(\vartheta,\xi_d)}{\partial \xi_d}\Big| \le C \quad \text{if } 0 < |\xi_d| \le 1/2. \tag{2.7}$$

The proof of (2.7) follows the lines above. Differentiation with respect to ξ_d gives another factor of $-i\log r$ in the formulas (2.4). In the estimation of $\mathcal{E}_1^R(\vartheta,\xi_d)$ this yields an additional factor of $\log|\langle\theta,\vartheta\rangle|^{-1}$ which is harmless in view of our assumption $\Omega \in L^q(S^{d-2})$. In the estimation of $\mathcal{E}_2^R(\vartheta,\xi_d)$ we shall only need to consider the term corresponding to (2.6) since we assume that $|\xi_d| \le 1/2$, and we get boundedness of the derivative (again the calculation yields an additional factor of $\log |\langle\theta,\vartheta\rangle|^{-1}$). The term corresponding to $\mathcal{E}_3^R(\vartheta,\xi_d)$ has to be handled with some care; it is a difference of $\widetilde{\mathcal{E}}_{3,2}^R(\vartheta,\xi_d)$ and $\widetilde{\mathcal{E}}_{3,1}^R(\vartheta,\xi_d)$ given by

$$\widetilde{\mathcal{E}}_{3,1}^R(\vartheta,\xi_d) = -i\int_{S^{d-2}}\Omega(\theta)\int_{r=|\langle\theta,\vartheta\rangle|^{-1}}^{R} e^{-i\xi_d\log r}\frac{\log r}{r}dr\,d\sigma(\theta),$$
$$\widetilde{\mathcal{E}}_{3,2}^R(\vartheta,\xi_d) = -i\int_{|\langle\theta,\vartheta\rangle|\le 1/R}\Omega(\theta)\int_{r=|\langle\theta,\vartheta\rangle|^{-1}}^{R} e^{-i\xi_d\log r}\frac{\log r}{r}dr\,d\sigma(\theta).$$

Now for $\xi_d \neq 0$

$$\int_{r=a}^{R} e^{-i\xi_d\log r}\frac{\log r}{r}dr\; = i\xi_d^{-1}R^{-i\xi_d}(\log R - i\xi_d^{-1}) - i\xi_d^{-1}a^{-i\xi_d}(\log a - i\xi_d^{-1}).$$

Using this for $a = |\langle\theta,\vartheta\rangle|^{-1}$ we may copy the argument for $\mathcal{E}_{3,1}^R(\vartheta,\xi_d)$, $\mathcal{E}_{3,2}^R(\vartheta,\xi_d)$ above, producing an additional factor of ξ_d^{-1}. Moreover the limiting argument above can be carried over as long as we stay away from $\xi_d = 0$. This yields (2.7).

3.1. The model multiplier in two dimensions. We now give a proof of Proposition 1.2. Clearly $h \in M_2$ since h is bounded. Let $1 < p < 2$ and assume that χ is not identically zero. We argue by contradiction and assume that $h \in M^p$. Our proof is related to an argument by Littman, McCarthy and Rivière [9].

We may choose an interval $I = (\alpha_0, \alpha_1)$ so that $\chi(\eta) \neq 0$ if η belongs to the closure of I. Let $\Phi \in \mathcal{S}(\mathbb{R})$ so that the Fourier transform $\widehat{\Phi}$ is compactly supported in I but does not identically vanish. Let β be a C^∞ function so that β is supported in $\{\tau : |\tau| \le 1\}$, $\beta(\tau) = 1$ if $|\tau| \le 1/2$.

Let

$$g_N(\tau,\eta) = \sum_{k=10}^{N}\frac{\widehat{\Phi}(\eta)}{\chi(\eta)}\beta(\tau - e^{2^k})e^{-i\eta(2^k-\log\tau)}.$$

Then it is easy to see by the sharp form of the Marcinkiewicz multiplier theorem ([13, p. 109]) that

$$\|g_N\|_{M^p} \le C_p \text{ for } 1 < p < \infty.$$

Let

$$h_N(\tau,\eta) = \sum_{k=10}^{N}\widehat{\Phi}(\eta)\beta(\tau - e^{2^k})e^{-i\eta 2^k},$$

then $h_N = g_N h$ and therefore

$$\|h_N\|_{M^p} \le C_p\|h\|_{M^p}.$$

However we shall show that

$$\|h_N\|_{M^p} \ge cN^{1/p-1/2} \tag{3.1}$$

so h cannot be in M^p.

Define f_N by

$$\widehat{f_N}(\tau,\eta) = \sum_{k=10}^{N}\beta(\tau - e^{2^k})\widehat{\Psi}(\eta)$$

where $\widehat{\Psi}$ is compactly supported but equals 1 on the support of $\widehat{\Phi}$, so $\Phi = \Phi * \Psi$.

Then by Littlewood-Paley theory

$$\|f_N\|_p \approx \Big\| \Big(\sum_{k=10}^{N} |\mathcal{F}^{-1}[\beta]|^2 \Big)^{1/2} \Big\|_p \approx N^{1/2}.$$

But

$$\mathcal{F}^{-1}[h_N \widehat{f_N}](x) = \sum_{k=10}^{N} \mathcal{F}^{-1}[\beta^2](x_1) e^{ix_1 e^{2^k}} \Phi(x_2 - 2^k)$$

and since $\Phi \neq 0$ is a Schwartz function it is easy to see that

$$\|\mathcal{F}^{-1}[h_N \widehat{f_N}]\|_p \geq cN^{1/p}.$$

This yields (3.1) and therefore the desired contradiction. The above argument also proves the corresponding statement for the multiplier h_+ and then also for h_-.

3.2. Failure of L^p-boundedness in Theorem 1.1. We now show that if $\Gamma(t) = \log t$ and if T is bounded on $L^p(\mathbb{R}^d)$, then $p = 2$, assuming that Ω is not identically 0. By the Riesz-Thorin theorem we may assume that $1 < p < \infty$. Let χ_+ be the characteristic function of $(0, \infty)$. If m is the corresponding multiplier, then we know by de Leeuw's theorem [7] that for almost all $\vartheta \in S^{d-2}$ the function $(\tau, \eta) \to \chi_+(\tau) m(\tau\vartheta, \eta)$ is a Fourier multiplier on $L^p(\mathbb{R}^2)$.

Now $m(\tau\vartheta, \eta) = |\tau|^{i\eta} M(\vartheta, \eta)$ for $\tau > 0$, by Proposition 2.1. Let K_Ω be the kernel $\Omega(x'/|x'|)|x'|^{1-d}$ on $\mathbb{R}^{d-1}$. Then its Fourier transform in $\mathbb{R}^{d-1}$ is homogeneous of degree zero and equals $M(\xi'/|\xi'|, 0)$. The latter cannot be zero almost everywhere by uniqueness of Fourier transforms. Therefore there is $\vartheta \in S^{d-2}$ such that $m(\tau\vartheta, \eta)$ is a Fourier multiplier on $L^p(\mathbb{R}^2)$ and such that $M(\vartheta, 0) \neq 0$. Since M is continuous in η there is $0 < \epsilon < 1/2$ and $c > 0$ so that $|M(\vartheta, \eta)| \geq c$ for $\epsilon/2 \leq \eta \leq \epsilon$. Let χ be a C^∞ function supported in $(\epsilon/2, \epsilon)$, not identically zero.

From (2.7) we see that $\eta \mapsto \chi(\eta)[M(\vartheta, \eta)]^{-1}$ is a Fourier multiplier on L^p, with bounds uniform in ϑ. Therefore $\chi(\eta)\chi_+(\tau)|\tau|^{i\eta}$ is a Fourier multiplier on $L^p(\mathbb{R}^2)$ and by Proposition 1.2 this implies that $p = 2$.

4. Examples for specific L^p spaces

In this section we give a proof of Theorem 1.3. For each p_0, with $1 < p_0 \leq 2$, we construct an even function $\Gamma \in C^\infty(\mathbb{R})$ such that $\Gamma(0) = 0$ and $\Gamma(t) = 0$ for $t \geq 1$, and such that the operator T as in (1.1) is bounded on $L^p(\mathbb{R}^d)$ if and only if $p_0 \leq p \leq p_0'$ or $\Omega = 0$ a.e.

Let $\zeta \in C^\infty(\mathbb{R})$ so that $\zeta(t) = 1$ if $t > 1/4$ and $\zeta(t) = 0$ if $t < -1/4$. Let $\delta = \{\delta_n\}$ be a sequence of positive numbers, so that $|\delta_n| \leq 1$ and $\lim_{n\to\infty} \delta_n = 0$.

Let $\{\gamma_n\}_{n=1}^\infty$ be a sequence of positive numbers such that $\gamma_{n+1} \leq \gamma_n/10$ for all $n \geq 1$. Our function Γ is then defined by

(4.1)

$$\Gamma(t) = \sum_{n=1}^{\infty} \gamma_n \zeta(2^{n^2+n}\delta_n^{-1}(|t| - 2^{-n^2}(1-\delta_n)))\zeta(2^{n^2+n}\delta_n^{-1}(2^{-n^2}(1+\delta_n) - |t|)).$$

Then for $n \geq 1$

$$\Gamma(t) = \begin{cases} \gamma_n & \text{if } 2^{-n^2}(1 - \delta_n + \delta_n 2^{-n-2}) \leq |t| \leq 2^{-n^2}(1 + \delta_n - \delta_n - \delta_n 2^{-n-2}), \\ 0 & \text{if } 2^{-(n+1)^2}(1 + \delta_{n+1} + \delta_{n+1} 2^{-n-3}) \leq |t| \leq 2^{-n^2}(1 - \delta_n - \delta_n 2^{-n-2}) \end{cases}$$

and $\Gamma(t) = 0$ for $|t| \geq 2$.

Theorem 4.1. *Let Γ be as in (4.1), T and Ω as in §1, $1 < p < \infty$ and let $s(p) = |1/p - 1/2|^{-1}$. Then T is bounded on L^p if and only if $\delta \in \ell^{s(p)}$ or $\Omega = 0$ almost everywhere.*

Theorem 1.3 is an immediate consequence, except for the fact that the even function Γ may not be smooth at the origin. This however can be achieved by an appropriate choice of γ_n, for example, $\gamma_n \le \gamma_{n-1} \exp(-2^n \delta_n^{-1})$ for all $n \ge 2$.

Proof of Theorem 4.1. Let $I_n = [2^{-n^2}(1-\delta_n), 2^{-n^2}(1+\delta_n)]$ and

$$T_n f(x) = \int_{|y'|\in I_n} f(x'-y', x_d - \gamma_n) \frac{\Omega(y')}{|y'|^{d-1}} dy'.$$

It is easy to see that $T = \sum_{n=1}^\infty T_n + \mathcal{H} + \sum_{n=1}^\infty K_n$ where the L^p operator norm of K_n is $O(2^{-n})$, for $1 \le p \le \infty$ and where $\mathcal{H}$ is the extension to $L^p(\mathbb{R}^d)$ of a variant of a Calderón-Zygmund operator acting in the x' variables; the L^p boundedness for $1 < p < \infty$ follows from [1]. It therefore suffices to examine the operator $\sum_n T_n$.

Let L_k denote the standard Littlewood-Paley operator on $\mathbb{R}^{d-1}$, i.e.,

$$\widehat{L_k f}(\xi) = \phi(2^{-k}|\xi'|)\hat{f}(\xi)$$

where ϕ is a C_0^∞ function supported on $\frac{1}{2} \le t \le 2$ such that $\sum_{k=-\infty}^\infty \phi(2^{-k}|t|) = 1$ for $t \ne 0$.

Then for some $\epsilon > 0$, depending on $p > 1$ and $q > 1$

$$\|L_{k+n} T_n\|_{L^p} \le A \min\{2^{-\epsilon|k|}, \delta_n\}; \tag{4.2}$$

see *e.g.* [6].

Define $\Delta_n = \sum_{j=n^2-n+1}^{n^2+n} L_j$, $\widetilde{\Delta}_n = \sum_{j=n^2-n-1}^{n^2+n+2} L_j$, so that $\Delta_n \widetilde{\Delta}_n = \Delta_n$. Observe by (4.2) that

$$\sum_{n=1}^\infty \|T_n - T_n \Delta_n\|_{L^p \to L^p} < \infty$$

for all $p \in (1,\infty)$. The L^p boundedness of T, under the assumption $\delta \in \ell^s$, follows by a well known argument using Littlewood-Paley theory (see [12] and [5]). For convenience we include the short proof. Without loss of generality assume $1 < p \le 2$. By Littlewood-Paley theory (or Calderón-Zygmund theory for vector-valued singular integrals [13, ch. II]) the inequality $\|\{\Delta_n f\}\|_{L^p(\ell^2)} \le C\|f\|_p$ holds for all $p \in (1,\infty)$, similarly the corresponding inequality involving $\widetilde{\Delta}_n$. Since the L^p operator norm of T_n is $O(\delta_n)$ we see that

$$\begin{aligned}
\Big\| \sum_n \widetilde{\Delta}_n T_n \Delta_n f \Big\|_p &\le C_p \big\|\{T_n \Delta_n f\}\big\|_{L^p(\ell^2)} \le C_p \big\|\{T_n \Delta_n f\}\big\|_{L^p(\ell^p)} \\
&= C_p \big\|\{T_n \Delta_n f\}\big\|_{\ell^p(L^p)} \le C_p \Big(\sum_n \|T_n\|_{L^p \to L^p}^p \|\Delta_n f\|_p^p \Big)^{1/p} \\
&\le C_p' \|\delta\|_{\ell^s} \big\|\{\Delta_n f\}\big\|_{\ell^2(L^p)} \le C_p' \|\delta\|_{\ell^s} \big\|\{\Delta_n f\}\big\|_{L^p(\ell^2)} \\
&\le C_p'' \|\delta\|_{\ell^s} \|f\|_p.
\end{aligned}$$

We now turn to the proof of the converse. We fix $p \in (1,2)$ and assume that T is bounded on L^p and that Ω does not vanish on a set of positive measure; we then have to prove that $\delta \in \ell^s$, $s = s(p)$.

Let

$$m_n(\xi') = \int_{|y'|\in I_n} e^{i\langle \xi', y'\rangle} \Omega(y'/|y'|)|y'|^{1-d} dy'.$$

Since by (4.1) the operator $\sum_n T_n$ is bounded on L^p,

$$m(\xi', \xi_d) = \sum_n e^{i\xi_d \gamma_n} m_n(\xi')$$

is a bounded multiplier on $L^p(\mathbb{R}^d)$. Since we assume that Ω does not vanish on some set of positive measure, it follows that there is an open set U on which the Fourier transform $\widehat{\Omega d\sigma}$ does not vanish, in fact we may assume that $|\widehat{\Omega d\sigma}(\xi)| \geq A > 0$ for $\xi \in U$. By de Leeuw's theorem [6] there is $\Xi \in U$ so that

$$u(\tau, \eta) = \sum_n e^{i\eta\gamma_n} m_n(\tau\Xi)$$

is a multiplier in $M^p(\mathbb{R}^2)$.

Since we assume that $\lim_{n\to\infty} \delta_n = 0$ we can choose a positive integer K so that the closed ball of radius δ_ℓ and center Ξ is contained in U for all $\ell \geq K$. Let $\beta \in C^\infty(\mathbb{R})$ with β supported in $[1/2, 2]$ so that $\beta(t) = 1$ in a neighborhood of 1. By the Marcinkiewicz multiplier theorem $\sum_{\ell=K}^N \beta(\tau - 2^{\ell^2})$ is in $M^r(\mathbb{R})$ for every r, $1 < r < \infty$, uniformly in N (here and in what follows we assume that $N \geq K$). Therefore the norms in $M^p(\mathbb{R}^2)$ of the multipliers $\sum_{\ell=K}^N \sum_n e^{i\eta\gamma_n} m_n(\tau\Xi)\beta(\tau - 2^{\ell^2})$ are uniformly bounded.

It follows from (4.2) that the $M_r(\mathbb{R}^2)$ norm of $m_n(\tau\Xi)\beta(\tau - 2^{\ell^2})$ is $O(2^{-\epsilon|\ell^2 - n^2|})$, where $\epsilon = \epsilon(r, q) > 0$ if $r > 1$, $q > 1$. Therefore $\sum_{\ell=K}^N \sum_{n\neq\ell} e^{i\eta\gamma_n} m_n(\tau\Xi)\beta(\tau - 2^{\ell^2})$ is a Fourier multiplier of $L^r(\mathbb{R}^2)$ for all $r \in (1, \infty)$ with bound uniformly in N. Consequently, by our assumption

$$v_N(\tau, \eta) = \sum_{\ell=K}^N e^{i\eta\gamma_\ell} m_\ell(\tau\Xi)\beta(\tau - 2^{\ell^2})$$

is a Fourier multiplier of $L^p(\mathbb{R}^2)$.

Now let

$$A_\ell = \int_{1-\delta_\ell}^{1+\delta_\ell} \int_{S^{d-2}} \Omega(\theta) e^{ir\langle \Xi, \theta\rangle} d\theta r^{-1} dr,$$
$$b_\ell(\tau) = \int_{1-\delta_\ell}^{1+\delta_\ell} \int_{S^{d-2}} \Omega(\theta) \big[e^{ir2^{-\ell^2}\tau\langle \Xi, \theta\rangle} - e^{ir\langle \Xi, \theta\rangle}\big] d\theta r^{-1} dr,$$

then

$$v_N(\tau, \eta) = \sum_{\ell=1}^N e^{i\eta\gamma_\ell} \big(A_\ell + b_\ell(\tau)\big).$$

Observe that for $\ell \geq K$

$$|A_\ell| \geq A \log\big(\frac{1+\delta_\ell}{1-\delta_\ell}\big) \geq A\delta_\ell. \tag{4.3}$$

Moreover $\beta(\cdot - 2^{\ell^2}) b_\ell$ is a Fourier multiplier of $L^1(\mathbb{R})$, with bound independent of ℓ. The L^∞ norm of this function is $O(2^{-\ell^2})$ and therefore by interpolation the multiplier $\sum_{\ell=K}^{N}\beta(\cdot-2^{\ell^2})b_\ell$ belongs to $M_r(\mathbb{R})$ for $r\in(1,\infty)$, with norm bounded in N. We conclude that

$$w_N(\tau,\eta)=\sum_{\ell=K}^{N}e^{i\eta\gamma_\ell}\beta(\tau-2^{\ell^2})A_\ell$$

belongs to $M^p(\mathbb{R}^2)$ with norm independent of N.

Let ψ be a nonnegative smooth function not identically zero, with support in $[-1/2,1/2]$ and let $\psi_N(y)=\gamma_{N+1}^{-1/p}\psi(\gamma_{N+1}^{-1}y)$.

Now let $\alpha=\{\alpha_\ell\}$ be a sequence in $\ell^{2/p}$, so that $\|\alpha\|_{\ell^{2/p}}\le 1$. Note that $2/p=(s/p)'$. We test w_N on f_N with

$$\widehat{f}_N(\tau,\eta)=\sum_{\ell=K}^{N}|\alpha_\ell|^{1/p}\beta(\tau-2^{\ell^2})\widehat{\psi}_N(\eta);$$

then by Littlewood-Paley theory

$$\|f_N\|_{L^p}\le C\Big\|\Big(\sum_{\ell=K}^{N}|\alpha_\ell|^{2/p}|\mathcal{F}^{-1}[\beta]|^2\Big)^{1/2}\Big\|_{L^p}\le C'$$

where C' is independent of N. On the other hand, for $(x,y)\in\mathbb{R}^2$,

$$\mathcal{F}^{-1}[w_N\widehat{f_N}](x,y)=\sum_{\ell=K}^{N}A_\ell|\alpha_\ell|^{1/p}\mathcal{F}^{-1}[\beta^2](x)e^{i2^{\ell^2}x}\psi_N(y-\gamma_\ell).$$

Since $\gamma_{N+1}\le\gamma_\ell/10$, $\ell=K,\dots,N$, the supports of the functions $\psi_N(y-\gamma_\ell)$ are disjoint. Therefore

$$\Big(\sum_{\ell=K}^{N}|A_\ell|^p|\alpha_\ell|\Big)^{1/p}\le C\|\mathcal{F}^{-1}[w_N\widehat{f_N}]\|_p\le C\|w_N\|_{M^p}\|f_N\|_p\le C'$$

uniformly in N. This implies by (4.3) that

$$\sup_{\|\alpha\|_{\ell^{(s/p)'}}\le 1}\sum_{\ell=K}^{\infty}|\delta_\ell|^p|\alpha_\ell|<\infty.$$

By the converse of Hölder's inequality it follows that $\{\delta_n^p\}\in\ell^{s/p}$ and therefore $\delta\in\ell^s$. □

5. Appendix: Odd extensions of convex curves in the plane

Here we include some observations concerning *odd* curves $(t,\gamma(t))$ where γ is convex in $(0,\infty)$. Our examples are modifications of those in [3] and [4]. For $r>0$, $\epsilon\ge 0$, and $j\ge 1$ set $\alpha_{\epsilon,j}=\tau 4^{-j}j^{\epsilon-1}$ for a small τ to be chosen later and

$$\gamma_{r,\epsilon}(t)=(2j)^r4^j+((2j+2)^r+\alpha_{\epsilon,j})(t-4^j)\text{ for }4^j\le t\le 4^j(1+j^{-\epsilon}).\tag{5.1}$$

For $4^j(1+j^\epsilon)\le t\le 4^{j+1}$, extend $\gamma_{r,\epsilon}$ so $\gamma''_{r,\epsilon}(t)$ is constant in this interval, $\gamma'_{r,\epsilon}$ is continuous at $4^j(1+j^{-\epsilon})$ and $\gamma_{r,\epsilon}(t)$ is continuous for $t\ge 4$. Similarly extend $\gamma_{r,\epsilon}$ to $[0,4]$ with constant positive curvature so that $\gamma_{r,\epsilon}(0)=0$. A calculation shows that $\gamma_{r,\epsilon}$ is convex for $t>0$. Finally extend $\gamma_{r,\epsilon}$ as an odd function. The perturbation by $\alpha_{\epsilon,j}$ in (5.1) is convenient in order that arguments in [4] to study maximal functions should apply to singular integral operators. We consider

$$H_{r,\epsilon}f(x,y) = \text{p.v.}\int f(x-t, y-\gamma_{r,\epsilon}(t))t^{-1}dt.$$

Proposition 5.1.
(i) For any $\epsilon \geq 0$ and $r > 0$, $\|H_{r,\epsilon}f\|_{L^2} \leq A\|f\|_{L^2}$.
(ii) If $p_0 = \frac{2\epsilon+2}{2\epsilon+1}$, then for any $r > 0$, $\|H_{r,\epsilon}f\|_{L^p} \leq A_p\|f\|_{L^p}$ for $p_0 < p < p_0'$.
(iii) If $r = 1$ and $\frac{4}{3} \leq p < 2$, $H_{r,\epsilon}$ is unbounded on L^p if $\epsilon < \frac{1}{p} - \frac{1}{2}$.
(iv) If $r = 1$ and $p \leq \frac{4}{3}$, $H_{r,\epsilon}$ is unbounded on L^p if $\epsilon \leq \frac{3}{p} - 2$.
(v) If r is a positive integer, then $H_{r,\epsilon}$ is unbounded on L^p if $p < \frac{r+2}{r+1+\epsilon}$.

Remarks. Consider the maximal function $\sup_{h>0} h^{-1}\int_0^h |f(x-t, y-\gamma_{r,\epsilon}(t))|dt$. Then the operator M is unbounded on L^p if $p < \frac{r+2}{r+1+\epsilon}$. This is a slight improvement over a result in [4]. More generally if $r = \frac{m}{n}$ with m and n positive integers, then one can show that M is unbounded if $p < \frac{m+2}{m+1+n\epsilon}$. One achieves this by restricting the values of j in the argument below to be nth powers. Obviously many questions remain open.

Proof of Proposition 5.1. Clearly (i) follows from [10] since $h_{r,\epsilon}(t) = t\gamma_{r,\epsilon}'(t) - \gamma_{r,\epsilon}(t)$ is doubling (see also [3], [16] for a more geometric proof of this result). In particular note that if $I_{j,\epsilon} = [4^j, 4^j(1+j^{-\epsilon})]$, then $\gamma_{r,\epsilon}(t) = s_j t - h_j$ where $s_j = (2j+2)^r + \alpha_{\epsilon,j}$ and $h_j = 4^j[(2j+2)^r - (2j)^r + \alpha_{\epsilon,j}]$.

Now set $\mathcal{I}_{j,\epsilon} = \{t : |t| \in I_{j,\epsilon}\}$ and let

$$G_j f(x,y) = \int_{\mathcal{I}_{j,\epsilon}} f(x-t, y-\gamma_{r,\epsilon}(t))t^{-1}dt.$$

Then $H_{r,\epsilon} = \sum_{j=1}^{\infty} G_j + E$. In view of the curvature properties of $\gamma_{r,\epsilon}$ in the complement of $\bigcup_j \mathcal{I}_{j,\epsilon}$ (where h is "infinitesimally doubling") the method of [3] may be applied to yield the L^q boundedness of E for all $q \in (1,\infty)$.

For the remaining assertions of the proposition it suffices to consider $G = \sum_j G_j$. To prove (ii) we consider the analytic family $G_z = \sum_j j^z G_j$. If $\text{Re}(z) < -1$, G_z is clearly bounded in L^1. (ii) follows by analytic interpolation if we can show that G_z is bounded in L^2 for $\text{Re}(z) < \epsilon$. This however follows by Fourier transform estimates following [11] or [16]. One derives the estimate

$$|\widehat{G_j}(\xi)| \leq C_1 \min\{j^{-\epsilon};\ 4^j|\xi_1 + \xi_2(s_j - 4^{-j}h_j)| + C_2|\xi_2|4^{-j}h_j;\ 4^{-j}|\xi_1 + \xi_2 s_j|^{-1}\}.$$

The first estimate is obvious, the second estimate uses the oddness of γ and the estimate $|\sin a| \leq |a|$ and the third uses an integration by parts. It is now straightforward to bound the sum $\sum_{j=1}^{\infty} |j^z\widehat{G_j}(\xi)|$ provided that $\text{Re}(z) < \epsilon$.

To obtain conclusion (v) we follow Christ [4]. We test G on the characteristic function f_N of a union of small rectangles $R_{(a,b)}$ centered at lattice points (a,b) with $0 \leq a \leq 2^N$ and $0 \leq b \leq N^r 2^N$,

$$R_{a,b} = \big\{(x,y) : a - N^{-r-1}\sigma \leq x \leq a + N^{-r-1}\sigma, b - N^{-1}\sigma \leq y \leq b + N^{-1}\sigma\big\}.$$

Here σ is small (to be chosen). We let for each pair of positive integers ℓ and j

$$S^{\ell,j} = \big\{(x,y) \mid 0 \leq x \leq 2^N,\ 0 \leq y \leq N^r 2^N, |y - (2j+2)^r x - \ell| \leq N^{-1}\sigma\big\}.$$

Then $|S^{\ell,j}| \geq \sigma 2^N(2N)^{-1}$ if $j \leq N/4$ and $\ell \leq N^r 2^N/10$, moreover if $j' \neq j$, $|S^{\ell,j} \cap S^{\ell',j'}| \leq A\sigma^2 N^{-2r-2}\big|j^{-r} - (j')^{-r}\big|^{-1} \leq A'\sigma^2 N^{-2}|j^r - (j')^r|^{-1}$.

Fixing ℓ, j, and j', the number of strips $S^{\ell',j'}$ that intersect $S^{\ell,j}$ is at most $2^N|j^r - (j')^r|$. Since there are at most N values of j', the measure of the union of all strips intersecting a given $S^{\ell,j}$ is at most $A\sigma|S^{\ell,j}|$, with A an absolute constant not depending on σ. We are going to restrict j to $N/5 \leq j \leq N/4$. We estimate Gf_N for points (x,y) in $S^{\ell,j}$ such that (x,y) is in no $S^{\ell',j'}$ with $j' \neq j$ and such that the vertical distance from (x,y) to the top of $S^{\ell,j}$ is between $10^{-5}\tau/N$ and $10^{-6}\tau/N$. If we first choose σ sufficiently small and then $\tau = \sigma/100$, we will be estimating Gf_N on a positive fraction of $S^{\ell,j}$. In evaluating Gf_N at such points (x,y) the contribution to Gf_N from pieces of $\gamma_{r,\epsilon}$ with slopes other than $(2j+2)^r$ is zero. The contribution Gf_N at such points comes from two strips

$$S^{\ell+(2j)^r 2^{2j},j} \quad \text{and} \quad S^{\ell-(2j)^r 2^{2j},j}.$$

The contribution from $S^{\ell-(2j)^r 2^{2j},j}$ is at least $10^{-2}j^{-\epsilon}N^{-r-1}$. The absolute value of the contribution from $S^{\ell+(2j)^r 2^{2j},j}$ is at most $10^{-3}j^{-\epsilon}N^{-r-1}$. Thus if G is bounded in L^p,

$$N^{-(r+1)p}j^{-p\epsilon}\Big|\bigcup_{\ell,j} S^{\ell,j}\Big| \leq A|\mathrm{supp}(f_N)|.$$

Therefore $N^{-(r+1)p}j^{-p\epsilon}NN^r2^N(2^N/N) \leq AN^r2^{2N}N^{-r-2}$ which implies for $N \to \infty$ the necessary condition $p \geq \frac{r+2}{\epsilon+r+1}$.

Note that (iv) is a special case of (v). Finally (iii) follows along the same lines as in §7 of [3]. Let

$$\begin{aligned} b_\eta(k) &= \int_{4^k}^{4^k(1+k^{-\epsilon})} \sin\{\eta[\alpha_{\epsilon,k}(t-4^k) - 4^{k+1}]\}\frac{dt}{t} \\ &= -(\log 2)k^{-\epsilon}\sin(4^{k+1}\eta) + O(k^{-1}). \end{aligned}$$

It then suffices to show that the sequence b_η does not belong to $M^p(\mathbb{Z})$ (the class of Fourier multipliers for Fourier series in $L^p(\mathbb{T})$) uniformly for $\pi \leq \eta \leq 3\pi$. The error $O(k^{-1})$ represents the Fourier coefficients of an L^2 function and belongs to $M_r(\mathbb{Z})$ for all $r \in [1,\infty]$. Now the argument in [3] shows $b_\eta \notin M^p(\mathbb{Z})$ if $\{k^{-\epsilon-1/p'}\log^{-1}k\} \notin \ell^2(\mathbb{Z})$ which is true if $\epsilon < 1/p - 1/2$. □

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