TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 351, Number 10, Pages 4113–4126
S 0002-9947(99)02299-0
Article electronically published on March 19, 1999

REMARKS ABOUT GLOBAL ANALYTIC HYPOELLIPTICITY

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Dedicated to Antonio Gilioli, in memoriam

ABSTRACT. We present a characterization of the operators

$$L = \partial/\partial t + (a(t) + ib(t))\partial/\partial x$$

which are globally analytic hypoelliptic on the torus. We give information about the global analytic hypoellipticity of certain overdetermined systems and of sums of squares.

0. INTRODUCTION

The purpose of this work is to study the property of global analytic hypoellipticity for certain partial differential operators on the torus. An operator P is said to be globally analytic hypoelliptic on $\mathbf{T}^n$ (GAH) if the conditions $u \in \mathcal{D}'(\mathbf{T}^n)$ and $Pu \in C^\omega(\mathbf{T}^n)$ imply $u \in C^\omega(\mathbf{T}^n)$.

The local version of this property was studied by Treves [T1] in the case of operators of principal type. For operators of the form $L = \partial/\partial t + (a(t) + ib(t))\partial/\partial x$, one has L analytic hypoelliptic if and only if b does not change sign (i.e., condition (P) holds) and b is not identically zero (i.e., condition (Q) holds); in particular, $a(t)$ plays no role.

In the case of real constant coefficients Greenfield [G] showed that $L = \partial/\partial t + \alpha\partial/\partial x$ is GAH on $\mathbf{T}^2$ if and only if α is an irrational number not too well approximable by rationals (in this paper we say that α is not an exponential Liouville number (EL); see Definition 2.2). In Section 2 we show that

$$L = \partial/\partial t + (a(t) + ib(t))\partial/\partial x$$

(where a and b are real-valued, real-analytic functions on the unit circle $\mathbf{S}^1$) is GAH on $\mathbf{T}^2$ if and only if $t \in \mathbf{S}^1 \mapsto b(t) \in \mathbf{R}$ does not change sign (this is a global version of (P)) and, when $b \equiv 0$, the real number $a_0 = (2\pi)^{-1}\int_0^{2\pi} a(t)dt$ is neither rational nor exponential Liouville. The hardest part of the proof of this result is the construction of singular solutions when (P) is violated; the main tool here is the steepest descent method, as described in [deB, Ch. 5].

In Section 3 we study involutive systems of vector fields on $\mathbf{T}^{n+1}$, of the form $\mathbb{L} = (L_1, \dots, L_n)$, where $L_j = \partial/\partial t_j + c_j(t_j)\partial/\partial x$, $j = 1, \dots, n$, with each $c_j = a_j + ib_j$ real-analytic. We show that $\mathbb{L}$ is GAH on $\mathbf{T}^{n+1}$ if and only if the set

Received by the editors July 19, 1996 and, in revised form, September 29, 1997.
1991 *Mathematics Subject Classification.* Primary 35H05.
Key words and phrases. Global analytic hypoellipticity, exponential Liouville numbers, exponential Liouville vectors, steepest descent, involutive systems, continued fractions.
The author was partially supported by CNPq.

$J \doteq \{j; b_j \text{ does not change sign}\} = \{j_1, \dots, j_\ell\}$ is nonempty and, when $b_j \equiv 0$, for all $j \in J$, the ℓ-tuple $(a_{0j})_{j \in J}$ (where $a_{0j} = (2\pi)^{-1} \int_0^{2\pi} a_j(t_j)dt_j$) is neither an exponential Liouville vector (see definition 3.2) nor an element of $\mathbf{Q}^\ell$. What is involved here is the problem of *simultaneous approximation*, that is approximations of a point in $\mathbf{R}^\ell$ by elements $(p_1/q, \dots, p_\ell/q)$ of $\mathbf{Q}^\ell$, with all entries having the same denominators.

Section 4 furnishes several examples illustrating the main theorems. We call attention to Example 4.9, where real numbers α, β, are constructed so that each of them is exponential Liouville, but (α, β) is not an exponential Liouville vector. One then has a system $\mathbb{L} = (L_1, L_2)$, where $L_1 = \partial/\partial t_1 - \alpha\partial/\partial x$, $L_2 = \partial/\partial t_2 - \beta\partial/\partial x$, such that $\mathbb{L}$ is GAH on $\mathbf{T}^3$, even though neither L_1 nor L_2 is GAH (these are not GAH even if we consider them as operators acting only in two variables). The same phenomenon may occur locally, but never in real structures of codimension ≥ 1. In our construction, the theory of continued fractions is heavily relied upon.

In Section 5 we comment on the connections of our work with [B], [BCM], [Ca-Ho], [Co-Hi], and [GPY]. A characterization of GAH for certain sums of squares arising from involutive systems of real vector fields is presented.

1. Asymptotic behavior of certain integrals

The following result is an important ingredient in the construction of singular solutions.

Lemma 1.1. *Consider the integral*

$$J(n) = \int_{-a}^{a} \exp[-n(F(t) + KG(t))]dt, \ n \in \mathbf{N},$$

where $a > 0$, $K > 0$ is a large parameter, and $F, G \in C^\omega(\mathbf{R})$ are such that $F(0) = \Re F'(0) = G(0) = G'(0) = 0$; $G''(0) = 1$; G is strictly decreasing (resp. increasing) on $-a \leq t < 0$ (resp. $0 < t \leq a$).

Then there exist $R_0 > 0$, $K_0 > 0$ such that for each $K \geq K_0$, the holomorphic function $z \mapsto F'(z) + KG'(z)$ has exactly one zero, z_0, in the (complex) disc $|z| \leq R_0$. Furthermore,

$$J(n) = \gamma n^{-1/2} \exp[-n(F(z_0) + KG(z_0))](1 + O(1/n)),$$

as $n \to \infty$, where γ is a nonzero constant.

We also have

$$|J(n)| = \gamma_1 n^{-1/2} \exp(-\varepsilon n)(1 + O(1/n)),$$

as $n \to \infty$, where $\gamma_1 = |\gamma| > 0$, and $\varepsilon = \Re(F(z_0) + KG(z_0))$; finally, $\varepsilon \geq 0$ and $\varepsilon = O(1/K)$, as $K \to \infty$.

Proof. The proof relies on applying the so-called steepest descent method; the main point is to deform the integration contour into the complex plane so that, in the new contour, a single point of minimum of $\Re(F(z) + KG(z))$ occurs.

Set $h(z) = F(z) + KG(z)$, and $H(z) = F(z)/K + G(z)$; then h and H are holomorphic in a neighborhood of $[-a, a]$. The assumptions imply that, if $R_0 > 0$ is small and $K_0 > 0$, then for all $K \geq K_0$, G' dominates F'/K on $|z| = R_0$; also R_0 can be chosen so that $z = 0$ is the only zero of $G'(z)$ on $|z| \leq R_0$. Now, by Rouché's theorem, $H'(z)$ and $h'(z)$ have exactly one zero, z_0, on $|z| \leq R_0$. Simple computations show that $|z_0| = O(1/K)$, and that $\varepsilon \doteq \Re h(z_0) = O(1/K)$.

We now use Taylor's formula to write $H(z) - H(z_0) = \widetilde{H}_2(z)(z - z_0)^2$, where $\widetilde{H}_2$ is holomorphic on $|z - z_0| < r \doteq R_0 - |z_0|$. Simple computations show that $\widetilde{H}_2$ has a square root, $H_2 = (\widetilde{H}_2)^{1/2}$, on $|z - z_0| < r'$, for some $0 < r' < r$; we also have $H_2(z_0) = (\widetilde{H}_2(z_0))^{1/2} = (1/\sqrt{2})(1 + O(1/K))$. We write $H_2(z)(z - z_0) = u + iv$, and so $H(z) - H(z_0) = (u + iv)^2$.

We are interested in the curves $\Re(H(z) - H(z_0)) = 0$, or $u = \pm v$.

Write $z = t + is$, $z_0 = t_0 + is_0$.

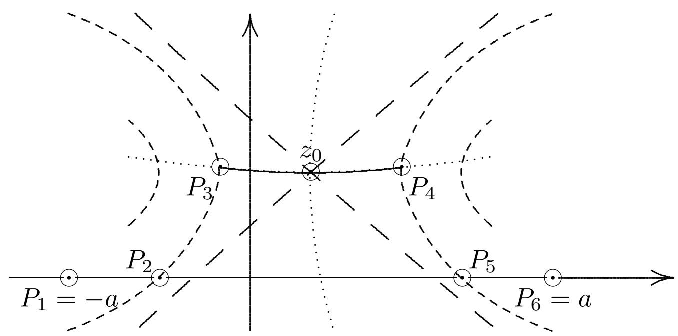
The above arguments show that $|H(z) - H(z_0) - \alpha(z - z_0)^2| \le \delta|z - z_0|^2$, where $0 < \delta = O(1/K)$, and $\alpha = 1/\sqrt{2}$ for $|z - z_0| \le r'$.

Thus, $u(t + is) = \pm v(t + is)$ implies $(1 - O(1/K))(t - t_0)^2 \le (s - s_0)^2 \le (1 + O(1/K))(t - t_0)^2$.

The point is that both curves $u = \pm v$ reach the real axis inside the disc $|z| \le R_0$. Furthermore, the same happens with the curves $\Re(H(z) - H(z_0)) = \delta'$, if $\delta' > 0$ is small enough.

Moreover, one of the curves $\Im(H(z) - H(z_0)) = 0$ is nearly horizontal.

This allows us to deform the contour $[-a, a]$ into $P_1P_2P_3z_0P_4P_5P_6$, where: $P_1 = -a$, $P_6 = a$; P_2 and P_5 are on the real axis; P_2P_3 and P_4P_5 are pieces of $\Re(H(z) - H(z_0)) = \delta'$; $P_3z_0P_4$ is an arc of the horizontal piece of $\Im(H(z) - H(z_0)) = 0$.



We can now use the steepest descent method: indeed, z_0 is the only point of minimum of $\Re H(z)$ along the deformed curve. This completes the proof of Lemma 1.1. $\square$

2. The case of a single vector field

Our concern here is the study of vector fields of the form

$$L = \partial_t + c(t)\partial_x \tag{2.1}$$

where $c \in C^\omega(\mathbf{S}^1)$, i.e., c is a real-analytic, 2π-periodic, complex-valued function of a single real variable. We may write $c(t) = a(t) + ib(t)$, with a and b real-valued, and we may also write

$$L = \partial_t + (a(t) + ib(t))\partial_x. \tag{2.2}$$

We are interested in the action of L on periodic (in both variables t, x) functions or distributions. More precisely, our aim is to give a characterization of those functions c for which L has the regularity property appearing in the following definition.

Definition 2.1. The vector field L is said to be globally analytic hypoelliptic on the torus $\mathbf{T}^2$ (briefly: L is GAH) if the conditions $u \in \mathcal{D}'(\mathbf{T}^2)$ and $Lu \in C^\omega(\mathbf{T}^2)$ imply $u \in C^\omega(\mathbf{T}^2)$.

We need one more definition, namely

Definition 2.2. An irrational number α is said to be an exponential Liouville number (briefly: α is EL) if there exists $\varepsilon > 0$ such that the inequality $|\alpha - p/q| \leq \exp(-\varepsilon q)$ has infinitely many rational solutions p/q, with $p \in \mathbf{Z}, q \in \mathbf{N}$.

We are now ready to state the main result of this section.

Theorem 2.3. *The vector field $L = \partial_t + (a(t) + ib(t))\partial_x$ is globally analytic hypoelliptic on the torus if and only if the following conditions are satisfied:*

(2.3) *b does not change sign;*

(2.4) *if $b \equiv 0$, then the real number $a_0 = (2\pi)^{-1} \int_0^{2\pi} a(t)dt$ is neither rational nor exponential Liouville.*

Before embarking on the proof of this theorem we will state, without proof, two results describing the possible solutions to the homogeneous equation $Lu = 0$, and, also, to the non-homogeneous equation

$$Lu = f \tag{2.5}$$

where $u \in \mathcal{D}'(\mathbf{T}^2)$ and $f \in C^\omega(\mathbf{T}^2)$.

Set $c_0 = (2\pi)^{-1} \int_0^{2\pi} c(t)dt$; then $c_0 = a_0 + ib_0$, where $a_0, b_0 \in \mathbf{R}$ are given by $a_0 = (2\pi)^{-1} \int_0^{2\pi} a(t)dt$, and $b_0 = (2\pi)^{-1} \int_0^{2\pi} b(t)dt$.

Lemma 2.4. *A distribution $u \in \mathcal{D}'(\mathbf{T}^2)$ is a solution to $Lu = 0$ if and only if one of the following situations occurs:*

(i) *$c_0 \notin \mathbf{Q}$ and u is a constant;*
(ii) *$c_0 = p/q$ (lowest terms), with $p \in \mathbf{Z}, q \in \mathbf{N}$ and u belongs to the closed span (in $\mathcal{D}'(\mathbf{T}^2)$) of the functions $\exp(ikq(x - \int_0^t c)), k \in \mathbf{Z}$.*

Lemma 2.5. *Assume that $c_0 \notin \mathbf{Q}$. If (2.5) has a solution, then necessarily*

$$\int_0^{2\pi} f_0(t)dt = 0.$$

If u is a solution to (2.5), then its partial Fourier coefficients are given by

$$u_0(t) = \int_0^t f_0(s)ds \ + \ \textit{constant}, \tag{2.6}$$

and, for $n \in \mathbf{Z} \setminus \{0\}$, by

$$u_n(t) = (1 - \exp(-i2\pi nc_0))^{-1} \int_0^{2\pi} \exp(-in \int_{t-s}^t c) f_n(t-s)ds \ ; \tag{2.7}$$

here $f_n(t)$ denotes the n^{th} partial (with respect to x) Fourier coefficient of f.

For $n \in \mathbf{Z} \setminus \{0\}$ we have the following alternative expression:

$$u_n(t) = (\exp(i2\pi nc_0) - 1)^{-1} \int_0^{2\pi} \exp(in \int_t^{t+s} c) f_n(t+s)ds \ . \tag{2.8}$$

We now point out the following consequence of Lemma 2.4, which already proves part of Theorem 2.3.

Corollary 2.6. *If $c_0 \in \mathbf{Q}$, then L is not* GAH.

Proof. Pick $t_0 \in \mathbf{S}^1$ such that $B(t_0) = \max\{B(t); t \in \mathbf{S}^1\}$. Write $a_0 = p/q$ and set $u(t,x) = \sum_{k=1}^{\infty} \exp(ikq(x - \int_0^t c + iB(t_0))$. It is easy to see that $u \in \mathcal{D}'(\mathbf{T}^2) \setminus C^\omega(\mathbf{T}^2)$ and $Lu = 0$. $\square$

Proof of Theorem 2.3. We begin by proving the necessity of conditions (2.3) and (2.4); the proof amounts to constructing singular solutions to $Lu = f$, when either (2.3) or (2.4) are not satisfied. Suppose first that (2.3) does not hold; thus $b(t) \not\equiv 0$ and $b(t)$ changes sign. We will divide the proof into four cases, according to the nature of the complex number $c_0 = a_0 + ib_0$.

Case 1. $b_0 < 0$. Here we choose $f(t,x) = (2\pi)^{-1} \sum_{n=1}^{\infty} f_n(t) \exp(inx)$ with

$$f_n(t) = (1 - \exp(-i2\pi n c_0)) \exp(-n(A + KG(t) - \varepsilon)) \tag{2.9}$$

where $G(t) = 1 - \cos(t - t_0 + s_0)$; in the sequel we will explain how to choose the real numbers $A, K, \varepsilon, t_0, s_0$.

A formal solution to $Lu = f$ is $u(t,x) = (2\pi)^{-1} \sum_{n=1}^{\infty} u_n(t) \exp(inx)$, where

$$u_n(t) = \int_0^{2\pi} \exp\left\{-n(A + KG(t-s) - \varepsilon + i \int_{t-s}^{t} c)\right\} ds. \tag{2.10}$$

Note that each $u_n \in C^\omega(\mathbf{S}^1)$.

We set $A = \max\left\{\int_{t-s}^{t} b; 0 \le s, t \le 2\pi\right\}$. This maximum is attained when, say, $t = t_0, s = s_0$; thus $A = \int_{t_0 - s_0}^{t_0} b$. We may assume that $b(0) > 0$ and that $0 < t_0, s_0, t_0 - s_0 < 2\pi$. In what follows it will be important to consider the points belonging to the set $Y = \left\{t; 0 \le t \le 2\pi \text{ and } \int_{t_0 - s_0}^{t} b = A\right\}$. Since b is C^ω and $\not\equiv 0$, Y is a finite set; we write $Y = \{t_0, t_1, \ldots, t_r\}$.

Note that $\int_{t_0}^{t_k} b = 0$ for each k; note also that b changes sign from $-$ to $+$ at $t = t_0 - s_0$, and from $+$ to $-$ at $t = t_k$, for $k = 0, \ldots, r$. This implies $\int_{t_0}^{t} b \le 0$ if $|t - t_k| \le \rho$, for $k = 0, \ldots, r$, if $\rho > 0$ is small. We also have $t_0 - s_0 \ne t_k$, for $k = 0, \ldots, r$.

We have, for all $n \in \mathbf{N}$, $0 < c_1 \doteq 1 - \exp(2\pi b_0) \le |1 + \exp(-i2\pi n c_0)| \le 1 + exp(2\pi b_0) \doteq c_2 < \infty$.

The estimate $|f_n(t)| \le c_2 \exp(-n(A - \varepsilon - K(\cosh(\delta) - 1)))$ holds for all $n \in \mathbf{N}, t \in \mathbf{S}^1 + i(-\delta, \delta)$.

We will choose $K > 0$ very large; the value of $\varepsilon \ge 0$ will come out from applying Lemma 1.1, and we will have $\varepsilon = O(K^{-1})$. We will also require $\delta = O(K^{-1})$. Thus $K\delta^2 = O(K^{-1})$, and $A - \varepsilon - K(\cosh(\delta) - 1) \ge A - \varepsilon - 2K\delta^2 = A - O(K^{-1}) \ge A/2 > 0$, for large K.

Thus $|f_n(t)| \le c_2 \exp(-An/2)$, for all $n \in \mathbf{N}, t \in \mathbf{S}^1 + i(-\delta, \delta)$; since each $f_n \in C^\omega(\mathbf{S}^1)$ we will have $f \in C^\omega(\mathbf{T}^2)$.

We claim that there exist $C > 0$, $\gamma_1 > 0$ such that

$$|u_n(t)| \le C n^{-1/2}, \ t \in \mathbf{S}^1, \ n \in \mathbf{N}, \tag{2.11}$$

and

$$|u_n(t_k)| = \gamma_1 n^{-1/2}(1 + O(n^{-1})) \ , \ k = 0, \ldots, r, \text{ as } n \to \infty \ . \tag{2.12}$$

Assume, for a moment, that the claim has been proved. Then $u \in \mathcal{D}'(\mathbf{T}^2)$ because of (2.11). On the other hand, $u \notin C^\omega(\mathbf{T}^2)$ because of (2.12); in fact, (t_k, x) belongs to the analytic singular support of u, for all $k = 0, \dots, r$, $x \in \mathbf{S}^1$.

We now proceed to prove our claim.

We may write

$u_n(t) = \tilde{u}_n(t) \exp\left\{-n[A - \varepsilon + i\int_{\sigma_0}^t c]\right\}$, where $\sigma_0 = t_0 - s_0$ and $\tilde{u}_n(t) = \int_{t-2\pi}^t \exp\left\{-n(K[1 - cos(\sigma - \sigma_0)] - i\int_{\sigma_0}^\sigma c)\right\} d\sigma$.

Pick $\rho > 0$ so that, when t belongs to $D \doteq \{t; |t - t_k| \le \rho,$ for some $k = 0, \dots, r\}$, we have $\int_{t_0}^t b \le 0$ and $t \ne \sigma_0$. For $t \in D$ we also have $t - 2\pi < \sigma_0 < t$.

Set $h(\sigma) = K[1 - \cos(\sigma - \sigma_0)] - i\int_{\sigma_0}^\sigma c$, $t - 2\pi \le \sigma \le t$.

It is easy to see that, given C_1, C_2, there exists $K_0 > 0$ such that, for all $K \ge K_0$, we have: $\Re h$ is strictly decreasing (resp. increasing) on $[\sigma_0 - \pi/2, \sigma_0] \cap [t - 2\pi, t]$ (resp. $[\sigma_0, \sigma_0 + \pi/2] \cap [t - 2\pi, t]$). Furthermore, $\Re h \ge C_1 + C_2 \max\{|b(t)|; 0 \le t \le 2\pi\}$ on $\{|\sigma - \sigma_0| \ge \pi/2\} \cap [t - 2\pi, t]$.

We will first analyze the growth of $|u_n(t)|$, as $n \to \infty$, when $t \in D$.

Pick σ_1, σ_2 so that $\sigma_1 < \sigma_0 < \sigma_2$, $[\sigma_1, \sigma_2] \subset [t - 2\pi, t] \cap [\sigma_0 - \pi/2, \sigma_0 + \pi/2]$, and $[\sigma_1, \sigma_2] \cap D = \emptyset$.

Set $J_n = \int_{\sigma_1}^{\sigma_2} \exp(-nh(\sigma)) d\sigma$.

Lemma 1.1 applies to give $J_n = \gamma n^{-1/2}(1 + O(1/n)) \exp(-nh(z_0))$.

Set

$$\varepsilon = \Re\left\{K[1 - \cos(z_0 - \sigma_0)] - i\int_{\sigma_0}^{z_0} c\right\} . \tag{2.13}$$

By taking K_0 sufficiently large we can be sure that (for all $K \ge K_0$), when we deform the integration contour in order to bypass (via z_0) the point σ_0, we get back to the real axis at points ${\sigma'}_1, {\sigma'}_2$ with $\sigma_1 < {\sigma'}_1 < \sigma_0 < {\sigma'}_2 < \sigma_2$; we then have $\Re h(\sigma) \ge \varepsilon' > \varepsilon$, for all $\sigma \in \{|\sigma - \sigma_0| \ge \rho\} \cap [t - 2\pi, t]$. Thus

$$\left|\left\{\int_{t-2\pi}^{\sigma_1} + \int_{\sigma_2}^t\right\} (\exp(-nh(\sigma)))d\sigma\right| = O(\exp(-\varepsilon' n)) ,$$

as $n \to \infty$, uniformly in $t \in D$.

Hence, with $\gamma_1 = |\gamma|$, and for $t \in D$,

$$\begin{aligned} |u_n(t)| &= O(\exp(-\varepsilon' n)) \\ &\quad + \gamma_1 n^{-1/2}(1 + O(1/n)) \left|\exp\left\{-n\left(A - \varepsilon + i\int_{\sigma_0}^t c + h(z_0)\right)\right\}\right| . \end{aligned}$$

Now $\Re\left\{A - \varepsilon + i\int_{\sigma_0}^t c + h(z_0)\right\} = A - \int_{\sigma_0}^t b = A - \int_{\sigma_0}^{t_0} b - \int_{t_0}^t b = -\int_{t_0}^t b.$

Recall that $-\int_{t_0}^t b \ge 0$ if $t \in D$, and $-\int_{t_0}^t b = 0$ if $t = t_k, k = 0, \dots, r$.

Thus, for each t_k, we have

$$|u_n(t_k)| = O(\exp(-\varepsilon' n)) + \gamma_1 n^{-1/2}(1 + O(1/n)), \text{ as } n \to \infty ,$$

which implies the validity of (2.12).

On the other hand, we have

$$|u_n(t)| = O(\exp(-\varepsilon' n)) + \gamma_1 n^{-1/2}(1 + O(1/n)) \exp\left(n\int_{t_0}^t b\right), \tag{2.14}$$

for $t \in D$, as $n \to \infty$, which implies the validity of (2.11) when $t \in D$.

It remains to estimate $|u_n(t)|$ when $t \in [0, 2\pi] \setminus D$, i.e., when $|t - t_k| \geq \rho$, for all $k = 0, \dots, r$.

There exist $K_0 > 0, \eta > 0$ such that, for all $K > K_0$,

$$\Re\left(A - \varepsilon + i\int_{\sigma_0}^{t} c\right) \geq \eta/2, \quad t \in [0, 2\pi] \setminus D. \tag{2.15}$$

Indeed take $\eta > 0$ such that $\int_{\sigma_0}^{t} b \leq A - \eta$, for $t \in [0, 2\pi] \setminus D$. Then

$$\Re\left(A - \varepsilon + i\int_{\sigma_0}^{t} c\right) = A - \varepsilon - \int_{\sigma_0}^{t} b \geq \eta - \varepsilon \geq \eta/2,$$

since $\varepsilon = O(1/K)$.

We claim that, for K_0 sufficiently large, and for all $K \geq K_0$,

$$\Re h(\sigma) \geq 0\ ,\ \text{for all}\ \sigma \in [t - 2\pi, t],\ t \in [0, 2\pi] \setminus D\ . \tag{2.16}$$

We divide the proof of (2.16) into three cases, according to the location of σ; note that we always have $-2\pi \leq \sigma \leq 2\pi$ in (2.16).

Let ρ' be such that $\int_{\sigma_0}^{\sigma} b \geq 0$ if $|\sigma - \sigma_0| \leq \rho'$.

If $|\sigma - \sigma_0| \leq \rho'$ we get (2.16) because $K[1 - \cos(\sigma - \sigma_0)] \geq 0$ for all σ.

Now if $|\sigma - (\sigma_0 - 2\pi)| \leq \rho'$ we have $\int_{\sigma_0}^{\sigma} b = \int_{\sigma_0}^{\sigma_0 - 2\pi} b + \int_{\sigma_0 - 2\pi}^{\sigma} b = -2\pi b_0 + \int_{\sigma_0 - 2\pi}^{\sigma} b > 0$, and (2.16) again follows.

Finally if σ is such that $-2\pi \leq \sigma \leq 2\pi$, $|\sigma - \sigma_0| \geq \rho'$, and $|\sigma - (\sigma_0 - 2\pi)| \geq \rho'$ we have $1 - \cos(\sigma - \sigma_0) \geq \tilde{\eta}$, for some $\tilde{\eta} > 0$; thus $K\tilde{\eta} + \int_{\sigma_0}^{\sigma} b \geq K\tilde{\eta}/2$, provided K is large. The proof of (2.16) is complete.

Now the conjunction of (2.15) and (2.16) shows that

$$|u_n(t)| = O(-\eta n/2),\ t \in [0, 2\pi] \setminus D,\ \text{as}\ n \to \infty\ . \tag{2.17}$$

Finally (2.14) and (2.17) together imply (2.11); this completes the proof in Case 1.

Case 2. $b_0 > 0$. Here the proof is entirely analogous to that of Case 1. We use (2.8) instead of (2.7), we let n vary in $\mathbf{N}$, take $A = -\min\left\{\int_{t-s}^{t} b; {\scriptstyle 0 \leq s,t \leq 2\pi}\right\}$, and so on.

Case 3. $b_0 = 0$ and $a_0 \in \mathbf{R} \setminus \mathbf{Q}$. The proof is again similar to that of Case 1. Note that, for all $n \in \mathbf{Z}$, $|1 - \exp(-i2\pi n c_0)| = |1 - \exp(-i2\pi n a_0)| \leq 2$.

Case 4. $b_0 = 0$ and $a_0 \in \mathbf{Q}$. This was taken care of in Corollary 2.6.

This concludes the proof of the necessity of (2.3).

Suppose now that (2.3) holds but (2.4) does not. Thus $b(t) \equiv 0$, and a_0 is either rational or exponential Liouville.

We use the automorphism of $\mathcal{D}'(\mathbf{T}^2)$ (and, also, of $C^\omega(\mathbf{T}^2)$) defined by $Su = v$, where the partial Fourier coefficients are related by

$$v_n(t) = u_n(t)\exp\left[in\left(\int_0^t a(s)ds - a_0 t\right)\right],\ n \in \mathbf{Z}\ .$$

The equation $Lu = f$ becomes $(\partial_t + a_0\partial_x)v = g$, where $Su = v, Sf = g$. The conclusion is that, when $b \equiv 0$, L is GAH if and only if $\partial_t + a_0\partial_x$ is GAH if and only if a_0 is neither rational nor exponential Liouville ([G]). This concludes the proof of the necessity of (2.4), and also proves the sufficiency when $b \equiv 0$.

To complete the proof of sufficiency it remains to consider the case where $b \not\equiv 0$ and b does not change sign. Here the results of Treves [T1] imply that L is actually (locally) analytic hypoelliptic, hence also GAH.

The proof of Theorem 2.3 is complete. ◻

3. A class of overdetermined systems

Here we study systems of vector fields of the form $\mathbb{L} = (L_1, \dots, L_n)$, where

$$L_j = \frac{\partial}{\partial t_j} + c_j(t_j)\frac{\partial}{\partial x},\ j = 1, \dots, n\ , \tag{3.1}$$

with each $c_j \in C^\omega(\mathbf{S}^1)$.

We use the notations: $t = (t_1, \dots, t_n) \in \mathbf{T}^n$, $x \in \mathbf{S}^1$, $(t,x) \in \mathbf{T}^{n+1}$, $t_j \in \mathbf{S}^1_{t_j}$.

Note that each L_j is of the form studied in Section 2; note also that the only coupling occurs via $\partial/\partial x$.

Since each c_j depends only on t_j the system is automatically involutive.

We are interested in the equations $L_j u = f_j$, $j = 1, \dots, n$, where $u \in \mathcal{D}'(\mathbf{T}^{n+1})$ and each $f_j \in C^\omega(\mathbf{T}^{n+1})$.

Definition 3.1. The system $\mathbb{L}$ is said to be globally analytic hypoelliptic on the torus $\mathbf{T}^{n+1}$ (briefly: $\mathbb{L}$ is GAH) if the conditions $u \in \mathcal{D}'(\mathbf{T}^{n+1})$ and $f_j \in C^\omega(\mathbf{T}^{n+1})$, $j = 1, \dots, n$ imply $u \in C^\omega(\mathbf{T}^{n+1})$.

Definition 3.2. We say that $\alpha \in \mathbf{R}^n \setminus \mathbf{Q}^n$ is an exponential Liouville vector (briefly: α is EL) if there exists $\varepsilon > 0$ such that the inequality $|\alpha - \vec{p}/q| \le \exp(-\varepsilon q)$ has infinitely many solutions $\vec{p}/q$, with $\vec{p} \in \mathbf{Z}^n, q \in \mathbf{N}$.

We set $c_{j,0} = (2\pi)^{-1}\int_0^{2\pi} c_j(t_j)dt_j$; we also write $c_j = a_j + ib_j$ and $c_{j,0} = a_{j,0} + ib_{j,0}$.

We split the set $\{1, \dots, n\}$ as $\{1, \dots, n\} = J \cup K$, where $j \in J$ if and only if the function b_j does not change sign; we write $J = \{j_1, \dots, j_\ell\}$. We allow ℓ to be 0 (this means J is empty) or n (this means K is empty).

We are now ready to state the main result of this section.

Theorem 3.3. *Under the above assumptions and notations,* $\mathbb{L}$ *is* GAH *if and only if the following conditions are satisfied:*

(3.2) *J is nonempty;*

(3.3) *if $b_j \equiv 0$ for all $j \in J$, then $(a_{j_1,0}, \dots, a_{j_\ell,0})$ is neither* EL *nor an element of $\mathbf{Q}^\ell$.*

Proof. We begin by proving the necessity.

Suppose first that (3.2) is not satisfied, that is, b_j changes sign, for all $j = 1, \dots, n$.

The results of Section 2 apply to yield $u_j = u_j(t_j, x)$, with $u_j \in \mathcal{D}'(\mathbf{S}^1_{t_j} \times \mathbf{S}^1_x) \setminus C^\omega(\mathbf{S}^1_{t_j} \times \mathbf{S}^1_x)$ and $f_j \doteq L_j u_j \in C^\omega(\mathbf{S}^1_{t_j} \times \mathbf{S}^1_x)$, for $j = 1 \dots, n$.

In Section 2 we constructed these objects and proved that the following estimates hold, for all $j = 1, \dots, n$, and all $k \in \mathbf{N}$:

$$|\widehat{u}_j(t_j, k)| \le Ck^{-1/2},\ t_j \in \mathbf{S}^1, \tag{3.4}$$

$$|\widehat{u}_j(t_{0,j}, k)| = Ck^{-1/2}(1 + O(k^{-1})),\ \text{for some } t_{0,j} \in S^1, \tag{3.5}$$

$$|\widehat{f}_j(t_j,k)| \le C\exp(-Bk),\ t_j \in S^1 + i(-\delta,\delta), \text{for some } B>0\ . \tag{3.6}$$

We make the remark that a more careful analysis of the proof of (3.4) yields the following estimates:

$$|\widehat{u}_j(t_j,k)| \le Ck^{-1/2}\exp(\varepsilon k)\ ,\ t_j \in S^1 + i(-\delta,\delta) \tag{3.7}$$

where ε is small.

We set

$$u = u_1 * \cdots * u_n \tag{3.8}$$

where $*$ denotes the convolution in the x-variable alone.

Now, by using basic properties of the convolution product together with the estimates (3.4)–(3.7), one sees that $u \in \mathcal{D}'(\mathbf{T}^{n+1})$ and that each $L_j u \in C^\omega(\mathbf{T}^{n+1})$; this shows that (3.2) is indeed necessary.

Assume now that (3.2) holds but (3.3) does not. Then, after reordering the variables (if necessary) we may assume that there exists ℓ, with $1 \le \ell \le n$, such that

(i) $b_j \equiv 0$, for $j = 1,\dots,\ell$;
(ii) b_j changes sign, for $j = \ell+1,\dots,n$;
(iii) $\alpha_0 \doteq (a_{1,0},\dots,a_{\ell,0})$ is either EL or belongs to $\mathbf{Q}^\ell$.

We will split the variables as $t=(t',t'')$, where $t'=(t_1,\dots t_\ell)$, $t''=(t_{\ell+1},\dots,t_n)$.

We may write $a_j(t_j) = \frac{\partial}{\partial t_j}A(t') + a_{j,0}t_j\ , j=1,\dots,\ell$, with $A \in C^\omega(\mathbf{T}^\ell;\mathbf{R})$.

We will use the automorphism of $C^\omega(\mathbf{T}^{n+1})$ (and also of $\mathcal{D}'(\mathbf{T}^{n+1})$) defined by $Su = v$, where $\widehat{v}(t,k) = \widehat{u}(t,k)\exp(ikA(t'))$; with this, the equations $L_j u = f_j\ ,\ j=1,\dots,n$, become $\tilde{L}_j v = g_j\ ,\ j=1,\dots,n$, where $v = Su$, $g_j = Sf_j$, and $\tilde{L} = \frac{\partial}{\partial t_j} + a_{j,0}\frac{\partial}{\partial x}\ ,\ j=1,\dots,\ell$.

We set $\mathcal{A} = \{(p,k) \in \mathbf{Z}^\ell \times \mathbf{N}; \max_{1\le j\le \ell}|p_j + a_{0,j}k| < \exp(-\varepsilon(||p||+k))\}$.

Note that $\mathcal{A}$ is an infinite set; indeed, when a_0 is EL this follows from the definition of EL; when $a_0 \in \mathbf{Q}^\ell$, we may take $k = qm$, $p_j = -qma_{0,j}$, $m \in \mathbf{N}$, where $a_0 = \vec{r}/q$ (lowest terms).

We set $w(t',x) = \sum_{(p,k)\in\mathcal{A}} \exp(i(p\cdot t' + kx))$.

We have $w \in \mathcal{D}'(\mathbf{T}^\ell_{t'} \times \mathbf{S}^1_x) \setminus C^\omega(\mathbf{T}^\ell_{t'} \times \mathbf{S}^1_x)$, and $L_j w = h_j$, where

$$\widehat{h}_j(p,k) = \begin{cases} i(p_j + a_{0,j}k), & \text{if } (p,k) \in \mathcal{A}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus $h_j \in C^\omega(\mathbf{T}^\ell_{t'} \times \mathbf{S}^1_x)$, $j = 1,\dots,\ell$.

Now, as in the first part of this proof, for each $j = \ell+1,\dots,n$, there exists $u_j \in \mathcal{D}'(\mathbf{S}^1_{t_j} \times \mathbf{S}^1_x) \setminus C^\omega(\mathbf{S}^1_{t_j} \times \mathbf{S}^1_x)$ such that $L_j u_j \in C^\omega(\mathbf{S}^1_{t_j} \times \mathbf{S}^1_x)$.

We now define $u = w * u_{\ell+1} * \cdots * u_n$, and the proof ends just like in the proof of the necessity of (3.2).

We now prove the sufficiency of (3.2)–(3.3).

Case 1. There exists $j \in \{1,\dots,n\}$ such that b_j does not change sign and $b_j \not\equiv 0$. In this case $\mathbb{L}$ is even (locally) analytic hypoelliptic, hence GAH; indeed no local primitive of the 1-form $b_1dt_1 + \cdots + b_ndt_n$ has local extrema, and so the results of [BT] apply.

Case 2. Each b_j which does not change sign is $\equiv 0$. Then, after reordering the variables (if necessary) we may assume that there exists ℓ, with $1 \le \ell \le n$, such that

(i) $b_j \equiv 0$, for $j = 1, \dots, \ell$;
(ii) b_j changes sign, for $j = \ell + 1 \dots, n$;
(iii) $\alpha_0 \doteq (a_{1,0}, \dots, a_{\ell,0})$ is neither EL nor in $\mathbf{Q}^\ell$.

Let $u \in \mathcal{D}'(\mathbf{T}^{n+1})$ be given with $f_j \doteq L_j u \in C^\omega(\mathbf{T}^{n+1})$, for $j = 1, \dots, n$. We proceed to show that $u \in C^\omega(\mathbf{T}^{n+1})$.

As in the proof of the necessity of (3.3) we use the automorphism S and get the equations $\tilde{L}_j v = g_j, j = 1, \dots, n$, where $\tilde{L}_j = \frac{\partial}{\partial t_j} + a_{0,j} \frac{\partial}{\partial x}$, $j = 1, \dots, \ell$.

We use partial Fourier series in the variables (t', x) in the first ℓ equations and get

$$i(p_j + a_{0,j} k)\widehat{v}(t'', p, k) = \widehat{g}_j(t'', p, k)\ ,\ j = 1, \dots, \ell\ . \tag{3.9}$$

We get, for $(p, k) \in \mathbf{Z}^{\ell+1}$,

$$\widehat{v}(t'', p, k) = \widehat{g}_j(t'', p, k)/i(p_j + a_{0,j} k), \tag{3.10}$$

where j is any index with $1 \le j \le \ell$ and $p_j + a_{0,j} k \neq 0$ (such a j exists because $a_0 \notin \mathbf{Q}^\ell$).

When $(p, k) = (0, 0)$, we see that $\widehat{v}(t'', 0, 0)$ is not determined by (3.9), and we also see that $\widehat{g}_j(t'', 0, 0) \equiv 0$, $j = 1, \dots, \ell$.

Set $w(t, x) = v(t, x) - (2\pi)^{-\ell-1}\widehat{v}(t'', 0, 0)$. From (3.10) we get, since $g_1, \dots, g_\ell \in C^\omega$ and $\alpha_0 \notin \mathbf{Q}^\ell \cup \mathrm{E}L$, that there exist $\varepsilon > 0$, $C > 0$ such that $|\widehat{v}(t'', p, k)| \le C \exp(-\varepsilon(||p|| + |k|)$, for $(p, k) \in \mathbf{Z}^{\ell+1} \setminus \{0\}$, $t'' \in \mathbf{T}^{n-\ell}_{t''} + i\delta I^{n-\ell}$, where $I = (-1, 1)$. It follows that $w \in C^\omega(\mathbf{T}^{n+1})$.

Now, for $j = \ell + 1, \dots, m$, we have $L_j w = L_j v - (2\pi)^{-\ell-1} \frac{\partial}{\partial t_j}\widehat{v}(t'', 0, 0)$, and so $\frac{\partial}{\partial t_j}\widehat{v}(t'', 0, 0) \in C^\omega$, which implies $\widehat{v}(t'', 0, 0) \in C^\omega$. Thus $v = w + (2\pi)^{-\ell-1} v_{0,0} \in C^\omega$, and also $u \in C^\omega(\mathbf{T}^{n+1})$.

□

4. Examples

We begin with examples of vector fields on the torus $\mathbf{T}^2$.

Example 4.1. $L = \partial/\partial t + [\alpha \sin t + i(\beta - \cos t)]\partial/\partial x$, $\alpha, \beta \in \mathbf{R}$. Here L is GAH if and only if $|\beta| \ge 1$.

Example 4.2. $L = \partial/\partial t + (\sqrt{2} + \alpha \sin t)\partial/\partial x$, $\alpha \in \mathbf{R}$. This L is GAH because $\sqrt{2}$ is an algebraic number of degree 2 hence, by Liouville's theorem, satisfies $|\sqrt{2} - p/q| \ge Cq^{-2}$, for all $p \in \mathbf{Z}, q \in \mathbf{N}$, and some $C > 0$.

Example 4.3. $L = \partial/\partial t + (2 + \alpha \sin t)\partial/\partial x$, $\alpha \in \mathbf{R}$. This L is not GAH.

Example 4.4. $L = \partial/\partial t + (\alpha + \beta \cos t)\partial/\partial x$, where $\alpha, \beta \in \mathbf{R}$ and α has a continued fraction $\mathbb{K}_{n=1}^{\infty}(1/a_n)$ with $a_{n+1} > \exp q_n$, for all $n \in \mathbf{N}$ (here q_n is the denominator of the n^{th} convergent to α; see Example 4.9 for more details about continued fractions). This α is EL and so L is not GAH (for another, more explicit, example one may take $a_n = n!$, $n \in \mathbf{N}$; see [G]).

We now give examples of systems of two vector fields $\mathbb{L} = (L_1, L_2)$ on $\mathbf{T}^3$; Theorem 3.3 implies that only the first two are GAH.

Example 4.5. $L_1 = \partial/\partial t_1 + i(1 - \cos t_1)\partial/\partial x$,
$L_2 = \partial/\partial t_2 + i(1 - 2\cos t_2)\partial/\partial x$.

Example 4.6. $L_1 = \partial/\partial t_1 + \sqrt{2}\partial/\partial x$,
$L_2 = \partial/\partial t_2 + i(1 - 2\cos t_2)\partial/\partial x$.

Example 4.7. $L_1 = \partial/\partial t_1 + i(1 - 2\cos t_1)\partial/\partial x$,
$L_2 = \partial/\partial t_2 + i(1 - 2\cos t_2)\partial/\partial x$.

Example 4.8. $L_1 = \partial/\partial t_1 + \alpha\partial/\partial x$,
$L_2 = \partial/\partial t_2 + i(1 - 2\cos t_2)\partial/\partial x$,
where α is as in Example 4.4.

In order to motivate the next example, we begin by remarking that, for each j, the operator $L_j = \partial/\partial t_j + c_j(t_j)\partial/\partial x$ may be considered as an operator acting on functions or distributions depending only on two variables, namely $(t_j, x) \in \mathbf{T}^2$. Thus we may ask whether L_j is GAH on $\mathbf{T}^2$ or not.

Note that in Examples 4.5 and 4.6, where $\mathbb{L}$ is GAH, one of the vector fields (namely L_1) was also GAH on $\mathbf{T}^2$. Hence it makes sense to ask if this is always the case; more precisely, does the fact that $\mathbb{L}$ is GAH on $\mathbf{T}^{n+1}$ imply that at least one of the vector fields L_j is GAH on $\mathbf{T}^2$? The answer is no, as the following example will show.

Example 4.9. $L_1 = \partial/\partial t_1 - \alpha\partial/\partial x$,
$L_2 = \partial/\partial t_2 - \beta\partial/\partial x$.

We are going to construct two exponential Liouville numbers α, β such that (α, β) is *not* an exponential Liouville vector. The conclusion will be that $\mathbb{L}$ is GAH on $\mathbf{T}^3$ even though neither L_1 nor L_2 is GAH on $\mathbf{T}^2$.

The numbers $\alpha, \beta \in (0,1)$ will be constructed by means of their (simple) continued fractions, namely

$$\alpha = \mathbb{K}_{n=1}^{\infty}(1/a_n),\ \beta = \mathbb{K}_{n=1}^{\infty}(1/b_n),$$

where each $a_n, b_n \in \mathbf{N}$.

The best rational approximations to an irrational number, say α, are in a certain sense, the *convergents* p_n/q_n; these are given recursively by $p_1 = 1$, $q_1 = a_1$, $p_2 = a_2$, $q_2 = a_2 a_1 + 1$, and, for $n \geq 3$, $p_n = a_n p_{n-1} + p_{n-2}$, $q_n = a_n q_{n-1} + q_{n-2}$.

Legendre's theorem says that if $|\alpha - p/q| < 1/2q^2$, then $p/q = p_n/q_n$, for some n.

On the other hand, the convergents satisfy the following:

$$\frac{1}{(2 + a_{n+1})q_n^2} \leq \frac{1}{q_n(q_n + q_{n+1})} \leq |\alpha - p_n/q_n| \leq \frac{1}{q_n q_{n+1}} \leq \frac{1}{a_{n+1}q_n^2}\ .$$

Let us agree to say that p/q is a *good approximation* to α when $|\alpha - p/q| < 1/3q^2$.

Note that, when $a_{n+1} = 1$, p_n/q_n is not a good approximation to α. On the other hand, when $a_{n+1} = [\exp q_n] + 1$ ($[x]$ denoting the largest integer not exceeding the real number x), as will be the case below for certain values of n, p_n/q_n will be a (very) good approximation to α; we will need to know that, for large $t \in \mathbf{N}$, tp_n/tq_n is *not* a good approximation to α. We have, if $t \geq t_n \doteq 1 + \max\{2, [\exp(q_n/2)]\}$,

$$\begin{aligned} |\alpha - tp_n/tq_n| &= |\alpha - p_n/q_n| \geq \frac{1}{(2 + a_{n+1})q_n^2} \\ &\geq \frac{1}{(3 + \exp q_n))q_n^2} \geq \frac{1}{(3 + t^2)q_n^2} \\ &\geq \frac{1}{2t^2 q_n^2} > \frac{1}{3(tq_n)^2}\ ; \end{aligned}$$

hence, for such t, tp_n/tq_n is not a good approximation to α.

The convergents to β (resp.α) will be denoted r_n/s_n (resp.p_n/q_n).

To construct α (resp. β) we will choose most of the a_n (resp. b_n) equal to 1; the remaining a_n (resp. b_n) will be very large. The continued fractions (with a new notation) are

$$\begin{aligned}\alpha &= [1, 1, a_{k+1}, 1, \ldots\ldots\ldots\ldots 1, a_{k_3+1}, 1, \ldots],\\ \beta &= [1, \ldots\ldots\ldots\ldots 1, b_{k_2+1}, 1, \ldots\ldots\ldots\ldots],\end{aligned}$$

where $k_1 < k_2 < \cdots$, and

$$\begin{aligned}a_{k_\ell+1} &= [\exp q_{k_\ell}] + 1,\ \ \ell = 1, 3, 5, \ldots\ ,\\ b_{k_\ell+1} &= [\exp s_{k_\ell}] + 1,\ \ \ell = 2, 4, 6, \ldots\ .\end{aligned}$$

These conditions imply

$$|\alpha - p_{k_\ell}/q_{k_\ell}| \leq \frac{1}{q_{k_\ell} q_{k_\ell+1}} \leq \frac{1}{a_{k_\ell+1} q_{k_\ell}^2} \leq \exp(-q_{k_\ell})$$

and, similarly, $|\beta - r_{k_\ell}/s_{k_\ell}| \leq \exp(-s_{k_\ell})$, hence both α and β are EL.

In order to achieve the goal of having (α, β) not EL we must make a very careful choice of the sequence $k_1, k_2, \ldots$; basically, it will have to grow very fast.

We define $k_1 < k_2 < k_3 < \cdots$ recursively by setting $k_1 = 2$, and for $\ell \geq 2$, we require k_ℓ to be such that

$$s_{k_\ell} \geq {q'}_{\ell-1} \doteq q_{k_{\ell-1}}([\exp(q_{k_{\ell-1}}/2)] + 1),\ \text{ if } \ell \text{ is even},$$

and such that

$$q_{k_\ell} \geq {s'}_{\ell-1} \doteq s_{k_{\ell-1}}([\exp(s_{k_{\ell-1}}/2)] + 1),\ \text{ if } \ell \text{ is odd}.$$

We get $q_{k_1} < {q'}_1 < s_{k_2} < {s'}_2 < q_{k_3} < {q'}_3 < \cdots$. Set

$$I_\ell = \{q \in \mathbf{N} : q_{k_\ell} \leq q \leq {q'}_\ell\}, \quad \ell = 1, 3, 5, \ldots,$$

and

$$J_\ell = \{q \in \mathbf{N} : s_{k_\ell} \leq q \leq {s'}_\ell\}, \quad \ell = 2, 4, 6, \ldots.$$

Note that $I_\ell \cap J_m = \emptyset$, for all $\ell = 1, 3, 5, \ldots$, $m = 2, 4, 6, \ldots$.

We claim that the good approximations to α (resp. β) have denominators belonging to some I_ℓ (resp. J_m); this will imply that there are no good approximations to $((\alpha, \beta))$ with the same denominator, i.e., one has $|(\alpha, \beta) - (p/q, r/q)| \geq 1/(3q^2)$, for all $p, r \in \mathbf{Z}$, $q \in \mathbf{N}$; this will imply that (α, β) is not even a Liouville vector; indeed, for any $\varepsilon > 0$, the inequality $\exp(-\varepsilon q) > 1/3q^2$ has (at most) a finite number of solutions $q \in \mathbf{N}$; in other words, one has $|(\alpha, \beta) - (p/q, r/q)| \geq \exp(-\varepsilon q)$, except for a finite number of $(p/q, r/q)$.

It remains to prove our claim; we will prove it only for α. We have, by Legendre's theorem, $|\alpha - p/q| \geq 1/2q^2$, provided $p/q \neq p_n/q_n$, $n = 1, 2, \ldots$.

Now, when $a_{n+1} = 1$, we have $|\alpha - p_n/q_n| \geq 1/3q_n^2$; thus we also have, for all $t \in \mathbf{N}$, $|\alpha - tp_n/tq_n| \geq 1/3(tq_n)^2$.

Finally, when $n = k_\ell$, $\ell = 1, 3, \ldots$, we have seen that $t \geq t_\ell$ implies $|\alpha - tp_{k_\ell}/tq_{k_\ell}| \geq 1/3(tq_{k_\ell})^2$.

The conclusion is that if p/q is a good approximation to α, then $p = tp_{k_\ell}$, $q = tq_{k_\ell}$, for some $\ell = 1, 3, \ldots$ and some $t \in \mathbf{N}$ with $1 \leq t \leq t_\ell$; in other words, if p/q is a good approximation to α, then necessarily $q_{k_\ell} \leq q \leq {q'}_\ell$, for some $\ell = 1, 3, \ldots$.

This concludes the analysis of Example 4.9.

Example 4.10 (systems with constant coefficients)**.** $L_j = \partial/\partial t_j + (a_j + ib_j)\partial/\partial x$, $j = 1, \dots, n$. Here $\mathbb{L}$ is GAH if and only if either some $b_j \neq 0$ or else $(a_1, \dots, a_n)$ is neither EL nor an element of $\mathbf{Q}^n$.

Example 4.11 (systems of real vector fields)**.** $L_j = \partial/\partial t_j + a_j(t)\partial/\partial x$, $j = 1, \dots, n$, where each a_j is real-analytic, real-valued; we also assume that the system is involutive, i.e., $\partial a_j/\partial t_k = \partial a_k/\partial t_j$, for all j, k. Here, the conjugation with $\exp(in\tilde{A}(t))$, where $\tilde{A}(t) = \sum_{j=1}^n \int_0^t a_j(s)ds_j - a_0 \cdot t$, where $a_0 = (a_{01}, \dots, a_{0n})$ and $a_{0j} = (2\pi)^{-1}\int_0^{2\pi} a_j(t)dt_j$, reduces $\mathbb{L}$ to the constant coefficient system $\tilde{\mathbb{L}} = (\tilde{L_1}, \dots, \tilde{L_n})$, where $\tilde{L}_j = \partial/\partial t_j + a_{0j}\partial/\partial x$. Thus $\mathbb{L}$ is GAH if and only if a_0 is neither EL nor an element of $\mathbf{Q}^n$.

Example 4.12. $L_j = \partial/\partial t_j + ib_j(t_j)\partial/\partial x$, $j = 1, \dots, n$ with each $b_j \not\equiv 0$. Here $\mathbb{L}$ is GAH if and only if some b_j does not change sign. Also, $\mathbb{L}$ is GAH if and only if $\mathbb{L}$ is (locally) analytic hypoelliptic.

5. Concluding remarks

The paper [GPY] studies global regularity for several classes of operators, especially second-order ones. The authors announce a result about global Gevrey hypoellipticity, for all Gevrey indices σ with $1 \leq \sigma < \infty$, for the first-order operators (2.1). They prove the sufficiency in all cases and the necessity when $1 < \sigma < \infty$, where cut-off functions can be used. Our proof of Theorem 2.3 may be viewed as a completion of the proof of Theorem 3.4 in [GPY].

The article [Ca-Ho] contains a theorem with the same statement as that of our Theorem 2.3. However, what is proved there is that (2.3)–(2.4) are equivalent to the following notion of GAH: the conditions $u \in C_c^\infty(\mathbf{S}^1_t; \mathcal{D}'(\mathbf{S}^1_x))$ and $Lu \in C_c^\infty(\mathbf{S}^1_t; C^\omega(\mathbf{S}^1_x))$ imply $u \in C_c^\infty(\mathbf{S}^1_t; C^\omega(\mathbf{S}^1_x))$. In this context, one is free to use cut-off functions in the t-variable; this renders the construction of singular solutions a simple matter.

A crucial point in the construction of singular solutions, namely in the case when $b(t)$ changes sign, was the choice of a right-hand side $f \in C^\omega(\mathbf{T}^2)$ which extended holomorphically, in the x-variable, to a strip of finite width, namely $|\Im x| < A - \varepsilon$. We remark that, were $b(t)$ and $f = Lu$ entire, then the distribution u would automatically be entire as well (L^{-1} destroys only a finite width). Thus we can say that, when $b(t) \not\equiv 0$, L is always globally *entirely* hypoelliptic.

The paper [Co-Hi] studies GAH for sums of squares of real vector fields, under the assumption that each point is of finite type. We consider the sum of squares $P = L_1^2 + \cdots + L_n^2$, where $\mathbb{L} = (L_1, \dots, L_n)$ is an involutive system of real vector fields, and remark that it is always of infinite type.

We claim that P is GAH on $\mathbf{T}^{n+1}$ if and only if a_0 is neither EL nor an element of $\mathbf{Q}^n$. In view of Example 4.11, this is the same as saying that P is GAH if and only if $\mathbb{L}$ is GAH. It is well-known that P GAH implies $\mathbb{L}$ GAH; we must prove the converse.

Let $u \in \mathcal{D}'(\mathbf{T}^{n+1})$ be such that $Pu = f \in C^\omega(\mathbf{T}^{n+1})$. Set $v_j = L_j u$, $j = 1, \dots, n$. We see that $v_1, \dots, v_n$, must verify $L_1 v_1 + \cdots + L_n v_n = f$ and $L_j v_k - L_k v_j = 0$, $j, k = 1, \dots, n$.

It suffices to show that these equations have, up to a constant, a unique distribution solution, $(v_1, \dots, v_n)$ which, furthermore, is real-analytic.

For simplicity, we will assume $n = 2$, the case of general n being similar. We must solve $L_1 v_1 + L_2 v_2 = f$, $-L_2 v_1 + L_1 v_2 = 0$.

Set $Sv = w$, $Sf = g$, where S is the automorphism in Example 4.11. We now have to solve $\tilde{L}_1 w_1 + \tilde{L}_2 w_2 = g$, $-\tilde{L}_2 w_1 + \tilde{L}_1 w_2 = 0$.

By taking Fourier series (in all variables) we see that we must solve

$$\begin{cases} [k_1 + a_{01} j]\widehat{w}_1(j,k) + [k_2 + a_{02} j]\widehat{w}_2(j,k) = -i\widehat{g}(j,k), \\ -[k_2 + a_{02} j]\widehat{w}_1(j,k) + [k_1 + a_{01} j]\widehat{w}_2(j,k) = 0. \end{cases}$$

We have, for any $\varepsilon > 0$, $D_{jk} \doteq (k_1 + a_{01} j)^2 + (k_2 + a_{02} j)^2 \geq \exp(-\varepsilon(|j| + k))$, for $|j| + |k|$ large.

It is now easy to finish the proof.

A study of global hypoellipticity (GH), i.e., C^∞ rather than C^ω regularity, was carried out in [BCM] in a more general context. The techniques of the present paper, coupled with the use of cut-off functions, furnish results of GH for a class not covered in [BCM]. In fact, if we allow $b_j(t_j)$ to be C^∞ we get the perfect analogue of Theorem 3.3: it suffices to replace GH for GAH and Liouville vector for EL.

Perturbations of a non-GAH vector field by a term of order zero may turn out to be GAH, in contrast with what happens in the usual (local) analytic hypoellipticity for operators of principal type (see [T1]). Actually, for any $\alpha \in \mathbf{R}$, most (in the sense of Lebesgue measure) perturbations $\partial_t - \alpha\partial_x - \lambda$, $\lambda \in \mathbf{R}$, are indeed GAH. Furthermore, by mimmicking the constructions (via continued fractions) in [B], one can produce two exponential Liouville numbers α, β such that $\partial_t - \alpha\partial_x - 1/2$ is GAH but $\partial_t - \beta\partial_x - 1/2$ is not GAH.

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