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QUADRATIC DIFFERENTIAL EQUATIONS IN $\mathbb{Z}_2$-GRADED ALGEBRAS

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Abstract. Quadratic differential equations whose associated algebra has an automorphism of order two are studied. Under hypotheses that naturally generalize the cases where the even or odd part of the algebra is one dimensional, the following are examined: structure theory of the associated algebra (ideal structure, simplicity, solvability, and nilpotence), derivations and first integrals, trajectories given by derivations, and Floquet decompositions.

1. Introduction

A quadratic differential equation is a system of the form

$$\dot{Z} = Q(Z) \tag{1.1}$$

where $Q : \mathbb{R}^n \to \mathbb{R}^n$ is homogeneous of degree 2, i.e. $Q(aZ) = a^2Q(Z)$ for all $a \in \mathbb{R}$, $Z \in \mathbb{R}^n$. (The restriction to homogeneous equations involves no loss of generality; nonhomogeneous equations of the form $\dot{Z} = C + T(Z) + Q(Z)$ where $C \in \mathbb{R}^n$ is fixed and $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear can always be "homogenized" into an equation of the form (1.1) in $\mathbb{R}^{n+1}$ [5], [11, p.22].) Associated to a homogeneous quadratic mapping Q on $\mathbb{R}^n$ is a bilinear mapping $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ defined by $Z \cdot W := \frac{1}{2}[Q(Z + W) - Q(Z) - Q(W)]$. Sometimes $Z \cdot W$ is written ZW. Thinking of such a bilinear mapping as a multiplication gives $\mathbb{R}^n$ the structure of a nonassociative algebra [10], which we will denote by $(\mathfrak{A}, \cdot)$. Markus [8] introduced the idea of viewing (1.1) as being the equation $\dot{Z} = Z^2$ occurring in $\mathfrak{A}$, and then using the structure of this associated algebra to study solutions to the equation. This is a program analogous to a more familiar situation: one understands solutions of linear differential equations by studying the theory of vector spaces acted on by a single linear transformation (cf. Walcher [11, p. i]). For more studies of (1.1) from this point of view, see [2], [3], [4], [5], [6], [11]. We will assume throughout that all algebras are commutative, i.e. $ZW = WZ$ for all Z, W.

An *automorphism* of $(\mathfrak{A}, \cdot)$ is a linear transformation $A : \mathfrak{A} \to \mathfrak{A}$ satisfying $A(ZW) = A(Z) \cdot A(W)$ for all $Z, W \in \mathfrak{A}$. Automorphisms of $(\mathfrak{A}, \cdot)$ are linear symmetries of the corresponding differential equation (1.1) [5], [6], [11, Chap. 9]. For a study of systems of the form (1.1) with a finite group of automorphisms, see [3]; for more general polynomial systems, see [7]. Our interest in this paper is in those algebras having an automorphism A of order two ($A^2 = \mathrm{id}, A \neq \mathrm{id}$). In this case, the algebra $\mathfrak{A}$ naturally decomposes into eigenspaces $\mathfrak{A}_i := \{Z \in \mathfrak{A} | A(Z) = (-1)^i Z\}$ for $i = 0, 1$, and it is well-known and trivial to check that $\mathfrak{A}_i \cdot \mathfrak{A}_j \subseteq \mathfrak{A}_{i+j}$ with indices

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added modulo two, where for subspaces $\mathcal{B}, \mathcal{C} \subseteq \mathfrak{A}$, $\mathcal{B}\cdot\mathcal{C}$ is the subspace of $\mathfrak{A}$ spanned by all products $Z \cdot W$, $Z \in \mathcal{B}$, $W \in \mathcal{C}$. Thus for $Z = X + Y$, $X \in \mathfrak{A}_0$, $Y \in \mathfrak{A}_1$, we have a decomposition of the equation (1.1) into a system of the form

$$\begin{aligned} \dot{X} &= X^2 + Y^2, \\ \dot{Y} &= 2XY. \end{aligned} \tag{1.2}$$

Conversely if an algebra $\mathfrak{A}$ has a decomposition $\mathfrak{A} = \mathfrak{A}_0 \oplus \mathfrak{A}_1$ such that $\mathfrak{A}_i \cdot \mathfrak{A}_j \subseteq \mathfrak{A}_{i+j}$ with indices added modulo two, then the mapping $A : \mathfrak{A} \to \mathfrak{A}$ defined by $A(X+Y) := X - Y$ for $X \in \mathfrak{A}_0$, $Y \in \mathfrak{A}_1$ is an automorphism of $(\mathfrak{A}, \cdot)$ of order two. We say that the algebra $\mathfrak{A} = \mathfrak{A}_0 \oplus \mathfrak{A}_1$ is $\mathbb{Z}_2$-graded. Thus our objects of study in this paper are quadratic differential equations (1.2) in $\mathbb{Z}_2$-graded algebras.

Note in (1.2) that if $\mathfrak{A}_1 \cdot \mathfrak{A}_1 = 0$, then $Y^2 = 0$, so the system reduces to solving a quadratic differential equation on $\mathfrak{A}_0$ and then a (nonautonomous) linear equation on $\mathfrak{A}_1$. Also, if $\mathfrak{A}_0 \cdot \mathfrak{A}_1 = 0$, then $2XY = 0$ in (1.2), so Y is constant and the system reduces to a (nonhomogeneous) quadratic differential equation in $\mathfrak{A}_0$. Hence we will assume throughout that $\mathfrak{A}_1 \cdot \mathfrak{A}_1 \neq 0$ and $\mathfrak{A}_0 \cdot \mathfrak{A}_1 \neq 0$.

Such commonly studied quadratic differential equations as the Lorenz model of thermal convection (after homogenization) and the Euler equations for the motion of a rotating rigid body in the absence of external forces are in algebras that are $\mathbb{Z}_2$-graded. Generalizing the Euler equations slightly gives the following interesting example.

Example 1.3. For the system

$$\begin{aligned} \dot{z}_1 &= \alpha z_2 z_3, \\ \dot{z}_2 &= \beta z_1 z_3, \\ \dot{z}_3 &= \gamma z_1 z_2, \end{aligned}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha\beta\gamma \neq 0$, we have

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \alpha(z_2 w_3 + z_3 w_2) \\ \beta(z_1 w_3 + z_3 w_1) \\ \gamma(z_1 w_2 + z_2 w_1) \end{bmatrix}$$

and A_1, A_2, A_3 are automorphisms of $(\mathfrak{A}, \cdot)$, where

$$A_1 \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} := \begin{pmatrix} z_1 \\ -z_2 \\ -z_3 \end{pmatrix}, \qquad A_2 \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} := \begin{pmatrix} -z_1 \\ z_2 \\ -z_3 \end{pmatrix}, \qquad A_3 \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} := \begin{pmatrix} -z_1 \\ -z_2 \\ z_3 \end{pmatrix}.$$

In fact it is easy to construct differential equations occurring in $\mathbb{Z}_2$-graded algebras: suppose $\mathcal{B}$ is a nonassociative algebra, V is a vector space, C is a symmetric bilinear form on V, $Q \in \mathcal{B}$, $f : \mathcal{B} \to \mathbb{R}$ is linear, and M is a linear endomorphism of V. Then for $X \in \mathcal{B}$, $Y \in V$ the system

$$\begin{aligned} \dot{X} &= X^2 + C(Y,Y)Q, \\ \dot{Y} &= 2f(X)M(Y) \end{aligned} \tag{1.4}$$

is quadratic and so gives rise to an algebra structure on $\mathfrak{A} = \mathcal{B} \oplus V$ (written as column vectors), where $\mathfrak{A}_0 = \left\{ \binom{X}{0} : X \in \mathcal{B} \right\}$ and $\mathfrak{A}_1 = \left\{ \binom{0}{Y} : Y \in V \right\}$ relative to the automorphism A of $\mathfrak{A}$ defined by $A\binom{X}{Y} = \binom{X}{-Y}$ for $X \in \mathcal{B}$, $Y \in V$. Note that the multiplication in $\mathfrak{A}_0$ is the same as that in $\mathcal{B}$. The algebra defined from (1.4) will be denoted by $\mathfrak{A}(\mathcal{B}, V, C, Q, f, M)$. Clearly this is an important class of algebras, since it contains all $\mathbb{Z}_2$-graded algebras for which $\dim \mathfrak{A}_0 = 1$ or $\dim \mathfrak{A}_1 = 1$, and hence contains all two and three dimensional $\mathbb{Z}_2$-graded algebras. Thus Example 1.3 is a naturally occurring $\mathfrak{A}(\mathcal{B}, V, C, Q, f, M)$ relative to any of the automorphisms $A_i, i = 1, 2, 3$. In order to ensure that $\mathfrak{A}_1 \cdot \mathfrak{A}_1 \neq 0$ and $\mathfrak{A}_0 \cdot \mathfrak{A}_1 \neq 0$, we will assume throughout that $C \not\equiv 0,\ Q \neq 0,\ f \not\equiv 0$, and $M \not\equiv 0$.

Example 1.5. Let $\mathcal{B}$ be the space of real $n \times n$ symmetric matrices with the Jordan algebra product $X_1 \cdot X_2 := \frac{1}{2}(X_1 B X_2 + X_2 B X_1)$, where $B \in \mathcal{B}$ is fixed. Let $V = so(n)$, the space of real $n \times n$ skew symmetric matrices. Fix $Q, S \in \mathcal{B}$ and define $M \in \operatorname{End} V$ by $M(Y) := \frac{1}{2}(SY + YS)$, define C on $\mathcal{B}$ by $C(Y, Y) := \operatorname{tr}(Y^2)$, and $f : \mathcal{B} \to \mathbb{R}$ by $f(X) = \operatorname{tr}(X)$. Then for $X \in \mathcal{B},\ Y \in V$ the system

$$\begin{aligned} \dot{X} &= XBX + \operatorname{tr}(Y^2)Q, \\ \dot{Y} &= \operatorname{tr}(X)(SY + YS) \end{aligned} \tag{1.5.1}$$

is of the form (1.4), so $\mathfrak{A} = \mathfrak{A}(\mathcal{B}, V, C, Q, f, M)$.

Example 1.6. Let $\mathfrak{G}$ be a real Lie algebra (see, e.g., [9] for appropriate definitions). Fix $L \in \mathfrak{G}$ and let $\mathcal{B} = \mathfrak{G}$ with multiplication defined by

$$X_1 \cdot X_2 := \frac{1}{2}([[L, X_1], X_2] + [[L, X_2], X_1]).$$

Let $f : \mathcal{B} \to \mathbb{R}$ be a nontrivial linear form, $V = \mathfrak{G}$ and $\kappa : \mathcal{B} \times \mathcal{B} \to \mathbb{R}$ the Killing form. Fix $Q, S \in \mathfrak{G}$ and define $M \in \operatorname{End} V$ by $M(Y) := [S, Y]$. Define a system and corresponding product on $\mathfrak{A} = \mathcal{B} \oplus V = \mathfrak{G} \oplus \mathfrak{G}$ by

$$\begin{aligned} \dot{X} &= [[L, X], X] + \kappa(Y, Y)Q, \\ \dot{Y} &= 2f(X)[S, Y]. \end{aligned} \tag{1.6.1}$$

Then this has the form (1.4), so $\mathfrak{A} = \mathfrak{A}(\mathcal{B}, V, \kappa, Q, f, M)$. Note that if κ is nondegenerate (so that $\mathfrak{G}$ is semisimple), there exists a $U \in \mathfrak{G}$ such that $f(X) = \kappa(U, X)$ for all $X \in \mathfrak{G}$. In this case the product in $\mathfrak{A}$ is determined by the four elements L, Q, U and S in $\mathfrak{G}$.

The next proposition allows us to reduce (1.4) to solving a differential equation just on $\mathfrak{A}_0$.

Proposition 1.7. *Suppose $X(t) + Y(t)$ is the solution to (1.4), where $X(t) \in \mathfrak{A}_0$ and $Y(t) \in \mathfrak{A}_1$ for all t and $X(0) = P_0,\ Y(0) = P_1$ is the initial condition.*

(i) $Y(t) = \exp(\Theta(t)M)P_1$, *where* $\Theta(t) = 2\int_0^t f(X(s))ds$.

(ii) *If there is an $a \in \mathbb{R}$ such that $C(MY_1, Y_2) + C(Y_1, MY_2) = aC(Y_1, Y_2)$ for all $Y_1, Y_2 \in \mathfrak{A}_1$, then $X(t)$ is the solution to the following second order cubic equation:*

$$\ddot{X} = 2X\dot{X} + 2af(X)(\dot{X} - X^2). \tag{1.7.1}$$

Note that the hypothesis of Proposition 1.7(ii) is satisfied in Example 1.6 with $a = 0$ and is satisfied in Example 1.5 if S is a scalar matrix, but is not satisfied in Example 1.3.

While Proposition 1.7 is interesting, for our purposes the original first order system (1.4) is more tractable.

We conclude this introduction with an outline of the sequel. In §2 we give necessary and sufficient conditions for $\mathfrak{A}(\mathcal{B}, V, C, Q, f, M)$ to be simple and to be semisimple as a $\mathbb{Z}_2$-graded algebra. In §3 we give necessary and sufficient conditions for $\mathfrak{A}(\mathcal{B}, V, C, Q, f, M)$ to be nilpotent. In §4 we discuss the existence of a certain kind of first integral for (1.2) and show that for (1.4) $C(Y,Y)$ is a first integral if and only if the trivial extension of M to all of $\mathfrak{A}(\mathcal{B}, V, C, Q, f, M)$ is a derivation. Finally, in §5 we consider the case where $X(t)$ is periodic and discuss how to find the explicit Floquet decomposition of $Y(t)$ in certain circumstances.

2. Simplicity and Semisimplicity of $\mathfrak{A}(\mathcal{B}, V, C, Q, f, M)$

Throughout this section $\mathfrak{A} = \mathfrak{A}(\mathcal{B}, V, C, Q, f, M)$, $X(t)+Y(t)$ will be the solution to (1.4) in $\mathfrak{A}$ with $X(t) \in \mathfrak{A}_0, Y(t) \in \mathfrak{A}_1$ for all t, and $\operatorname{Rad} C := \{Y \in V | C(Y,W) = 0 \ \forall W \in V\}$.

Recall from [10] that a subspace $\mathcal{S}$ of a commutative nonassociative algebra $\mathcal{C}$ is a *subalgebra* if $\mathcal{S} \cdot \mathcal{S} \subseteq \mathcal{S}$ and is an *ideal*, denoted $\mathcal{S} \trianglelefteq \mathcal{C}$, if $\mathcal{S} \cdot \mathcal{C} \subseteq \mathcal{S}$. If $S \subseteq \mathcal{C}$, then $\mathcal{C}(S)$ denotes the smallest subalgebra of $\mathcal{C}$ containing S. It is well known [11] that if $X(0) + Y(0) = P$ and $\mathcal{S}$ is a subalgebra of $\mathfrak{A}$ with $P \in \mathcal{S}$, then $X(t) + Y(t) \in \mathcal{S}$ for all t. Then the following proposition is trivial to prove.

Proposition 2.1. *Suppose $X(0) = P_0$ and $Y(0) = P_1$. Then $X(t) + Y(t) \in \mathfrak{A}(\{P_0, P_1, Q\})$ for all t. Moreover,*

$$\mathfrak{A}(\{P_0, P_1, Q\}) = \mathfrak{A}(\mathcal{B}(\{P_0, Q\}), W, C|_W, Q, f|_{\mathcal{B}(\{P_0,Q\})}, M|_W),$$

where W is the smallest subspace of V with $P_1 \in W$ and $MW \subseteq W$.

Hence we can always assume $\mathcal{B}$ is generated by two elements and that V is spanned by $\{M^j P_1 | j \in \mathbb{N}\}$, if necessary.

Since ideals of $\mathfrak{A}$ can be used to reduce (1.4) to the study of systems with fewer variables (see [11] for how this is done), we now turn to the question of when $\mathfrak{A}$ has ideals other than 0 and $\mathfrak{A}$. Recall that $\mathfrak{A}$ is *simple* if $\mathfrak{A} \cdot \mathfrak{A} \neq 0$ and the only ideals of $\mathfrak{A}$ are 0 and $\mathfrak{A}$, and $\mathfrak{A}$ is *simple as a $\mathbb{Z}_2$-graded algebra* if $\mathcal{I} \trianglelefteq \mathfrak{A}$ with $\mathcal{I} = (\mathcal{I} \cap \mathfrak{A}_0) + (\mathcal{I} \cap \mathfrak{A}_1)$ implies $\mathcal{I} = 0$ or $\mathcal{I} = \mathfrak{A}$ and $\mathfrak{A} \cdot \mathfrak{A} \neq 0$. The next lemma is easy to check.

Lemma 2.2. *(i) Suppose $I \subseteq V$ is the image of M. Then*

$$\mathcal{J}_1 \trianglelefteq \mathfrak{A} \quad \textit{where } \mathcal{J}_1 := \left\{ \begin{pmatrix} X \\ Y \end{pmatrix} \in \mathfrak{A} \,\middle|\, \begin{matrix} X \in \mathcal{B} \\ Y \in I \end{matrix} \right\}.$$

(ii) Suppose $0 \neq \mathcal{L} \trianglelefteq \mathcal{B}$ with $f|_{\mathcal{L}} \equiv 0$. Then

$$\mathcal{J}_2 \trianglelefteq \mathfrak{A} \quad \textit{where } \mathcal{J}_2 := \left\{ \begin{pmatrix} X \\ 0 \end{pmatrix} \in \mathfrak{A} \,\middle|\, X \in \mathcal{L} \right\}.$$

(iii) Suppose $\mathcal{L} \trianglelefteq \mathcal{B}$ with $Q \in \mathcal{L}$. Then

$$\mathcal{J}_3 \trianglelefteq \mathfrak{A} \quad \textit{where } \mathcal{J}_3 := \left\{ \begin{pmatrix} X \\ Y \end{pmatrix} \in \mathfrak{A} \,\middle|\, \begin{matrix} X \in \mathcal{L} \\ Y \in V \end{matrix} \right\}.$$

(iv) Suppose K is a subspace of $\operatorname{Rad} C$ with $MK \subseteq K$. Then

$$\mathcal{J}_4 \trianglelefteq \mathfrak{A} \quad \textit{where } \mathcal{J}_4 := \left\{ \begin{pmatrix} 0 \\ Y \end{pmatrix} \in \mathfrak{A} \,\middle|\, Y \in K \right\}.$$

Since $\mathcal{J}_i = (\mathcal{J}_i \cap \mathfrak{A}_0) + (\mathcal{J}_i \cap \mathfrak{A}_1)$ for $i = 1, 2, 3, 4$ in Lemma 2.2, the hypotheses of the following theorem are clearly necessary for $\mathfrak{A}$ to be simple as a $\mathbb{Z}_2$-graded algebra. Recall that we are assuming $f \not\equiv 0, M \not\equiv 0$, and $C \not\equiv 0$.

Theorem 2.3. *$\mathfrak{A}$ is simple as a $\mathbb{Z}_2$-graded algebra iff all of the following are satisfied:*

(1) *$M \in GL(V)$.*
(2) *$\mathcal{L} \trianglelefteq \mathcal{B}$ with $f|_{\mathcal{L}} \equiv 0$ implies $\mathcal{L} = 0$.*
(3) *If $\mathcal{L} \trianglelefteq \mathcal{B}$ is such that $Q \in \mathcal{L}$, then $\mathcal{L} = \mathcal{B}$.*
(4) *If K is a subspace of $\operatorname{Rad} C$ such that $MK \subseteq K$, then $K = 0$.*

Proof. Suppose (1), (2), (3), and (4) are all satisfied and suppose $0 \neq \mathcal{I} \trianglelefteq \mathfrak{A}$ with $\mathcal{I} = (\mathcal{I} \cap \mathfrak{A}_0) + (\mathcal{I} \cap \mathfrak{A}_1)$. Thus if $\binom{X}{Y} \in \mathcal{I}$ then $\binom{X}{0}, \binom{0}{Y} \in \mathcal{I}$, and hence if $\binom{Z}{W} \in \mathfrak{A}$, it follows that $\binom{XZ}{0}, \binom{C(Y,W)Q}{0}, \binom{0}{f(Z)MY}, \binom{0}{f(X)MW} \in \mathcal{I}$. Let $\mathcal{L} = \{X \in \mathcal{B} | \binom{X}{0} \in \mathcal{I} \cap \mathfrak{A}_0\}$ and $K = \{Y \in V | \binom{0}{Y} \in \mathcal{I} \cap \mathfrak{A}_1\}$. Hence $\mathcal{L} \trianglelefteq \mathcal{B}$. Since $C(Y,W)Q \in \mathcal{L}$ for all $Y \in K$, $W \in V$, either $Q \in \mathcal{L}$ or $K \subseteq \operatorname{Rad}\ C$ with $MK \subseteq K$. By (3) and (4), either $\mathcal{L} = \mathcal{B}$ or $K = 0$. Since $f(X)MW \in K$ for all $X \in \mathcal{L}$, $W \in V$, either $MV \subseteq K$ or $f|_{\mathcal{L}} \equiv 0$. By (1) and (2), either $V = K$ or $\mathcal{L} = 0$. Now if $\mathcal{L} \neq \mathcal{B}$, then $K = 0$; hence $V \neq K$, and thus $\mathcal{L} = 0$. Otherwise, if $\mathcal{L} = \mathcal{B}$, then $\mathcal{L} \neq 0$ and thus $V = K$. Thus either $\mathcal{I} = 0$ or $\mathcal{I} = \mathfrak{A}$, and therefore $\mathfrak{A}$ is simple as a $\mathbb{Z}_2$-graded algebra. □

Corollary 2.4. *Suppose $\mathcal{B}$ is simple. Then $\mathfrak{A}$ is simple as a $\mathbb{Z}_2$-graded algebra iff (1) and (4) of Theorem 2.3 are satisfied.*

Note that (4) of Theorem 2.3 is automatically satisfied if C is nondegenerate. Hence it is easy to check using Corollary 2.4 that the algebra in Example 1.3 is simple as a $\mathbb{Z}_2$-graded algebra.

Clearly, if $\mathfrak{A}$ is simple, then $\mathfrak{A}$ is simple as a $\mathbb{Z}_2$-graded algebra. The next example shows that the converse is not true.

Example 2.5. Suppose $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta > 0$. For the system

$$\dot{x} = \alpha x^2 + \beta y^2,$$
$$\dot{y} = 2\alpha xy$$

$\mathfrak{A} = \mathbb{R}^2$ with multiplication defined by $\binom{x}{y} \cdot \binom{z}{w} = \binom{\alpha xz + \beta yw}{\alpha xw + \alpha yz}$ and $A\binom{x}{y} := \binom{x}{-y}$ is an automorphism of order two, $\mathcal{B} = \mathbb{R}$ with $x \cdot x := \alpha x^2$, $V = \mathbb{R}$, $C(y,y) := \beta y^2$, $Q = 1$, $f(x) := x$, and $M(y) := \alpha y$. Hence by Corollary 2.4, $\mathfrak{A}$ is simple as a $\mathbb{Z}_2$-graded algebra. But it is easy to check that $0 \neq \mathcal{I} \trianglelefteq \mathfrak{A}$ and $\mathcal{I} \neq \mathfrak{A}$, where

$$\mathcal{I} := \left\{ \begin{pmatrix} x \\ \sqrt{\alpha\beta^{-1}}x \end{pmatrix} | x \in \mathbb{R} \right\}.$$

However, Example 2.5 is the only case of $\mathfrak{A}$ being $\mathbb{Z}_2$-graded simple but not simple, as the next theorem shows.

Theorem 2.6. *Suppose $\mathfrak{A}$ is simple as a $\mathbb{Z}_2$-graded algebra, but $\mathfrak{A}$ is not simple. Then $\mathfrak{A}$ is isomorphic to the algebra constructed in Example 2.5.*

Proof. Choose $0 \neq \mathcal{I} \trianglelefteq \mathfrak{A}$ and $\mathcal{I} \neq \mathfrak{A}$ of minimal dimension. Let A be the automorphism of $\mathfrak{A}$ giving the $\mathbb{Z}_2$-grading. Then $A(\mathcal{I}) \trianglelefteq \mathfrak{A}$, $\mathcal{I} \neq \mathfrak{A}$ and $\mathcal{I} \cap A(\mathcal{I}) \trianglelefteq \mathfrak{A}$, $\mathcal{I} \cap A(\mathcal{I}) \neq \mathfrak{A}$ implies $\mathcal{I} \cap A(\mathcal{I}) = 0$, since otherwise $A(\mathcal{I}) = \mathcal{I}$, contradicting the simplicity of $\mathfrak{A}$ as a $\mathbb{Z}_2$-graded algebra. Let $\mathcal{J} = \mathcal{I} + A(\mathcal{I})$. Then $\mathcal{J} \trianglelefteq \mathfrak{A}$ and $A(\mathcal{J}) = \mathcal{J}$, so $\mathcal{J} = \mathfrak{A}$. Thus $\mathfrak{A} = \mathcal{I} \oplus A(\mathcal{I})$.

Let $\mathcal{L} := \{X \in \mathcal{B} | \binom{X}{0} \in \mathcal{I}\}$ and $K := \{Y \in V | \binom{0}{Y} \in \mathcal{I}\}$. Then $\mathcal{L} \trianglelefteq \mathcal{B}$. If $\mathcal{L} = \mathcal{B}$, then $\mathcal{I} = \mathfrak{A}$, contradicting the choice of $\mathcal{I}$. Hence $Q \notin \mathcal{L}$ by Theorem 2.3(2). Now for all $Y \in K$ and $W \in V$,

$$\begin{pmatrix} 0 \\ Y \end{pmatrix}\begin{pmatrix} 0 \\ W \end{pmatrix} = \begin{pmatrix} C(Y,W)Q \\ 0 \end{pmatrix} \in \mathcal{I},$$

and thus $C(Y,W)Q \in \mathcal{L}$, which implies $C(Y,W) = 0$.

Hence $K \subseteq \operatorname{Rad} C$. For all $Z \in \mathcal{B}$ and $Y \in K$,

$$\begin{pmatrix} Z \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ f(Z)MY \end{pmatrix} \in \mathcal{I},$$

and thus $f(Z)MY \in K$. Hence $MK \subseteq K$, and by Theorem 2.3(4), $K = 0$. For all $X \in \mathcal{L}$ and $W \in V$,

$$\begin{pmatrix} X \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ W \end{pmatrix} = \begin{pmatrix} 0 \\ f(X)MW \end{pmatrix} \in \mathcal{I},$$

and since $K = 0$, $f(X)MW = 0$. By Theorem 2.3(1), $f(X) = 0$ for all $X \in \mathcal{L}$. By Theorem 2.3(2), $\mathcal{L} = 0$.

Now since $\mathfrak{A} = \mathcal{I} \oplus A(\mathcal{I})$, for every $X \in \mathcal{B}$ there are $\binom{Z}{W}, \binom{S}{T} \in \mathcal{I}$ with $\binom{X}{0} = \binom{Z}{W} + A\binom{S}{T} = \binom{Z}{W} + \binom{S}{-T}$, so $W = T$, and thus $Z = S$ since $\mathcal{L} = 0$. Hence $Z = \frac{1}{2}X$. Therefore for every $X \in \mathcal{B}$, there is a unique $Y \in V$ with $\binom{X}{Y} \in \mathcal{I}$. Similarly, since $K = 0$, for every $Y \in V$ there is a unique $X \in \mathcal{B}$ with $\binom{X}{Y} \in \mathcal{I}$. Thus $\dim \mathcal{B} = \dim V$.

For $\binom{X}{Y} \in \mathcal{I}$ and $W \in V$,

$$\begin{pmatrix} X \\ Y \end{pmatrix}\begin{pmatrix} 0 \\ W \end{pmatrix} = \begin{pmatrix} C(Y,W)Q \\ f(X)MW \end{pmatrix} \in \mathcal{I}.$$

If $f(X) = 0$, then since $\mathcal{L} = 0$, it follows that $C(Y,W) = 0$, i.e., $Y \in \operatorname{Rad}\ C$. Conversely, if $Y \in \operatorname{Rad}\ C$, then since $K = 0$ and M is nonsingular, it follows that $f(X) = 0$. Hence $\binom{X}{Y} \in \mathcal{I}$ satisfies $f(X) = 0$ if and only if $Y \in \operatorname{Rad}\ C$. Therefore $\dim \ker f = \dim \operatorname{Rad} C$.

Finally, choose $\binom{X}{Y} \in \mathcal{I}$ such that $f(X) \neq 0$. Then, for all $W \in \operatorname{Rad} C$, $\binom{X}{Y}\binom{0}{W} = \binom{0}{f(X)M(W)} \in \mathcal{I}$ implies $M(W) = 0$, and so $W = 0$. Thus $\dim \ker f = 0$, and so $\dim \mathcal{B} = 1 = \dim V$. Thus $\mathcal{B} = \mathbb{R}$ with $x \cdot x = ax^2$ for some $a \in \mathbb{R}$; $V = \mathbb{R}$ with $C(y,y) = by^2$ for some $0 \neq b \in \mathbb{R}$; $Q = 1; M(y) = cy$ for some $0 \neq c \in \mathbb{R}$; and $f(x) = x$. Now there is a unique $0 \neq d \in \mathbb{R}$ such that $\binom{1}{d} \in \mathcal{I}$. Since $\dim \mathcal{I} = 1$, it is easy to check that $a = bd^2 \neq 0$ and $c = 2a$. Hence $\mathfrak{A}$ is the algebra constructed in Example 2.5. □

Corollary 2.7. *Suppose* $\dim \mathfrak{A} > 2$. *Then* $\mathfrak{A}$ *is simple iff all of the following are satisfied:*

(1) $M \in GL(V)$.
(2) $\mathcal{L} \trianglelefteq \mathcal{B}$ *with* $f|_{\mathcal{L}} \equiv 0$ *implies* $\mathcal{L} = 0$.
(3) *If* $\mathcal{L} \trianglelefteq \mathcal{B}$ *such that* $Q \in \mathcal{L}$, *then* $\mathcal{L} = \mathcal{B}$.
(4) *If* K *is a subspace of* $\operatorname{Rad} C$ *such that* $MK \subseteq K$, *then* $K = 0$.

Thus the algebra in Example 1.3 is simple.

Finally we turn to the question of when $\mathfrak{A}$ is semisimple. Recall from [11] that a nonassociative algebra $\mathcal{C}$ is *semisimple* if $\mathcal{C}$ is the direct sum of simple ideals. If $\mathcal{C}$ is semisimple and $\mathcal{I} \trianglelefteq \mathcal{C}$, there exists a $\mathcal{J} \trianglelefteq \mathcal{C}$ such that $\mathcal{C} = \mathcal{I} \oplus \mathcal{J}$. If $\mathfrak{A}$ is semisimple, then (1.4) reduces to solving a quadratic equation on each of the simple components.

Proposition 2.8. *Suppose $\mathfrak{A}$ is semisimple. Then $M \in GL(V)$, and if K is a subspace of* $\operatorname{Rad} C$ *such that $MK \subseteq K$, then $K = 0$.*

Proof. That $M \in GL(V)$ follows from $\mathfrak{A} \cdot \mathfrak{A} = \mathfrak{A}$. Now suppose K is a subspace of $\operatorname{Rad} C$ such that $MK \subseteq K$. Recall from Lemma 2.2(iv) that $\mathcal{J}_4 \trianglelefteq \mathfrak{A}$, where $\mathcal{J}_4 := \left\{\binom{0}{Y} \in \mathfrak{A} \middle| Y \in K\right\}$. Since $\mathfrak{A}$ is semisimple, there is an $\mathcal{I} \trianglelefteq \mathfrak{A}$ with $\mathfrak{A} = \mathcal{J}_4 \oplus \mathcal{I}$. Now for every $Z \in \mathcal{B}$ there is a $W \in V$ such that $\binom{Z}{W} \in \mathcal{I}$. Then if $\binom{0}{Y} \in \mathcal{J}_4$, $\binom{Z}{W} \in \mathcal{I}$ it follows that

$$\begin{pmatrix} 0 \\ Y \end{pmatrix} \begin{pmatrix} Z \\ W \end{pmatrix} = \begin{pmatrix} 0 \\ f(Z)M(Y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

since $\mathcal{J}_4 \cap \mathcal{I} = 0$. Then since $f \not\equiv 0, M(Y) = 0$ so $M \in GL(V)$ implies $Y = 0$, i.e. $K = 0$. □

Corollary 2.9. *Suppose $\mathfrak{A}_0$ is simple. Then $\mathfrak{A}$ is semisimple iff either $\mathfrak{A}$ is simple or $\mathfrak{A}$ is the algebra constructed in Example 2.5.*

Proposition 2.10. *Suppose $\mathfrak{A}$ is semisimple.*

(i) *If $\mathcal{L} \trianglelefteq \mathcal{B}$ is such that $f|_{\mathcal{L}} \equiv 0$, then there is an $\mathcal{M} \trianglelefteq \mathcal{B}$ such that $\mathcal{B} = \mathcal{L} \oplus \mathcal{M}$.*
(ii) *Suppose $\mathcal{L} \trianglelefteq \mathcal{B}$ with $Q \in \mathcal{L}$. Then there is an $\mathcal{M} \trianglelefteq \mathcal{B}$ such that $\mathcal{B} = \mathcal{L} \oplus \mathcal{M}$ and $f|_{\mathcal{M}} \equiv 0$.*

Proof. (i) Recall from Lemma 2.2(ii) that $\mathcal{J}_2 \trianglelefteq \mathfrak{A}$, where $\mathcal{J}_2 := \left\{\binom{X}{0} \middle| X \in \mathcal{L}\right\}$. Since $\mathfrak{A}$ is semisimple, there is an $\mathcal{I} \trianglelefteq \mathfrak{A}$ such that $\mathfrak{A} = \mathcal{J}_2 \oplus \mathcal{I}$. Let $\mathcal{M} := \left\{Z \in \mathcal{B} \middle| \binom{Z}{0} \in \mathcal{I}\right\}$. Clearly $\mathcal{B} = \mathcal{L} \oplus \mathcal{M}$ as a vector space, and $\mathcal{J}_2\mathcal{I} = 0$ implies $\mathcal{L}\mathcal{M} = 0$, so $\mathcal{M} \trianglelefteq \mathcal{B}$.

(ii): Recall from Lemma 2.2(iii) that

$$\mathcal{J}_3 \trianglelefteq \mathfrak{A}, \quad \text{where } \mathcal{J}_3 := \left\{ \begin{pmatrix} X \\ Y \end{pmatrix} \in \mathfrak{A} \middle| \begin{matrix} X \in \mathcal{L} \\ Y \in V \end{matrix} \right\}.$$

Again there is an $\mathcal{I} \trianglelefteq \mathfrak{A}$ such that $\mathfrak{A} = \mathcal{J}_3 \oplus \mathcal{I}$. Let $\mathcal{M} := \{Z \in \mathcal{B} \big| \binom{Z}{W} \in \mathcal{I}$ for some $W \in V\}$. Now if $\binom{Z}{W} \in \mathcal{I}$, then $\binom{0}{Y} \in \mathcal{J}_3$ for all $Y \in V$, so

$$\begin{pmatrix} 0 \\ Y \end{pmatrix} \begin{pmatrix} Z \\ W \end{pmatrix} = \begin{pmatrix} C(Y,W)Q \\ f(Z)M(Y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which implies $f(Z) = 0$ for all $Z \in \mathcal{M}$ by Proposition 2.8. Hence $f|_{\mathcal{M}} \equiv 0$. Since $\mathcal{B} = \mathcal{L} \oplus \mathcal{M}$ as a vector space, this implies $f|_{\mathcal{L}} \not\equiv 0$. Choose $X \in \mathcal{L}$ such that $f(X) \neq 0$. Then for all $\binom{Z}{W} \in \mathcal{I}, \binom{X}{0}\binom{Z}{W} = \binom{XZ}{f(X)M(W)} = \binom{0}{0}$ implies $W = 0$ by Proposition 2.8. Hence $\mathcal{M} \trianglelefteq \mathcal{B}$. □

It is not clear whether the conditions in Proposition 2.8 and 2.10 are sufficient to guarantee that $\mathfrak{A}$ is semisimple. However, things are much easier to see with regard to being semisimple as a $\mathbb{Z}_2$-graded algebra, where $\mathcal{C}$ is *semisimple as a $\mathbb{Z}_2$-graded algebra* if $\mathcal{C}$ is the direct sum of ideals each of which is simple as a $\mathbb{Z}_2$-graded algebra.

Theorem 2.11. *$\mathfrak{A}$ is semisimple as a $\mathbb{Z}_2$-graded algebra if and only if all of the following are satisfied:*

(1) *Suppose $\mathcal{L} \trianglelefteq \mathcal{B}$ generated by Q. There is an $\mathcal{M} \trianglelefteq \mathcal{B}$ such that $\mathcal{M}$ is semisimple, $\mathcal{B} = \mathcal{L} \oplus \mathcal{M}$ and $f|_{\mathcal{M}} \equiv 0$.*
(2) *If $\mathcal{K} \trianglelefteq \mathcal{L}$ with $f|_{\mathcal{K}} \equiv 0$, then $\mathcal{K} = 0$.*
(3) *$M \in GL(V)$.*

(4) *If K is a subspace of* $\operatorname{Rad} C$ *such that* $MK \subseteq K$*, then* $K = 0$.

Proof. Suppose (1), (2), (3), and (4) are satisfied. Let $\mathcal{I} = \mathfrak{A}(\mathcal{L}, V, C, Q, f|_{\mathcal{L}}, M)$. Then $\mathcal{I} \trianglelefteq \mathfrak{A}$, and $\mathcal{I}$ is simple as a $\mathbb{Z}_2$-graded algebra by Theorem 2.3. Moreover, $\mathfrak{A} = \mathcal{I} \oplus \mathcal{J}$, where $\mathcal{J} = \left\{ \binom{X}{0} \middle| X \in \mathcal{M} \right\}$. $\mathcal{J} \trianglelefteq \mathfrak{A}$ with $\mathcal{J} \simeq \mathcal{M}$, so $\mathcal{J}$ is semisimple, and since $A|_{\mathcal{J}} = \mathrm{id}$, $\mathcal{J}$ is semisimple as a $\mathbb{Z}_2$-graded algebra. Hence $\mathfrak{A}$ is semisimple as a $\mathbb{Z}_2$-graded algebra.

Conversely, suppose $\mathfrak{A}$ is semisimple as a $\mathbb{Z}_2$-graded algebra. There is an $\mathcal{I} \trianglelefteq \mathfrak{A}$ which is simple as a $\mathbb{Z}_2$-graded algebra and for which $\mathcal{I} \cap \mathfrak{A}_1 \neq 0$. Let

$$\mathcal{L} = \left\{ X \in \mathcal{B} \middle| \begin{pmatrix} X \\ 0 \end{pmatrix} \in \mathcal{I} \right\}, \qquad V_1 = \left\{ Y \in V \middle| \begin{pmatrix} 0 \\ Y \end{pmatrix} \in \mathcal{I} \right\},$$

$$C_1 = C|_{V_1 \times V_1}, \qquad Q_1 = \begin{cases} Q & \text{if } Q \in \mathcal{L}, \\ 0 & \text{otherwise,} \end{cases}$$

$$f_1 = f|_{\mathcal{L}}, \qquad M_1 = \begin{cases} M|_{V_1} & \text{if } f|_{\mathcal{L}} \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mathcal{L} \trianglelefteq \mathcal{B}$. Since $\mathcal{I} \cap \mathfrak{A}_1 \neq 0$, $V_1 \neq 0$, and thus $\mathcal{I} = \mathfrak{A}(\mathcal{L}, V_1, C_1, Q_1, f_1, M_1)$; so $f_1 \not\equiv 0, M_1 \not\equiv 0, Q_1 \neq 0$ and $C_1 \not\equiv 0$, since otherwise $\mathcal{I}$ is not simple as a $\mathbb{Z}_2$-graded algebra. Hence by Theorem 2.6, $\mathcal{I}$ is semisimple. If $\mathcal{J} \trianglelefteq \mathfrak{A}$, which is simple as a $\mathbb{Z}_2$-graded algebra and $\mathcal{J} \cap \mathfrak{A}_1 = 0$, then $\mathcal{J}$ is isomorphic to a simple ideal of $\mathcal{B}$. Thus $\mathfrak{A}$ is semisimple. Hence (3) and (4) follow from Proposition 2.8. Since $\mathcal{I}$ is simple, $\mathcal{L} \trianglelefteq \mathcal{B}$ must be generated by Q and (2) must be true by Theorem 2.3. But then, by Theorem 2.3, $\mathcal{I} = \mathfrak{A}(\mathcal{L}, V, C, Q, f|_{\mathcal{L}}, M)$ so $\mathfrak{A} = \mathcal{I} \oplus \mathcal{J}$, where $\mathcal{J} \trianglelefteq \mathfrak{A}$ and $\mathcal{J} \cap \mathfrak{A}_1 = 0$. Let $\mathcal{M} = \left\{ X \in \mathcal{B} \middle| \binom{X}{0} \in \mathcal{J} \right\}$. Then $\mathcal{J} \simeq \mathcal{M}$, so $\mathcal{M}$ is semisimple, $\mathcal{B} = \mathcal{L} \oplus \mathcal{M}$, $\mathcal{M} \trianglelefteq \mathcal{B}$, and $f|_{\mathcal{M}} \equiv 0$. □

3. Nilpotence and Solvability

In this section $\mathfrak{A} = \mathfrak{A}(\mathcal{B}, V, C, Q, f, M)$ throughout, A is the automorphism giving the $\mathbb{Z}_2$-grading, $\operatorname{Rad} C$ is the radical of C, and $X(t) + Y(t)$ is the solution to (1.4) in $\mathfrak{A}$ with $X(t) \in \mathfrak{A}_0, Y(t) \in \mathfrak{A}_1$ for all t. Recall from [10] that a commutative nonassociative algebra $\mathcal{C}$ is *nilpotent* if $\mathcal{C}^n = 0$ for some $n \in \mathbb{N}$, where $\mathcal{C}^j$ is defined recursively by $\mathcal{C}^0 := \mathcal{C}$ and $\mathcal{C}^{j+1} := \mathcal{C} \cdot \mathcal{C}^j$. Clearly, $\mathcal{C}^{j+1} \subseteq \mathcal{C}^j$ and $\mathcal{C}^j \trianglelefteq \mathcal{C}$ for all j. Moreover, if σ is any automorphism of $\mathcal{C}$, $\sigma(\mathcal{C}^j) = \mathcal{C}^j$. The importance of nilpotence is that if $\mathfrak{A}$ is nilpotent, then $X(t) + Y(t)$ is a polynomial in t and hence exists for all $t \in \mathbb{R}$ [11].

Proposition 3.1. *For all* $j \in \mathbb{N}$*:*

(i) $\left\{ \binom{X}{Y} \in \mathfrak{A} \middle| \begin{matrix} X \in \mathcal{B}^j \\ Y \in M^j(V) \end{matrix} \right\} \subseteq \mathfrak{A}^j$.

(ii) *If* $f|_{\mathcal{B}^j} \not\equiv 0$*, then* $\left\{ \binom{0}{Y} \in \mathfrak{A} \middle| Y \in M(V) \right\} \subseteq \mathfrak{A}^{j+1}$.

(iii) *If* $M^j(V) \not\subseteq \operatorname{Rad} C$*, then* $\binom{Q}{0} \in \mathfrak{A}^{j+1}$.

(iv) *If* $f(Q) \neq 0$ *and* $\binom{Q}{0} \in \mathfrak{A}^j$*, then* $\left\{ \binom{0}{Y} \in \mathfrak{A} \middle| Y \in M(V) \right\} \subseteq \mathfrak{A}^{j+1}$.

(v) *If* $M(V) \not\subseteq \operatorname{Rad} C$ *and* $\left\{ \binom{0}{Y} \in \mathfrak{A} \middle| Y \in M(V) \right\} \subseteq \mathfrak{A}^j$*, then* $\binom{Q}{0} \in \mathfrak{A}^{j+1}$.

Theorem 3.2. *Suppose* $\mathfrak{A}$ *is nilpotent. Then*

(i) M *is a nilpotent endomorphism.*

(ii) $\mathcal{B}$ *is nilpotent.*

(iii) *Either* $f(Q) = 0$ *or* $M(V) \subseteq \operatorname{Rad} C$

Proof. (i) and (ii) follow from Proposition 3.1(i), and (iii) follows from Proposition 3.1(iv) and (v). □

Theorem 3.3. *Suppose* M *is a nilpotent endomorphism,* $\mathcal{B}$ *is nilpotent, and* $M(V) \subseteq \operatorname{Rad} C$. *Then* $\mathfrak{A}$ *is nilpotent.*

Proof. It is easy to see by induction that for $j \geq 2$

$$\mathfrak{A}^j \subseteq \left\{ \begin{pmatrix} X \\ Y \end{pmatrix} \in \mathfrak{A} \middle| \begin{matrix} X \in \mathcal{B}^{j-1} \\ Y \in M(V) \end{matrix} \right\}.$$

Thus if $\mathcal{B}^n = 0$,

$$\mathfrak{A}^{n+1} \subseteq \left\{ \begin{pmatrix} 0 \\ Y \end{pmatrix} \in \mathfrak{A} \middle| Y \in M(V) \right\}$$

so by induction

$$\mathfrak{A}^{n+j} \subseteq \left\{ \begin{pmatrix} 0 \\ Y \end{pmatrix} \in \mathfrak{A} \middle| Y \in M^j(V) \right\}.$$

Since M is a nilpotent endomorphism, there is an $m \in \mathbb{N}$ such that $M^m(V) = 0$; so $\mathfrak{A}^{n+m} = 0$. □

It is not clear whether $\mathfrak{A}$ is nilpotent if $\mathcal{B}$ is nilpotent, $f(Q) = 0$, and M is a nilpotent endomorphism. However, strengthening the hypotheses somewhat gives the desired result.

Theorem 3.4. *Suppose* M *is a nilpotent linear transformation,* $\mathcal{B}$ *is nilpotent, and* $f|_{\mathcal{L}} \equiv 0$*, where* $\mathcal{L}$ *is the ideal of* $\mathcal{B}$ *generated by* Q*. Then* $\mathfrak{A}$ *is nilpotent.*

Proof. By induction

$$\mathfrak{A}^j \subseteq \left\{ \begin{pmatrix} X \\ Y \end{pmatrix} \in \mathfrak{A} \middle| \begin{matrix} X \in \mathcal{B}^j + \mathcal{L} \\ Y \in M(V) \end{matrix} \right\} \quad \text{for } j \geq 1.$$

If $\mathcal{B}^n = 0$,

$$\mathfrak{A}^n \subseteq \left\{ \begin{pmatrix} X \\ Y \end{pmatrix} \in \mathfrak{A} \middle| \begin{matrix} X \in \mathcal{L} \\ Y \in M(V) \end{matrix} \right\}.$$

Then by induction

$$\mathfrak{A}^{n+j} \subseteq \left\{ \begin{pmatrix} X \\ Y \end{pmatrix} \in \mathfrak{A} \middle| \begin{matrix} X \in \mathcal{L} \\ Y \in M^{j+1}(V) \end{matrix} \right\}.$$

Since M is a nilpotent endomorphism, there is an $m \in \mathbb{N}$ such that $M^m(V) = 0$, so $\mathfrak{A}^{n+m} \subseteq \left\{ \binom{X}{0} \middle| X \in \mathcal{L} \right\}$. Using induction again gives

$$\mathfrak{A}^{n+m+j} \subseteq \left\{ \begin{pmatrix} X \\ 0 \end{pmatrix} \middle| X \in \mathcal{B}^j \cap \mathcal{L} \right\},$$

so $\mathfrak{A}^{n+m+n} = 0$. □

Thus in Example 1.6 if $\mathfrak{G}$ is nilpotent and $f|_{\mathcal{L}} \equiv 0$ where $\mathcal{L}$ is the ideal generated by Q, then $\mathfrak{A}$ is nilpotent.

Note that if $\mathcal{C}$ is a commutative nonassociative algebra such that $\mathcal{C}^{n-1} \neq 0$ and $\mathcal{C}^n = 0$, then $\mathcal{C}^{n-1} \subseteq Z(\mathcal{C})$, where $Z(\mathcal{C}) := \{x \in \mathcal{C} | xy = 0 \text{ for all } y \in \mathcal{C}\}$ is the *center* of $\mathcal{C}$. $\mathcal{C}$ is an *abelian algebra* (or *zero algebra*) if $\mathcal{C} = Z(\mathcal{C})$. It is easy to determine $Z(\mathfrak{A})$, since $A(Z(\mathfrak{A})) = Z(\mathfrak{A})$.

Proposition 3.5.

$$Z(\mathfrak{A}) = \left\{ \begin{pmatrix} X \\ Y \end{pmatrix} \in \mathfrak{A} \Big| \begin{matrix} X \in Z(\mathcal{B}),\ f(X) = 0 \\ Y \in \operatorname{Rad} C \cap \ker M \end{matrix} \right\}.$$

We briefly discuss the solvability of $\mathfrak{A}$. Recall from [10] that a nonassociative commutative algebra $\mathcal{C}$ is *solvable* if $\mathcal{C}^{(n)} = 0$ for some $n \in \mathbb{N}$, where $\mathcal{C}^{(j)}$ is defined recursively by $\mathcal{C}^{(0)} := \mathcal{C}$ and $\mathcal{C}^{(j+1)} := \mathcal{C}^{(j)} \cdot \mathcal{C}^{(j)}$. Clearly $\mathcal{C}^{(j+1)} \subseteq \mathcal{C}^{(j)}$ for all j, but it is not necessarily the case that $\mathcal{C}^{(j)} \trianglelefteq \mathcal{C}$, although it is true that $\mathcal{C}^{(j+1)} \trianglelefteq \mathcal{C}^{(j)}$. If $\mathcal{C}$ is solvable and $\mathcal{C}^{(j)} \trianglelefteq \mathcal{C}$ for all j, then the quadratic differential equation in $\mathcal{C}$ can be solved by solving a sequence of linear equations; see [5]. We make no claim that the next result is the best possible, but it does generalize special cases in the literature; see [5].

Proposition 3.6. *If $\mathcal{B}$ is abelian and $M(V) \subseteq \operatorname{Rad} C$, then $\mathfrak{A}^{(3)} = 0$, so $\mathfrak{A}$ is solvable.*

Proof. Since

$$\begin{pmatrix} X \\ Y \end{pmatrix} \begin{pmatrix} Z \\ W \end{pmatrix} = \begin{pmatrix} C(Y,W)Q \\ f(Z)M(Y) + f(X)M(W) \end{pmatrix},$$

it follows that

$$\mathfrak{A}^{(1)} = \mathfrak{A} \cdot \mathfrak{A} = \left\{ \begin{pmatrix} aQ \\ Y \end{pmatrix} \Big| \begin{matrix} a \in \mathbb{R} \\ Y \in M(V) \end{matrix} \right\}.$$

If $Y, W \in M(V)$ and $a, b \in \mathbb{R}$, then

$$\begin{pmatrix} aQ \\ Y \end{pmatrix} \begin{pmatrix} bQ \\ W \end{pmatrix} = \begin{pmatrix} 0 \\ af(Q)M(W) + bf(Q)M(Y) \end{pmatrix}.$$

Hence $\mathfrak{A}^{(2)} = 0$ if $f(Q) = 0$, and $\mathfrak{A}^{(2)} = \left\{ \binom{0}{Y} \in \mathfrak{A} \big| Y \in M^2(V) \right\}$ otherwise. Now if $Y, W \in M^2(V)$, then $\binom{0}{Y}\binom{0}{W} = \binom{0}{0}$. Hence $\mathfrak{A}^{(3)} = 0$. □

Note that, under the hypotheses of Proposition 3.6, $\mathfrak{A}^{(j)} \trianglelefteq \mathfrak{A}$ for all j.

Example 3.7. For the system

$$\begin{aligned} \dot{x} &= y_1^2, \\ \dot{y}_1 &= 0, \\ \dot{y}_2 &= 2xy_2, \end{aligned}$$

$\mathfrak{A}$ is solvable by Proposition 3.6. Here $\mathcal{B} = \mathbb{R}$ with $x^2 = 0, Q = 1, V = \mathbb{R}^2$ with $C\left(\binom{y_1}{y_2}, \binom{y_1}{y_2}\right) = y_1^2, f(x) = x$, and $M\binom{y_1}{y_2} = \binom{0}{y_2}$.

4. Partial First Integrals and Derivations

Now we return to the general case of $\mathfrak{A}$ being the $\mathbb{Z}_2$-graded algebra coming from the system (1.2) whose solution is $X(t) + Y(t), X(t) \in \mathfrak{A}_0$ and $Y(t) \in \mathfrak{A}_1$ for all t, and with the $\mathbb{Z}_2$-grading on $\mathfrak{A}$ being defined by the automorphism A. Recall from [11] that a *first integral* of (1.2) is a function $\gamma : \mathfrak{A} \to \mathbb{R}$ which is constant on trajectories of (1.2), i.e. $\gamma(X(t) + Y(t)) =$ constant. A nonconstant first integral can be used to reduce the dimension of (1.2). A *partial first integral* of (1.2) is a function $\gamma : \mathfrak{A}_1 \to \mathbb{R}$ such that $\gamma(Y(t)) =$ constant. If γ is a partial first integral of (1.2), then $\tilde{\gamma} : \mathfrak{A} \to \mathbb{R}$ defined by $\tilde{\gamma}(X + Y) := \gamma(Y)$ for $X \in \mathfrak{A}_0, Y \in \mathfrak{A}_1$ is a first integral of $\mathfrak{A}$. The reason partial first integrals are of interest is that they are easy to find.

Proposition 4.1. *Suppose K is a symmetric bilinear form on $\mathfrak{A}_1$ such that $K(XY, Y) = 0$ for all $X \in \mathfrak{A}_0$ and $Y \in \mathfrak{A}_1$. Then $\gamma : \mathfrak{A}_1 \to \mathbb{R}$ defined by $\gamma(Y) := K(Y, Y)$ is a partial first integral of (1.2).*

Proof. We have

$$\begin{aligned} \frac{d}{dt}[\gamma(Y(t))] &= \frac{d}{dt}[K(Y(t), Y(t))] \\ &= 2K\left(\frac{dY}{dt}, Y(t)\right) = 2K(2X(t)Y(t), Y(t)) = 0, \end{aligned}$$

so $\gamma(Y(t)) =$ constant. □

If γ is a partial first integral defined from the symmetric bilinear form K, we write γ_K and call γ_K a *quadratic partial first integral*.

Choosing a basis of $\mathfrak{A}_1$ and writing the elements of $\mathfrak{A}_1$ as column vectors, for any symmetric bilinear form K on $\mathfrak{A}_1$ there is a symmetric matrix K such that $K(Y, W) = Y^t KW$ for all $Y, W \in \mathfrak{A}_1$. For $X \in \mathfrak{A}_0$ define the matrix $L(X) \in \operatorname{End} \mathfrak{A}_1$ by $L(X)Y := XY$. Then $K(XY, Y) = 0$ for all $X \in \mathfrak{A}_0, Y \in \mathfrak{A}_1$ iff $L(X)^t K + KL(X) = 0$ in $\operatorname{End} \mathfrak{A}_1$ for all $X \in \mathfrak{A}_0$. Hence finding quadratic partial first integrals is a simple linear algebra problem in $\mathfrak{A}_1$.

Proposition 4.2. *Suppose $L(X)^t = -L(X)$ for all $X \in \mathfrak{A}_0$. Then the set of quadratic partial first integrals forms a Jordan algebra where $\gamma_K \circ \gamma_N := \gamma_{\frac{1}{2}(KN+NK)}$.*

Proof. Since γ_K and γ_N are quadratic partial first integrals and $L(X)^t = -L(X)$ for all $X \in \mathfrak{A}_0$, it follows that $KL(X) = L(X)K$ and $NL(X) = L(X)N$ for all $X \in \mathfrak{A}_0$. Thus $L(X)^t(KN+NK)+(KN+NK)L(X) = -L(X)KN-L(X)NK+KNL(X) + NKL(X) = -KL(X)N - NL(X)K + KL(X)N + NL(X)K = 0$, so $\gamma_{\frac{1}{2}(KN+NK)}$ is a quadratic partial first integral. □

Proposition 4.3. *Suppose $N \in \operatorname{End} \mathfrak{A}_1$ is such that $L(X)N - NL(X) = 0$ for all $X \in \mathfrak{A}_0$ and $N^t = -N$. Then if γ_K is a quadratic partial first integral, $\gamma_{[K,N]} := \gamma_{KN-NK}$ is a quadratic partial first integral.*

Proof. Since $L(X)N = NL(X)$ for all $X \in \mathfrak{A}_0$ and $N^t = -N$, it follows that $L(X)^t N = NL(X)^t$ for all $X \in \mathfrak{A}_0$. Since γ_K is a quadratic partial first integral, $L(X)^t K = -KL(X)$, and so

$$\begin{aligned} &L(X)^t(KN - NK) + (KN - NK)L(X) \\ &\quad = -KL(X)N - L(X)^t NK + KL(X)N - NKL(X) \\ &\quad = -L(X)^t NK + NL(X)^t K = 0. \end{aligned}$$

$KN - NK$ is symmetric, so γ_{KN-NK} is a quadratic partial first integral. □

Proposition 4.4. *Suppose there are a linear function $f : \mathfrak{A}_0 \to \mathbb{R}$ and an $M \in \operatorname{End} \mathfrak{A}_1$ such that $XY = f(X)M(Y)$ for all $X \in \mathfrak{A}_0, Y \in \mathfrak{A}_1$. Then γ_K is a quadratic partial first integral iff $K(MY, Y) = 0$ for all $Y \in \mathfrak{A}_1$.*

Clearly, Proposition 4.4 is relevant to finding first integrals in $\mathfrak{A}(\mathcal{B}, V, C, Q, f, M)$ as is the next proposition.

Proposition 4.5. *Suppose there are a linear function $f : \mathfrak{A}_0 \to \mathbb{R}$ and an $M \in \operatorname{End}\mathfrak{A}_1$ such that $XY = f(X)M(Y)$ for all $X \in \mathfrak{A}_0, Y \in \mathfrak{A}_1$. Suppose γ_K is a quadratic partial first integral. Then γ_{N_j} is a quadratic partial first integral for all $j \in \mathbb{N}$, where $N_j(Y,W) := K(M^jY, M^jW)$ for all $Y, W \in \mathfrak{A}_1$.*

Proof. This is an immediate consequence of Proposition 4.4. □

The number of functionally independent quadratic partial first integrals produced by Proposition 4.5 for a given K depends on the degree of the minimal polynomial of M. The following example illustrates this.

Example 4.6. Suppose $\mathfrak{A}_1 = \mathbb{R}^4$ and there is a linear mapping $f : \mathfrak{A}_0 \to \mathbb{R}$ such that $XY = f(X)M(Y)$ for all $X \in \mathfrak{A}_0, Y \in \mathfrak{A}_1$, where

$$M = \begin{pmatrix} 0 & \lambda & 0 & 0 \\ -\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu \\ 0 & 0 & -\mu & 0 \end{pmatrix}$$

for some $\lambda, \mu \in \mathbb{R}^*$. Let K be the standard inner product on $\mathbb{R}^4$. Then γ_K is a quadratic partial first integral, so (whatever $\mathfrak{A}_0$ is) for $X \in \mathfrak{A}_0$ and $Y \in \mathfrak{A}_1$,

$$\gamma_0(X+Y) := y_1^2 + y_2^2 + y_3^2 + y_4^2$$

is a first integral on $\mathfrak{A}$, as are

$$\gamma_1(X+Y) := K(MY, MY) = \lambda^2(y_1^2 + y_2^2) + \mu^2(y_3^2 + y_4^2)$$

and

$$\gamma_2(X+Y) := K(M^2Y, M^2Y) = \lambda^4(y_1^2 + y_2^2) + \mu^4(y_3^2 + y_4^2).$$

γ_0 and γ_1 are functionally independent. But γ_0, γ_1 and γ_2 are not functionally independent: Since $M^4 + (\mu^2 + \lambda^2)M^2 + \mu^2\lambda^2 I = 0$, we have

$$\begin{aligned} \gamma_2(X+Y) &= K(M^2Y, M^2Y) = K(Y, M^4Y) \\ &= -(\mu^2 + \lambda^2)K(Y, M^2Y) - \mu^2\lambda^2 K(Y,Y) \\ &= (\mu^2 + \lambda^2)\gamma_1(X+Y) - \mu^2\lambda^2\gamma_0(X+Y). \end{aligned}$$

Note that taking appropriate linear combinations of γ_0 and γ_1 yields the simpler first integrals $X + Y \mapsto y_1^2 + y_2^2$ and $X + Y \mapsto y_3^2 + y_4^2$. This reflects the fact that the mappings $\mathfrak{A} \to \mathfrak{A} : X + Y \mapsto X + (\pm y_1 \pm y_2, \pm y_3, \pm y_4)$ are automorphisms of $\mathfrak{A}$.

We now consider the derivations of $\mathfrak{A}$. Recall from [10] that a *derivation* of a commutative nonassociative algebra $\mathcal{C}$ is a linear mapping $D : \mathcal{C} \to \mathcal{C}$ such that $D(x^2) = 2x \cdot D(x)$ for all $x \in \mathcal{C}$. If D is a derivation of $\mathcal{C}$, then $\exp(sD)$ is an automorphism of $\mathcal{C}$ for all $s \in \mathbb{R}$, i.e. the automorphism group contains a one parameter subgroup. This additional structure on $\mathcal{C}$ gives important information about the quadratic differential equation in $\mathcal{C}$; see [5], [6], [11, Ch 9] for discussion along these lines.

For $N \in \operatorname{End}\mathfrak{A}_1$, define $D_N : \mathfrak{A} \to \mathfrak{A}$ by $D_N\binom{X}{Y} := \binom{0}{N(Y)}$.

Proposition 4.7. *Suppose there are a linear function $f : \mathfrak{A}_0 \to \mathbb{R}$ and an $M \in \operatorname{End}\mathfrak{A}_1$ such that $XY = f(X)M(Y)$ for all $X \in \mathfrak{A}_0, Y \in \mathfrak{A}_1$. Then D_N is a derivation of $\mathfrak{A}$ if and only if $NM = MN$ and $Y \cdot N(Y) = 0$ for all $Y \in \mathfrak{A}_1$. In particular, D_M is a derivation of $\mathfrak{A}$ if and only if $Y \cdot M(Y) = 0$.*

In $\mathfrak{A}(\mathcal{B}, V, C, Q, f, M)$, we have $Y \cdot N(Y) := C(Y, N(Y))Q$, so comparing Proposition 4.4 and Proposition 4.7 gives the following result.

Proposition 4.8. *Suppose $M \in \operatorname{End} V$. The following are equivalent.*

(1) *D_M is a derivation of $\mathfrak{A}(\mathcal{B}, V, C, Q, f, M)$.*
(2) *M is C-skew; that is, $C(MY, Y) = 0$ for all $Y \in V$.*
(3) *γ_C is a quadratic partial first integral of (1.4).*

Example 4.9. In Example 1.6 D_S is a derivation of $\mathfrak{A}$, so $X + Y \mapsto \kappa(Y, Y)$ is a first integral of (1.6.1).

If any (and hence all) of the statements in Proposition 4.8 hold, then the differential equation in $\mathfrak{A}_0$ reduces to $\dot{X} = X^2 + kQ$, where k is a constant (and it suffices to consider those values of k in the range of γ_C). By Proposition 1.7(i) the differential equation in $\mathfrak{A}_1$ is nonautonomous and linear with an elementary solution.

Example 4.10. Suppose $\mathcal{B} = \mathbb{R}$ and assume M is C-skew. Then the differential equation in $\mathfrak{A}_0$ becomes $\dot{x} = \lambda x^2 + k$, where $k \in \mathbb{R}$, whose solution (depending on the signs of k and λ) is elementary. This (very general) example includes as a special case Example 3.10 of [5].

5. Trajectories Given by Derivations and Floquet Decompositions

In this section we continue with the hypotheses and notation of the previous section. Under certain circumstances, the action of an exponentiated derivation $\exp(tD)$ on an initial point P can give the solution to (1.1) through P; that is, $Z(t) = \exp(tD) \cdot P$ is the solution to (1.1) iff D is a derivation and $D(P) = P^2$. This last equation can be viewed as a system of polynomial equations in $\mathfrak{A}$ to be solved; see [5], [6] for a thorough discussion and examples. In this section we study the existence and properties of solutions of this type for quadratic differential equations in $\mathbb{Z}_2$-graded algebras, i.e. of the form (1.2).

Proposition 5.1. *Suppose $X(0) = P_0, Y(0) = P_1$, and D is a derivation of $\mathfrak{A}$ such that $D(\mathfrak{A}_i) \subseteq \mathfrak{A}_i$ for $i = 0, 1$ and $X(t) = \exp(tD) \cdot P_0$ for all t. Then $D(P_0) = P_0^2 + P_1^2$ and, for all t,*

$$Y(t) = \exp(tD) \cdot \exp(t(2L(P_0) - D)) \cdot P_1. \tag{5.1.1}$$

Proof. We have

$$\begin{aligned}(\exp tD)D \cdot P_0 &= \dot{X} = X^2 + Y^2 \\ &= (\exp(tD) \cdot P_0)^2 + Y^2 \\ &= \exp(tD) \cdot P_0^2 + Y^2\end{aligned}$$

since $\exp(tD)$ is an automorphism of $\mathfrak{A}$. Setting $t = 0$ in this equation gives $D(P_0) = P_0^2 + P_1^2$ as a necessary condition for $X(t)$ to have the given form. Further,

$$\begin{aligned}\dot{Y} = 2XY &= 2L(\exp(tD) \cdot P_0)Y \\ &= 2\exp(tD) \cdot L(P_0) \cdot \exp(-tD)Y\end{aligned}$$

so $\exp(-tD)\dot{Y} = 2L(P_0)\exp(-tD)Y$. Set $W = \exp(-tD)Y$. Then

$$\begin{aligned}\dot{W} &= \exp(-tD)\dot{Y} - D\exp(-tD)Y \\ &= 2L(P_0)W - DW \\ &= (2L(P_0) - D)W\end{aligned}$$

and $W(0) = P_1$. Thus $W(t) = \exp(t(2L(P_0) - D)) \cdot P_1$, so that $Y(t) = \exp(tD) \cdot \exp(t(2L(P_0) - D)) \cdot P_1$. ☐

Suppose now that the hypotheses of Proposition 5.1 are satisfied and $\exp(tD)|_{\mathfrak{A}_0}$ is a periodic matrix. Then the equation $\dot{Y} = 2L(\exp(tD) \cdot P_0)Y$ in $\mathfrak{A}_1$ is a linear system with periodic coefficients, and thus the solution $Y(t)$ has a Floquet decomposition $Y(t) = \Phi(t)(\exp tR) \cdot P_1$ where $\Phi(t)$ is a periodic matrix and R is a constant matrix whose eigenvalues are the Floquet exponents [1]. The exact Floquet decomposition is usually difficult to compute in practice, but if the hypotheses of Proposition 5.1 are satisfied with $\exp(tD)|_{\mathfrak{A}_0}$ periodic, then (5.1.1) is exactly the Floquet decomposition of Y. In particular, if the equation $\dot{Y} = 2L(\exp(tD) \cdot P_0)Y$ in $\mathfrak{A}_1$ is assumed to have a unique equilibrium point (the origin of $\mathfrak{A}_1$), then the stability of the origin can be determined from the eigenvalues of the matrix $2L(P_0) - D|_{\mathfrak{A}_1}$. See [6] for further remarks on the relationship between trajectories given by derivations and Floquet theory.

Finally, we have the following result.

Theorem 5.2. *Suppose $X(0) = P_0, Y(0) = P_1$, and D is a derivation of $\mathfrak{A}$ such that $D(\mathfrak{A}_i) \subseteq \mathfrak{A}_i$ for $i = 0, 1$ and $X(t) = \exp(tD) \cdot P_0$ for all t. Suppose further that $\exp(tD)|_{\mathfrak{A}_0}$ is periodic. Then either*

(1) *there are $Z, W \in \mathfrak{A}(\{P_0, P_1\}) \cap \mathfrak{A}_1$ such that $Z \neq 0 \neq W$ but $ZW = 0$, or*
(2) *$D(P_1) = 2P_0P_1$.*

Proof. If (1) is not true, then by Theorem 4.4(i) of [3], $Y(t)$ is periodic and $Y(t + 2T) = Y(t)$ for all t, where T is the period of $X(t)$. Then by Proposition 5.1, $\exp(2T(2L(P_0) - D))P_1 = P_1$. Hence $(2L(P_0) - D)P_1 = 0$. ☐

It is also possible to get the Floquet decomposition of $Y(t)$ for (1.4) from the following result.

Proposition 5.3. *Suppose there are a linear mapping $f : \mathfrak{A}_0 \to \mathbb{R}$ and an $M \in \operatorname{End}\mathfrak{A}_1$ such that $XY = f(X)M(Y)$ for all $X \in \mathfrak{A}_0, Y \in \mathfrak{A}_1$. Suppose $X(t)$ is periodic of period T. Then the Floquet decomposition of $Y(t)$ is*

$$Y(t) = P(t)e^{tR} \cdot P_1 \tag{5.3.1}$$

where $P(T) = \exp\left[\left(\Theta(t) - \frac{t}{T}\Theta(T)\right)M\right], \Theta(t) = 2\int_0^t f(X(s))ds$, and $R = \frac{\Theta(T)}{T}M$.

Proof. This is simple, using the fact that the fundamental matrix of the system $\dot{Y} = 2f(X(t))Y$ is $e^{\Theta(t)M}$. ☐

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