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HOMOLOGY OF THE UNIVERSAL COVERING OF A CO-H-SPACE

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ABSTRACT. The problem 10 posed by Tudor Ganea is known as *the Ganea conjecture* on *a co-H-space*, a space with co-H-structure. Many efforts are devoted to show the Ganea conjecture under additional assumptions on the given co-H-structure. In this paper, we show a homological property of co-H-spaces in a slightly general situation. As a corollary, we get the Ganea conjecture for spaces up to dimension 3.

1. INTRODUCTION

In this paper we work in the category of connected CW complexes with base points. A connected CW complex X is called a *co-H-space* when the homotopy set $[X, Y]$ has a multiplication with two-sided unit which is natural with respect to Y, where we denote by $[K, L]$ the set of homotopy classes of base point preserving mappings from K to L. We call the natural multiplication the *co-H-structure*. Ganea problem 10 is as follows: Does a connected space with co-H-structure have the homotopy type of a one-point-sum of a wedge sum of circles and a simply connected space?

It is known by S. Eilenberg and T. Ganea [5] that if a co-H-space X is paracompact and normal, $B\pi_1(X)$ has the homotopy type of a wedge sum of circles, say B. Thus we have two mappings $i\colon B \to X$ and $j\colon X \to B$, where i induces an isomorphism of the fundamental groups and j is the classifying mapping of the universal covering $\tau\colon \tilde{X} \to X$. Clearly, we may choose the mappings so that $ji \simeq 1_X$, the identity, and hence, B is a retract of X up to homotopy.

Under the assumption, we have the cofibre $c\colon X \to C$ of $i\colon B \to X$. Then $c\colon X \to C$ has a homotopy splitting (see Theorem 3.3). By using the co-H-structure, one can easily get a canonical homology equivalence $X \to B \vee C$ which induces also an isomorphism of fundamental groups. These properties, however, would *not* guarantee that the two spaces have the same homotopy type (see G.W. Whitehead [16] pp.183-184).

Berstein and Dror [2] showed that the conjecture is true provided that the co-action induced from the co-H-structure is associative. Hilton, Mislin and Roitberg [9] showed that the conjecture is true provided that the co-H-structure gives a natural algebraic loop structure on the homotopy sets $[X, \quad]$. In addition, we can

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easily see that two conditions given by the above authors are valid for all co-H-spaces if the Ganea conjecture is true.

Our approach is completely different. In this paper, we use the existence of the co-H-structure for a co-H-space X of up to dimension 3 to deform the homotopy splitting: $C \to X$ to get a homotopy equivalence $B \vee C \to X$ in light of a theorem of Seshadri, Cohn and Bass on algebras. To proceed further, we need to exclude the Whitehead products from the boundary of the cells.

We should give a comment on the work of Dr. Komatsu on the theory of links [11]. In the proof of [11], we can find out an argument which gives a proof of Ganea's conjecture for co-H-spaces whose cohomology is concentrated in one dimension other than 1, using Fox's free differential calculus.

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2. Main Theorem

A co-pairing introduced by Oda [12] is a mapping from a space A to a space $B \vee C$, whose projections to B and C are called its co-axes. Let us introduce a notion of *co-action*: a co-action of B under A along $f\colon A \to B$ is a co-pairing $\mu\colon A \to B \vee A$ with co-axes f and the identity 1_A.

Our main theorem is as follows:

Theorem 2.1. *Let X be a finite CW complex and $j\colon X \to B$ the classifying mapping of the universal covering $\tau\colon \tilde{X} \to X$. If there exists a co-action of B under X along j, then $H_*(\tilde{X};\mathbb{Z}) \cong \mathbb{Z}\pi \otimes H_*(X;\mathbb{Z})$, $\pi = \pi_1(X)$ for $* > 1$.*

Corollary 2.2. *Let X be a finite co-H-space. If $H_{n+1}(X;\mathbb{Z})$ is concentrated in dimensions 1, n and $n+1$ for $1 < n$ with $H_{n+1}(X;\mathbb{Z})$ torsion free, then X splits into a one-point sum of S^1's, S^n's, S^{n+1}'s and $S^n(m)$'s, where we denote by $S^k(\ell)$ the Moore space $S^k \cup_{\ell\iota_k} e^{k+1}$.*

Hence, the Ganea conjecture is true for complexes whose homology is concentrated in dimensions ≤ 3.

The key lemma of the theorem is based on the work done by Seshadri [15], Cohn [3] and Bass [1], that every finitely generated projective A-module is free when A is a *fir*, a free ideal ring.

The algebraic version of the Ganea conjecture could be described as follows.

Conjecture 2.3. *Let A be an algebra over a principal ideal domain R. If an A-module P is a direct summand of an induced A-module $A \otimes_R M$ from an R-module M, then P itself is induced.*

We do not know anything about the general case but for the case when $R = \mathbb{Z}$, the ring of integers, A is the group ring of a free group over $\mathbb{Z}$ and P is finitely generated.

From now on, we reserve the symbol X for a path-connected CW complex of finite type, π for its fundamental group, $j\colon X \to B\pi$ the classifying mapping of the universal covering $\tau\colon \tilde{X} \to X$. Let us denote by $H_q(-)$ the q-th (ordinary) integral homology group and by $\pi_q(-)$ the q-th homotopy group for $q \geq 0$.

3. Preliminaries

First, we recall the following well-known result (by Ganea [7]).

Fact 3.1. *The following three conditions are equivalent for a CW complex W:*
i) *W is a co-H-space.*
ii) *LS-cat(W) ≤ 1, i.e., there is a mapping $W \to W \vee W$ so that the composition with the first and second projections $W \vee W \to W$ are homotopic to the identity.*
iii) *The evaluation $\Sigma\Omega W \to W$ has a homotopy section.*

Let us call a group a co-H-group if the group, say G, admits a homomorphism $G \to G * G$ so that the compositions with the first and second projections $G * G \to G$ are the identity of G, where we denote by $G * G$ the free product. Then the fundamental group of a co-H-space is clearly a co-H-group. Let us recall the following

Proposition 3.2 (Kan [10], Eilenberg-Ganea [5]). *For G a (discrete) group, the following four conditions are equivalent.*
i) *The classifying space BG is a co-H-space.*
ii) *G is a co-H-group.*
iii) *G is a free group.*
iv) *BG has the homotopy type of a wedge sum of circles.*

Proof. By Van Kampen's Theorem, $BG \vee BG$ has the homotopy type of $B(G*G)$. Hence, i) is equivalent to ii). And also iii) is equivalent to iv), because $B\mathbb{Z}$ has the homotopy type of a circle. Here, iv) implies i), since a circle is a co-H-space.

So we are left to show that i) implies iii). It is obtained by Proposition 3 in [5], when BG is paracompact and normal. Here, let us give another simple proof of the implication as follows: Since BG is a co-H-space, $\Sigma\Omega BG$ dominates BG by Fact 3.1. Since $\Sigma\Omega BG \simeq \Sigma G$ has the homotopy type of a wedge sum of circles, its fundamental group is a free group, which could have infinitely many generators. Hence G is a subgroup of a free group, and is itself a free group (see Crowell-Fox [4]). Thus i) implies iii) and this completes the proof of the proposition. □

Hence by Proposition 3.2, the fundamental group π is a free group and the classifying space $B\pi$ has the homotopy type of a wedge sum of circles, say B, if there exists a co-action of $B\pi$ under X along j. From now on, we always assume the existence of such co-action and denote by $i\colon B \to X$ a mapping representing the generators of the fundamental group of X and by $j\colon X \to B\pi \simeq B$ the classifying mapping of the universal covering $\tau\colon \tilde{X} \to X$, so that ji is a homotopy equivalence. Hence we may assume that $ji \simeq 1_B$. Let us denote by $X \to C$ the mapping cone of $i\colon B \to X$. Under the above notations, we have the following

Theorem 3.3. *There is a natural mapping $p\colon B \vee D(X) \to X$ with a (homotopy) section $s\colon X \to B \vee D(X)$, for some simply connected space $D(X)$.*

Proof. Let E be the homotopy pull-back of a mapping $\Delta_j = (j \times 1_X)\Delta_X\colon X \to B \times X$ and the inclusion $k\colon B \vee X \to B \times X$:

$$\begin{array}{ccc} E & \xrightarrow{\hat{\Delta}_j} & B \vee X \\ {\scriptstyle \hat{p}}\downarrow & & {\scriptstyle k}\downarrow \\ X & \xrightarrow[\Delta_j]{} & B \times X \end{array}$$

where E can be described as the set $\{(x, \ell_B, \ell_X) \mid \ell_B\colon I \to B,\ \ell_X\colon I \to X,\ \ell_B(0) = j(x),\ \ell_X(0) = x,\ (\ell_B(1), \ell_X(1)) \in B \vee X\}$.

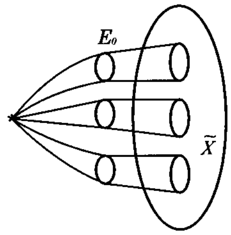

FIGURE 1

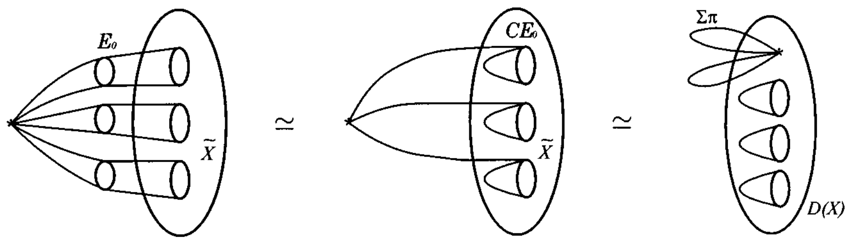

FIGURE 2

We remark here that the presence of a co-action of B under X along j implies the existence of the homotopy section of $\hat{p}$.

We can find out subspaces E_1, E_2 and E_0 in E as follows:

$$E_1 = \{(x, \ell_B, \ell_X) \in E \mid \ell_X(1) = *\} \supset \{(x, c(j(x)), \ell_X) \in E \mid \ell_X(1) = *\} \cong F^{1_X}, \tag{3.1}$$

$$E_2 = \{(x, \ell_B, \ell_X) \in E \mid \ell_B(1) = *\} \supset \{(x, \ell_B, c(x)) \in E \mid \ell_B(1) = *\} \cong F^{j} \quad \text{and} \tag{3.2}$$

$$E_0 = \{(x, \ell_B, \ell_X) \in E \mid \ell_B(1) = *,\ \ell_X(1) = *\} = F^{\Delta_j} \tag{3.3}$$

where $c(y)$ denotes the constant mapping at y and F^f denotes the homotopy fibre of f. Then we have $E = E_1 \cup E_2$ and $E_1 \cap E_2 = E_0$. We can easily show that $F^{1_X} \simeq \{*\}$ and $F^j \simeq \tilde{X}$ are deformation retracts of E_1 and E_2, respectively. Hence E has the homotopy type of the push-out $F^{1_X} \cup \{I \times E_0\} \cup F^j \simeq * \cup \{I \times E_0\} \cup \tilde{X}$, which is an unreduced mapping cone as shown in Figure 1.

Since E_0 is the homotopy fibre of Δ_j, we have the following exact sequence of homotopy groups (sets):

$$\cdots \to \pi_1(X) = \pi \xrightarrow{\Delta} \pi \times \pi = \pi_1(B \times X) \to \pi_0(E_0) \to *.$$

Hence we have that the set $\pi_0(E_0)$ is in one-to-one correspondence with the group π.

Therefore E_0 is a topological sum $\coprod_{\omega\in\pi} E_0^\omega$ of connected component E_0^ω that corresponds to ω. Then it follows that the unreduced mapping cone $\{*\} \cup \{I \times E_0\} \cup \tilde{X}$ is naturally homotopy equivalent to $\Sigma\pi \vee D(X)$, where $D(X)$ is the reduced mapping cone of a mapping from $\bigvee_{\omega\in\pi} E_0^\omega$ to $\tilde{X}$. Since $\bigvee_{\omega\in\pi} E_0^\omega$ is path-connected and $\tilde{X}$ is 1-connected, we obtain by Van Kampen's Theorem that

$D(X)$ is 1-connected and is natural with respect to X. Thus E has the homotopy type of $\Sigma\pi \vee D(X)$, where $\Sigma\pi$ is a wedge sum of circles. (See Figure 2.)

Using the projection $\hat{p}\colon \Sigma\pi \vee D(X) \simeq E \to X$, we can define $p\colon B \vee D(X) \to X$ by $p|_B = i$ and $p|_{D(X)} = \hat{p}|_{D(X)}$. Then by the definition of $D(X)$, p is natural with respect to X up to homotopy.

A restriction of $\hat{p}$ to a circle gives an element of the fundamental group $\pi_1(X) \cong \pi_1(B)$, and hence it factors through $i\colon B \to X$. Thus the restriction of $\hat{p}$ to $\Sigma\pi$ factors through $i\colon B \to X$. Thus the mapping $\hat{p}$ factors through $p\colon B \vee \tilde{X} \to X$ as $\hat{p} = p\hat{k}$ for some mapping $\hat{k}\colon E \simeq \Sigma\pi \vee D(X) \to B \vee \tilde{X}$.

On the other hand, $\hat{p}$ has a homotopy section $\hat{s}$, since X admits a co-action of B. By putting $s = \hat{k}\hat{s}$, we see that s is a homotopy section of p:

$$\begin{array}{ccccc} X & \xrightarrow{\hat{s}} & E & \xrightarrow{\hat{p}} & X \\ {\scriptstyle =}\downarrow & & {\scriptstyle \hat{k}}\downarrow & & \downarrow{\scriptstyle =} \\ X & \xrightarrow{s} & B \vee D(X) & \xrightarrow{p} & X \end{array}$$

This implies the theorem. □

By collapsing B in X and $B \vee D(X)$, we obtain the following

Corollary 3.4. *C is a retract of $D(X)$.*

We fix the homotopy section s of $p\colon B \vee D(X) \to X$. Let μ_0 be the composition $(1_B \vee p)(\mu_B \vee 1_{D(X)})s\colon X \to B \vee X$ where μ_B is the canonical co-multiplication of B a wedge sum of circles. Thus we have the following

Corollary 3.5. *If there exists a co-action of B under X along $j\colon X \to B$, then there is a co-action $\mu_0\colon X \to B \vee X$, such that the induced homomorphism $\mu_{0*}\colon \pi_1(X) = \pi \to \pi * \pi = \pi_1(B \vee X)$ is the standard (see Kan* [10]*) co-monoid structure of the free group π.*

Let us consider the universal covering $\tau\colon \tilde{X} \to X$.

Corollary 3.6. *For $q \geq 2$, there is a 1-connected finite complex D_q such that $H_q(\tilde{X})$ is a direct summand of $Z\pi \otimes H_q(D_q)$.*

Proof. By Theorem 3.3, $B \vee D(X)$ dominates X: $ps = 1_X$. Since the $(q+1)$-skeleton $X^{(q+1)}$ of X is a finite complex, the image $s(X^{(q+1)})$ is in a finite subcomplex $B \vee D_q$ of $B \vee D(X)$ with D_q 1-connected. On the other hand, $ps|_{X^{(q+1)}}$ is homotopic to the canonical inclusion $X^{(q+1)} \to X$. Since the universal cover $\widetilde{B \vee D_q}$ of $B \vee D_q$ has the homotopy type of a wedge sum of D_q's indexed by π, $H_q(\tilde{X})$ is a direct summand of $H_q(\widetilde{B \vee D_q}) \cong Z\pi \otimes H_q(D_q)$. □

4. Proof of the Main Theorem

Let us first consider the universal covering $\tau\colon \tilde{X} \to X$. Then the fundamental group $\pi = \langle \Lambda | \rangle$ acts on $\tilde{X}$, and hence on $H_*(\tilde{X})$, as a deck transformation group. The following is a generalization of Proposition 1.10 in [2].

Proposition 4.1.

$$\tilde{H}_q(\tilde{X})/\pi \cong \tilde{H}_q(C),$$

where we denote by $H_*(\tilde{X})/\pi$ the module $\mathbb{Z} \otimes_{\mathbb{Z}\pi} H_*(\tilde{X})$.

Proof. By Theorem 3.3, there is a homotopy section s of p which induces the following commutative diagram up to homotopy.

$$\begin{array}{ccccc} \tilde{X} & \xrightarrow{\tau} & X & \xrightarrow{j} & B \\ {\scriptstyle \tilde{s}}\downarrow & & {\scriptstyle s}\downarrow & & \downarrow{=} \\ \bigvee_{\omega\in\pi} D(X)^{\omega} & \xrightarrow{\tilde{\tau}} & B\vee D(X) & \xrightarrow{\tau_B} & B \end{array}$$

where τ_B is the first projection and τ and $\tilde{\tau}$ is the universal covering up to homotopy.

The E^1-terms of the Leray-Serre spectral sequence (see [16]) for the above fibrations j and τ_B are isomorphic to the cellular chain complexes with the local coefficients in $H_*(\tilde{X})$ and $\mathbb{Z}\pi\otimes H_*(\tilde{X})$ given as

$$E^1_{0,v}(j) = C_0(B, H_v(\tilde{X})) \cong H_v(\tilde{X}), \tag{4.1}$$

$$E^1_{0,v}(\tau_B) = C_0(B, \mathbb{Z}\pi\otimes H_v(D(X))) \cong \mathbb{Z}\pi\otimes H_v(D(X)), \tag{4.2}$$

$$E^1_{1,v}(j) = C_1(B, H_v(\tilde{X})) \cong \sum_{g\in\Lambda}\langle g\rangle H_v(\tilde{X}), \tag{4.3}$$

$$E^1_{1,v}(\tau_B) = C_1(B, \mathbb{Z}\pi\otimes H_v(D(X))) \cong \sum_{g\in\Lambda}\langle g\rangle \mathbb{Z}\pi\otimes H_v(D(X)), \tag{4.4}$$

$$E^1_{u,v}(j) = 0, \quad \text{unless } 0\le u\le 1, \tag{4.5}$$

$$E^1_{u,v}(\tau_B) = 0, \quad \text{unless } 0\le u\le 1, \tag{4.6}$$

where $v\ge 0$, Λ the set of generators of π, and the first differentials $d^1(j)$ and $d^1(\tau_B)\colon E^1_{1,v}\to E^1_{0,v}$ are given by the formulae

$$d^1(j)(\langle g\rangle\tilde{x}) = (g-1)\tilde{x}, \tag{4.7}$$

$$d^1(\tau_B)(\langle g\rangle w) = (g-1)w \tag{4.8}$$

where $\tilde{x}\in H_v(\tilde{X})$, $w\in\mathbb{Z}\pi\otimes H_v(\tilde{X})$ and $g\in\Lambda$. Since $E^1_{u,v}=0$ for $u\ge 2$, the spectral sequences collapse from the E^2-terms.

By comparing with the E^∞-term, we obtain that $d^1(\tau_B)$ is injective. By Theorem 3.3, s induces a split monomorphism of E^1-terms, and hence $d^1(j)$ is also injective. Thus we obtain the following equations:

$$E^2_{0,v}(j) = H_v(\tilde{X})/\pi, \text{ when } v\ge 0, \tag{4.9}$$

$$E^2_{1,v}(j) = H_{v+1}(B), \text{ when } v\ge 0, \tag{4.10}$$

$$E^2_{u,v}(j) = 0, \text{ unless } 0\le u\le 1 \text{ or } v<0. \tag{4.11}$$

Since the spectral sequence for j collapses, this implies the proposition. □

Remark 4.2. The above proof also shows that the augmentation ideal I of $\mathbb{Z}\pi$ can be described as the direct sum of $(g-1)\mathbb{Z}\pi$'s indexed by $g\in\Lambda$, where each summand is the injection image of $\mathbb{Z}\pi$.

Lemma 4.3. *Let M be a $\mathbb{Z}\pi$-module which is a direct summand of an induced module from a finitely generated $\mathbb{Z}$-module. Then M is itself an induced $\mathbb{Z}\pi$-module.*

Assuming Lemma 4.3, we obtain the following

Theorem 4.4. *Let X be a connected CW complex of finite type. If X admits a co-action of B along j, we have the following isomorphism of $\mathbb{Z}\pi$-modules:*

$$\tilde{H}_q(\tilde{X}) = \mathbb{Z}\pi \otimes G_q \quad \textit{as } \mathbb{Z}\pi\textit{-modules},$$

where G_q is a submodule of $\tilde{H}_q(\tilde{X})$ isomorphic to $\tilde{H}_q(C)$. And the covering projection is equivalent to the canonical projection:

$$\tilde{H}_q(\tilde{X}) = \mathbb{Z}\pi \otimes G_q \to \{\mathbb{Z}\pi \otimes G_q\}/\pi \cong G_q \cong \tilde{H}_q(C).$$

Proof. By Corollary 3.6, it follows that $H_q(\tilde{X})$ is a direct summand of an induced module $\mathbb{Z}\pi \otimes H_q(D_q)$ from a finitely generated $\mathbb{Z}$-module $H_q(D_q)$. Then by Lemma 4.3 we obtain that $H_q(\tilde{X})$ is isomorphic with $\mathbb{Z}\pi \otimes G_q$ as $\mathbb{Z}\pi$-modules for some $\mathbb{Z}$-module G_q. Here the module G_q is isomorphic with $\tilde{H}_q(C)$, by Proposition 4.1. The latter part is clear and this completes the proof of the theorem. □

Since $H_*(X) \cong H_*(C)$ for $* > 1$, Theorem 4.4 implies the main theorem. So we are left to show Lemma 4.3. First, let us consider the p-torsion part:

Proposition 4.5. *Let p be a prime and M_p a $\mathbb{Z}\pi$-module which is a direct summand of an induced module from a finitely generated p-torsion module F_p. Then M_p can be described as a direct sum: $M_p \cong M_p[1] \oplus M_p[2] \oplus ... \oplus M_p[h]$, where $M_p[i]$ is an induced $\mathbb{Z}\pi$-module from a free $\mathbb{Z}/p^i\mathbb{Z}$-module.*

Proof. By the hypothesis, M_p is a direct summand of $\mathbb{Z}\pi \otimes F_p$ and F_p can be described as a direct sum $F_p[t] \oplus F_p[t+1] \oplus ... \oplus F_p[h]$, $1 \le t \le h$, where $F_p[i]$ is a free $\mathbb{Z}/p^i\mathbb{Z}$-module.

Let us introduce a notion of *excess*: *excess*$(F_p) = 2h - t \ge 1$ if $F_p[t] \ne 0$ and $F_p[h] \ne 0$. We show the theorem by induction on excess: Let $k \ge 1$. By the induction hypothesis, we may suppose that we are done in case when $2h - t \le k - 1$. First, let us assume that $h = t$. Then $F_p = F_p[h]$ is a free $\mathbb{Z}/p^h\mathbb{Z}$-module, M_p is a projective $\{\mathbb{Z}/p^h\mathbb{Z}\}\pi$-module. Hence by Bass [1], Cohn [3] or Seshadri [15], M_p is a free $\{\mathbb{Z}/p^h\mathbb{Z}\}\pi$-module. By putting $M_p[h] = M_p$, we obtain the conclusion in this case.

Next assume that $h > t$, and hence $k > h$. We introduce $\mathbb{Z}\pi$-submodules of M_p

$$M_p(i) = \{x \in M_p \mid p^i x = 0\}, \tag{4.12}$$

$$M_p(i,j) = M_p(i)/\{p^j M_p \cap M_p(i)\}, \tag{4.13}$$

and submodules of F_p

$$F_p(i) = \{x \in F_p \mid p^i x = 0\}, \tag{4.14}$$

$$F_p(i,j) = F_p(i)/\{p^j F_p \cap F_p(i)\}. \tag{4.15}$$

Then by the hypothesis, we have $M_p(h) = M_p$.

We have that $\{\mathbb{Z}/p^i\mathbb{Z}\}\pi$-modules $M_p(i)$ and $M_p(i,j)$ are direct summands of $\mathbb{Z}\pi \otimes F_p(i)$ and $\mathbb{Z}\pi \otimes F_p(i,j)$.

Let us now consider the following commutative diagram:

$$\begin{array}{ccccc}
p^t M_p & \longrightarrow & M_p & \xrightarrow{\rho_t} & M_p/p^t M_p = M_p \otimes \mathbb{Z}/p^t\mathbb{Z} \\
\uparrow & & \uparrow & & \uparrow \\
p^t M_p \cap M_p(t) & \longrightarrow & M_p(t) & \xrightarrow{\rho'_t} & M_p(t)/\{p^t M_p \cap M_p(t)\} = M_p(t,t)
\end{array}$$

where ρ_t and ρ'_t denote the canonical projections and all other lines denote the inclusions.

Let us recall that $k = 2h - t > h$ under the hypothesis. Since $F_p(t,t)$ is given as $F_p[t] \oplus \{pF_p[t+1]\}/(p^t) \oplus \ldots \oplus \{p^{h-t}F_p[h]\}/(p^t) \cong \sum_{i=\max(1,2t-h)}^{t} F_p(t,t)[i]$, we have that $excess(F_p(t,t)) \leq 2t-(2t-h) = h < k$ and hence, by the induction hypothesis, that $M_p(t,t)$ admits a direct sum decomposition $M_p(t,t) \cong \sum_{i=1}^{t} M_p(t,t)[i]$ as $\mathbb{Z}\pi$-modules, where $M_p(t,t)[i]$ is a free $\{\mathbb{Z}/p^i\mathbb{Z}\}\pi$-module and hence an induced $\mathbb{Z}\pi$-module from a free $\mathbb{Z}/p^i\mathbb{Z}$-module.

Since $M_p(t)$ is also a $\{\mathbb{Z}/p^t\mathbb{Z}\}\pi$-module, the canonical projection from $M_p(t)$ to $M_p(t,t)[t]$ has a splitting $s_t \colon M_p(t,t)[t] \to M_p(t)$.

Let us define $M_p[t]$ as the image $s_t(M_p(t,t)[t]) \subseteq M_p(t) \subseteq M_p$, which is an induced $\mathbb{Z}\pi$-module from a free $\mathbb{Z}/p^t\mathbb{Z}$-module. Then the image $\rho_t(M_p[t])$ is nothing but $M_p(t,t)[t]$ in $M_p(t,t) \subseteq M_p/p^tM_p$, since $M_p[t]$ is in $M_p(t)$ and the restriction of ρ_t to $M_p(t)$ coincides with $\rho'_t \colon M_p(t) \to M_p(t,t)$. This implies that $M_p[t]$ is a direct summand of M_p.

Thus we obtain a direct sum decomposition $M_p \cong M_p[t] \oplus M'_p$ as $\mathbb{Z}\pi$-modules, where M'_p is the kernel of the projection from M_p to $M_p[t]$. On the other hand, we have a direct sum decomposition $F_p(h) \cong F_p[t] \oplus F'_p$, where $F'_p = F_p[t+1] \oplus \ldots \oplus F_p[h]$.

By the definition of $M_p[t]$, every generator in M'_p has the order $\geq p^{t+1}$, because a generator with the order p^t has non-zero projection in $M_p[t]$.

Let us recall that M_p is a direct summand of $\mathbb{Z}\pi \otimes F_p$, and hence there is a retraction from $\mathbb{Z}\pi \otimes F_p$ to M'_p. Since an element in $F_p[t]$ has the order $< p^{t+1}$, the retraction restricted to $\mathbb{Z}\pi \otimes F_p[t]$ is trivial after reduced modulo p. Hence the retraction restricted to $\mathbb{Z}\pi \otimes F'_p$ has to be a surjection after reduced modulo p, and hence is a surjection before reducing modulo p. Thus M'_p is a direct summand of $\mathbb{Z}\pi \otimes F'_p$, where $excess(F'_p) \leq h - 1 + h - t = k - 1 < k$.

By the induction hypothesis, we have a direct sum decomposition of M'_p:

$$M'_p \cong M_p[t+1] \oplus M_p[t+2] \oplus \ldots \oplus M_p[h], \text{ as } \mathbb{Z}\pi\text{-modules},$$

where $M_p[i]$ is an induced $\mathbb{Z}\pi$-module from a free $\mathbb{Z}/p^i\mathbb{Z}$-module for $t+1 \leq i \leq h$. Thus we obtain a direct sum decomposition of M_p: $M_p \cong M_p[t] \oplus M_p[t+1] \oplus \ldots \oplus M_p[h]$ as desired. This completes the proof of the proposition. □

Now we prove Lemma 4.3.

Proof. Let us assume that M is a direct summand of $\mathbb{Z}\pi \otimes F$, F a $\mathbb{Z}$-module. Then $M/torsion$ is a direct summand of $\mathbb{Z}\pi \otimes \{F/torsion\}$.

Since $F/torsion$ is a free $\mathbb{Z}$-module, the free part of M is a projective $\mathbb{Z}\pi$-module, and hence is free by Bass [1]. Thus we have the following isomorphism of $\mathbb{Z}\pi$-modules:

$$M/torsion \cong \mathbb{Z}\pi \otimes M_0, \ M_0 \text{ is a free } \mathbb{Z}\text{-module}.$$

To proceed, let us consider the p-torsion part M_p and F_p of M and F, respectively:

$$M \cong M_0 \oplus \sum_{p\colon \text{ all primes}} M_p, \tag{4.16}$$

$$F \cong F_0 \oplus \sum_{p\colon \text{ all primes}} F_p. \tag{4.17}$$

Since F_p is finitely generated, M_p satisfies the hypothesis of Proposition 4.5. Thus M_p is also an induced $\mathbb{Z}\pi$-module from a finitely generated p-torsion module. □

5. The Proof of Corollary 2.2

Let us assume that B is a subspace of X. Then $\pi_q(X, B)$ is isomorphic with $\pi_q(\tilde{X}, \tilde{B})$. By the assumption, the Hurewicz homomorphism $\pi_n(\tilde{X}, \tilde{B}) \to H_n(\tilde{X}, \tilde{B}) = H_n(\tilde{X})$ is an isomorphism. Hence by Theorem 2.1, $\pi_n(X, B) \cong \mathbb{Z}\pi \otimes H_n(X)$, where $H_n(X)$ can be described as a direct sum of cyclic groups $\Sigma_a \mathbb{Z}e_a \oplus \Sigma_b \mathbb{Z}/m_b\mathbb{Z}f_b$. Let $\alpha_a\colon S^n \to X$ and $\beta_b\colon S^n \to X$ be mappings corresponding to e_a and f_b.

Then $m_b f_b = 0$ implies that β_b is extendable to the Moore space $S^n(m_b) = S^n \cup_{m_b} e^{n+1}$, say $\gamma_b\colon S^n(m_b) \to X$. Hence we have a mapping $\phi_1\colon X_n = B \vee \bigvee_a {S^n}_a \vee \bigvee_b S^n(m_b) \to X$, which induces clearly an isomorphism of homology groups of the universal coverings up to dimension n. Thus ϕ_1 is n-connective.

Let us assume that X_n is a subspace of X. Then $\pi_q(X, X_n)$ is isomorphic with $\pi_q(\tilde{X}, \tilde{X_n})$. By the assumption, the Hurewicz homomorphism $\pi_{n+1}(\tilde{X}, \tilde{X_n}) \to H_{n+1}(\tilde{X}, \tilde{X_n}) = H_{n+1}(\tilde{X})$ is an isomorphism. Hence by Theorem 2.1, $\pi_{n+1}(X, X_n) \cong \mathbb{Z}\pi \otimes H_{n+1}(X)$, where $H_{n+1}(X)$ can be described as a direct sum of cyclic groups $\Sigma_{a'} \mathbb{Z}e'_{a'}$. Let $\alpha'_{a'} \in \pi_{n+1}(X, X_n)$ be mappings corresponding to $e'_{a'}$.

Let us consider the following exact sequence:

$$\cdots \to \pi_{n+1}(X) \to \pi_{n+1}(X, X_n) \xrightarrow{\partial} \pi_n(X_n) \to \pi_n(X) \to \cdots.$$

Here, by the theorem of Hurewicz, both the modules $\pi_n(X_n)$ and $\pi_n(X)$ are isomorphic with $H_n(\tilde{X})$, and hence, the above homomorphism ∂ is trivial. So, we may assume that $\alpha'_{a'}$ is a mapping from S^{n+1} to X.

Hence we have a mapping

$$\phi_2\colon X_{n+1} = X_n \vee \bigvee_{a'} {S^{n+1}}_{a'} = B \vee \bigvee_a {S^n}_a \vee \bigvee_b S^n(m_b) \vee \bigvee_{a'} {S^{n+1}}_{a'} \to X,$$

which induces a homology equivalence of the universal coverings. Thus ϕ_2 is a homotopy equivalence, which completes the proof of the corollary.

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