SYMMETRY OF PROPERLY EMBEDDED
SPECIAL WEINGARTEN SURFACES IN $\mathbf{H}^3$

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ABSTRACT. In this paper we prove some existence and uniqueness results about special Weingarten surfaces in hyperbolic space.

1. INTRODUCTION

In this paper we consider C^2 surfaces M in $\mathbf{H}^3$ oriented by a global unit normal field N whose mean curvature H and extrinsic Gaussian curvature K_e satisfy a Weingarten relation of the form

$$(1) \qquad H = f(H^2 - K_e).$$

We shall require that f is a C^1 function defined on $[0, +\infty[$, satisfying

$$(2) \qquad \forall t \in [0, +\infty[, \ 4tf'^2(t) < 1.$$

We will say that f is elliptic if f satisfies inequality (2). If M satisfies relation (1) for f elliptic, we will call M either a *special Weingarten surface* or an *f-surface*. In euclidean space they have been studied by Hopf [10], Hartman and Wintner [9], Chern [5]. More recently, there has been great progress, as much in the euclidean case as in the hyperbolic, Bryant [4], Braga Brito [3], Rosenberg [15]. In section 2 we will briefly describe this and we will develop the fundamental basis of the theory. Both authors have constructed a family of embedded (and also a family of non-embedded) complete rotational Weingarten surfaces in $\mathbf{R}^3$. They have also proved uniqueness of such surfaces (see [16], [17] and [18]).

An interesting question in this subject is whether or not an f-surface inherits the symmetries of its boundary: When $f = H$ (constant) and $\mid H \mid \leq 1$, Nelli and Rosenberg [14] solved the boundary value problem for compact embedded surfaces and spherical boundary. The sharp related theorem for M *immersed* was proved by Barbosa and the first author in [2]. Here, we shall generalize Rosenberg and Nelli's result to f-surfaces when $f^2 \leq 1$.

In [1], it is stated that when the boundary consists of two circles invariant under a 1-parameter group of rotations of $\mathbf{H}^3$, then stability implies the surface is also invariant under rotations. Barbosa and the first author have asked whether the assumption of stability could be replaced by embeddedness [1]. In this direction, we are able to obtain the following result: *Assume $\partial M = C_1 \cup C_2$, where C_1 and C_2 are circles of the same radius. Suppose that ∂M is invariant under a rotation. Then there is a positive real number d_0 such that if $\mid H \mid = 1$ and $dist(C_1, C_2) \geq 2d_0$,*

then M is a piece of a Catenoid Cousin. In fact, we prove an analogous result for a wider class of f-surfaces (see Theorem 1, section 3).

We mention that symmetry results proved by several authors concerning H-surfaces whose asymptotic boundary consists of a point, a circle or the union of two circles ([6], [7], [13]) are valid for f-surfaces, since the maximum principle still holds (see section 2). On the other hand, others characterizations of f-surfaces with boundary a circle, proved in [3] when the ambient space is euclidean space, are also stated in section 2.

Finally, we will give existence and uniqueness results for rotational f-surfaces in section 4, under further hypotheses on f. Moreover, we will prove that the geometric behaviour of such surfaces is the same as that of the related well-known rotational H-surfaces in $\mathbf{H}^3$ (see Theorems 2 and 3).

2. Some basic properties: the structure of principal lines and the maximum principle. Immediate consequences and generalizations

In this section we shall establish some basic properties of special Weingarten surfaces. We will state some results of Bryant's theory [4] on the structure of the principal line distribution and we will write down the maximum principle. We will point out that one can apply Alexandrov techniques to obtain immediate consequences generalizing well-known results for the case $H = \text{const}$.

2.1. Principal lines distribution (R. Bryant's theory). Let M be an immersed special Weingarten surface satisfying (1) with respect to a global unit normal field N.

In a pioneering paper, R. Bryant had an amazing idea of constructing a new metric ds^2 by means of the function f given in the Weingarten relation (1) to obtain constant mean curvature type results, as follows:

Proposition 1 ([4]). *Let z be an isothermal coordinate of the Bryant metric ds^2 in U. If $ds^2 = \lambda^2 \mid dz \mid^2$ then the function $\phi(z) = \lambda^2(z)\left(\frac{l-n}{2} - im\right)$ is holomorphic, where l, m, and n are the coefficients of the second fundamental form with respect to a local oriented orthonormal frame field. Furthermore, the holomorphic quadratic form $\Phi(z) = \phi(z)(dz)^2$ is defined globally on M.*

Proposition 2 ([4]). *Either M is totally umbilic, or else the umbilic locus consists entirely of isolated points of strictly negative index.*

We remark that when f is analytic Proposition 2 follows from the proof of theorem 3.2 on p. 142 of Hopf's famous book [10]. Notice that z_0 is an umbilic point of M if and only if $\phi(z_0) = 0$. Thus, in a neighborhood of $z_0 = 0$ the holomorphic quadratic form is $z^n(dz)^2$; hence the index at z_0 is $\frac{-n}{2}$.

We emphasize that R. Bryant proved (in view of Propositions 1, 2) the following generalization of Hopf's theorem: *A closed genus zero special Weingarten surface satisfying (1) is a round sphere.*

Remark 1. We can derive from Bryant's construction and straightforward computations that the principal line distribution is given by $\text{Im}\,\Phi = 0$.

2.2 Maximum principle. In this section we will establish interior and boundary maximum principles for special Weingarten surfaces in hyperbolic space. We shall apply these maximum principles in subsequent sections to derive several uniqueness and symmetry results.

Lemma 1 (Interior maximum principle). *Suppose M_1, M_2 are C^2 surfaces in $\mathbf{H}^3$ which are given locally as graphs of C^2 functions u, v. Suppose the tangent planes of both M_1 and M_2 agree at a point p.*

Let $H(N_1)$ and $H(N_2)$ be the mean curvature functions of u and v with respect to unit normals N_1 and N_2 that agree at p. Let K_i be the extrinsic Gaussian curvature of $M_i, i = 1, 2$. Suppose the M_i satisfy

$$H(N_i) = f({H_i}^2 - K_i), \qquad i = 1, 2,$$

for f elliptic.

Then, if $u \leq v$ near p, also $M_1 = M_2$ near p.

Lemma 2 (Boundary maximum principle). *Consider M_1, M_2 as in the statement of the interior maximum principle with C^2 boundaries B_1, B_2 given by restritions of u and v. Suppose $T_p M_1 = T_p M_2$ and $T_p B_1 = T_p B_2$ with p in the interior of both B_1 and B_2. Suppose M_1, M_2 satisfy (1) and (2) with respect to the same normal N at p.*

Then $u \leq v$ near p implies $M_1 = M_2$ near p.

The analogous statement in euclidean space is proved in [3], so we omit the proof here.

2.3. Immediate consequences. In this section we will derive some applications extending known results of the constant mean curvature theory.

Proposition: Alexandrov theorem for special Weingarten surfaces in hyperbolic space. *If M is a closed embedded special Weingarten surface in $\mathbf{H}^3$ satisfying (1) for f elliptic, then M is a sphere.*

The proof makes use of the Alexandrov reflection principle, with totally geodesic hyperbolic planes in place of ordinary planes in the Euclidean case. For Alexandrov techniques the reader is referred to [3], [13] and [15].

The Alexandrov reflection principle is a beautiful and fundamental tool for several nice theorems in the theory of constant mean curvature surfaces. It provides several analogous results for special Weingarten surfaces. At this point, we will make use of the unit ball model for the hyperbolic space $\mathbf{H}^3$. Thus, $\overline{\mathbf{H}^3}$ will denote the closed unit ball and $S^2(\infty)$ will denote the unit sphere. We will recall some basic definitions given in [7]. Let M be an embedded surface in $\mathbf{H}^3$. We denote by $\partial_\infty M$ the intersection of the closure of M with $S^2(\infty)$, and we will call $\partial_\infty M$ the asymptotic boundary of M. We will say M is C^2-*regular* at infinity if $\overline{M} \subset \overline{\mathbf{H}^3}$ is a C^2 surface with boundary in $\overline{\mathbf{H}^3}$ and $\partial_\infty M$ is a C^2 curve of $S^2(\infty)$.

Now we set up the following symmetry statements. *Let M be a connected complete properly embedded surface in $\mathbf{H}^3$. Suppose M is a special Weingarten surface satisfying (1) for f elliptic. Assume M is C^2-regular at infinity. Then:*

(1) *If the asymptotic boundary is one point, then M is a horosphere.*
(2) *If the asymptotic boundary is one circle, then M is a geodesic plane or a equidistant surface.*

(3) *If the asymptotic boundary is the union of two disjoint circles, then M is a surface of revolution.*

The proof is an application of the Alexandrov principle as in the constant mean curvature case, so we omit the proof here. Let us give the references for the above statements in the case $H =$ const. If $H = 0$ and M is immersed, statement 3 was proved by Levitt and Rosenberg [13]. Statements 1 and 2 was derived by Do Carmo and Lawson [6], while statement 3 for $H = \text{const} \neq 0$ was deduced by Do Carmo, Gomes and Thorbergsson [7].

Another amazing symmetry result for $H =$ const is the Hsiang theorem, which implies that if M is cylindrically bounded, complete and embedded then M has rotational symmetry [11]. We now assert the following: *Let M be a connected properly embedded complete special Weingarten surface in $\mathbf{H}^3$ satisfying (1) for f elliptic. If M is cylindrically bounded then M is a surface of revolution.*

The proof is similar to the constant mean curvature case.

3. Symmetry arising from the boundary

In this section we shall establish several symmetry results for special Weingarten surfaces arising from the symmetry of the boundary. Some of them are not known even in the constant mean curvature case. Others are generalizations of well-know results for $H =$ const or simple extensions of results obtained recently in euclidean space.

Let C_1, C_2 be two circles in $\mathbf{H}^3$ with the same radius. Suppose $C_1 \cup C_2$ is invariant under a rotation; that is, C_1, C_2 are on the same cylinder C. We will denote by $\text{int}(C)$ the component of $\mathbf{H}^3 - C$ containing the axis of C. The distance $d(C_1, C_2)$ between C_1 and C_2 is the distance between the "parallel" geodesic planes determined by C_1 and C_2.

It is well-known (see [11]) that if $C_1 \cup C_2$ is the boundary of a minimal surface of revolution then $d(C_1, C_2) \leq 2d_0$, where d_0 is the maximum value of the function $x(y_0)$ given by

$$x(y_0) = \int_{y_0}^{+\infty} \frac{\sinh y_0 \cosh y_0}{\cosh y} \sqrt{\frac{1}{\sinh^2 y \cosh^2 y - \sinh^2 y_0 \cosh^2 y_0}} \, dy.$$

Clearly the above function is bounded, and numerical computation shows that $d_0 \approx 0.5$. (The above formula follows from the first integral of the second order differential equation satisfied by the generating curve [11].)

From now on, we will denote by d_0 the positive real number defined above.

We shall need the following lemma:

Lemma 3. *Let M be a compact connected embedded special Weingarten surface in $\mathbf{H}^3$ satisfying (1) for f elliptic. Assume ∂M is the union of two distinct circles C_1, C_2 lying in a cylinder C of axis γ. Then, if $M \subset \overline{\text{int}(C)}$ and $M \cap C = \partial M$, M has a plane of symmetry.*

Moreover, if f does not change sign and if $d(C_1, C_2) \geq 2d_0$, then M lies between the parallel planes containing C_1 and C_2.

Proof. We denote by P the plane of symmetry of $C_1 \cup C_2$, that is, the plane such that C_1 is the symmetric image of C_2 with respect to P. We will prove P is a plane of symmetry of M as follows: We start by moving P along γ by doing hyperbolic translations. This movement gives rise to a 1-parameter family $\{P_t\}$ of

geodesic planes starting from $P = P_0$ and cutting γ orthogonally. We may choose a parameter t such that t is the oriented distance between P_t and P. We claim that our assumptions in the first statement allow us to apply Alexandrov reflection, by the means of the family $\{P_t\}$, to conclude that M inherits the symmetry of ∂M. Actually, to explain how to do this, we will denote by $M_t{}^*$ the reflection on P_t for $t \neq 0$ of the components of $M - P_t$ lying in the connected component of $\mathbf{H}^3 - P_t$ not containing P. We begin the standard procedure by moving P until P_t is disjoint of M, i.e., $P_t \cap M = \emptyset$. Then moving P_t back towards P, doing Alexandrov reflection during this movement, we find a first point of contact between M and $M_t{}^*$. Since C is invariant under reflection on P_t, then, if by absurd $P_t \neq P$, this first point of contact cannot occurs at a boundary point of ∂M, because $\partial M \subset C$ and $M \cap C = \partial M$. So, if $t \neq 0$ we get a tangent point of contact, and both M and $M_t{}^*$ have the correct orientation at this point. We arrive to a contradiction, for the maximum principle says that $P_t, t \neq 0$, is a plane of symmetry of M. Then $P_t = P$, that is, P is a plane of symmetry of M, as required.

To prove the second statement we will proceed as follows. Let P_{d_1} and P_{-d_1}, $d_1 \geq d_0$, be the geodesic planes passing through C_1 and C_2. Denote by $\mathcal{H}$ the region bounded by P_{d_1} and P_{-d_1}, and by $P_t{}^+$ the connected component of $\mathbf{H}^3 - P_t$ not containing P. Under those conventions, notice that $M \cup (P_{d_1}^+ \cap C) \cup (P_{-d_1}^+ \cap C)$ is the boundary of a region V in $\mathbf{H}^3$; V is contained in $\overline{\mathrm{int}(C)}$, and ∂V is not smooth over ∂M. Note that, since f does not change sign, either the mean curvature vector $\overrightarrow{H}$ of M is pointing into the interior of V or else $\overrightarrow{H}$ is pointing towards the exterior of V. We claim $\overrightarrow{H}$ is an inward pointing normal vector. Indeed, set $p = \gamma \cap P$ and let $\alpha \subset P$ be a geodesic line cutting γ orthogonally at p. Let $\{\zeta_t\}$ be the 1-parameter family of minimal surfaces of revolution with axis γ and generating curve c_t such that α is the symmetry line of c_t and $t = d(\zeta_t, \gamma)$. Note that, as $d(C_1, C_2) \geq 2d_0$, the family $\{\zeta_t\}$ is inside $\mathcal{H}$. To prove the claim, we move ζ_t from infinity towards M, making $t \to 0$, to reach a first interior point of contact. By the comparison principle $\overrightarrow{H}$ points into V, as required. To conclude the proof, if $M \cap \mathrm{ext}(\mathcal{H}) \neq \emptyset$, one derives a contradiction by the same argument as in the preceding paragraph, making use of the family $\{P_t\}$ coming from infinity towards $\partial \mathcal{H}$.

We remark that there always exists an umbilic surface in the class of any Weingarten surface satisfying (1) for f elliptic: If $f(0) = 0$, any geodesic plane is in the class. If $f^2(0) = 1$, any horosphere is in the class. If $f^2(0) < 1$, any equidistant surface with mean curvature $f(0)$ is in the class. If $f^2(0) > 1$, any sphere of radius ρ with $\coth \rho = |f(0)|$ is in the same class of M.

Theorem 1. *Let M be a connected properly embedded special Weingarten surface in $\mathbf{H}^3$ satisfying (1) for f elliptic. Assume $\partial M \neq \emptyset$. Suppose $f^2 \leq 1$. We have the following:*

(1) If $\partial_\infty M = \emptyset$ and ∂M is a circle, then M is totally umbilic.

(2) Assume that M is compact and ∂M is the union of two circles C_1, C_2 of the same radius with $C_1 \cup C_2$ invariant under a rotation. If $d(C_1, C_2) \geq 2d_0$, $|f(0)| = 1$, and f does not change sign, then M is part of a complete embedded special surface of revolution. Furthermore, the generating curve attains one and only one global minimum , and each end is asymptotically umbilic in the sense of Theorem 2, section 4.

Proof of Statement 1. We suppose $\partial_\infty M = \emptyset$, and then M is compact.

Let P be the geodesic plane in $\mathbf{H}^3$ containing ∂M. We consider the halfspace model $\mathbf{H}^3 = \{(x, y, z), z > 0\}$. Without loss of generality, we can suppose $P = \{x = 0\}$. Let H_0 and $H_0{}^*$ be the unique horospheres in $\mathbf{H}^3$ such that $H_0 \cap P = H_0{}^* \cap P = \partial M$. Denote by D_0, $D_0{}^*$, $D_0 \subset H_0$, $D_0{}^* \subset H_0{}^*$, the compact pieces with boundary ∂M. It follows from [2] that M is inside the region of $\mathbf{H}^3$ with boundary $D_0 \cup D_0{}^*$. Now, we claim M is contained in one side of P. Indeed, if $f(0) = 0$, the maximum principle shows that M is the geodesic disk $D \subset P$ whose boundary is equal to ∂M. But if $f(0) \neq 0$, we will arrive to a contradiction by the following argument. Take the family $\{S\}$ of umbilic surfaces contained in $\mathbf{H}^3 - P$ which is in the same class of M, i.e., $H = f(0)$, by choosing an appropriate normal field N. We also require that the mean curvature vector of each surface S points into the region of $\mathbf{H}^3 - S$ not containing P (this makes sense since $f(0) \neq 0$). Now it is clear that if $M \cap \{x < 0\} \neq \emptyset$ and $M \cap \{x > 0\} \neq \emptyset$, we may compare of M with a member S of the family in a suitable way to obtain a contradiction by the maximum principle. This proves the previous claim, and we conclude M is in one side of P. Finally, the standard Alexandrov principle, using the family of geodesic planes orthogonal to P, allows us to conclude that M inherits the symmetry of ∂M, i.e., M is invariant under a 1-parameter group of rotations. Clearly, the curvature lines are meridians and parallels. Note that either M is totally umbilic or else the singularities of the lines of curvature are isolated, by Bryant's theorem (see Proposition 2). Also note that the only possible singularities of the lines of curvatures are the points on the rotation axis, and there the singularity has nonnegative index. Consequently M is totally umbilic.

Proof of Statement 2. Let C be the cylinder with axis γ containing $C_1 \cup C_2$. Let $\{C_t\}$, $t \geq 0$, be the 1-parameter family of cylinders with the same axis γ, with $t = d(C_t, \gamma)$. We assert that $M \subset \overline{\text{int}(C)}$ and $M \cap C = \partial M$. Indeed, if this is not the case, then as $t \to +\infty$ one may find a "last" cylinder C_{t_1} such that $C_{t_1} \cap M \neq \emptyset$ and $C_t \cap M = \emptyset$ for $t > t_1$. If $M \cap (\mathbf{H}^3 - \overline{\text{int}(C)}) \neq \emptyset$, then $t_1 > 0$; hence we get a contradiction with the standard comparison principle (because the mean curvature of M is not greater than 1). For the same reasons, $M \cap C = \partial M$. It can be inferred from a straightforward application of the Alexandrov principle and Lemma 3 that M is a surface of revolution whose generating curve attains a local minimum. Suppose without loss of generality that γ is the z-axis (in the upper halfspace model). Assume C_1, C_2 belong to the horospheres $\Theta_{z_1} = \{z = z_1\}$ and $\Theta_{z_2} = \{z = z_2\}$, respectively, with $z_1 < z_2$. Consider the 1-parameter family of horospheres $\Theta_t = \{z = t\}$ for $t > z_2$. Recall that we may move this family along γ by making use of hyperbolic translations. Now, doing $t \downarrow z_2$ coming from infinity towards M, it follows from the maximum principle and from the proof of Lemma 3 that M is entirely contained inside the region $\{z \leq z_2\}$ of $\mathbf{H}^3$. By taking Lemma 3 into account again, one may conclude that M is still inside the region $\{z \geq z_1\}$. Finally, doing Alexandrov reflection, one may derive our previous claim, as desired. Recall that it follows from the proof of Lemma 3 that $\overrightarrow{H}$ is an inward pointing normal vector. Thus we have $H = f \leq 0$ on M with respect to the outward normal orientation. Therefore we are able to apply Theorem 2, section 4, to conclude that the generating curve attains one and only one global minimum since $f(0) = -1$. Furthermore in Theorem 2, section 4, it is proved that each end of M is asymptotically umbilic. This concludes the proof of Theorem 1.

Remark 2. Statement 1 of Theorem 1 when $H = \text{const}$ and $H^2 \leq 1$ was obtained in [14]; the sharp result (assuming M immersed) followed in [2]. The general situation for $H = \text{const}$, $H^2 > 1$, as in the euclidean case, is still not known. Partial results were obtained in [2]. See also J. L. Barbosa and R. Sa Earp, *Prescribed mean curvature hypersurfaces in H^{n+1} with convex planar boundary.* II, Séminaire de Théorie Spectrale et Géométrie, Vol. 16, Univ. Grenoble, 1998, pp. 43–79. CMP 99:07.

The following corollary is immediate:

Corollary 1. *Let M be a compact embedded surface in $\mathbf{H}^3$ with constant mean curvature 1. Suppose ∂M is the union of two circles C_1, C_2 of the same radius, with $C_1 \cup C_2$ invariant under a rotation.*
If $d(C_1, C_2) \geq 2d_0$, then M is a piece of a Catenoid Cousin.

Proposition 3. *Let M be a compact embedded special Weingarten surface in $\mathbf{H}^3$ satistying (1) for f elliptic. Assume there exists a cylinder C such that $\partial M \subset C$ and $M \subset \overline{\text{int}(C)}$. Suppose f satisfies, $f < 0$, $f(t) + \sqrt{t+1}$ is an increasing function for $t \geq 0$, and $f(c) + \sqrt{c+1} = 0$, where $H = -\sqrt{c+1}$ is the mean curvature of C with respect to the outward normal orientation (that is, C is f-special; see Remark 4).*
Then if ∂M is the union of two circles C_1, C_2 invariant under the group of rotations of C, and if $d(C_1, C_2) \geq 2d_0$, M is part of a complete embedded periodic special surface of revolution.

Proof. Let us consider the half-space model of $\mathbf{H}^3$. First, notice that our assumptions imply that C is the *unique* cylinder in the same class of M i.e., C satisfies the Weingarten relation (1). Thus, ellipticity yields $M \cap C = \partial M$. Since our hypotheses imply $f(0) < -1$, we may use spheres of the same class as M, above M, to show as in Theorem 1 that M lies between the parallel euclidean planes containing C_1 and C_2 respectively (which are hyperbolic horospheres, denoted by Θ_{z_1} and Θ_{z_2} in the proof of Theorem 1). So we may apply Lemma 3 and the maximum principle to show as in statement 2 of Theorem 1, that M is invariant under the group of rotations keeping C invariant. Furthermore, since Lemma 3 shows that M has a plane of symmetry, we deduce that the graph generating M has a local minimum. By Remark 6, section 4, we conclude M is part of a complete embedded periodic special surface of revolution.

To conclude this section we shall figure out a characterization of geodesic disks of spheres proved by Braga Brito and the first author in [3], when the ambient space is euclidean space. The proof is the same in both euclidean and hyperbolic space, so we will omit it here. *Let M be an immersed disk type special Weingarten surface in $\mathbf{H}^3$ satisfying (1) for f elliptic. Assume ∂M is a circle of radius ρ. Suppose f is analytic with $f(0) > 0$. Then:*
   a) *$f(0) \leq \coth \rho$.*
   b) *If $f(0) = \coth \rho$, M is a half sphere.*

Remark 3. There is another characterization with same statement and proof in both euclidean and hyperbolic space derived in [3]: *If M is a disk type embedded special Weingarten surface in $\mathbf{H}^3$ satisfying (1) for f elliptic, if ∂M is a circle of radius ρ, if f is positive $f > 0$, and if M cuts transversely, along ∂M, the geodesic plane P containing ∂M, then M is a geodesic disk of a sphere.*

4. Rotational special Weingarten surfaces. Existence and uniqueness

In this section, we are interested in special Weingarten surfaces of revolution in the hyperbolic space $\mathbf{H}^3$. We choose the model of the 3-ball; hence our surfaces are generated by a curve of the 2-hyperbolic disc $\mathbf{D} = \{(u,v) \in \mathbf{R}^2 \mid u^2 + v^2 < 1\}$. We choose the following coordinates (x,y), x, $y \in \mathbf{R}$, of $\mathbf{D}$: Let $p \in \mathbf{D}$, and let γ be the unique geodesic passing through p and orthogonal to the horizontal geodesic $\{v = 0\}$. Let q be the intersection between the two geodesics. Then x is equal to the oriented hyperbolic distance between q and 0, and y is the oriented hyperbolic length of γ between p and q. This means that $x \geq 0$ (resp. $y \geq 0$) if and only if $u \geq 0$ (resp. $v \geq 0$), where (u,v) are the euclidean coordinates of p. Note that the coordinate curves $\{x = \text{const}\}$ are the geodesics orthogonal to the u-axis and $\{y = y_0\}$ are the equidistant-curves of the u-axis, namely the arcs of circles passing through the points $(-1,0)$ and $(1,0)$ making an angle α with $\partial\mathbf{D}$ such that $\cos(\alpha) = \tanh(y_0)$. It is well-known that the metric of $\mathbf{D}$ with (x,y)-coordinates is

$$ds^2 = \cosh^2(y)dx^2 + dy^2.$$

Now, let $\gamma \subset \mathbf{D}$ be a curve and let us call M the revolution surface generated by γ, where the revolution axis is always the x-axis (which is also the u-axis). Suppose that γ is the graph of a positive C^2 function $y = y(x)$. By the *outward normal orientation* on M we mean the orientation given on γ by the unit normal field pointing toward the direction of increasing y. Straightforward computations show that the principal curvatures of M with respect to the outward normal orientation are

$$\lambda_1(x) = \frac{y'' \cosh(y) - 2\sinh(y)y'^2 - \sinh(y)\cosh^2(y)}{(\cosh^2(y) + y'^2)^{3/2}},$$

$$\lambda_2(x) = -\frac{\cosh^2(y)}{\sinh(y)(\cosh^2(y) + y'^2)^{1/2}}.$$

Note that $\lambda_1(x)$ is the hyperbolic curvature of γ as a planar curve. From now on we will always assume that f is an elliptic function (see equation (2)). Let M be a surface of $\mathbf{H}^3$ and let N be a normal unit field over M. Recall that M is an f-surface (with respect to N) if and only if the principal curvatures of M satisfy the relation (see equation (1)):

$$\frac{\lambda_1 + \lambda_2}{2}(N) = f([\frac{\lambda_1 - \lambda_2}{2}]^2).$$

Recall that for every elliptic function f any totally umbilic hypersurfaces with mean curvature $f(0)$ are f-surfaces. Namely, if $\mid f(0) \mid > 1$ they are the compact spheres of hyperbolic radius r with $\coth(r) = \mid f(0) \mid$, and if $\mid f(0) \mid \leq 1$ they are the intersections of spheres with the 3-ball which intersect the boundary $\partial\mathbf{H}^3$ at an angle α such that $\cos(\alpha) = f(0)$.

Now, let M be a surface of revolution of $\mathbf{H}^3$ and let γ be the plane curve generating M. Let us assume that γ is the graph of a positive C^2 function y. Then M

is an f-surface if and only if f satisfies the relation $F(y, y', y'') = 0$, where

$$F(y, y', y'')$$
$$= \frac{y'' \cosh(y) \sinh(y) - (2 \sinh^2(y) + \cosh^2(y))y'^2 - \cosh^2(y)(\sinh^2(y) + \cosh^2(y))}{2 \sinh(y)(\cosh^2(y) + y'^2)^{3/2}}$$
$$- f([\frac{y'' \cosh(y) \sinh(y) + (1 - \sinh^2(y))y'^2 + \cosh^2(y)}{2 \sinh(y)(\cosh^2(y) + y'^2)^{3/2}}]^2).$$

From the ellipticity of f, we deduce that $\partial F/\partial y'' > 0$, so F is a strictly increasing function of the third variable.

Proposition 4. *Let f be an elliptic function. Then there exists an f-surface of revolution, possibly noncomplete, if and only if f satisfies the following condition:*

$$\lim_{t \to +\infty} (t - f(t^2)) > 0.$$

Proof. Suppose there exists an f-surface of revolution M. We can assume that part of M is generated by the graph of a strictly positive function $y(x)$. It follows that there exist real numbers $y_0 > 0$, y_0' and y_0'' such that $F(y_0, y_0', y_0'') = 0$. But this is equivalent to

$$t - f(t^2) = \frac{\cosh^2(y_0)}{\sinh(y_0)(\cosh^2(y_0) + y_0'^2)^{1/2}},$$

where

$$t = \frac{y_0'' \cosh(y_0) \sinh(y_0) + (1 - \sinh^2(y_0))y_0'^2 + \cosh^2(y_0)}{2 \sinh(y_0)(\cosh^2(y_0) + y_0'^2)^{3/2}}.$$

As f is elliptic, we know that $(t - f(t^2))$ is a strictly increasing function. Hence

$$\lim_{t \to +\infty} (t - f(t^2)) > \frac{\cosh^2(y_0)}{\sinh(y_0)(\cosh^2(y_0) + y_0'^2)^{1/2}},$$

and so $\lim_{t \to +\infty}(t - f(t^2)) > 0$.

Conversely, assume that $\lim_{t \to +\infty}(t - f(t^2)) > 0$. Note that

$$\lim_{t \to -\infty} (t - f(t^2)) < \lim_{t \to +\infty} (t - f(t^2)).$$

Clearly, there exist strictly positive real numbers y_0, $y_0' > 0$ such that

$$\lim_{t \to -\infty} (t - f(t^2)) < \frac{\cosh^2(y_0)}{\sinh(y_0)(\cosh^2(y_0) + y_0'^2)^{1/2}} < \lim_{t \to +\infty} (t - f(t^2)).$$

On the other hand, $F(y_0, y_0', .)$ is an increasing function. Therefore, we conclude from the ellipticity that

$$\lim_{s \to -\infty} F(y_0, y_0', s) < 0 \quad \text{and} \quad \lim_{s \to +\infty} F(y_0, y_0', s) > 0.$$

This let us conclude that there exists a real number y_0'' such that $F(y_0, y_0', y_0'') = 0$. Then, using the implicit function theorem, we can prove that there exists a real C^1 function h, defined on a neighborhood of (y_0, y_0'), satisfying

$$F(y, y', y'') = 0 \Leftrightarrow y'' = h(y, y'), \quad \text{and} \quad h(y_0, y_0') = y_0''.$$

Now Picard's theorem shows that there exists a unique function $y(x)$ satisfying the last ODE with the initial conditions $y(0) = y_0$ and $y'(0) = y_0'$. Clearly, the graph of this function generates an f-surface, as desired.

From now on, we only consider the outward normal orientation for a graph of $\mathbf{D}$, and we will always suppose that $f(0) \leq 0$. For later use, we state the following lemmas.

Lemma 4. *Let f be an elliptic function and let y be a C^2 real function whose graph γ generates an f-surface M. Suppose that $y'(x_0) = 0$ for a real number x_0. Then γ is symmetric with respect to the geodesic orthogonal to γ at the point $(x_0, y(x_0))$ (hence M is symmetric with respect to the totally geodesic hyperplane of $\mathbf{H}^3$ orthogonal to $\mathbf{D}$ which contains this geodesic).*

Proof. Observe that the functions y and $z(x) = y(2x_0 - x)$ satisfy the same second order differential equation with the same initial conditions at x_0, since $y(x_0) = z(x_0)$ and $y'(x_0) = 0 = z'(x_0)$. Actually (1) can be written locally in the form $y'' = h(y, y')$, see the proof of Proposition 4. So we have $y = z$.

Lemma 5. *Let f be an elliptic function and let y be a C^2 real function whose graph γ generates an f-surface M. Suppose that M has an umbilic point. Then M is totally umbilic.*

Proof. This is a direct consequence of Proposition 2, section 2. If M admits an umbilic point outside the revolution axis, then M has at least a circle of umbilic points. Since Proposition 2, section 2, shows that either M is totally umbilic or umbilic points are isolated, we conclude in this case that M is totally umbilic. If the umbilic point stays on the x-axis, note that the index of this umbilic point (if it is isolated) is $+1$, but Proposition 2, section 2, shows that this index should be negative and we again conclude that M is totally umbilic.

We remark that we can give a direct proof of Lemma 5. For this, it suffices to show that there is a C^2 curve tangent to γ at the umbilic point generating a totally umbilic surface with constant mean curvature $f(0)$ with respect to the outward normal orientation.

Lemma 6. *Let f be an elliptic function and let y be a C^2 positive monotone function whose graph generates a non-totally-umbilic f-surface M of $\mathbf{H}^3$. Then*

$$\lambda_2' = y' \coth(y)(\lambda_1 - \lambda_2).$$

Consequently, the principal curvatures of M also are monotone and λ_1 is increasing if and only if λ_2 is decreasing.

Proof. A simple computation shows that the principal curvatures of M satisfy the formula in the statement. Then, Lemma 5 shows that λ_2 is a monotone function. Moreover, by differentiating the relation (1) we obtain

$$\lambda_1' + \lambda_2' = (\lambda_1' - \lambda_2').(\lambda_1 - \lambda_2).f'([\frac{\lambda_1 - \lambda_2}{2}]^2).$$

Using the ellipticity of f, we infer that

$$\mid \lambda_1' + \lambda_2' \mid < \mid \lambda_1' - \lambda_2' \mid,$$

which finishes the proof.

Now, recall that a cylinder of $\mathbf{H}^3$ is a complete surface of revolution generated by the graph of a constant function $y(x) = \tau > 0$, which will be denoted by C_τ.

Note that the principal curvatures of C_τ are $\lambda_1 = -\tanh(\tau)$ and $\lambda_2 = -\coth(\tau)$. Then, if f is an elliptic function, the cylinder C_τ is a f-surface if and only if

$$\frac{\tanh(\tau) + \coth(\tau)}{2} + f([\frac{\tanh(\tau) - \coth(\tau)}{2}]^2) = 0.$$

Proposition 5. *Let f be an elliptic function with $f' \geq 0$. Then there exists an f-special cylinder if and only if $f(0) < -1$. Furthermore, this cylinder is unique.*

Proof. As f is elliptic, the function $t + f(t^2)$ is increasing. Moreover, as f is increasing, the function

$$g(t) = \frac{\tanh(t) + \coth(t)}{2} + f([\frac{\tanh(t) - \coth(t)}{2}]^2)$$

is decreasing. Let us assume first that $f(0) < -1$. Note that $\lim_{t\to+\infty} g(t) = 1 + f(0) < 0$ and $\lim_{t\to0} g(t) = \lim_{t\to+\infty}(t + f(t^2)) = +\infty$. So, there exists a unique positive real number τ satisfying $g(\tau) = 0$, i.e., there exists a unique f-special cylinder.

Conversely, suppose there exists an f-special cylinder C_τ. Consequently $g(\tau) = 0$, and we deduce that (as f is increasing)

$$f(0) \leq -\frac{\tanh(\tau) + \coth(\tau)}{2} < -1.$$

As before, we may conclude that C_τ is the unique f-special cylinder.

Remark 4. Note that in the euclidean case ellipticity of f ensures uniqueness of the f-special cylinder (see [18]). In the hyperbolic case, if we do not assume f increasing, there may exist many, even infinitely many, f-special cylinders. To see this, set

$$X = X(t) = [\frac{\tanh(t) - \coth(t)}{2}]^2.$$

We then have $g(t) = h(X) = \sqrt{X+1} + f(X)$ with $X \in [0, +\infty[$. Observe that each zero of h gives an f-special cylinder. Moreover, we can write h as an integral:

$$h(X) = \int_0^X (f'(t) + \frac{1}{2\sqrt{t+1}})dt + f(0) + 1.$$

Then

$$h(X) = 0 \Leftrightarrow \int_0^X (f'(t) + \frac{1}{2\sqrt{t+1}})dt = -(f(0) + 1).$$

Now observe that the last integral is the (oriented) area of the finite plane region limited by the graph of the two functions $f'(t)$ and $-\frac{1}{2\sqrt{t+1}}$ for t between 0 and X. Also, f elliptic means only that the graph of f' stays between the graph of the two functions $\frac{1}{2\sqrt{t}}$ and $-\frac{1}{2\sqrt{t}}$. So it is not difficult to find elliptic functions f such that the associated function h has as many zeros as we want (even infinitely many). Finally, observe that the weaker condition $f'(t) \geq -\frac{1}{2\sqrt{t+1}}$ ensures uniqueness of the f-special cylinder, since the function h is increasing.

Theorem 2. *Let f be an elliptic function with $f(0) \leq 0$ and $\lim_{t\to+\infty}(t + f(t^2)) > 0$. Let $\tau > 0$ be a positive real number satisfying*

$$\frac{\tanh(\tau) + \coth(\tau)}{2} + f([\frac{\tanh(\tau) - \coth(\tau)}{2}]^2) > 0,$$

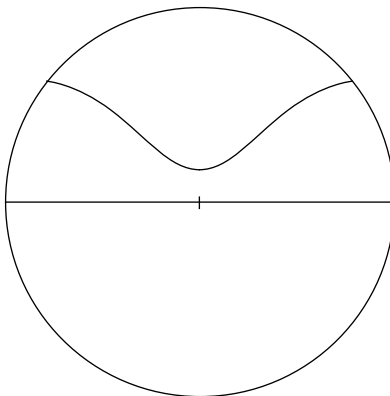FIGURE 1

$$\coth(\tau) < \lim_{t \to +\infty} (t - f(t^2)).$$

Then there exists a unique complete curve γ_τ which is the graph of a C^2 function $y_\tau(x)$, admitting a minimum at $(0, \tau)$, symmetric with respect to the y-axis and generating a complete embedded f-surface of revolution M_τ.

Moreover, if $f' \geq 0$, the curve γ_τ has the following behaviour.

1) If $-1 < f(0) \leq 0$, then y_τ is defined on an open interval $]-x_\tau, x_\tau[$ with $0 < x_\tau < +\infty$; y_r is increasing on $[0, x_\tau[$ and satisfies $\lim_{x \to x_\tau} y_\tau = +\infty$. Also γ_τ has a limit direction tangent and makes an angle $\theta \in]0, \pi/2]$ with $\partial \mathbf{D}$ such that $\cos(\theta) = |f(0)|$ (see figure 1). Furthermore, both the principal curvatures of M_τ go to $f(0)$ when x goes to $\pm x_\tau$. Consequently M_τ is asymptotically umbilic.

2) If $f(0) = -1$, then y_τ is defined on $]-\infty, +\infty[$, is increasing on $[0, +\infty[$ and satisfies $\lim_{x \to \infty} y_\tau = +\infty$. Also γ_τ has a limit tangent direction and is tangent to $\partial \mathbf{D}$ (see figure 2). Furthermore, both principal curvatures of M_τ go to -1 when x goes to $\pm \infty$. It follows that M_τ is asymptotically umbilic.

3) If $f(0) < -1$, then y_τ is defined on $]-\infty, +\infty[$ and is periodic (see figure 3).

In the same way, if $-1 \leq f \leq 0$ (without supposing $f' \geq 0$) then cases 1) and 2) still hold.

Remark 5. (a) Note that in each case γ_τ has the same geometrical behaviour as the curve generating a complete embedded surface of $\mathbf{H}^3$ with constant mean curvature $f(0)$.

(b) We deduce that if $-1 < f(0) \leq 0$ and $f' \geq 0$ (or $-1 \leq f \leq 0$), then each end of the f-surfaces given by Theorem 2 is C^1 asymptotic to an end of a totally umbilic surface of $\mathbf{H}^3$ with constant mean curvature $f(0)$.

Proof of Theorem 2. We have seen that the graph of a positive function y_τ generates an f-surface if and only if $F(y_\tau, y'_\tau, y''_\tau) = 0$. Also, f elliptic implies that F is strictly increasing with respect to y''_τ.

A computation shows that

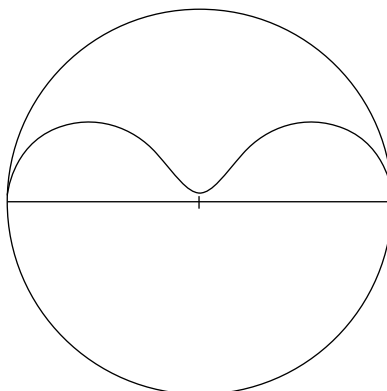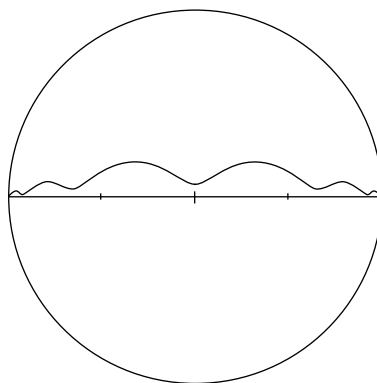$$F(\tau, 0, 0) = -\frac{\tanh(\tau) + \coth(\tau)}{2} - f([\frac{\tanh(\tau) - \coth(\tau)}{2}]^2) < 0,$$

FIGURE 2

FIGURE 3

and

$$\lim_{t \to +\infty} F(\tau, 0, t) = \lim_{t \to +\infty} (t - f(t^2)) - \coth(\tau) > 0.$$

Therefore, there is an unique positive real number $y_0'' > 0$ satisfying $F(\tau, 0, y_0'') = 0$. Now, the implicit function theorem shows that there exists a C^1 real function h defined in a neighborhood of $(\tau, 0)$ satisfying

$$F(y_\tau, y_\tau', y_\tau'') = 0 \Leftrightarrow y_\tau'' = h(y_\tau, y_\tau'), \quad \text{and} \quad h(\tau, 0) = y_0''.$$

Furthermore, Picard's theorem shows that the above differential equation has a unique solution y_τ satisfying $y_\tau(0) = \tau$ and $y_\tau'(0) = 0$. Lemma 4 shows that $y_\tau(-x) = y_\tau(x)$, so we may suppose y_τ is defined on an interval $]-x_1, x_1[$. If $x_1 = +\infty$ we are done, so suppose $0 < x_1 < +\infty$. Note that $y_\tau'(x) > 0$ for $x > 0$ near 0. If y_τ' had another zero after 0, y_τ would have another symmetry; but then y_τ would be a periodic function, and its graph γ_τ would be complete, as we wish. So suppose that $y_\tau'(x) > 0$ for $x > 0$; consequently $y_\tau(x)$ has a positive limit y_1 as x goes to x_1. If $y_1 = +\infty$ the graph γ_τ is complete, so suppose that $0 < y_1 < +\infty$.

Observe that $\lambda_2(0) = -\coth(\tau)$ and

$$\lambda_1(0) = \frac{y_0''}{\cosh^2(\tau)} - \tanh(\tau),$$

and so $\lambda_2(0) < \lambda_1(0)$. Since the f-surface generated by γ_τ is not totally umbilic, Lemma 5 shows that $\lambda_2(x) < \lambda_1(x)$ for every x. Moreover, Lemma 6 says that $\lambda_2(x)$ is strictly increasing and $\lambda_1(x)$ is strictly decreasing on $[0, x_1[$, so they have finite limit when x goes to x_1. Hence y_τ' has a positive limit y_1'.

Note also that, as f is elliptic, the function

$$G(\lambda_1, \lambda_2) = \frac{\lambda_1 + \lambda_2}{2} - f([\frac{\lambda_1 - \lambda_2}{2}]^2)$$

is strictly increasing with respect to λ_1 and λ_2, and $G(f(0), f(0)) = 0$. We get from this and from the above observations that $\lambda_2(x) < f(0) < \lambda_1(x)$. If $f(0) < 0$ the first inequality shows that $y_1' < +\infty$. If $f(0) = 0$ we also have $y_1' < +\infty$, as otherwise the graph γ_τ would be tangent and into one side of the geodesic $\{x = x_1\}$. However, those two curves generate two f-surfaces of revolution. This last situation contradicts the maximum principle with boundary.

Thus in all cases we have $y_1' < +\infty$, and as λ_1 has a finite limit we derive that y_τ'' also has a finite limit y_1''. As F is a continuous function we have $F(y_1, y_1', y_1'') = 0$. So, using the implicit function theorem as above, we can extend y_τ beyond x_1. Repeating this argument we see that we obtain a complete symmetric curve γ_τ generating an f-surface, which is the graph of a C^2 function y_τ defined on $]-x_\tau, x_\tau[$, $0 < x_\tau \leq +\infty$, as claimed. This completes the proof of the first assertion.

Proof of Statement (3). Now let us assume $f' \geq 0$. Suppose first that $f(0) < -1$. Note that in this case, we have $x_\tau = +\infty$. Otherwise, we could find an f-special sphere with mean curvature $f(0)$ tangent to and on one side of M_τ, which contradicts the maximum principle. Then y_τ is defined for all real numbers $x \in \mathbf{R}$. Now we are going to show that y_τ is periodic. To the contrary, assume that y_τ is an increasing function on $[0, +\infty[$ and consequently has a positive limit $c > 0$ when x goes to $+\infty$. The last argument with f-special spheres shows that $c \neq +\infty$. Hence, $0 < c < +\infty$. Recall that λ_2 is strictly increasing and λ_1 is strictly decreasing, and that $\lambda_2(x) < f(0) < \lambda_1(x)$. Hence, λ_2 has a strictly negative limit and we deduce that $y_\tau'(x) \to 0$. Furthermore, since λ_1 has a finite limit we get that y_τ'' also has a limit. Then $y_\tau''(x) \to 0$. It follows that

$$\lim_{x \to +\infty} \lambda_1 = -\tanh(c), \quad \lim_{x \to +\infty} \lambda_2 = -\coth(c).$$

We conclude that the cylinder C_c is an f-surface. Observe that, until now, we have not used the hypothesis $f' \geq 0$. As f is increasing, we know that the function

$$g(t) = \frac{\tanh(t) + \coth(t)}{2} + f([\frac{\tanh(t) - \coth(t)}{2}]^2)$$

is decreasing. Consequently, $t < c$ implies

$$-f([\frac{\tanh(t) - \coth(t)}{2}]^2) < \frac{\tanh(t) + \coth(t)}{2}.$$

Also let us note that the above observations about the principal curvatures show that $\lambda_1 - \lambda_2$ is a decreasing function for $x > 0$. We deduce that $[\frac{\lambda_1 - \lambda_2}{2}]^2$ is also a

decreasing function. Then for every $x > 0$ we have

$$[\frac{\lambda_1 - \lambda_2}{2}]^2 > [\frac{\tanh(c) - \coth(c)}{2}]^2,$$

from which we derive

$$-f([\frac{\lambda_1 - \lambda_2}{2}]^2) < \sqrt{[\frac{\lambda_1 - \lambda_2}{2}]^2 + 1}$$

for $x > 0$, since

$$\frac{\tanh(t) + \coth(t)}{2} = \sqrt{[\frac{\tanh(t) - \coth(t)}{2}]^2 + 1}.$$

The last inequality implies that

$$0 < \frac{\lambda_1 + \lambda_2}{2} + \sqrt{[\frac{\lambda_1 - \lambda_2}{2}]^2 + 1}.$$

A computation shows that this is equivalent, for x big enough, to $\lambda_1.\lambda_2 < 1$. Replacing λ_1 and λ_2 in the last inequality by their expressions as functions of y_τ and doing further simplifications, we get, for x big enough,

$$(*) \qquad\qquad 0 < y_\tau'' \cosh^3(y_\tau) + y_\tau'^4 \sinh(y_\tau).$$

On the other hand, note that $y_\tau'(x) \cosh(y_\tau(x)) > 0$ for every $x > 0$, and moreover

$$y_\tau'(0) \cosh(y_\tau(0)) = 0 \quad \text{and} \quad \lim_{x \to +\infty} y_\tau'(x) \cosh(y_\tau(x)) = 0.$$

From the above, we deduce that there is a sequence $x_n > 0$, with $\lim_{n \to +\infty} x_n = +\infty$, such that

$$(y_\tau' \cosh(y_\tau))'(x_n) < 0.$$

This means that $(y_\tau'' \cosh(y_\tau))(x_n) < -(y_\tau'^2 \sinh(y_\tau))(x_n)$. Then, for every n:

$$(y_\tau'' \cosh^3(y_\tau) + y_\tau'^4 \sinh(y_\tau))(x_n) < (\sinh(y_\tau) y_\tau'^2 (y_\tau'^2 - \cosh^2(y_\tau))(x_n)$$

which is < 0 for n big enough. But this gives a contradiction with $(*)$. We conclude that y_τ has to be periodic, as desired.

For the remaining cases, $-1 \le f(0) \le 0$, we need to consider the angle $\sigma \in\ ]-\pi/2, \pi/2[$ between γ_τ and the coordinate curves $\{y = \text{const}\}$. Notice that the orientation chosen is such that $\cos \sigma = \cosh(y)/(\cosh^2(y) + y'^2)^{1/2}$. It follows that

$$\lambda_2(x) = \frac{-\cos \sigma}{\tanh(y)}.$$

Proof of Statement (2). Let us suppose now that $f(0) = -1$ (and $f' \ge 0$). Therefore $x_\tau = +\infty$ or, equivalently, y_τ is defined on $]-\infty, +\infty[$. For otherwise, using the family of horocycles issuing from the point $(u, v) = (1, 0)$ in $\partial \mathbf{D}$, we may reach a tangent point between γ_τ and one of those horocycles with γ_τ on one side of the horocycle. But those two curves generate two f-surfaces, so we would have a contradiction with the maximum principle. Observe also that if $\lim_{x \to +\infty} y_\tau(x) = c < +\infty$, the cylinder C_c would be an f-surface. This gives a contradiction, since the mean curvature of C_c is strictly less than -1 (with respect to the exterior normal orientation), and $f \ge -1$ because f is increasing and $f(0) = -1$. So this allows us to conclude that $\lim_{x \to +\infty} y_\tau(x) = +\infty$. As $\lambda_2(x)$ is a strictly increasing function and bounded from above by -1, we deduce that $\lim_{x \to +\infty} \lambda_2 = \lim_{x \to +\infty} -\cos \sigma \le -1$; hence $\lim_{x \to +\infty} \cos \sigma = 1$. This means that γ_τ is tangent to $\partial \mathbf{D}$ when x goes to $\pm\infty$.

We also have that $\lim_{x \to +\infty} \lambda_2 = -1$. Since the function

$$G(\lambda_1, \lambda_2) = \frac{\lambda_1 + \lambda_2}{2} - f([\frac{\lambda_1 - \lambda_2}{2}]^2)$$

is increasing with respect to λ_1 and λ_2 with $G(f(0), f(0)) = 0$, we also deduce that $\lim_{x \to +\infty} \lambda_1 = -1$. We conclude that M_τ is asymptotically umbilic.

Proof of Statement (1). Let us assume $-1 < f(0) \le 0$ (and $f' \ge 0$). Consider the family of umbilic f-surfaces of $\mathbf{H}^3$ invariant under rotation about the y-axis. Namely, this is the family of pieces of sphere making an angle $\alpha \in]0, \pi/2]$ with $\partial \mathbf{H}^3$ such that $\cos(\alpha) = -f(0)$ and each spherical piece stands above the totally geodesic plane with the same boundary. Suppose that $x_\tau = +\infty$. Assume first that $\lim_{x \to +\infty} y_\tau(x) = +\infty$. Then we could find one of the previous spherical pieces tangent to and above M_τ, which contradicts the maximum principle. If $\lim_{x \to +\infty} y_\tau(x) = c$ with $c < +\infty$, we would conclude as before that the cylinder C_c is an f-surface. But any cylinder has mean curvature strictly less than -1 (with respect to the outward orientation), and f is bigger than -1. So we deduce that $x_\tau < +\infty$. As we know that γ_τ is a complete curve, we have $\lim_{x \to x_\tau} y_\tau = +\infty$. As before, since λ_2 has a finite limit at x_τ, we conclude that $\cos \sigma$ also has a limit at x_τ. Let us call σ_τ the limit angle. This means that γ_τ has a limit tangent direction and that the angle between γ_τ and $\partial \mathbf{D}$ is σ_τ.

Using the family of umbilic f-surfaces (with respect to the outward normal orientation) invariant under rotation about the x-axis, the maximum principle shows that $\sigma_\tau \le \alpha$. Moreover, using the first family of umbilic f-surfaces (namely, those invariant under rotation about the y-axis), the maximum principle shows that $\sigma_\tau \ge \alpha$. We obtain that $\sigma_\tau = \alpha$; hence $\cos(\sigma_\tau) = -f(0)$. As λ_2 goes to $-\cos(\sigma_\tau)$ when x goes to x_τ, we deduce that $\lim_{x \to x_\tau} \lambda_2 = f(0)$. The same argument as in the case $f(0) = -1$ shows that we also have $\lim_{x \to x_\tau} \lambda_1 = f(0)$. Thus, M_τ is again asymptotically umbilic.

Finally, assume that $-1 \le f \le 0$. Observe that in this case there is no f-special cylinder, since the mean curvature of any cylinder is strictly less than -1. Therefore, we can follow the same proof as before for the cases $f(0) = -1$ and $-1 < f(0) \le 0$. This concludes the proof of Theorem 2.

Remark 6. (a) It is easy to show that if a plane curve which attains a minimum $(0, \tau)$, $\tau > 0$, generates an f-surface of revolution M, then τ satisfies the conditions of Theorem 2 (so our hypothesis are necessary). Therefore, the proof of the first part of Theorem 2 also shows that M can be extended as a complete embedded f-surface of revolution. Namely, M is part of one surface M_τ given by Theorem 2.

(b) In the case $f(0) < -1$, for the geometrical description of γ_τ, note that we only used the fact that g is a decreasing function. It follows that we only need the hypothesis $f'(t) > -1/2\sqrt{t+1}$ to show that γ_τ is periodic.

In order to prove uniqueness of f-surfaces given by Theorem 2, we shall need the following lemma.

Lemma 7. *Let f be an elliptic function with $f' \ge 0$. Let $y = y(x)$ be a C^2 decreasing and strictly positive function defined on an interval $]x_0, +\infty[$. Then the surface of revolution M generated by the graph of $y(x)$ is not an f-surface.*

Proof. Let $c = \lim_{x \to +\infty} y(x)$. We have $0 \le c < +\infty$. Let us suppose that M is an f-surface. Observe that Lemma 6 implies that the principal curvatures of M are monotone functions, and that λ_1 is increasing if and only if λ_2 is decreasing.

Let us first suppose $c = 0$. Therefore y' does not have $-\infty$ as limit when x goes to $+\infty$. We deduce that $\lim_{x \to +\infty} \lambda_2 = -\infty$. It follows that λ_2 is decreasing, so $\lim_{x \to +\infty} (\lambda_2 - \lambda_1) = -\infty$. Note that

$$-\lambda_1 = \frac{\lambda_2 - \lambda_1}{2} - f([\frac{\lambda_2 - \lambda_1}{2}]^2).$$

Then

$$\lim_{x \to +\infty} \lambda_1 = \lim_{t \to +\infty} (t + f(t^2)).$$

As f is increasing, it follows that $\lim_{x \to +\infty} \lambda_1(x) = +\infty$. Consequently the expression of λ_1 in terms of y implies that $\lim_{x \to +\infty} y''(x) = +\infty$. But the last fact yields a contradiction, because y' is negative. Hence, $c \ne 0$.

Let us assume now that $c > 0$. As $\lim_{x \to +\infty} y' \ne -\infty$ and λ_2 has a limit, we conclude that y' has a finite limit, and then $\lim_{x \to +\infty} y' = 0$. Similarly, we can show that $\lim_{x \to +\infty} y''(x) = 0$. It follows that

$$\lim_{x \to +\infty} \lambda_1(x) = -\tanh(c), \qquad \lim_{x \to +\infty} \lambda_2(x) = -\coth(c).$$

Consequently, we have $\lambda_1(x) > \lambda_2(x)$, and Lemma 6 shows that λ_2 is decreasing and $\lambda_1(x)$ is increasing. Hence $[\lambda_1(x) - \lambda_2(x)]^2$ is an increasing function, and we get $[\lambda_1(x) - \lambda_2(x)]^2 < [\tanh(c) - \coth(c)]^2$. As f is increasing, we have

$$f([\frac{\lambda_1(x) - \lambda_2(x)}{2}]^2) < f([\frac{\tanh(c) - \coth(c)}{2}]^2).$$

On the other hand, a continuity argument shows that the cylinder C_c is an f-surface. This allows us to conclude that

$$\frac{\lambda_1(x) + \lambda_2(x)}{2} < \frac{-\tanh(c) - \coth(c)}{2}.$$

Hence the mean curvature of M is smaller than the mean curvature of C_c, with respect to the outward normal orientation. But it is well-known (see [8] and [11]) that for every $H < -1$ there exists a family of complete embedded Delaunay type surfaces of revolution with constant mean curvature H. This means that those surfaces are periodic and vary continuously. Now, it is also well-known (see [12], Lemma 6.4, or [15], Corollary 4.1.1) that by doing a little perturbation of C_c in this family, we can get a surface with the same constant mean curvature as C_c which is tangent to and stays under M. As the mean curvature of M is smaller, this gives a contradiction with the usual maximum principle (that is, the maximum principle for mean curvature). Thus M cannot be an f-surface.

Remark 7. Observe that Lemma 7 remains true if we replace the hypothesis $f' \ge 0$ by $|f| \le 1$. Indeed, we would again have $\lim_{t \to +\infty} (t + f(t^2)) = +\infty$, so this eliminates the case $c = 0$ (keeping the notation of the proof). Observe also that, in this case, the mean curvature of any f-surface is always bigger than -1 and smaller than 1. Then there is no f-special cylinder. This eliminates the case $c > 0$.

Let M be a complete embedded non-compact surface of revolution in $\mathbf{H}^3$ which does not intersects the axis of rotation (x-axis). Hence, $\mathbf{H}^3 - M$ has two connected components, one of which contains the x-axis. Then we call the unit normal field

along M pointing toward the component which does not contain the x-axis the *exterior normal orientation*. Observe that this definition coincides with the outward normal orientation in the case when M is generated by a graph.

Theorem 3. *Let f be an elliptic function satisfying one of the two following conditions:*

(1) $f' \geq 0$ and $f(0) \leq 0$;

(2) $-1 \leq f \leq 0$.

Let M be a complete and embedded f-surface of revolution that is not totally umbilic.

Then M is one of the surfaces given by Theorem 2.

Proof. Let us assume first that f satisfies condition (1).

Let $\gamma \subset \mathbf{D}$ be the curve generating M. Observe that if γ intersects the x-axis then, for regularity, this intersection must be orthogonal. But this point would be an umbilic point. Hence, as M is not a totally umbilic surface, Lemma 5 shows that this situation is impossible. So we can suppose that γ stays in the part $\{y > 0\}$ of $\mathbf{D}$.

Let us prove that γ is a graph. If it were not, γ would have a vertical point p (this means the tangent of γ at p is vertical). Let γ^- be the component of $\gamma - p$ which begins under p. Up to a symmetry, we can suppose that a neighborhood of γ^- near p stays in the region $\{x \geq x(p)\}$. Observe that the y-coordinate of γ^- is decreasing near p. Suppose that γ^- has a horizontal point (this means a point where the tangent is horizontal), and let q be the first one. Lemma 4 implies that γ is symmetric with respect to the vertical geodesic $\{x = x(q)\}$, and then q is a local minimum. Observe that, if γ^- had another horizontal point, then the next such point after q would be a local maximum. Hence γ would be a periodic curve (see Lemma 4) and q a global minimum for γ. This implies that the exterior normal orientation of γ (or M) at q points in the direction of increasing y. But then near q the curve γ is a graph with q as minimum. As γ generates an f-surface M, the proof of Theorem 2 shows that M should be one of the surfaces given there (see Remark 6(a)). But this is absurd, since no surface given by Theorem 2 is generated by a curve which admits a vertical point. So γ^- has no other horizontal point after q. Hence we deduce that q is a global minimum for γ, which is absurd as we have seen before. It follows that γ^- cannot have a horizontal point after p.

Now if γ^- were a graph, this graph would be decreasing and defined on the interval $]x(p), +\infty[$ (as γ is complete). But Lemma 7 shows this is not possible. So we conclude that γ^- is not a graph. This allows us to deduce that γ^- has another vertical point after p. Let p' be the first one. Note that hence γ^- is a decreasing graph between p and p'. Combining this with Lemma 6, we find that λ_2 is strictly monotone between those two vertical points. Now, as $y(p)$, $y(p') > 0$, we have $\lambda_2(p) = \lambda_2(p') = 0$, which gives a contradiction.

Thus γ is a complete graph. If γ had no horizontal points, then, up to a symmetry, we could assume that γ is a decreasing graph defined on an interval $]x_0, +\infty[$, $-\infty \leq x_0$. But Lemma 7 shows this is impossible. If γ had a unique horizontal point, γ would be symmetric (see Lemma 4), and then this point would be a global maximum or minimum. Lemma 7 shows that this point cannot be a global maximum. Therefore γ admits a global minimum, and hence M is one of the surfaces given by Theorem 2 (see Remark 6(a)). Finally, if γ has many horizontal

points, Lemma 4 shows that γ is periodic and so admits a global minimum, which concludes the proof in the first case.

In the second case the proof is analogous, since the principal tool, Lemma 7, is again true in this new context. See Remark 7.

Remark 8. In the euclidean case we do not need to assume $f(0) \leq 0$. This is implied by the other hypothesis in Theorem 3; see [16]. In the hyperbolic case we must assume this hypothesis, since J. Gomes [8] has shown existence of an embedded complete surface of revolution in $\mathbf{H}^3$ with constant and strictly positive mean curvature (with respect to the exterior normal orientation).

REFERENCES

1. J. L. Barbosa, R. Sa Earp. New results on prescribed mean curvature hypersurfaces in space forms. *An Acad. Bras. Ci.*, **67**, 1,1-5, (1995).
2. J. L. Barbosa, R. Sa Earp. Prescribed mean curvature hypersurfaces in $H^{n+1}(-1)$ with convex planar boundary I. *Geom. Dedicata*, **71**, 61-74, (1998). MR **99d:**53064
3. F.G. Braga Brito, R. Sa Earp. On the structure of certain Weingarten surfaces with boundary a circle. *An. Fac. Sci. Toulouse,* **VI**, No 2, 243-255, (1997). MR **99a:**53005
4. R. Bryant. Complex analysis and a class of Weingarten surfaces. Preprint.
5. S.S. Chern. On special W-surface. *Proc. Amer. Math. Soc.*, **6**, 783-786, (1955). MR **17:**657h
6. M.P. Do Carmo, H.B. Lawson Jr. On Alexandrov-Bernstein theorems in hyperbolic space. *Duke Math. J.*, **50**, 995–1003, (1983). MR **85f:**53009
7. M.P. Do Carmo, J.M. Gomes, G. Thorbergsson. The influence of the boundary behaviour on hypersurfaces with constant mean curvature in H^{n+1}. *Comm. Math. Helvitici*, **61**, 429-491, (1986). MR **87j:**53087
8. J.M. Gomes. Sobre hirpersuperfícies com curvatura média constante no espaço hiperbólico. Tese de doutorado. IMPA, (1985).
9. P. Hartman, W. Wintner. Umbilical points and W-surfaces. *Amer. J. Math.*, **76**, 502-508, (1954). MR **16:**68a
10. H. Hopf. Differential geometry in the large. *Lecture Notes in Math.*, Springer-Verlag **1000**, (1983). MR **85b:**53001
11. W.Y. Hsiang. On generalization of theorems of A. D. Alexandrov and C. Delaunay on hypersurfaces of constant mean curvature. *Duke Math. J.*, **49**, 485–496, (1982). MR **84k:**53007
12. N.J. Korevaar, R. Kusner, W.H. Meeks III, B. Solomon. Constant mean curvature surfaces in hyperbolic space. *Amer. J. Math.*, **114**, 1-43, (1992). MR **92k:**53116
13. G. Levitt, H. Rosenberg. Symmetry of constant mean curvature hypersurfaces in hyperbolic space. *Duke Math. J.*, **52**, 53–59, (1985). MR **86h:**53063
14. B. Nelli, H. Rosenberg. Some remarks on embedded hypersurfaces in Hyperbolic Space of constant mean curvature and spherical boundary. *Ann. Global Anal. Geom.*, **13**, 23-30, (1995). MR **96c:**53097
15. H. Rosenberg, R. Sa Earp. The geometry of properly embedded special surfaces in $\mathbf{R}^3$, *e.g.*, surfaces satisfying $aH + bK = 1$, where a and b are positive. *Duke Math. J.*, **73**, 291–306, (1994). MR **95b:**53010
16. R. Sa Earp et E. Toubiana. A note on special surfaces in $\mathbf{R}^3$. *Mat. Contemp.*, **4**, 108-118, (1993). MR **95j:**53008
17. R. Sa Earp et E. Toubiana. Sur les surfaces de Weingarten spéciales de type minimal. *Bol. Soc. Bras. Mat.*, **26**, No. 2, 129-148, (1995). MR **96j:**53006
18. R. Sa Earp et E. Toubiana. Classification des surfaces de type Delaunay et applications. *Amer. J. Math.* **221**, 671–700, (1999).

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