TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 351, Number 12, Pages 4713–4726
S 0002-9947(99)02523-4
Article electronically published on August 25, 1999

HOMOGENEOUS SPACES WITH INVARIANT PROJECTIVELY FLAT AFFINE CONNECTIONS

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Abstract. We characterize invariant projectively flat affine connections in terms of affine representations of Lie algebras, and show that a homogeneous space admits an invariant projectively flat affine connection if and only if it has an equivariant centro-affine immersion. We give a correspondence between semi-simple symmetric spaces with invariant projectively flat affine connections and central-simple Jordan algebras.

Introduction

An affine connection is said to be projectively flat if it is locally projectively equivalent to a flat affine connection. In this paper we study invariant projectively flat affine connections.

Applying the theory of projective normal Cartan connections [A] gave a correspondence between the set of invariant projectively flat affine connections on G/K and the set of projective equivalence classes of Lie algebra homomorphisms from the Lie algebra $\mathfrak{g}$ of G to $\mathfrak{sl}(n+1,\mathbf{R})$ where $n = \dim G/K$. Using this correspondence [A] classified irreducible classical Riemannian symmetric spaces with invariant projectively flat affine connections.

The following facts are fundamental for projectively flat affine connections [NS]:

A. *A torsion-free and Ricci-symmetric affine connection D on an n-dimensional manifold is projectively flat if and only if*

(i) *the curvature tensor R and the Ricci tensor Ric satisfy*

$$R(X,Y)Z = \frac{1}{n-1}\{Ric(Y,Z)X - Ric(X,Z)Y\},$$

(ii) *the Ricci tensor satisfies the Codazzi equation, that is,*

$$(D_X Ric)(Y,Z) = (D_Y Ric)(X,Z).$$

B. *The induced connections of centro-affine hypersurface immersions are projectively flat.*

Being motivated the above facts [NP] gave a correspondence between Lie groups admitting bi-invariant projectively flat affine connections and associative algebras with unit, and classified all such spaces.

Received by the editors March 15, 1996.
1991 *Mathematics Subject Classification.* Primary 53C05, 53C30, 53C35, 53A15; Secondary 17C20.
Key words and phrases. Invariant projectively flat affine connections, centro-affine immersions, homogeneous spaces, symmetric spaces, Jordan algebras.

Along the same line as in [NP], in section 1 we relate the existence of invariant projectively flat affine connections to that of affine representations of Lie algebras due to [Ks]. As an immediate consequence we find that a homogeneous space G/K admits an invariant projectively flat affine connection if and only if G/K has an equivariant centro-affine hypersurface immersion. It seems that our method is more elementary and direct than in [A]. In section 2 we give a correspondence between n-dimensional Lie groups with left invariant projectively flat affine connections and $(n+1)$-dimensional left symmetric algebras with unit. In section 3 we show that semi-simple symmetric spaces with invariant projectively flat affine connections correspond to central-simple Jordan algebras and are realized as centro-affine hypersurfaces in the algebras (cf. [Ka]). Riemannian semi-simple symmetric spaces with invariant projectively flat affine connections correspond to simple formal real Jordan algebras. The classification of central-simple Jordan algebras and simple formal real Jordan algebras were given in [BK]. In section 4 for a better understanding we explain our correspondence by typical examples.

Acknowledgment

This work was done while I was staying at Brown University. I would like to express my sincere thanks to Professor K. Nomizu, who kindly acquainted me with the importance of projectively flat affine connections.

1. Invariant projectively flat affine connections

Let G be a simply connected Lie group and K a connected closed subgroup of G. Assume that G acts effectively on G/K. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively. We enlarge $\mathfrak{g}$ as follows

$$\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbf{R}E,$$
$$[\tilde{\mathfrak{g}}, E] = \{0\}.$$

Then our first main result is the following.

Theorem 1.1. *An n-dimensional simply connected effective homogeneous space G/K admits a G-invariant projectively flat affine connection if and only if $\tilde{\mathfrak{g}}$ has an affine representation $(\tilde{f}, \tilde{q})$ on an $(n+1)$-dimensional real vector space $\tilde{V}$, that is,*

(i) *$\tilde{f}$ is a representation of $\tilde{\mathfrak{g}}$ on $\tilde{V}$,*
(ii) *$\tilde{q}$ is a linear mapping from $\tilde{\mathfrak{g}}$ to $\tilde{V}$ such that*

$$\tilde{q}([\tilde{X}, \tilde{Y}]) = \tilde{f}(\tilde{X})\tilde{q}(\tilde{Y}) - \tilde{f}(\tilde{Y})\tilde{q}(\tilde{X}) \quad \text{for} \quad \tilde{X}, \tilde{Y} \in \tilde{\mathfrak{g}},$$

with the following properties:

(iii) *$\tilde{q}$ is surjective and the kernel is $\mathfrak{k}$,*
(iv) *$\tilde{f}(E)$ is the identity mapping $I_{\tilde{V}}$ of $\tilde{V}$ and $\tilde{q}(E) \neq 0$.*

Corollary 1.2. *Let G/K be a simply connected homogeneous space. Then the following conditions are equivalent.*

(i) *G/K admits an invariant projectively flat affine connection.*
(ii) *G/K has an equivariant centro-affine hypersurface immersion into a real affine space.*

Proof of Theorem 1.1. Suppose that G/K admits a G-invariant projectively flat affine connection D. For $X \in \mathfrak{g}$, we denote by X^* a vector field on G/K induced by $\exp(-tX)$. Set

$$A_{X^*}Y^* = -D_{Y^*}X^*.$$

Then it is known

$$A_{X^*}Y^* = (L_{X^*} - D_{X^*})Y^*,$$

where L_{X^*} denote Lie differentiation by X^*, and

$$A_{[X,Y]^*} = [A_{X^*}, A_{Y^*}] - R(X^*, Y^*),$$

where R is the curvature tensor for D [KN, p. 235]. Let V be the tangent space of G/K at $o = \{K\}$. We set

$$f(X) = (A_{X^*})_o,$$
$$q(X) = X^*_o,$$

where the subscript o of tensor fields means the values of the tensor fields at o. Then we have

$$f([X,Y]) = [f(X), f(Y)] - R(X^*, Y^*)_o,$$
$$\ker q = \mathfrak{k}.$$

Since D is projectively flat it follows that

$$R(X^*, Y^*) = \gamma(Y^*, \ \ )X^* - \gamma(X^*, \ \ )Y^*,$$

where $\gamma = \dfrac{1}{n-1} Ric$. Thus we have

$$f([X,Y]) = [f(X), f(Y)] - \gamma_o(q(Y), \ \ )q(X) + \gamma_o(q(X), \ \ )q(Y). \tag{1.1}$$

Since the torsion tensor of R vanishes, it follows that

$$q([X,Y]) = f(X)q(Y) - f(Y)q(X). \tag{1.2}$$

In fact,

$$\begin{aligned} q([X,Y]) &= [X^*, Y^*]_o \\ &= (-A_{Y^*}X^* + A_{X^*}Y^*)_o \\ &= -f(Y)q(X) + f(X)q(Y). \end{aligned}$$

Since γ is G-invariant, we obtain

$$\begin{aligned} &(D_{X^*}\gamma)(Y^*, Z^*) \\ &= X^*(\gamma(Y^*, Z^*)) - \gamma(D_{X^*}Y^*, Z^*) - \gamma(Y^*, D_{X^*}Z^*) \\ &= \gamma(L_{X^*}Y^*, Z^*) + \gamma(Y^*, L_{X^*}Z^*) - \gamma(D_{X^*}Y^*, Z^*) - \gamma(Y^*, D_{X^*}Z^*) \\ &= \gamma(A_{X^*}Y^*, Z^*) + \gamma(Y^*, A_{X^*}Z^*). \end{aligned}$$

Using the Codazzi equation for the Ricci tensor,

$$(D_{X^*}\gamma)(Y^*, Z^*) = (D_{Y^*}\gamma)(X^*, Z^*),$$

we have

$$\begin{aligned} &\gamma_o(f(X)q(Y), q(Z)) + \gamma_o(q(Y), f(X)q(Z)) \\ &\quad = \gamma_o(f(Y)q(X), q(Z)) + \gamma_o(q(X), f(Y)q(Z)). \end{aligned} \tag{1.3}$$

We enlarge the vector space V so that

$$\tilde{V} = V \oplus \mathbf{R}e.$$

For $X \in \mathfrak{g}$ we define an endomorphism $\tilde{f}(X)$ of $\tilde{V}$ by

$$\begin{aligned}\tilde{f}(X)q(Z) &= f(X)q(Z) - \gamma_o(q(X), q(Z))e,\\ \tilde{f}(X)e &= q(X).\end{aligned}$$

Using (1.1), (1.2) and (1.3), we have

$$\begin{aligned}&[\tilde{f}(X), \tilde{f}(Y)]q(Z)\\ &= [f(X), f(Y)]q(Z) - \gamma_o(q(Y), q(Z))q(X) + \gamma_o(q(X), q(Z))q(Y)\\ &\quad -\{\gamma_o(q(X), f(Y)q(Z)) - \gamma_o(q(Y), f(X)q(Z))\}e\\ &= f([X, Y])q(Z) - \{\gamma_o(f(X)q(Y), q(Z)) - \gamma_o(f(Y)q(X), q(Z))\}e\\ &= f([X, Y])q(Z) - \gamma_o(q([X, Y]), q(Z))e\\ &= \tilde{f}([X, Y])q(Z),\end{aligned}$$

and

$$\begin{aligned}&[\tilde{f}(X), \tilde{f}(Y)]e\\ &= \tilde{f}(X)q(Y) - \tilde{f}(Y)q(X) = f(X)q(Y) - f(Y)q(X) = q([X, Y])\\ &= \tilde{f}([X, Y])e.\end{aligned}$$

These imply that $(\tilde{f}, q)$ is an affine representation of $\mathfrak{g}$ on $\tilde{V}$, that is,

$$\begin{aligned}\tilde{f}([X, Y]) &= [\tilde{f}(X), \tilde{f}(Y)],\\ q([X, Y]) &= \tilde{f}(X)q(Y) - \tilde{f}(Y)q(X),\end{aligned}$$

for $X,\ Y \in \mathfrak{g}$. We extend this affine representation $(\tilde{f}, q)$ of $\mathfrak{g}$ by

$$\begin{aligned}\tilde{f}(\tilde{X}) &= \begin{cases} \tilde{f}(X), & \tilde{X} = X \in \mathfrak{g},\\ I_{\tilde{V}}, & \tilde{X} = E,\end{cases}\\ \tilde{q}(\tilde{X}) &= \begin{cases} q(X), & \tilde{X} = X \in \mathfrak{g},\\ e, & \tilde{X} = E.\end{cases}\end{aligned}$$

Then $(\tilde{f}, \tilde{q})$ is an affine representation of $\tilde{\mathfrak{g}}$ on $\tilde{V}$ with required properties.

Conversely, suppose that $\tilde{\mathfrak{g}}$ admits an affine representation $(\tilde{f}, \tilde{q})$ on $\tilde{V}$ satisfying (iii)(iv). Using an affine coordinate system $\{x^1, ..., x^{n+1}\}$ on $\tilde{V}$ we can express an affine mapping $\tilde{v} \longrightarrow \tilde{f}(\tilde{X})\tilde{v} + \tilde{q}(\tilde{X})$ by an $(n+2) \times (n+2)$ matrix representation

$$a(\tilde{X}) = \begin{bmatrix} \tilde{f}(\tilde{X})^i_j & \tilde{q}(\tilde{X})^i \\ 0 & 0 \end{bmatrix},$$

where $[\tilde{f}(\tilde{X})^i_j]$ is an $(n+1)\times(n+1)$ matrix and $[\tilde{q}(\tilde{X})^i]$ is a $(n+1)$ row vector. Then $\tilde{X} \longrightarrow a(\tilde{X})$ is an injective Lie algebra homomorphism from $\tilde{\mathfrak{g}}$ into the Lie algebra of all $(n+2) \times (n+2)$ matrices. We set $\tilde{\mathfrak{g}}_a = a(\tilde{\mathfrak{g}})$, $\mathfrak{g}_a = a(\mathfrak{g})$ and $\mathfrak{c}_a = a(\mathbf{R}E)$. We denote by $\tilde{G}_a$, G_a and C_a the linear Lie subgroup of $GL(n+2, \mathbf{R})$ generated by $\tilde{\mathfrak{g}}_a$, $\mathfrak{g}_a$ and $\mathfrak{c}_a$, respectively. An element $\tilde{s} \in \tilde{G}_a$ is expressed by

$$\tilde{s} = \begin{bmatrix} \tilde{\mathbf{f}}(\tilde{s}) & \tilde{\mathbf{q}}(\tilde{s}) \\ 0 & 1 \end{bmatrix},$$

where $\tilde{\mathbf{f}}(\tilde{s})$ and $\tilde{\mathbf{q}}(\tilde{s})$ are the linear part and the translation part of $\tilde{s}$, respectively. Let $\tilde{\Omega}_a$ and M_a be the orbit of $\tilde{G}_a$ and G_a through the origin o respectively. Then we have

$$\tilde{\Omega}_a = \tilde{\mathbf{q}}(\tilde{G}_a) = C_a G_a/K_a = C_a M_a,$$
$$M_a = \tilde{\mathbf{q}}(G_a) = G_a/K_a,$$

where $K_a = \{s \in G_a \mid \tilde{\mathbf{q}}(s) = 0\}$, and its Lie algebra is $a(\mathfrak{k})$. Since $\tilde{q}(\tilde{\mathfrak{g}}) = \tilde{V}$, $\tilde{\Omega}_a$ is an open orbit in $\tilde{V}$. For $\tilde{X} \in \tilde{\mathfrak{g}}$ we denote by $\tilde{X}^*$ a vector field on $\tilde{\Omega}_a$ induced by $\exp a(-t\tilde{X})$. Since $\tilde{\Omega}_a = C_a M_a$ is an open set, a curve $\exp a(-tE)m$ through $m \in M_a$ is transversal to M_a at m. Hence E^* is transversal to M_a.

Let $\tilde{D}$ be the canonical flat affine connection on $\tilde{V}$. As in affine differential geometry [NS], we can define the induced affine connection D on M_a and the affine fundamental form h by

$$\tilde{D}_{X^*}Y^* = D_{X^*}Y^* + h(X^*, Y^*)E^*,$$

for $X, Y \in \mathfrak{g}$. Then, D and h are invariant by G_a, because $\tilde{D}$ and E^* are invariant by $\tilde{G}_a$. Since $E^* = -\sum_i (x^i + \tilde{q}^i(E))\partial/\partial x^i$, M_a is a centro-affine hypersurface with center $-\tilde{q}(E)$. Hence the induced connection D is projectively flat [NS]. Since G is simply connected, there exists a covering homomorphism

$$\rho : G \longrightarrow G_a$$

such that $d\rho(X) = a(X)$. K being the identity component of $\rho^{-1}(K_a)$, we have a covering mapping

$$G/K \longrightarrow G/\rho^{-1}(K_a) \cong G_a/K_a$$

induced by ρ. Hence G/K admits a G-invariant projectively flat affine connection. □

Proof of Corollary 1.2. (i) $\Longrightarrow$ (ii) follows from the above arguments. The induced affine connection of a centro-affine immersion being projectively flat [NS], we have (ii) $\Longrightarrow$ (i). □

2. The case of Lie groups

Let V be an algebra over $\mathbf{R}$ with multiplication uv. We set

$$[uvw] = u(vw) - (uv)w.$$

If the algebra V satisfies

$$[uvw] = [vuw], \tag{2.1}$$

then V is said to be a *left symmetric algebra* [V2]. The following theorem was essentially known to Koszul and Vinberg.

Theorem 2.1. *There is a natural one-one correspondence between*

(i) *n-dimensional simply connected Lie groups with left invariant flat affine connections up to affine diffeomorphism;*
(ii) *n-dimensional left symmetric algebras over $\mathbf{R}$ up to algebraic isomorphism.*

In this section we prove the following.

Theorem 2.2. *There is a natural one-one correspondence between*

(i) *n-dimensional simply connected Lie groups with left invariant projectively flat affine connections up to equivariant projective diffeomorphism;*

(ii) *$(n+1)$-dimensional left symmetric algebras over* $\mathbf{R}$ *with unit up to algebraic isomorphism.*

Proof. Using the same notation as in section 1 we can find by Theorem 1.1 an affine representation $(\tilde{f}, \tilde{q})$ of the Lie algebra $\tilde{\mathfrak{g}}$ on an $(n+1)$-dimensional real vector space $\tilde{V}$ satisfying the conditions (iii)(iv). Since $\tilde{q} : \tilde{\mathfrak{g}} \longrightarrow \tilde{V}$ is an isomorphism we define a multiplication law in $\tilde{V}$ by

$$uv = \tilde{f}(\tilde{q}^{-1}(u))v.$$

Denoting by L_u the left multiplication by u we have

$$[L_u, L_v] = L_{uv-vu}, \tag{2.2}$$

$$ue = eu = u, \tag{2.3}$$

where $e = \tilde{q}(E)$. In fact, since

$$\tilde{q}([\tilde{q}^{-1}(u), \tilde{q}^{-1}(v)]) = \tilde{f}(\tilde{q}^{-1}(u))v - \tilde{f}(\tilde{q}^{-1}(v))u = uv - vu,$$

we have

$$\begin{aligned} L_{uv-vu} &= \tilde{f}(\tilde{q}^{-1}(uv - vu)) = \tilde{f}([\tilde{q}^{-1}(u), \tilde{q}^{-1}(v)]) \\ &= [\tilde{f}(\tilde{q}^{-1}(u)), \tilde{f}(\tilde{q}^{-1}(v))] = [L_u, L_v], \end{aligned}$$

and

$$ue = eu + \tilde{q}([\tilde{q}^{-1}(u), \tilde{q}^{-1}(e)]) = eu = \tilde{f}(E)u = u.$$

By (2.2) we have

$$[uvw] = [vuw].$$

Thus the algebra $\tilde{V}$ is an $(n+1)$-dimensional left symmetric algebra with unit e.

Conversely suppose that $\tilde{V}$ is an $(n+1)$-dimensional left symmetric algebra with unit e. Let $V = \{v \in \tilde{V} \mid \mathrm{Tr} L_v = 0\}$. Since $\tilde{v} - \{1/(n+1)\mathrm{Tr} L_{\tilde{v}}\}e \in V$, it follows that $\tilde{V} = V \oplus \mathbf{R}e$. We set

$$\begin{aligned} \mathfrak{g}(V) &= \{L_v \mid v \in V\}, \\ \mathfrak{g}(\tilde{V}) &= \{L_{\tilde{v}} \mid \tilde{v} \in \tilde{V}\}. \end{aligned}$$

Then by (2.2), $\mathfrak{g}(V)$ and $\mathfrak{g}(\tilde{V})$ are Lie algebras, and we have

$$\mathfrak{g}(\tilde{V}) = \mathfrak{g}(V) \oplus \mathbf{R} I_{\tilde{V}}.$$

Setting $\tilde{f}(L_{\tilde{v}}) = L_{\tilde{v}}$ and $\tilde{q}(L_{\tilde{v}}) = \tilde{v}$, we obtain an affine representation $(\tilde{f}, \tilde{q})$ of $\mathfrak{g}(\tilde{V})$ satisfying the conditions (iii)(iv) of Theorem 1.1. Thus the simply connected Lie group with Lie algebra $\mathfrak{g}(V)$ admits a left invariant projectively flat affine connection. □

Remark. [NP] gave a correspondence between n-dimensional Lie groups with bi-invariant projectively flat affine connections and $(n+1)$-dimensional associative algebras with unit, and classified all such spaces.

3. The case of symmetric spaces

In this section we give a correspondence between semi-simple symmetric spaces with invariant projectively flat affine connections and central-simple Jordan algebras with unit.

Let (G, K) be an effective symmetric pair where G is semi-simple and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the canonical decomposition, that is,

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}.$$

Suppose that G/K admits a G-invariant projectively flat affine connection. As in section 1 we enlarge $\mathfrak{g}$ so that

$$\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbf{R}E,$$
$$[\tilde{\mathfrak{g}}, E] = \{0\},$$

and set $\tilde{\mathfrak{k}} = \mathfrak{k}$, $\tilde{\mathfrak{m}} = \mathfrak{m} \oplus \mathbf{R}E$. Then

$$[\tilde{\mathfrak{k}}, \tilde{\mathfrak{m}}] \subset \tilde{\mathfrak{m}}, \quad [\tilde{\mathfrak{m}}, \tilde{\mathfrak{m}}] \subset \tilde{\mathfrak{k}}.$$

By Theorem 1.1 there exists an affine representation $(\tilde{f}, \tilde{q})$ of $\tilde{\mathfrak{g}}$ on an $(n+1)$-dimensional real vector space $\tilde{V}$ where $n = \dim G/K$. The restriction of $\tilde{q}$ to $\tilde{\mathfrak{m}}$ being an isomorphism, for each $u \in \tilde{V}$ there exists a unique element $X_u \in \tilde{\mathfrak{m}}$ such that $\tilde{q}(X_u) = u$. We put

$$L_u = \tilde{f}(X_u),$$

and define a multiplication law in $\tilde{V}$ by

$$u \cdot v = L_u v.$$

Then the algebra $\tilde{V}$ is commutative and has unit $e = \tilde{q}(E)$. In fact

$$u \cdot v - v \cdot u = \tilde{f}(X_u)\tilde{q}(X_v) - \tilde{f}(X_v)\tilde{q}(X_u) = \tilde{q}([X_u, X_v]) = 0,$$
$$e \cdot u = \tilde{f}(E)u = u.$$

Lemma 3.1. *For $W \in \tilde{\mathfrak{k}}$, $\tilde{f}(W)$ is a derivation of the algebra $\tilde{V}$.*

Proof. Since

$$[W, X_u] \in \tilde{\mathfrak{m}},$$
$$\tilde{q}([W, X_u]) = \tilde{f}(W)u - \tilde{f}(X_u)\tilde{q}(W) = \tilde{f}(W)u,$$

we have

$$[W, X_u] = X_{\tilde{f}(W)u}.$$

Thus we get

$$\begin{aligned}(\tilde{f}(W)u) \cdot v &= \tilde{f}([W, X_u])v \\ &= \tilde{f}(W)\tilde{f}(X_u)v - \tilde{f}(X_u)\tilde{f}(W)v \\ &= \tilde{f}(W)(u \cdot v) - u \cdot (\tilde{f}(W)v).\end{aligned}$$

□

Define a symmetric bilinear form τ on $\tilde{V}$ by $\tau(u, v) = \mathrm{Tr} L_{u \cdot v}$.

Lemma 3.2. *We have*

(i) $[[L_u, L_v], L_w] = L_{[u \cdot w \cdot v]}$, *where* $[u \cdot w \cdot v] = u \cdot (w \cdot v) - (u \cdot w) \cdot v$.
(ii) $\tau(u \cdot v, w) = \tau(v, u \cdot w)$.

Proof. Since

$$\begin{aligned}\tilde{q}([[X_u, X_v], X_w]) &= \tilde{f}([X_u, X_v])\tilde{q}(X_w) - \tilde{f}(X_w)\tilde{q}([X_u, X_v]) \\ &= [L_u, L_v]w \\ &= [u \cdot w \cdot v],\end{aligned}$$

we have

$$[[X_u, X_v], X_w] = X_{[u\cdot w\cdot v]}.$$

This implies (i). Using (i) we obtain

$$\begin{aligned}\tau(u \cdot v, w) - \tau(v, u \cdot w) &= \mathrm{Tr}L_{(u\cdot v)\cdot w} - \mathrm{Tr}L_{v\cdot(u\cdot w)} \\ &= -\mathrm{Tr}L_{[v\cdot u\cdot w]} \\ &= -\mathrm{Tr}[[L_v, L_w], L_u] \\ &= 0.\end{aligned}$$

□

Lemma 3.3. *τ is non-degenerate.*

Proof. We set

$$\tilde{V}_0 = \{v_0 \in \tilde{V} \mid \tau(v_0, v) = 0 \text{ for all } v \in \tilde{V}\}.$$

For $v_0 \in \tilde{V}_0$, $v \in \tilde{V}$ and $W \in \tilde{\mathfrak{k}}$ we have

$$\tilde{q}([W, X_{v_o\cdot v}]) = \tilde{f}(W)\tilde{q}(X_{v_0\cdot v}) - \tilde{f}(X_{v_0\cdot v})\tilde{q}(W) = \tilde{f}(W)(v_0 \cdot v).$$

Hence we know

$$\begin{aligned}[\tilde{f}(W), \tilde{f}(X_{v_0\cdot v})] &= \tilde{f}(X_{\tilde{f}(W)(v_0\cdot v)}) \\ &= L_{\tilde{f}(W)(v_0\cdot v)} \\ &= L_{(\tilde{f}(W)v_0)\cdot v} + L_{v_0\cdot(\tilde{f}(W)v)}.\end{aligned}$$

Thus we obtain

$$\begin{aligned}0 &= \mathrm{Tr}[\tilde{f}(W), \tilde{f}(X_{v_0\cdot v})] \\ &= \mathrm{Tr}L_{(\tilde{f}(W)v_0)\cdot v} + \mathrm{Tr}L_{v_0\cdot(\tilde{f}(W)v)} \\ &= \tau(\tilde{f}(W)v_0, v).\end{aligned}$$

Hence we get

$$\tilde{f}(\tilde{\mathfrak{k}})\tilde{V}_0 \subset \tilde{V}_0.$$

For $v_0 \in \tilde{V}_0$, $v \in \tilde{V}$ and $X \in \tilde{\mathfrak{m}}$ we have

$$\tau(\tilde{f}(X)v_0, v) = \tau(\tilde{q}(X) \cdot v_0, v) = \tau(v_0, \tilde{q}(X) \cdot v) = 0.$$

This means

$$\tilde{f}(\tilde{\mathfrak{m}})\tilde{V}_0 \subset \tilde{V}_0.$$

These show $\tilde{f}(\tilde{\mathfrak{g}})\tilde{V}_0 \subset \tilde{V}_0$, and

$$\tilde{f}(\mathfrak{g})\tilde{V}_0 \subset \tilde{V}_0.$$

Since $\mathfrak{g}$ is semi-simple, the representation $\tilde{f}$ of $\mathfrak{g}$ on $\tilde{V}$ is completely reducible.

Therefore there exists a complementary subspace $\tilde{V}_1$ of $\tilde{V}$ such that

$$\tilde{V} = \tilde{V}_0 \oplus \tilde{V}_1,$$
$$\tilde{f}(\mathfrak{g})\tilde{V}_1 \subset \tilde{V}_1.$$

Since $\tilde{f}(E) = I_{\tilde{V}}$ we have

$$\tilde{f}(\tilde{\mathfrak{g}})\tilde{V}_i \subset \tilde{V}_i \quad (i = 1, 2).$$

Thus we get

$$\tilde{V} \cdot \tilde{V}_i \subset \tilde{V}_i \quad (i = 1, 2).$$

Denoting $e = e_0 + e_1$ where $e_i \in \tilde{V}_i$ we know

$$L_{e_i} v_j = \delta_{ij} v_j \quad \text{for } v_j \in \tilde{V}_j,$$

where δ_{ij} is Kronecker's delta. Hence $\dim V_0 =$ trace of L_{e_0} on $\tilde{V}_0 =$ trace of L_{e_0} on $\tilde{V} = \mathrm{Tr} L_{e_0 \cdot e_0} = \tau(e_0, e_0) = 0$. This implies that τ is non-degenerate. □

Let us recall the definition of Jordan algebra. An algebra $\tilde{V}$ over $\mathbf{R}$ is said to be a *Jordan algebra* if, for all $u, v \in \tilde{V}$,

$$\begin{aligned} u \cdot v &= v \cdot u, \\ u \cdot (u^2 \cdot v) &= u^2 \cdot (u \cdot v). \end{aligned}$$

The following lemma is due to [V1].

Lemma 3.4. *Let $\tilde{V}$ be a commutative algebra with a multiplication $u \cdot v = L_u v$. Suppose*

(a) $[[L_u, L_v], L_w] = L_{[u \cdot w \cdot v]}$,
(b) *the bilinear form $\tau(u, v) = \mathrm{Tr}\, L_{u \cdot v}$ is non-degenerate.*

Then $\tilde{V}$ is a semi-simple Jordan algebra.

Therefore our algebra $\tilde{V}$ is a semi-simple Jordan algebra.

Lemma 3.5. *The representation $\tilde{f}$ of $\tilde{\mathfrak{g}}$ on $\tilde{V}$ is faithful.*

Proof. We set

$$\ker_{\mathfrak{g}} \tilde{f} = \{X \in \mathfrak{g} \mid \tilde{f}(X) = 0\}.$$

We denote by $d_{\tilde{f}}$ the coboundary operator for the cohomology of the Lie algebra $\mathfrak{g}$ with coefficients in $(\tilde{V}, \tilde{f})$. Regarding $\tilde{q}$ as a 1-dimensional $(\tilde{V}, \tilde{f})$-cochain, we have $(d_{\tilde{f}}\tilde{q})(X, Y) = \tilde{f}(X)\tilde{q}(Y) - \tilde{f}(Y)\tilde{q}(X) - \tilde{q}([X, Y]) = 0$ for $X, Y \in \mathfrak{g}$. Since $\mathfrak{g}$ is semi-simple, there exists an element $\tilde{e} \in \tilde{V}$ such that $\tilde{q} = d_{\tilde{f}}\tilde{e}$. Thus we have

$$\tilde{q}(X) = \tilde{f}(X)\tilde{e} \quad \text{for } X \in \mathfrak{g}.$$

This shows that $\ker_{\mathfrak{g}} \tilde{f} \subset \tilde{\mathfrak{k}} = \mathfrak{k}$. By effectiveness we have $\ker_{\mathfrak{g}} \tilde{f} = \{0\}$. Suppose $\tilde{f}(\tilde{X}) = 0$, where $\tilde{X} = X + xE$ $(X \in \mathfrak{g})$. Then $\mathrm{Tr}\tilde{f}(X) = \mathrm{Tr}(-x\tilde{f}(E)) = -x \dim \tilde{V}$. $\mathfrak{g}$ being semi-simple we have $X \in [\mathfrak{g}, \mathfrak{g}]$, and so $\mathrm{Tr}\tilde{f}(X) = 0$. Thus $x = 0$ and $X \in \ker_{\mathfrak{g}} \tilde{f} = \{0\}$. Hence $\tilde{X} = 0$. □

Let $\mathfrak{m}(\tilde{V}) = \{L_v \mid v \in \tilde{V}\}$ and let $\mathfrak{k}(\tilde{V})$ be the vector subspace spanned by $[L_u, L_v]$ $(u, v \in \tilde{V})$. Then $\tilde{f}(\tilde{\mathfrak{m}}) = \mathfrak{m}(\tilde{V})$, and

$$\mathfrak{g}(\tilde{V}) = \mathfrak{k}(\tilde{V}) + \mathfrak{m}(\tilde{V}). \tag{3.1}$$

is a Lie algebra. By Lemma 3.1 an element in $\tilde{f}(\tilde{\mathfrak{k}})$ is a derivation of $\tilde{V}$. Since a derivation of a semi-simple Jordan algebra is inner [BK], we have

$$\tilde{f}(\tilde{\mathfrak{k}}) = \mathfrak{k}(\tilde{V}).$$

Thus

$$\tilde{f} : \tilde{\mathfrak{g}} = \tilde{\mathfrak{k}} + \tilde{\mathfrak{m}} \longrightarrow \mathfrak{g}(\tilde{V}) = \mathfrak{k}(\tilde{V}) + \mathfrak{m}(\tilde{V}) \tag{3.2}$$

is an isomorphism including decompositions.

The *center* $Z(\tilde{V})$ of a Jordan algebra $\tilde{V}$ is by definition [BK]

$$Z(\tilde{V}) = \{u \in \tilde{V} \mid [u \cdot v \cdot w] = [v \cdot u \cdot w] = [v \cdot w \cdot u] = 0 \text{ for all } v, w \in \tilde{V}\}.$$

A Jordan algebra $\tilde{V}$ with unit e is said to be *central-simple* if $\tilde{V}$ is simple and $Z(\tilde{V}) = \mathbf{R}e$.

Lemma 3.6. *Our Jordan algebra $\tilde{V}$ is central-simple.*

Proof. Let $c \in Z(\tilde{V})$. Then $0 = [c \cdot v \cdot u] = [L_c, L_u]v$ for all $u, v \in \tilde{V}$. Thus $[L_c, L_u] = 0$ for all $u \in \tilde{V}$. This together with (3.1) shows that L_c is contained in the center of $\mathfrak{g}(\tilde{V})$. By (3.2) the center of $\mathfrak{g}(\tilde{V})$ is equal to $\tilde{f}$ (the center of $\tilde{\mathfrak{g}}$). Since the center of $\tilde{\mathfrak{g}}$ is $\mathbf{R}E$, we know $L_c \in \tilde{f}(\mathbf{R}E) = \mathbf{R}L_e$. Thus $Z(\tilde{V}) = \mathbf{R}e$. $\tilde{V}$ being semi-simple we have a direct sum decomposition

$$\tilde{V} = \tilde{V}_1 \oplus \cdots \oplus \tilde{V}_k,$$

where $\tilde{V}_i$ are simple ideals of $\tilde{V}$. Let us denote $e = e_1 + \cdots + e_k$ where $e_i \in \tilde{V}_i$. Suppose $\tilde{V}_1 \neq \{0\}$. Then e_1 is the unit of $\tilde{V}_1$. Let $c_1 \neq 0 \in Z(\tilde{V}_1)$. We have

$$[c_1 \cdot \tilde{V}_1 \cdot \tilde{V}_1] = \{0\},$$
$$[c_1 \cdot \tilde{V}_i \cdot \tilde{V}_j] = \{0\}, \text{ if } i \neq 1 \text{ or } j \neq 1.$$

Thus

$$[c_1 \cdot \tilde{V} \cdot \tilde{V}] = \{0\}.$$

Analogously we have

$$[\tilde{V} \cdot c_1 \cdot \tilde{V}] = \{0\}, \quad [\tilde{V} \cdot \tilde{V} \cdot c_1] = \{0\}.$$

Thus $c_1 \in Z(\tilde{V}) = \mathbf{R}e$, and $c_1 = ae$ where $a \neq 0$. This means that $\tilde{V}_i = \{0\}$ if $i \neq 1$. Hence $\tilde{V}$ is simple. □

Summing up the above results we have

Theorem 3.7. *Let (G, K) be an effective symmetric pair where G is semi-simple. Suppose that the space G/K admits a G-invariant projectively flat affine connection. Then there exists a central-simple Jordan algebra $\tilde{V}$ with unit e such that*

(i) *$\tilde{V} = V \oplus \mathbf{R}e$ (direct sum as vector spaces).*
(ii) *Let $\mathfrak{m}(V) = \{L_u \mid u \in V\}$ and let $\mathfrak{k}(V)$ be the vector space spanned by $[L_u, L_v]$ where $u, v \in V$. Then $\mathfrak{g}(V) = \mathfrak{k}(V) + \mathfrak{m}(V)$ is a Lie algebra and is isomorphic to the Lie algebra $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ of G including decompositions.*

Conversely, we have

Theorem 3.8. *Let $\tilde{V}$ be a central-simple Jordan algebra with unit e. We set $V = \{v \in \tilde{V} \mid \mathrm{Tr}\, L_v = 0\}$. Let $\mathfrak{m}(V) = \{L_v \mid v \in V\}$ and let $\mathfrak{k}(V)$ be the vector space spanned by $[L_u, L_v]$ for $u, v \in V$. Then $\mathfrak{k}(V)$ and $\mathfrak{g}(V) = \mathfrak{k}(V) + \mathfrak{m}(V)$ are linear Lie algebras. Let $G(V)$ and $K(V)$ be linear Lie groups generated by $\mathfrak{g}(V)$ and $\mathfrak{k}(V)$, respectively. Then $(G(V), K(V))$ is a symmetric pair, where $G(V)$ is a semi-simple Lie group, and $G(V)/K(V)$ admits a $G(V)$-invariant projectively flat affine connection.*

Proof. It is known [BK] that $\mathfrak{g}(V)$ is a semi-simple Lie algebra and

$$[\mathfrak{k}(V), \mathfrak{m}(V)] \subset \mathfrak{m}(V), \qquad [\mathfrak{m}(V), \mathfrak{m}(V)] \subset \mathfrak{k}(V).$$

Let $\mathfrak{m}(\tilde{V}) = \{L_{\tilde{v}} \mid \tilde{v} \in \tilde{V}\}$ and let $\mathfrak{k}(\tilde{V})$ be the vector space spanned by $[L_{\tilde{u}}, L_{\tilde{v}}]$ for $\tilde{u}, \tilde{v} \in \tilde{V}$. We put

$$\mathfrak{g}(\tilde{V}) = \mathfrak{k}(\tilde{V}) + \mathfrak{m}(\tilde{V}).$$

Then $\mathfrak{g}(\tilde{V})$ is a Lie algebra and

$$\mathfrak{g}(\tilde{V}) = \mathfrak{g}(V) + \mathbf{R} I_{\tilde{V}};$$

cf. [BK]. We define a representation $\tilde{f}$ of $\mathfrak{g}(\tilde{V})$ on $\tilde{V}$ by $\tilde{f}(\tilde{X}) = \tilde{X}$ for $\tilde{X} \in \mathfrak{g}(\tilde{V})$ and a linear mapping $\tilde{q}$ from $\mathfrak{g}(\tilde{V})$ to $\tilde{V}$ by $\tilde{q}(W + L_{\tilde{v}}) = \tilde{v}$ for $W \in \mathfrak{k}(\tilde{V})$, $L_{\tilde{v}} \in \mathfrak{m}(\tilde{V})$. Then $(\tilde{f}, \tilde{q})$ is an affine representation of $\mathfrak{g}(\tilde{V})$ on $\tilde{V}$ satisfying the conditions of Theorem 1.1. Therefore the space $G(V)/K(V)$ admits a $G(V)$-invariant projectively flat affine connection. □

Remark. Let $G(\tilde{V})$ denote the linear Lie group generated by $\mathfrak{g}(\tilde{V})$. Then $G(\tilde{V})$ is the identity component of the structure group of $\tilde{V}$, and the orbit $\tilde{\Omega} = G(\tilde{V})e$ is a ω-domain [BK], [K]. We have

$$\tilde{\Omega} = \mathbf{R}^+ G(V)e = \mathbf{R}^+ G(V)/K(V).$$

Thus $\tilde{\Omega}$ is a cone obtained from $G(V)/K(V)$ by positive dilations at the origin 0.

Remark. For the classification of central-simple Jordan algebras see [BK].

4. Examples

Using typical examples we explain our correspondence between semi-simple symmetric spaces with invariant projectively flat affine connections and central-simple Jordan algebras.

Example 4.1. Quadratic surface $SO(p, n+1-p)/SO(p, n-p)$ $(0 \le p \le n)$.

Denoting by I_p the unit matrix of degree p we set

$$\begin{aligned} J &= \begin{bmatrix} -I_p & 0 \\ 0 & I_{n-p} \end{bmatrix}, \\ \tilde{J} &= \begin{bmatrix} -I_p & 0 \\ 0 & I_{n+1-p} \end{bmatrix} = \begin{bmatrix} J & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Let M_p^n be the connected component of the set defined by

$$\{x \in \mathbf{R}^{n+1} \mid {}^t x \tilde{J} x = 1\}$$

containing ${}^te = [0, \cdots, 0, 1]$. Then M_0^n is a sphere, and M_n^n is a hyperbolic space. Let

$$SO(p, n-p) = \{s \in SL(n, \mathbf{R}) \mid {}^tsJs = J\}.$$

Then we know

$$M_p^n = SO(p, n+1-p)/SO(p, n-p).$$

The Lie algebra $\mathfrak{o}(p, n-p)$ of $SO(p, n-p)$ is

$$\mathfrak{o}(p, n-p) = \{A \in \mathfrak{gl}(n, \mathbf{R}) \mid {}^tAJ + JA = 0\},$$

and the Lie algebra $\mathfrak{g}$ of $SO(p, n+1-p)$ is

$$\mathfrak{o}(p, n+1-p) = \left\{ \begin{bmatrix} A & a \\ -{}^t(Ja) & 0 \end{bmatrix} \mid A \in \mathfrak{o}(p, n-p),\ a \in \mathbf{R}^n \right\}.$$

Let ι be an involutive automorphism of $\mathfrak{g}$ defined by

$$\iota\left(\begin{bmatrix} A & a \\ -{}^t(Ja) & 0 \end{bmatrix} \right) = \begin{bmatrix} A & -a \\ {}^t(Ja) & 0 \end{bmatrix}.$$

Then the canonical decomposition of $\mathfrak{g}$ with respect to ι is

$$\begin{aligned} \mathfrak{g} &= \mathfrak{k} + \mathfrak{m}, \\ \mathfrak{k} &= \left\{ \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \mid A \in \mathfrak{o}(p, n-p) \right\}, \\ \mathfrak{m} &= \left\{ \begin{bmatrix} 0 & a \\ -{}^t(Ja) & 0 \end{bmatrix} \mid a \in \mathbf{R}^n \right\}. \end{aligned}$$

We set

$$\begin{aligned} \tilde{\mathfrak{g}} &= \mathfrak{g} \oplus \mathbf{R}E, \\ \tilde{\mathfrak{m}} &= \mathfrak{m} \oplus \mathbf{R}E, \end{aligned}$$

where $E = I_{n+1}$. We define an affine representation $(\tilde{f}, \tilde{q})$ of $\tilde{\mathfrak{g}}$ on $\mathbf{R}^{n+1}$ by $\tilde{f}(\tilde{X}) = \tilde{X}$, $\tilde{q}(\tilde{X}) = \tilde{X}e$. Then the affine representation $(\tilde{f}, \tilde{q})$ satisfies the conditions of Theorem 1.1. Using the notation in section 3, for $\tilde{u} = {}^t[u_1, \cdots, u_n, u_{n+1}] \in \mathbf{R}^{n+1}$ we have

$$X_{\tilde{u}} = \begin{bmatrix} u_{n+1}I_n & u \\ -{}^t(Ju) & u_{n+1} \end{bmatrix},$$

where $u = {}^t[u_1, \ldots, u_n] \in \mathbf{R}^n$. Hence

$$\begin{bmatrix} u \\ u_{n+1} \end{bmatrix} \cdot \begin{bmatrix} v \\ v_{n+1} \end{bmatrix} = X_{\tilde{u}}\tilde{v} = \begin{bmatrix} u_{n+1}v + v_{n+1}u \\ -{}^tuJv + u_{n+1}v_{n+1} \end{bmatrix}.$$

Thus the Jordan algebra $\mathbf{R}^{n+1}$ with this multiplication coincides with the Jordan algebra $[X; \mu, e]$ associated to the bilinear form $\mu(\tilde{u}, \tilde{v}) = -{}^tuJv + u_{n+1}v_{n+1}$ on $X = \mathbf{R}^{n+1}$, and e [BK, p. 193]. The corresponding ω-domain Ω_p^{n+1} is a cone in $\mathbf{R}^{n+1}$ given by

$$\{x \in \mathbf{R}^{n+1} \mid {}^tx\tilde{J}x > 0\},$$

and $\Omega_p^{n+1} = \mathbf{R}^+ M_p^n$.

Example 4.2. $SL(n, \mathbf{R})/SO(p, n-p)$ $(0 \le p \le n)$.

Let ι be an involutive automorphism of $\mathfrak{g} = \mathfrak{sl}(n, \mathbf{R})$ defined by

$$\iota(X) = -J\, {}^tXJ,$$

where J is the same as in Example 4.1. Then the canonical decomposition of $\mathfrak{g}$ with respect to ι is

$$\begin{aligned} \mathfrak{g} &= \mathfrak{k} + \mathfrak{m}, \\ \mathfrak{k} &= \{A \in \mathfrak{g} \mid \iota(A) = A\} = \mathfrak{o}(p, n-p), \\ \mathfrak{m} &= \{A \in \mathfrak{g} \mid \iota(A) = -A\}. \end{aligned}$$

Denoting $E = I_n$ we set

$$\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbf{R}E = \mathfrak{gl}(n, \mathbf{R}).$$

Let $\tilde{V} = \{v \in \tilde{\mathfrak{g}} \mid \ {}^tv = v\}$. We define an affine representation $(\tilde{f}, \tilde{q})$ of $\tilde{\mathfrak{g}}$ on $\tilde{V}$ by $\tilde{f}(X)v = -\,{}^tXv - vX$, $\tilde{q}(X) = -\,{}^tXJ - JX$. Then the affine representation $(\tilde{f}, \tilde{q})$ satisfies the conditions of Theorem 1.1. Using the notation of section 3 we have

$$X_u = -\frac{1}{2}Ju, \quad \text{for } u \in \tilde{V}.$$

Hence

$$u \cdot v = \tilde{f}(X_u)v = \frac{1}{2}(uJv + vJu).$$

The Jordan algebra with this multiplication is a mutation of the Jordan algebra $\tilde{V}$ with standard multiplication

$$u \circ v = \frac{1}{2}(uv + vu)$$

[BK], and the corresponding ω-domain is a cone $\Omega(p, n-p)$ in $\tilde{V}$ consisting of all symmetric matrices of signature $(p, n-p)$. We know

$$\begin{aligned} \Omega(p, n-p) &= \{\ {}^tgJg \mid g \in GL(n, \mathbf{R})\} \\ &= GL^+(n, \mathbf{R})/SO(p, n-p) = \mathbf{R}^+SL(n, \mathbf{R})/SO(p, n-p), \end{aligned}$$

and $SL(n, \mathbf{R})/SO(p, n-p)$ is a level surface in $\Omega(p, n-p)$ defined by $\det u = 1$ (cf. [Sa]).

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