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RESOLUTIONS OF MONOMIAL IDEALS AND COHOMOLOGY OVER EXTERIOR ALGEBRAS

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Abstract. This paper studies the homology of finite modules over the exterior algebra E of a vector space V. To such a module M we associate an algebraic set $V_E(M) \subseteq V$, consisting of those $v \in V$ that have a non-minimal annihilator in M. A cohomological description of its defining ideal leads, among other things, to complementary expressions for its dimension, linked by a 'depth formula'. Explicit results are obtained for $M = E/J$, when J is generated by products of elements of a basis $e_1, \dots, e_n$ of V. A (infinite) minimal free resolution of E/J is constructed from a (finite) minimal resolution of S/I, where I is the squarefree monomial ideal generated by 'the same' products of the variables in the polynomial ring $S = K[x_1, \dots, x_n]$. It is proved that $V_E(E/J)$ is the union of the coordinate subspaces of V, spanned by subsets of $\{ e_1, \dots, e_n \}$ determined by the Betti numbers of S/I over S.

Introduction

Let V be a vector space with basis $e_1, \dots, e_n$ over a field K, and let $E = \bigwedge(V)$ be the exterior algebra over V. The standard basis elements $e_{k_1} \wedge \cdots \wedge e_{k_s}$ of E, $k_1 < \cdots < k_s$, are called monomials in E. An ideal $J \subseteq E$ generated by monomials is called a monomial ideal. We study the (co)homological algebra of such ideals.

Along with J, we consider the corresponding squarefree monomial ideal I in the polynomial ring $S = K[x_1, \dots, x_n]$. Each S–module F_i in a minimal multigraded free resolution F of S/I can be written in the form

$$F_i = \bigoplus_{j=1}^{\beta_i} S(-a_{ij}) \qquad \text{with uniquely determined} \quad a_{ij} \in \mathbb{N}^n .$$

A well known formula of Hochster [12] on the multigraded Betti numbers of squarefree monomial ideals shows that F is itself squarefree, in the sense that the coordinates of all shifts a_{ij} are equal to 0 or 1. Furthermore, there exist interesting non-minimal squarefree resolutions, for example the Taylor resolution [15].

Given any squarefree resolution F of the monomial ideal $I \subseteq S$, we choose a homogeneous basis B of F and construct a multigraded free resolution G of the monomial ideal J in the exterior algebra E. The resolution depends on B, but different choices of multihomogeneous bases lead to isomorphic complexes; if F is minimal, then so is G. The construction is given in Section 1.

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Section 2 contains applications. An explicit formula gives the multigraded Betti numbers of the monomial ideal $J \subseteq E$ in terms of those of I. As a consequence, some interesting properties of J, like the linearity of its minimal resolution or the independence of its Betti numbers from the characteristic of the base field K, are seen to be equivalent to the corresponding properties of I. We also show that if I is a Gotzmann ideal in S, then J is a Gotzmann ideal in E. Our method yields exterior algebra analogues of the Taylor [15] and Eliahou-Kervaire [10] resolutions.

In Section 3 we associate with each finite E-module M an algebraic set $V_E(M) \subseteq V$. As for modular representations of finite groups, which provide the model, there are two constructions: in terms of the action of the graded ring $\operatorname{Ext}_E(K,K)$ on $\operatorname{Ext}_E(M,K)$, following Quillen [14], or in terms of the action of V on M, mimicking Carlson [7]. We prove that they yield the same result. Along with other properties of $V_E(M)$, this parallels results over group algebras; techniques developed for that case have been successfully extended to other Hopf algebras, but they do not always apply here, because E is *not* a Hopf algebra (in the category of rings). Our approach is similar to that used in [4] to study modules over complete intersections, and takes advantage of the simple structure of $\operatorname{Ext}_E(K,K)$; by Cartan [8] it is the symmetric algebra of $\operatorname{Hom}_K(V,K)$. In particular, we prove that the dimension of $V_E(M)$ is complementary to the (appropriately defined) depth of M over E.

When Δ is a simplicial complex and $J = J_\Delta$ is the ideal in E generated by all monomials $e_{k_1} \wedge \cdots \wedge e_{k_s}$ such that $\{ k_1, \ldots, k_s \} \notin \Delta$, the K-algebra $K\langle\Delta\rangle = E/J_\Delta$ is called the indicator algebra of Δ. It has proved to be important in the study of the f-vector of Δ; see for example [3]. The corresponding squarefree ideal $I = I_\Delta$ in S defines the more familiar Stanley-Reisner ring $K[\Delta] = S/I_\Delta$. In Section 4 we prove that $V_E(K\langle\Delta\rangle)$ is a union of coordinate subspaces of V, determined by the supports of the shifts of a minimal free resolution of the Stanley-Reisner ring $K[\Delta]$ over S. This has consequences for the simplicial cohomology of Δ.

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1. The main construction

In the rest of the paper we fix some—mostly standard—notation.

An n-tuple $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ is *squarefree* if $0 \le a_j \le 1$ for $j = 1, \ldots, n$. For $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ we set $|a| = a_1 + \cdots + a_n$, and $\operatorname{supp}(a) = \{ j \mid a_j \neq 0 \}$; by convention, $\operatorname{supp}(0) = \varnothing$, and $[n] = \{ 1, \ldots, n \}$. For an element u of an n-graded vector space $M = \bigoplus_{a \in \mathbb{Z}^n} M_a$, the notation $\deg(u) = a$ is equivalent to $u \in M_a$; we set $\operatorname{supp}(\deg(u)) = \operatorname{supp}(u)$ and $|\deg(u)| = |u|$. The decomposition $M = \bigoplus_{j \in \mathbb{Z}} M_j$, where $M_j = \bigoplus_{a \in \mathbb{Z}^n, |a| = j} M_a$, turns M into a graded vector space.

Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring on n commuting variables, and let $E = K\langle e_1, \ldots, e_n \rangle$ be the exterior algebra on n alternating variables. They are n-graded by $\deg(x_j) = \deg(e_j) = \varepsilon_j = (0, \ldots, 0, 1, 0 \ldots, 0)$, with 1 in the jth position. For $\sigma \subseteq [n]$ we set $x^\sigma = x_{k_1} \cdots x_{k_s}$ and $e_\sigma = e_{k_1} \wedge \cdots \wedge e_{k_s}$, where $\sigma = \{ k_1, \ldots, k_s \}$ with $k_1 < \cdots < k_s$; we say that e_σ is a *monomial* in E. For $a \in \mathbb{N}^n$ we set $x^a = x_1^{a_1} \cdots x_n^{a_n}$ and $e_a = e_{\operatorname{supp}(a)}$.

The following simple observation is used in many computations.

Observation 1.0. For monomials $u, v \in E$ with $\operatorname{supp}(v) \subseteq \operatorname{supp}(u)$ there exists a unique monomial $u' \in E$ such that $vu' = u$; we then set $v^{-1}u = u'$. For monomials $u, v, w, z \in E$ the equalities below hold whenever the left hand side is defined:

$$(v^{-1}u)w = v^{-1}(uw) \qquad \text{and} \qquad (z^{-1}v)(v^{-1}u) = z^{-1}u\,.$$

Construction 1.1. Let (F, θ) be a *squarefree complex* of n-graded S-modules, meaning that each F_i has a basis B_i with $\deg(f)$ squarefree for all $f \in B_i$.

Let P_i be an n-graded K-vector space with basis B_i, and set $B = \bigsqcup_i B_i$. Let C_j be the n-graded right E-module with basis $\{\, y^{(a)} \mid a \in \mathbb{N}^n\,,\ \deg\big(y^{(a)}\big) = a\,,\ |a| = j\,\}$. The tensor product $C_j \otimes_K P_i$ becomes a right n-graded E–module, by

$$\begin{aligned} \deg(y^{(a)} \otimes f) &= a + b\,; \\ (y^{(a)} \otimes f)e &= (-1)^{|b|} y^{(a)} e \otimes f\,, \end{aligned} \qquad \text{where} \quad b = \deg(f)\,.$$

Let G_ℓ be the residue module of $\bigoplus_{\ell = j+i} C_j \otimes_K P_i$ by the submodule generated by $\{\, y^{(a)} \otimes f \mid \operatorname{supp}(a) \not\subseteq \operatorname{supp}(f)\,\}$, and write $y^{(a)}f$ for the image of $y^{(a)} \otimes f$ in G_ℓ. Thus, G_ℓ is the n-graded right E-module with basis

$$Y_\ell = \left\{ y^{(a)} f \;\middle|\; \begin{array}{l} a \in \mathbb{N}^n\,,\ f \in B_i\,,\ \operatorname{supp}(a) \subseteq \operatorname{supp}(f) \\ \ell = |a| + i\,,\ \deg(y^{(a)} f) = a + \deg(f) \end{array} \right\}.$$

If in the complex (F, θ) the differential of $f \in B_i$ has the form

$$\theta(f) = \sum_{j\colon\, f_j \in B_{i-1}} \lambda_j x^{b - b_j} f_j \qquad \text{with} \quad \lambda_j \in K\,,\ b = \deg(f)\,,\ b_j = \deg(f_j)\,,$$

then define homomorphisms $G_\ell \to G_{\ell-1}$ of n-graded E-modules by

$$\begin{aligned} \delta(y^{(a)} f) &= (-1)^{|b|} \sum_{k \in \operatorname{supp}(a)} y^{(a - \varepsilon_k)} f e_k, \\ \vartheta(y^{(a)} f) &= (-1)^{|a|} \sum_{j\colon\, f_j \in B_{i-1}} y^{(a)} f_j \lambda_j e_{b_j}^{-1} e_b\,. \end{aligned}$$

and set $\partial = \delta + \vartheta \colon G_\ell \to G_{\ell-1}$.

Proposition 1.2. *The preceding construction yields a complex (G, ∂) of right n-graded E-modules. If (G', ∂') is the complex obtained from homogeneous bases B'_i of F_i, then $G' \cong G$ as complexes of n-graded E-modules.*

Hochster's formula [12] for the Betti numbers of a squarefree monomial ideal $I \subseteq S$ shows that its minimal free resolution (F, θ) is squarefree. In that case, we can say more about the complex (G, ∂) described above.

Theorem 1.3. *Let Σ be a set of subsets of $[n]$, let $I \subseteq S = K[x_1, \ldots, x_n]$ be the ideal generated by the squarefree monomials $\{\, x^\sigma \mid \sigma \in \Sigma\,\}$, and let $J \subseteq E = K\langle e_1, \ldots, e_n\rangle$ be the ideal generated by the monomials $\{\, e_\sigma \mid \sigma \in \Sigma\,\}$.*

If (F, θ) is a (minimal) free resolution of S/I over S, then the complex (G, ∂) of Construction 1.1 *is a (minimal) free resolution of E/J over E.*

Proof of the proposition. To show that $\partial^2 = 0$ we establish equalities

$$\delta^2 = 0\,; \qquad \vartheta^2 = 0\,; \qquad \delta\vartheta = -\vartheta\delta\,.$$

The first one comes from an easy direct computation.

Writing $\theta(f_j) = \sum_{k\colon\, g_k \in B_{i-2}} \mu_{kj} x^{b_j - c_k} g_k \in F_{i-2}$, we have

$$\begin{aligned} \theta^2(f) &= \sum_j \lambda_j x^{b - b_j} \theta(f_j) = \sum_j \lambda_j x^{b - b_j} \sum_k \mu_{kj} x^{b_j - c_k} g_k \\ &= \sum_k \Big(\sum_j \mu_{kj} \lambda_j \Big) x^{b - c_k} g_k = 0\,. \end{aligned}$$

Thus, $\sum_j \mu_{kj}\lambda_j = 0$, so we get the second equality from:

$$\begin{aligned}\vartheta^2(y^{(a)}f) &= (-1)^{|a|}\sum_j \vartheta(y^{(a)}f_j)(\lambda_j e_{b_j}^{-1}e_b)\\ &= \sum_j\bigg(\sum_k y^{(a)}g_k(\mu_{kj}e_{c_k}^{-1}e_{b_j})\bigg)(\lambda_j e_{b_j}^{-1}e_b)\\ &= \sum_k y^{(a)}g_k\bigg(\sum_j \mu_{kj}\lambda_j(e_{c_k}^{-1}e_{b_j})(e_{b_j}^{-1}e_b)\bigg)\\ &= \sum_k y^{(a)}g_k\bigg(\sum_j \mu_{kj}\lambda_j\bigg)e_{c_k}^{-1}e_b = 0\,.\end{aligned}$$

Note that if $f \in B$ with $\deg(f) = b$ and $e \in E$ with $\deg(e) = c$, then

$$\left.\begin{aligned}\delta(y^{(a)}fe) &= \delta(y^{(a)}f)e\\ \vartheta(y^{(a)}fe) &= \vartheta(y^{(a)}f)e\end{aligned}\right.\qquad \text{provided}\quad \operatorname{supp}(a) \subseteq \operatorname{supp}(b) + \operatorname{supp}(c)\,.$$

When $\operatorname{supp}(a) \subseteq \operatorname{supp}(b)$, these formulas hold by definition. If $\operatorname{supp}(a) \not\subseteq \operatorname{supp}(b)$, then $y^{(a)}f = 0$, so we check that the right hand sides vanish. On the one hand, $\delta(y^{(a)}fe) = \pm\sum_{k\in\operatorname{supp}(a)} y^{(a-\varepsilon_k)}fe_k e$; if $\operatorname{supp}(a-\varepsilon_k) \not\subseteq \operatorname{supp}(b)$, then $y^{(a-\varepsilon_k)}f = 0$; otherwise, $k \in \operatorname{supp}(a) \setminus \operatorname{supp}(f)$, hence $k \in \operatorname{supp}(c)$, so $e_k e = 0$. On the other hand, $\vartheta(y^{(a)}f) = \pm\sum_j y^{(a)}g_j(\lambda_j e_{b_j}^{-1}e_b)$ with $g_j \in B$. Since $\operatorname{supp}(g_j) \subseteq \operatorname{supp}(f)$, for all j we have $\operatorname{supp}(a) \not\subseteq \operatorname{supp}(g_j)$, and hence $y^{(a)}g_j = 0$.

The third equality now results from the computation:

$$\begin{aligned}\vartheta(\delta(y^{(a)}f)) &= (-1)^{|b|}\vartheta\bigg(\sum_{k\in\operatorname{supp}(a)} y^{(a-\varepsilon_k)}fe_k\bigg) = (-1)^{|b|}\sum_{k\in\operatorname{supp}(a)}\vartheta(y^{(a-\varepsilon_k)}f)e_k\\ &= (-1)^{|b|+|a|-1}\sum_{k\in\operatorname{supp}(a)}\bigg(\sum_{j\colon f_j\in B_{i-1}} y^{(a-\varepsilon_k)}f_j\lambda_j e_{b_j}^{-1}e_b\bigg)e_k\\ &= (-1)^{|a|-1}\sum_{j\colon f_j\in B_{i-1}}\bigg(\sum_{k\in\operatorname{supp}(a)}(-1)^{|b_j|}y^{(a-\varepsilon_k)}f_je_k\bigg)\lambda_j e_{b_j}^{-1}e_b\\ &= (-1)^{|a|-1}\sum_{j\colon f_j\in B_{i-1}}\delta(y^{(a)}f_j)\lambda_j e_{b_j}^{-1}e_b\\ &= (-1)^{|a|-1}\delta\bigg(\sum_{j\colon f_j\in B_{i-1}} y^{(a)}f_j\lambda_j e_{b_j}^{-1}e_b\bigg) = -\delta(\vartheta(y^{(a)}f))\,.\end{aligned}$$

When (G', ∂') is a complex obtained from a homogeneous basis B' of F, write each $f' \in B'_i$ in the form $f' = \sum_{j\colon f_j\in B_i}\lambda_j x^{b'-b_j}f_j$ with $b' = \deg(f')$ and $b_j = \deg(f_j)$, and define homomorphisms of E-modules $\gamma_i\colon G'_i \to G_i$ by

$$\gamma_i(y^{(a)}f') = \sum_{j\colon f_j\in B_i} y^{(a)}f\lambda_j e_{b_j}^{-1}e_{b'}\,.$$

Computations similar to (and more straightforward than) those above show that $\gamma(\vartheta'(y^{(a)}f')) = \vartheta(\gamma(y^{(a)}f'))$ and $\gamma(\delta'(y^{(a)}f')) = \delta(\gamma(y^{(a)}f'))$, so γ is a chain map. It is clearly bijective, so we have the desired isomorphism. □

Proof of the theorem. Let (F,θ) be an n-graded free resolution of S/I over S, and let (G,∂) be the complex obtained from it by Construction 1.1. To show that it is a resolution of E/J, we construct a K-linear chain homotopy χ such that

$$(*) \qquad \chi\partial + \partial\chi = \mathrm{id}_{\widetilde{G}}$$

where $\widetilde{G}$ is the complex obtained from G by replacing G_0 with J.

Since F is exact, there is a homogeneous K-linear chain homotopy τ such that

$$\tau\theta + \theta\tau = \mathrm{id}_{\widetilde{F}}$$

where $\widetilde{F}$ is the complex obtained from F by replacing F_0 with I.

Thus, for $f \in B$ with $\deg(f) = b$ and $\sigma \subseteq [n]$ such that $\mathrm{supp}(b) \cap \sigma = \varnothing$, we have

$$\tau(fx^\sigma) = \sum_k \mu_k x^\sigma x^{b-a_k} h_k \quad \text{where } \mu_k \in K\,,\ h_k \in B\,,\ a_k = \deg(h_k)\,.$$

We define a K-linear map χ on the K-basis of $\tilde{G}$ described in Construction 1.1 by

$$\chi(y^{(a)} f e_\sigma) = \begin{cases} \sum_k h_k \mu_k e_{a_k}^{-1}(e_b e_\sigma) & \text{if } a = 0 \text{ and } \mathrm{supp}(b) \cap \sigma = \varnothing & (1)\\ (-1)^{r+|b|} y^{\varepsilon_s} f e_{\sigma\setminus\{s\}} & \text{if } a = 0 < \min(\mathrm{supp}(b)\cap\sigma) = s & (2)\\ 0 & \text{if } a \neq 0 \text{ and } \mathrm{supp}(b) \cap \sigma = \varnothing & (3)\\ 0 & \text{if } 0 < \min(a) < \min(\mathrm{supp}(b)\cap\sigma) & (4)\\ (-1)^{r+|b|} y^{(a+\varepsilon_s)} f e_{\sigma\setminus\{s\}} & \text{if } \min(a) \geq \min(\mathrm{supp}(b)\cap\sigma) = s & (5) \end{cases}$$

where $b = \mathrm{supp}(f)$ and $r = |\{\, k \in \sigma \mid k < \min(\mathrm{supp}(b)\cap\sigma)\,\}|\,.$

We establish $(*)$, by four separate computations. To simplify notation, we set

$$s(c) = \mathrm{supp}(c) \quad \text{for } c \in \mathbb{N}^n \qquad \text{and} \qquad u_j = \lambda_j e_{b_j}^{-1} e_b \quad \text{for } j \in [n]\,.$$

(1) One has $\partial(fe_\sigma) = \sum_j f_j(u_j)e_\sigma$. Since $s(u_j) = s(b) \setminus s(b_j)$ for every j, we get

$$s(u_j) \cap \sigma = \varnothing \quad \text{and} \quad s(b_j) \cap s(u_j e_\sigma) = s(b_j) \cap \sigma = \varnothing\,.$$

Write $\tau(f_j x^\sigma x^{b-b_j}) = \sum_\ell g_\ell \nu_{\ell j} x^\sigma x^{b-c_\ell}$ with $g_\ell \in B$, $\nu_{\ell j} \in K$ and $c_\ell = \deg(g_\ell)$. As $e_{b_j} u_j e_\sigma = \lambda_j e_b e_\sigma$, one has $\chi(f_j u_j e_\sigma) = \lambda_j \sum_\ell g_\ell \nu_{\ell j} e_{c_\ell}^{-1}(e_b e_\sigma)$, therefore

$$\chi(\partial(fe_\sigma)) = \sum_\ell g_\ell \Bigg(\sum_j \lambda_j \nu_{\ell j} \Bigg) e_{c_\ell}^{-1}(e_b e_\sigma)\,.$$

On the other hand, if $\theta(h_k) = \sum_\ell g_\ell \lambda_{\ell k} x^{a_k - c_\ell}$ with $\lambda_{\ell k} \in K$, then

$$\partial(\chi(fe_\sigma)) = \sum_k \sum_\ell g_\ell \mu_k \lambda_{\ell k} (e_{c_\ell}^{-1} e_{a_k})\big(e_{a_k}^{-1}(e_b e_\sigma)\big) = \sum_\ell g_\ell \Bigg(\sum_k \mu_k \lambda_{\ell k} \Bigg) e_{c_\ell}^{-1}(e_b e_\sigma).$$

Since $\theta\tau + \tau\theta = \mathrm{id}_F$, we see that there exists a ℓ_0 such that $g_{\ell_0} = f$, and

$$\sum_k \mu_k \lambda_{\ell k} + \sum_j \lambda_j \nu_{\ell j} = \begin{cases} 1 & \text{if } \ell = \ell_0\,; \\ 0 & \text{if } \ell \neq \ell_0\,. \end{cases}$$

This shows that $\partial\chi(fe_\sigma) + \chi\partial(fe_\sigma) = fe_\sigma$, as desired.

(2) and (5) In either case, $\partial\chi(y^{(a)} f e_\sigma)$ is equal to

$$(-1)^r \sum_{k \in s(a+\varepsilon_s)} y^{(a+\varepsilon_s-\varepsilon_k)} f e_k e_{\sigma\setminus\{s\}}$$
$$+ (-1)^{r+|b|+|a|+1} \sum_{j:\ s(b_j) \supseteq s(a+\varepsilon_s)} y^{(a+\varepsilon_s)} f_j u_j e_{\sigma\setminus\{s\}}\,.$$

Note that $y^{(a)}fe_\sigma$ appears above as a summand in the first sum for $k = s$. Now we compute $\chi(\partial(y^{(a)}fe_\sigma))$. If $s \notin s(b_j)$ for some j, then $s \in s(b) \setminus s(b_j) = s(u_j)$, therefore $u_je_s = 0$, so that in $\partial(y^{(a)}fe_\sigma)$ only the summands $y^{(a)}f_ju_je_\sigma$ with $s \in s(b_j)$ remain. In this case $\min(s(b_j) \cap s(u_je_\sigma)) = s$, hence

$$\chi(y^{(a)}f_ju_je_\sigma) = (-1)^{r+|b_j|+|u_j|}y^{(a+\varepsilon_s)}f_ju_je_{\sigma\setminus\{s\}}\,.$$

Since $|u_j| + |b_j| = |b|$, we see that the second sum in $\partial\chi(y^{(a)}fe_\sigma)$ appears in $\chi(\partial(y^{(a)}fe_\sigma))$ with the opposite sign. If $k \in s(a)$ and $k \notin \sigma$, then $k \geq \min(a) \geq s$, so $\min(s(b) \cap (\sigma \cup k)) = s$. As $\min(a - \varepsilon_k) \geq \min(a) \geq s$, we get

$$(-1)^{|b|}\chi(y^{(a-\varepsilon_k)}fe_ke_\sigma) = (-1)^{r+1}y^{(a+\varepsilon_s-\varepsilon_k)}fe_ke_{\sigma\setminus\{s\}}\,.$$

The desired equality follows.

(3) For each j with $s(a) \subseteq s(b_j)$, one has $s(b_j) \cap \sigma = \varnothing$, hence $\chi(y^{(a)}f_ju_je_\sigma) = 0$. Let $k \in s(a)$, $k \notin \sigma$ and consider $\chi(y^{(a-\varepsilon_k)}fe_ke_\sigma)$. We now have $s(b) \cap (\sigma \cup k)) = k$. If $k > \min(a)$, then $\min(a - \varepsilon_k) = \min(a)$, therefore $\chi(y^{(a-\varepsilon_k)}fe_ke_\sigma) = 0$. Let $k = \min(a)$. Then $\min(a - \varepsilon_k) \geq k$, hence $(-1)^{|b|}\chi(y^{(a-\varepsilon_k)}fe_ke_\sigma) = y^{(a)}fe_\sigma$. This proves the desired equality.

(4) For each j with $s(a) \subseteq s(b_j)$, one has $u_je_m = 0$ or $\min(s(b_j) \cap \sigma) = m$, so that in both cases $\chi(y^{(a)}f_ju_je_\sigma) = 0$. Let $k \in s(a)$, $k \notin \sigma$ and consider $\chi(y^{(a-\varepsilon_k)}fe_ke_\sigma)$. If $k > \min(a)$, then $\min(a-\varepsilon_k) = \min(a) < m$, therefore $\min(a) < \min(s(b) \cap (\sigma \cup k))$ and by definition $\chi(y^{(a-\varepsilon_k)}fe_ke_\sigma) = 0$. Let $k = \min(a)$. Then $\min(s(b) \cap (\sigma \cup k)) = k \leq \min(a - \varepsilon_k)$, therefore $(-1)^{|b|}\chi(y^{(a-\varepsilon_k)}fe_ke_\sigma) = y^{(a)}fe_\sigma$. This proves (∗). □

2. Applications

Recall that each finite n-graded module M over $A = E$ or $A = S$ has a unique up to isomorphism minimal resolution by free n-graded A-modules, and homogeneous A-linear homomorphisms. The *multigraded Betti* number $\beta^A_{ia}(M)$ is the number of basis elements of the ith free module in such a resolution, that are homogeneous of degree a. The *multigraded Poincaré series* of M over A is defined by

$$P^A_M(t, u) = \sum_{i\geq 0}\sum_{a\in\mathbb{N}^n}\beta^A_{ia}(M)t^iu^a\ .$$

For the rest of this section, I is an ideal generated by squarefree monomials in S, and J denotes the corresponding monomial ideal in E.

Counting ranks in the resolution of Theorem 1.3 we get a new proof of [3, (6.4)].

Proposition 2.1. *There is an equality of formal power series*

$$P^E_{E/J}(t, u) = \sum_{i\geq 0}\sum_{a\in\mathbb{N}^n}\beta^S_{ia}(S/I)\frac{t^iu^a}{\prod\limits_{j\in\operatorname{supp}(a)}(1 - tu_j)}\,.$$ □

We record a couple of immediate consequences of this formula.

Corollary 2.2. (1) *The multigraded Betti numbers of I are independent of the characteristic of the field K if and only if this is true for J.*

(2) *The ideal I has a linear free resolution over S if and only if the ideal J has a linear free resolution over E.* □

An important class of ideals in S with linear resolution are the Gotzmann ideals.

Recall that an ideal $L \subseteq A$, where $A = S$ or $A = E$, is called *Gotzmann* if it is generated by elements of the same degree, say d, and its span in degree $d+1$ is the smallest possible: $\operatorname{rank}_K L_{d+1} \le \operatorname{rank}_K L'_{d+1}$ holds for all graded ideals $L' \subseteq A$ with $\operatorname{rank}_K L'_d = \operatorname{rank}_K L_d$. It is a widely open question which monomial ideals are Gotzmann. From a combinatorial point of view, it is particularly interesting for ideals generated by squarefree monomials.

Proposition 2.3. *If the ideal $I \subseteq S$ is Gotzmann, then so is the ideal $J \subseteq E$.*

Note that the converse may fail: $J = (e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_4, e_1 \wedge e_3 \wedge e_4) \subseteq E$ is a Gotzmann ideal, but $I = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_4) \subseteq S$ is not.

Proof. Let $J' \subseteq E$ be an ideal generated in degree d, with $\operatorname{rank}_K J'_d = \operatorname{rank}_k J_d$.

The algebraic Kruskal-Katona Theorem [3, (4.4)] yields a monomial ideal J^{lex} generated in degree d, with $\operatorname{rank}_K J^{\text{lex}}_d = \operatorname{rank}_K J'_d$ and $\operatorname{rank}_K J^{\text{lex}}_{d+1} \le \operatorname{rank}_K J'_{d+1}$. For the squarefree monomial ideal $I' \subseteq S$ corresponding to J^{lex}, we have

$$\begin{aligned}\operatorname{rank}_K J_{d+1} &= n\beta_{0d}(J) - \beta_{1d+1}(J)\\ &= n\beta_{0d}(I) - (\beta_{1d+1}(I) + d\beta_{0d}(I))\\ &= \operatorname{rank}_K I_{d+1} - d \operatorname{rank}_K I_d\\ &\le \operatorname{rank}_K I'_{d+1} - d \operatorname{rank}_K I'_d\\ &= n\beta_{0d}(I') - (\beta_{1d+1}(I') + d\beta_{0d}(I'))\\ &= n\beta_{0d}(J^{\text{lex}}) - \beta_{1d+1}(J^{\text{lex}})\\ &= \operatorname{rank}_K J^{\text{lex}}_{d+1}\end{aligned}$$

where the inequality is the Gotzmann hypothesis on I, the second and penultimate equalities come from Proposition 2.1, the rest are read off from the corresponding minimal resolutions. Altogether, we get $\operatorname{rank}_K J_{d+1} \le \operatorname{rank}_K J'_{d+1}$, as desired. ☐

Applying Theorem 1.3 to the Taylor resolution of monomial ideals in polynomial rings (cf. [15] or [9, p. 439]), we obtain an analogue over exterior algebras.

For a set of monomials $\{ u_1, \dots, u_m \}$ and a subset $\tau \subseteq [m] = \{ 1, \dots, m \}$, we denote u_τ to be the least common multiple of the monomials $\{ u_j \mid j \in \tau \}$.

Proposition 2.4. *Let $J \subseteq E$ be an ideal generated by a set $\{ u_1, \dots, u_m \}$ of monomials. The right E-modules T_i with basis*

$$\{ y^{(a)} f_\tau \mid a \in \mathbb{N}^n ,\ |a| + |\tau| = i ,\ \tau \subseteq [m] ,\ \operatorname{supp}(a) \subseteq \operatorname{supp}(u_\tau) \}$$

where $\deg(y^{(a)} f_\tau) = a + \deg u_\tau$, *and the E-linear maps defined by*

$$\partial(y^{(a)} f_\tau) = (-1)^{|u_\tau|} \sum_{k \in \operatorname{supp}(a)} y^{(a-\varepsilon_k)} f_\tau e_k + \sum_{j:\ \operatorname{supp}(u_{\tau\setminus\{j\}}) \supseteq \operatorname{supp}(a)} (-1)^{r_j + |a|} y^{(a)} f_{\tau\setminus\{j\}} u^{-1}_{\tau\setminus\{j\}} u_\tau$$

where $r_j = |\{ t \in \tau \mid t < j \}|$, form an n-graded resolution of E/J. ☐

Example 2.5. When $J = (e_1, \ldots, e_n)$, the proposition provides a minimal n-graded resolution of $K = E/(e_1, \ldots, e_n)$ over E. Another one is the *Cartan resolution* (C, ∂), where C_i has a basis $\{ w^{(c)} \mid c \in \mathbb{N}^n, |c| = i \}$, and

$$d(w^{(c)}) = \sum_{k \in \operatorname{supp}(c)} w^{(c-\varepsilon_k)} e_k \,.$$

To get an isomorphism of complexes $\gamma\colon C \to T$, note that each $c \in \mathbb{N}^n$ can be written uniquely as $c = a + b$ with $\operatorname{supp}(c) = \operatorname{supp}(b)$ and b squarefree, and set

$$\gamma(w^{(c)}) = (-1)^{|b|(|a|+(|b|-1)/2)} y^{(a)} f_{\operatorname{supp}(b)} \,.$$

Our last application is to stable ideals, a notion extended in [3] from polynomial rings to exterior algebras: setting $\max(e_\sigma) = \max\{ i \mid i \in \sigma \}$, call a monomial ideal $J \subseteq E$ *stable* if $e_j e_{\sigma \setminus \{m\}} \in J$ for each $e_\sigma \in J$ and each $j < m = \max(e_\sigma)$.

For a monomial ideal $J \subseteq E$, we denote $G(J)$ the uniquely defined minimal generating set of J consisting of monomials. As in [10], it is easily seen that each monomial $u' \in J$ has a unique decomposition $u' = uw$ with $u \in G(J)$ and $\max(u) < \min(w)$. Applying Theorem 1.3 to the resolution of squarefree stable ideals in S given in [2], we get a resolution for stable monomial ideals in E.

Proposition 2.6. *If $J \subseteq E$ is a stable ideal, then E/J has a minimal resolution (G, ∂) by n-graded free E-modules G_ℓ with basis*

$$\left\{ y^{(a)} f_{\sigma,u} \;\middle|\; \begin{matrix} a \in \mathbb{N}^n \,, \ \sigma \subseteq [n] \,, \ u \in G(J) \\ \operatorname{supp}(a) \subseteq \sigma \cup \operatorname{supp}(u) \,, \ \sigma \cap \operatorname{supp}(u) = \varnothing \,, \ \max(\sigma) < \max(u) \\ i = |a| + |\sigma| + 1 \,, \ \deg(y^{(a)} f_{\sigma,u}) = a + \deg(e_\sigma) + \deg(u) \end{matrix} \right\}$$

and differentials $\partial_\ell\colon G_\ell \to G_{\ell-1}$ given by

$$\begin{aligned} \partial(y^{(a)} f_{\sigma,u}) = (-1)^{|u|+|\sigma|} & \sum_{\ell \in \operatorname{supp}(a)} y^{(a-\varepsilon_\ell)} f_{\sigma,u} e_\ell \\ & + (-1)^{|a|} \sum_{j \in \sigma} \left((-1)^{|\sigma|} y^{(a)} f_{\sigma \setminus \{j\},u} e_j + (-1)^{(|\sigma|-1)|w_j|} f_{\sigma \setminus \{j\},u_j} w_j \right) \end{aligned}$$

where $u_j \in G(J)$ is determined from the unique decomposition $ue_j = u_j w_j$ described above, and $y^{(b)} f_{\rho,v} = 0$ if $\max(\rho) > \max(v)$ or $\operatorname{supp}(b) \not\subseteq \rho \cup \operatorname{supp}(v)$. □

The preceding result was originally proved by different means in [3, (2.1)].

3. Cohomology

We study right modules over the exterior algebra E. Since the ideal $(V) \subset E$ is nilpotent, each (finite) E-module M has a unique up to isomorphism minimal free resolution F by (finite) free E-modules. The rank $\beta_i^E(M)$ of the free E-module F_i is known as the ith *Betti number* of M over E. The size of F is measured by the *complexity* of M over E, and is introduced as follows:

$$\operatorname{cx}_E M = \inf\{ c \in \mathbb{Z} \mid \beta_i^E(M) \le \alpha i^{c-1} \text{ for some } \alpha \in \mathbb{R} \text{ and all } i \ge 1 \} \,.$$

For each $v \in V = E_1$, the equality $v^2 = 0$ implies $Mv \subseteq \operatorname{Ann}_M(v)$. We say that v is *M-regular* if equality holds, or, equivalently, if the infinite complex of K-spaces

$$(M, \rho^v)\colon \quad \ldots \to M \xrightarrow{\rho^v} M \xrightarrow{\rho^v} M \to \ldots \qquad \text{where} \qquad \rho^v(y) = yv$$

has trivial homology $\mathrm{H}_*(M, \rho^v)$. Otherwise, we say that v is *M-singular*.

The set $V_E(M) \subseteq V$ of M-singular elements is called the *rank variety* of M.

If $M = \bigoplus_{a\in\mathbb{Z}} M_a$ is *graded*, regularity can also be introduced by the vanishing of the cohomology $\mathrm{H}^*(M,v)$ of the finite complex of K-vector spaces

$$(M,v): \qquad \dots \longrightarrow M_{a-1} \xrightarrow{\rho^v_{a-1}} M_a \xrightarrow{\rho^v_a} M_{a+1} \longrightarrow \dots .$$

Recall that when M and N are graded E-modules, their *graded* tensor product $M \otimes^{\mathrm{gr}}_K N$ and homomorphism space $\mathrm{Hom}^{\mathrm{gr}}_K(N,M)$ have *diagonal actions*:

$$\begin{aligned}(x\otimes y)e_\sigma &= \sum_{\tau\subseteq\sigma}(-1)^{k|\tau|}\,\mathrm{sgn}^\tau_{\sigma\setminus\tau}\, xe_\tau\otimes ye_{\sigma\setminus\tau}\\ (\gamma e_\sigma)(y) &= \sum_{\tau\subseteq\sigma}(-1)^{|\tau|(k+(|\tau|+1)/2)}\,\mathrm{sgn}^\tau_{\sigma\setminus\tau}\,\gamma(ye_\tau)e_{\sigma\setminus\tau}\end{aligned} \qquad \text{for} \quad y\in N_k \text{ and } \sigma\subseteq[n]$$

where $\mathrm{sgn}^\tau_{\sigma\setminus\tau}$ is the sign of the permutation $(\tau,\sigma\setminus\tau)$; that these are (graded) E-modules follows from the fact that E is a *super* Hopf algebra.

The properties of $V_E(M)$ are similar to those of the varieties of modular representations, but proofs are simpler; compare the account by Benson [5].

Theorem 3.1. *If the field K is algebraically closed, then the rank varieties of finite E–modules M, N satisfy the following properties.*

(1) $V_E(M)$ *is a cone* (*that is, a homogeneous algebraic subset*) *in* V.
(2) $\dim V_E(M) = \mathrm{cx}_E M$ *and* $2^{n-\mathrm{cx}_E M}$ *divides* $\mathrm{rank}_K M$.
(3) $V_E(M) = \{0\}$ *if and only if M is free.*
(4) $V_E(M) = V_E(N)$ *if M is a syzygy of N.*
(5) *If $M \subseteq N$, then each one of the three varieties $V_E(M)$, $V_E(N)$, $V_E(N/M)$, is contained in the union of the other two.*
(6) $V_E(M\oplus N) = V_E(M)\cup V_E(N)$.
(7) $V_E(M\otimes^{\mathrm{gr}}_K N) = V_E(M)\cap V_E(N) = V_E(\mathrm{Hom}^{\mathrm{gr}}_K(N,M))$ *if M, N are graded.*
(8) *Each cone in V is the rank variety of some graded E–module.*

As over commutative rings, the notion of regularity can be extended to sequences. Elements $v_1,\dots,v_r\in V$ form an *M-regular sequence* if v_i is $(M/M(v_1,\dots,v_{i-1}))$-regular for $1\le i\le r$, in other words, if $yv_i\in M(v_1,\dots,v_{i-1})$ implies that $y\in M(v_1,\dots,v_i)$ for $1\le i\le r$. It is clear that each M-regular sequence can be extended to a maximal one. The supremum of the lengths of M-regular sequences is called the *depth* of M over E, and denoted $\mathrm{depth}_E M$.

Parts of the preceding theorem depend on a depth-formula for modules over exterior algebras that is similar to the extension of the classical Auslander-Buchsbaum equality to modules over complete intersections, obtained in [4].

Theorem 3.2. *If the field K is infinite and M is a finite E-module, then each maximal M-regular sequence has $\mathrm{depth}_E M$ elements, and*

$$\mathrm{depth}_E M + \mathrm{cx}_E M = n\,.$$

Examples 3.3. (1) If $\mathrm{rank}_K M$ is odd, then $\mathrm{cx}_E M = n$.

Indeed, if $\mathrm{depth}_E M > 0$, then taking an M-regular $v\in V$ we get $\mathrm{rank}_K M = \mathrm{rank}_K(\mathrm{Ann}_M(v)) + \mathrm{rank}_K(Mv) = 2\,\mathrm{rank}_K(Mv)$, so $\mathrm{rank}_K M$ is even.

(2) The depth equality fails when K is finite and $n\ge 2$.

Indeed, if $v\in V\setminus\{0\}$, then $E\xrightarrow{\lambda_v} E\xrightarrow{\lambda_v} E$ with $\lambda_v(e)=ve$ is an exact complex of E–modules, so $\mathrm{cx}_E(E/(v)) = 1$, and hence $M=\bigoplus_{v\in V} E/(v)$ has complexity 1; on the other hand, it is clear that $V_E(M) = V$, hence $\mathrm{depth}_R M = 0$.

To begin the proofs, we record some simple facts on regularity.

Remarks 3.4. Let M be an E-module.

(1) When $v^2 = 0$, any $K[v]$-module is a direct sum of copies of $K[v]$ and $K[v]/(v)$. Thus, $v \in V = E_1$ is regular if and only if M is free over the subalgebra $K[v] \subseteq E$.

(2) For $v \in V$, let $\pi\colon E \to E/(v)$ and $\rho\colon M \to M/Mv$ be canonical homomorphisms. If v is M-regular, then they induce isomorphisms

$$\begin{matrix} \operatorname{Ext}^i_\pi(\rho, K)\colon \operatorname{Ext}^i_{E/(v)}(M/Mv, K) \cong \operatorname{Ext}^i_E(M, K) \\ \operatorname{Tor}_i^\pi(\rho, K)\colon \operatorname{Tor}_i^E(M, K) \cong \operatorname{Tor}_i^{E/(v)}(M/Mv, K) \end{matrix} \qquad \text{for } i \geq 0\,.$$

Indeed, M is free over $K[v]$ by (2), so if G is a free resolution of M over E, then G/Gv is a free resolution of M/Mv over $E/(v)$. Thus, $\operatorname{Ext}^*_\pi(\rho, K)$ and $\operatorname{Tor}_*^\pi(\rho, K)$ are the maps induced in homology by the isomorphisms of complexes $\operatorname{Hom}_{E/(v)}(G/Gv, K) \cong \operatorname{Hom}_E(G, K)$ and $G \otimes_E K \cong (G/Gv) \otimes_{E/(v)} K$, respectively.

(3) Regularity of a sequence $\boldsymbol{v} = v_1, \dots, v_d \in V$ is detected by its *Cartan complex* $C(\boldsymbol{v}; M)$, defined by $C_i(\boldsymbol{v}, M) = \bigoplus_{a \in \mathbb{N}^n, |a| = i} w^{(a)} M$ with $w^{(a)} M \cong M$ for each $a \in \mathbb{N}^n$ and $\partial(w^{(a)} u) = \sum_{\ell \in \operatorname{supp}(a)} w^{(a - \varepsilon_\ell)} u e_\ell$ for $u \in M$.

We set $\mathrm{H}(\boldsymbol{v}; M) = \mathrm{H}\big(C(\boldsymbol{v}; M)\big)$, and note that the following are equivalent:

(i) $\boldsymbol{v}$ is M-regular.
(ii) M is a free module over $K[v_1, \dots, v_d]$.
(iii) $\mathrm{H}_1(\boldsymbol{v}; M) = 0$.
(iv) $\mathrm{H}_i(\boldsymbol{v}; M) = 0$ for $i \geq 1$.

Indeed, let E' be an exterior algebra on alternating variables $e'_1, \dots, e'_d$, and let $\varphi\colon E' \to E$ be the homomorphism of K-algebras with $\varphi(e'_i) = v_i$ for $i = 1, \dots, r$. If C' is the Cartan resolution of the right E'–module K (cf. Remark 2.5), then $C(\boldsymbol{v}; M) = C' \otimes_E M$, so $\mathrm{H}_i(\boldsymbol{v}; M) = \operatorname{Tor}_i^{E'}(K, M)$. Thus, (i) $\implies$ (iv) by iterated use of (2). If (iii) holds, then $\operatorname{Tor}_1^{E'}(K, M) = 0$. Computing Tor from a minimal free resolution of M over E' we see that M' is free over E'; it follows that φ is an isomorphism, so (iii) $\implies$ (ii) holds. Finally, (ii) $\implies$ (i) is trivial.

(4) By (3), each permutation of an M-regular sequence is itself M-regular.

To study the geometry of $V_E(M)$ we use product structures in cohomology. We recall the basics, referring to Mac Lane [13] or Bourbaki [6] for details.

Construction 3.5. For E-modules M, L, N and $i, j \in \mathbb{Z}$, *composition pairings*

$$\operatorname{Ext}^j_E(L, N) \times \operatorname{Ext}^i_E(M, L) \to \operatorname{Ext}^{i+j}_E(M, N)$$

are introduced as follows. Let C and G be E-free resolutions of L and M, respectively, and represent elements in $\operatorname{Ext}^i_E(M, L)$ and $\operatorname{Ext}^j_E(L, N)$ by E-linear homomorphisms $\varkappa\colon G_i \to L$ with $\varkappa \partial_{i+1} = 0$ and $\xi\colon C_j \to N$ with $\xi \partial_{j+1} = 0$. Choosing a lifting of $\varkappa$ to an E-linear chain map $\widetilde{\varkappa}\colon G \to C$ of degree $-i$, define the product $\operatorname{cl}(\xi)\operatorname{cl}(\varkappa)$ to be the class of the composition $\xi \widetilde{\varkappa}_{i+j}\colon G_{i+j} \to N$.

The pairings are K-bilinear, associative, and natural (hence, independent of the choices made above). They make $\operatorname{Ext}^*_E(K, K) = \bigoplus_{i=0}^\infty \operatorname{Ext}^i_E(K, K)$ into a graded algebra, and $\operatorname{Ext}^*_E(M, K) = \bigoplus_{i=0}^\infty \operatorname{Ext}^i_E(M, K)$ into a graded left module over it.

Proposition 3.6. *There is a natural isomorphism of graded K-algebras in V*

$$\operatorname{Ext}^*_E(K, K) \cong \operatorname{Sym}^*_K(V^\vee) \quad \textit{where} \quad V^\vee = \operatorname{Hom}_K(V, K)\,.$$

If M is a finite E-module, then the $\operatorname{Ext}^*_E(K, K)$-module $\operatorname{Ext}^*_E(M, K)$ is finite.

Proof. Cartan's resolution (C,∂) of K over E (cf. Example 2.5) is minimal, so

$$\operatorname{Ext}^i(K,K)=\operatorname{H}^i\left(\operatorname{Hom}_E(C,K)\right)=\operatorname{Hom}_E\left(\bigoplus_{a\in\mathbb{N}^n,|a|=i} Ew^{(a)},K\right).$$

The homomorphisms of E-modules $\{\,\chi^a\colon C_i\to K\mid a\in\mathbb{N}^n,\ |a|=i\,\}$, such that $\chi^a\big(w^{(b)}\big)=1$ for $b=a$ and $\chi^a\big(w^{(b)}\big)=0$ for $b\in\mathbb{N}^n$ with $|b|=i$ and $b\neq a$ form a K-basis of $\operatorname{Hom}_E(C,K)$. The E-linear maps

$$\widetilde{\chi}^a_{i+j}\colon C_{i+j}\to C_j\quad\text{defined by}\quad \widetilde{\chi}^a_{i+j}\big(w^{(b)}\big)=\begin{cases} w^{(b-a)} & \text{if}\quad b-a\in\mathbb{N}^n\,;\\ 0 & \text{otherwise}\,,\end{cases}$$

define a lifting of χ^a to a chain map $C\to C$. This means that $\chi^a\chi^b=\chi^{a+b}$ for all $b\in\mathbb{N}^n$, so $\operatorname{Ext}^*_E(K,K)$ is the polynomial ring on $\chi_1=\chi^{\varepsilon_1},\dots,\chi_n=\chi^{\varepsilon_n}$.

To see that the $\operatorname{Ext}^*_E(K,K)$-module $\operatorname{Ext}^*_E(M,K)$ is finite we argue by induction on $q=\max\{\,r\mid ME_r\neq 0\,\}$. If $q=1$, then $M\cong K^s$ for some s and the assertion is clear. If $q>1$, then $M'=M(V)\neq 0$, so the exact sequence $0\to M'\to M\to M''\to 0$ of E-modules yields an exact sequence of $\operatorname{Ext}^*_E(K,K)$-modules

$$\operatorname{Ext}^*_E(M',K)\to\operatorname{Ext}^*_E(M,K)\to\operatorname{Ext}^*_E(M'',K)\tag{3.6.1}$$

in which those on the outside are noetherian by the induction hypothesis. □

Remark 3.7. If $\chi_1,\dots,\chi_n$ is the basis of $V^\vee$ dual to the basis $e_1,\dots,e_n$ of V, then we identify $\operatorname{Ext}^*_E(K,K)$ with the graded polynomial ring $\mathcal{S}=K[\chi_1,\dots,\chi_n]$ in which each χ_i has degree 1; the elements of $\mathcal{S}$ act as functions on V.

Applied to the $\mathcal{S}$-module $\operatorname{Ext}^*_E(M,K)$, the Hilbert-Serre theorem yields:

Corollary 3.8. *The Krull dimension of the $\mathcal{S}$-module $\operatorname{Ext}^*_E(M,K)$ is equal to $\operatorname{cx}_E M$, and there exists a polynomial $p_M(t)\in\mathbb{Z}[t]$ with $p_M(1)>0$, such that*

$$P^E_M(t)=\frac{p_M(t)}{(1-t)^c}\qquad\text{with}\qquad c=\operatorname{cx}_E M\,.$$ □

Now we give a basic cohomological description of the rank variety.

Theorem 3.9. *If K is algebraically closed and M is a finite E-module, then*

$$V_E(M)=\left\{\,v\in V\mid \xi(v)=0\ \textit{for all}\ \xi\in\operatorname{Ann}_{\mathcal{S}}\left(\operatorname{Ext}^*_E(M,K)\right)\right\}.$$

Proof. Let $\mathcal{I}=\operatorname{Ann}_{\mathcal{S}}\left(\operatorname{Ext}^*_E(M,K)\right)$. For $v\in V$, set $V^v=\operatorname{Ker}\left(V^\vee\to(vK)^\vee\right)$, and let $\mathcal{P}_v$ denote the homogeneous prime ideal (V^v) of $\mathcal{S}$. By the Nullstellensatz, we have to prove that $\mathcal{I}\subseteq\mathcal{P}_v$ if and only if v is M-singular.

If v is singular, then by Remark 3.4 (1) we have an isomorphism of $K[v]$-modules $M\cong K[v]^p\oplus K^q$ with $q>0$. The inclusion $\iota\colon K[v]\hookrightarrow E$ induces a diagram:

$$\begin{array}{ccc}
\operatorname{Ext}^*_E(K,K)\otimes_K\operatorname{Ext}^*_E(M,K) & \longrightarrow & \operatorname{Ext}^*_E(M,K)\\
\Big\downarrow{\scriptstyle \operatorname{Ext}^*_\iota(K,K)\otimes\operatorname{Ext}^*_\iota(M,K)} & & \Big\downarrow{\scriptstyle \operatorname{Ext}^*_\iota(M,K)}\\
\operatorname{Ext}^*_{K[v]}(K,K)\otimes_K\operatorname{Ext}^*_{K[v]}(M,K) & \longrightarrow & \operatorname{Ext}^*_{K[v]}(M,K)\,.
\end{array}$$

It commutates by naturality of composition products, so $\operatorname{Ext}^*_\iota(K,K)(\mathcal{I})$ annihilates

$$\operatorname{Ext}^*_{K[v]}(M,K)\cong K^p\oplus\operatorname{Ext}^*_{K[v]}(K,K)^q\,.$$

It is then equal to 0, that is, $\mathcal{I}\subseteq\operatorname{Ker}\operatorname{Ext}^*_\iota(K,K)=\mathcal{P}_v$.

If v is regular, then $\pi\colon E \to E/(v)$ and $\rho\colon M \to M/Mv$ induce a diagram

$$\begin{array}{ccc} \operatorname{Ext}^*_E(K,K) \otimes_K \operatorname{Ext}^*_E(M,K) & \longrightarrow & \operatorname{Ext}^*_E(M,K) \\ {\scriptstyle \operatorname{Ext}^*_\pi(K,K)\otimes \operatorname{Ext}^*_\pi(\rho,K)}\Big\uparrow & & \Big\uparrow{\scriptstyle \operatorname{Ext}^*_\pi(\rho,K)} \\ \operatorname{Ext}^*_{E/(v)}(K,K) \otimes_K \operatorname{Ext}^*_{E/(v)}(M/Mv,K) & \longrightarrow & \operatorname{Ext}^*_{E/(v)}(M/Mv,K)\,. \end{array}$$

It is commutative by naturality, and $\operatorname{Ext}^*_\pi(\rho,K)$ is an isomorphism by Remark 3.4 (2). Since $\operatorname{Ext}^*_{E/(v)}(M/Mv,K)$ is a finite $\operatorname{Ext}^*_{E/(v)}(K,K)$-module by Proposition 3.6, we conclude that $\operatorname{Ext}^*_E(M,K)$ is also. It follows that the composition

$$\operatorname{Ext}^*_{E/(v)}(K,K) \xrightarrow{\operatorname{Ext}^*_\pi(K,K)} \operatorname{Ext}^*_E(K,K) = \mathcal{S} \to \mathcal{S}/\mathcal{I}$$

is a finite homomorphism of rings. Assuming that $\mathcal{P}_v \supseteq \mathcal{I}$, we conclude that

$$\operatorname{Sym}^*_K[V^v] \cong \operatorname{Ext}^*_{E/(v)}(K,K) \to \mathcal{S}/\mathcal{P}_v = \operatorname{Ext}^*_{K[v]}(K,K) \cong \operatorname{Sym}^*_K[(Kv)^\vee]$$

is a finite homomorphism; this is absurd, since it maps V^v to 0. □

Proof of Theorem 3.2. Let $\boldsymbol{v} = v_1,\dots,v_d$ be an arbitrary maximal M-regular sequence in V. We want to prove that $\operatorname{depth}_E M = d$ and $\operatorname{cx}_E M = n-d$.

We first assume that K is algebraically closed; the elements in a regular sequence being K-linearly independent, we have $d \le n$, so we can induce on d. An equality $d=0$ means that each element of V is M-singular, that is, $\operatorname{depth}_E M = 0$; on the other hand, Theorem 3.9 yields $\operatorname{cx}_R M = \dim V_E(M) = \dim V = n$.

If $d > 0$, then the images of $v_2,\dots,v_d$ in $E/(v_1)$ form a maximal (M/Mv_1)-regular sequence. The induction hypothesis yields $\operatorname{depth}_E(M/Mv_1) = d-1$ and

$$\operatorname{cx}_{E/(v_1)}(M/Mv_1) = (n-1)-(d-1) = n-d\,.$$

As $\operatorname{cx}_{E/(v_1)}(M/Mv_1) = \operatorname{cx}_E M$ by Remark 3.4 (2), we are done.

Now let K be an arbitrary infinite field. Taking an algebraic closure $\bar K$ of K, we consider the finite module $\bar M = M\otimes_K \bar K$ over the exterior algebra $\bar E = E\otimes_K \bar K$ of the $\bar K$-vector space $\bar V = V\otimes_K \bar K$. Due to the flatness of $\bar E$ over E, we see that (considered as a sequence in $\bar V$) any M-regular sequence in V is $\bar M$-regular, and that $\beta_i^{\bar E}(\bar M) = \beta_i^E(M)$ for each i. This yields

$$\operatorname{depth}_E M \le \operatorname{depth}_{\bar E} \bar M = d \qquad\text{and}\qquad \operatorname{cx}_E M = \operatorname{cx}_{\bar E}\bar M = n-d\,.$$

Assuming that the $\bar M$-regular sequence $\boldsymbol{v}$ is not maximal, we can find in $\bar V/\bar K\boldsymbol{v}$ an element v that is $\big(\bar M/\bar M(\boldsymbol{v})\big)$-regular. As the set of regular elements is Zariski-open and K is infinite, we can even pick v in $V/(\boldsymbol{v})$, and get an M-regular sequence $\boldsymbol{v}, v$. This is absurd, so $\boldsymbol{v}$ is a maximal $\bar M$-regular sequence and we have

$$d \le \operatorname{depth}_E M \le \operatorname{depth}_{\bar E}\bar M = d\,.$$

It follows that $\operatorname{depth}_E M = d$ and $\operatorname{depth}_E M + \operatorname{cx}_E M = n$, as desired. □

Lemma 3.10. *For each $\xi \in \operatorname{Ext}^i_E(K,K)$ there is a graded E-module L_ξ such that*

$$V_E(L_\xi) = \{\, v\in V \mid \xi(v) = 0 \,\}\,.$$

Proof. In the Cartan resolution C of K over E, set $D_i = \partial_i(C_i)$, let $\bar\xi\colon D_i \to K$ be the E-linear map that corresponds to ξ under the isomorphisms

$$\operatorname{Ext}^i(K,K) = \operatorname{Hom}_E(C_i,K) \cong \operatorname{Hom}_E(D_i,K)$$

and set $L_\xi = \operatorname{Ker}\bar\xi$. The exact sequence of E-modules

$$0 \to L_\xi \to D_i \to K \to 0$$

induces an exact sequence of graded modules over $\mathcal{S} = \operatorname{Ext}^*(K, K)$,

$$\mathcal{S} \xrightarrow{\bar{\xi}^*} \operatorname{Ext}^*_E(D_i, K) \to \operatorname{Ext}^*_E(L_\xi, K) \xrightarrow{\eth} \mathcal{S}(1) \xrightarrow{\bar{\xi}^*(1)} \operatorname{Ext}^*_E(D_i, K)(1)$$

where $\bar{\xi}^* = \operatorname{Ext}^*_E(\bar{\xi}, K)$ maps $1 \in \mathcal{S}^0$ to $\xi \in \operatorname{Ext}^i_E(D_i, K) = \mathcal{S}^i$. Thus, $\bar{\xi}^*$ and $\bar{\xi}^*(1)$ are injective, yielding $\operatorname{Ext}^*_E(L_\xi, K) \cong \mathcal{S}^{\geq i}(i)/\mathcal{S}\xi$. As $\sqrt{\mathcal{S}^{\geq i}(i)/\mathcal{S}\xi} = \sqrt{\mathcal{S}\xi}$, we conclude from Theorem 3.9 that $V_E(L_\xi)$ has the desired form. ☐

Proof of Theorem 3.1. (1) Note that $\operatorname{rank}_K(Mv) \leq \operatorname{rank}_K\big(\operatorname{Ann}_M(v)\big)$ for each $v \in V$, and the inequality is strict precisely when v is M-singular. Setting $m = \operatorname{rank}_K M$, we rewrite the inequality as $\operatorname{rank}_K(\rho^v) < m - \operatorname{rank}_K(\rho^v)$, that is, as $\operatorname{rank}_K(\rho^v) < m/2$. Thus, $V_E(M)$ is the zero-set of the minors of order $\lceil m/2 \rceil$ of a matrix representing multiplication by a generic element of V. Clearly, $v \in V_E(M)$ implies $\lambda v \in V_E(M)$ for each $\lambda \in K$, so the variety is homogeneous.

(2) Let $\operatorname{cx}_E M = c$. By Corollary 3.8 and elementary dimension theory, the number c is equal to the Krull dimension of the ring $\mathcal{S}/\operatorname{Ann}_{\mathcal{S}}\big(\operatorname{Ext}^*_E(M, K)\big)$, which is the dimension of the variety $V_E(M)$.

Theorem 3.2 yields an M-regular sequence $v_1, \ldots, v_{n-c}$ in V, so M is free over $E' = K[v_1, \ldots, v_{n-c}]$ by Remark 3.4, so $\operatorname{rank}_K M = 2^{n-c} \operatorname{rank}_{E'} M$.

(3) If $V_E(M) = \{0\}$, then $\operatorname{cx}_E M = 0$, so the preceding argument works with $r = n$, and shows that M is free over $K[v_1, \ldots, v_n] = E$. Conversely, if M is free over E the non-zero elements of V are obviously M-regular, hence $V_E(M) = \{0\}$.

(5) An exact sequence of E-modules $0 \to M \to N \to M/N \to 0$ induces an exact sequence of complexes of vector spaces

$$0 \to (M, \rho^v) \to (N, \rho^v) \to (M/N, \rho^v) \to 0$$

and hence an exact sequence of homology spaces

$$\mathrm{H}_*(M, \rho^v) \to \mathrm{H}_*(N, \rho^v) \to \mathrm{H}_*(M/N, \rho^v) \to \mathrm{H}_*(M, \rho^v) \to \mathrm{H}_*(N, \rho^v)$$

which implies that the desired assertions follow immediately.

(4) It suffices to consider the case when M and N appear in an exact sequence $0 \to M \to P \to N \to 0$ with a free E-module P. By (5) and (3) we then have

$$V_E(M) \subseteq V_E(N) \cup V_E(P) = V_E(N) \subseteq V_E(M) \cup V_E(P) = V_E(M)\,.$$

(6) follows immediately from the definitions.

(7) Recall that $v \in V$ acts on $M \otimes^{\mathrm{gr}}_K M$ by the formula $(x \otimes y)v = x \otimes yv + (-1)^k xv \otimes y$, when $y \in N_k$. This means that $x \otimes y \mapsto y \otimes x$ is an isomorphism

$$(M \otimes^{\mathrm{gr}}_K N, v) \cong (N, v) \otimes_K (M, v)$$

where the tensor product on the right hand side is one of complexes of K-vector spaces. The Künneth formula then gives an isomorphism of graded vector spaces

$$\mathrm{H}^*(M \otimes^{\mathrm{gr}}_K N, v) \cong \mathrm{H}^*(N, v) \otimes_K \mathrm{H}^*(M, v)$$

from which we get $V_E(M \otimes^{\mathrm{gr}}_K N) = V_E(M) \cap V_E(N)$.

A similar argument yields $\mathrm{H}^*(\operatorname{Hom}^{\mathrm{gr}}_K(N, M), v) \cong \operatorname{Hom}_K(\mathrm{H}^*(N, v), \mathrm{H}^*(M, v))$, establishing the equality $V_E(\operatorname{Hom}^{\mathrm{gr}}_K(N, M)) = V_E(M) \cap V_E(N)$.

(8) Given a cone $W \subseteq V$, pick homogeneous polynomials $\xi_1, \ldots, \xi_s \in \mathcal{S}$ that define it, and note that $W = V_E\big(L_{\xi_1} \otimes^{\mathrm{gr}}_K \cdots \otimes^{\mathrm{gr}}_K L_{\xi_s}\big)$ by (7) and Lemma 3.10. ☐

4. Simplicial complexes

For $\sigma \subseteq [n]$, let $K\sigma$ denote the coordinate subspace spanned by $\{ e_j \mid j \in \sigma \}$. In an n-graded situation, we refine some results of the preceding section.

Proposition 4.1. *Let M be a finite n-graded E–module.*

(1) $\operatorname{Ext}_E^*(M,K)$ *is a finite $(1+n)$-graded left module over the polynomial ring $\mathcal{S} = K[\chi_1, \dots, \chi_n]$, in which χ_i has $(1+n)$-degree $(1, \varepsilon_i)$.*

(2) *There exists a polynomial $p_M(t, u_1, \dots, u_n) \in \mathbb{Z}[t, u_1, \dots, u_n]$ such that*

$$P_M^E(t, u_1, \dots, u_n) = \frac{p_M(t, u_1, \dots, u_n)}{\prod_{j=1}^n (1 - tu_j)} \; ;$$

if $M_a = 0$, then no monomial $t^i u^a$ appears in $p_M(t, u_1, \dots, u_n)$.

(3) *The variety $V_E(M)$ is a union of coordinate subspaces of V.*

(4) *Each union of coordinate subspaces is the variety of an n-graded E–module.*

Proof. (1) Take an n-graded free resolution G of M, and let $\operatorname{Ext}_E^{ia}(M,K)$ consist of those elements of $\operatorname{Ext}_E^i(M,K) = \operatorname{H}^i \operatorname{Hom}(G,K)$ that can be represented by a homomorphism $\varkappa\colon G_i \to K$, such that $\varkappa(G_{ib}) = 0$ when $a \neq b \in \mathbb{Z}^n$. Performing Construction 3.5 with this G and the n-graded Cartan resolution C of K (cf. Example 2.5) and using n-homogeneous maps, one gets bilinear pairings

$$\operatorname{Ext}_E^{jb}(K,K) \times \operatorname{Ext}_E^{ia}(M,K) \to \operatorname{Ext}_E^{i+j,a+b}(M,K) \quad \text{for all } i,j \in \mathbb{Z}\,;\, a,b \in \mathbb{Z}^n\,.$$

They make $\operatorname{Ext}_E^*(M,K)$ into a $(1+n)$-graded left module over $\operatorname{Ext}_E^*(K,K)$, and the identification $\operatorname{Ext}_E^*(K,K) = \mathcal{S}$ of Remark 3.7 is compatible with this grading.

(2) The expression for $P_M^R(t, u_1, \dots, u_n)$ comes from (1), by the multigraded version of the Hilbert-Serre theorem. The assertion on the monomials in the numerator is obvious when $M \cong \bigoplus_{i=1}^s K(a_i)$ with $a_i \in \mathbb{Z}^n$. Since (3.6.1) is an exact sequence of $(1+n)$-graded vector spaces, we conclude by induction on $\operatorname{rank}_K M$.

(3) The annihilator of the multigraded $\mathcal{S}$-module $\operatorname{Ext}_E^*(M,K)$ being a monomial ideal in $\chi_1, \dots, \chi_n$, its radical is an intersection of prime ideals generated by subsets of $\{ \chi_1, \dots, \chi_n \}$. The desired assertion follows from Theorem 3.9.

(4) Note that $\bigcap_{i=1}^s V_E(K\sigma_i) = V_E\big(\bigoplus_{i=1}^s E/(K\sigma_i)\big)$. □

Theorem 4.2. *If J is a monomial ideal in E, and I is the corresponding squarefree monomial ideal in S, then*

$$V_E(E/J) = \bigcup_{a \in \Sigma} K \operatorname{supp}(a)$$

where Σ is the set of shifts of a minimal free resolution of S/I over S, and so

$$\operatorname{cx}_E(E/J) = \max\{ |a| \mid a \in \Sigma \}\,.$$

The proof of the theorem is deferred to the end of the section.

Let Δ be a simplicial complex with n vertices, and set $K\langle \Delta \rangle = E/J$, where J is generated by $\{ e_\sigma \mid \sigma \notin \Delta \}$. We give a combinatorial interpretation of the complex

$$(K\langle\Delta\rangle, v)\colon \qquad 0 \to K\langle\Delta\rangle_1 \xrightarrow{\rho^v} K\langle\Delta\rangle_2 \xrightarrow{\rho^v} \dots\,.$$

For a subset $\rho \subseteq [n]$, we denote Δ_ρ the restriction of Δ to ρ, that is, the simplicial complex with faces $\sigma \in \Delta$ such that $\sigma \subseteq \rho$. Furthermore, for a face $\sigma \in \Delta$ we introduce the *link of σ in Δ_ρ* as the simplicial complex

$$\operatorname{lk}_{\Delta_\rho} \sigma = \langle \tau \in \Delta_\rho \mid \tau \cup \sigma \in \Delta \rangle.$$

For $v \in V$, $v = \sum_{i=1}^{n} \lambda_i e_i$, we call $\operatorname{supp}(v) = \{ i \mid \lambda_i \neq 0 \}$ the *support* of v.

Now the cohomology of $(K\langle\Delta\rangle, v)$ can be interpreted as follows:

Proposition 4.3. *The complex $(K\langle\Delta\rangle, v)$ only depends on $\rho = \operatorname{supp}(v)$, namely, it is isomorphic to $(K\langle\Delta\rangle, v_\rho)$ with $v_\rho = \sum_{j\in\rho} e_j$. Furthermore,*

$$\mathrm{H}^i(K\langle\Delta\rangle, v) \cong \bigoplus_{\sigma\in\Delta, \sigma\subseteq[n]\setminus\rho} \widetilde{\mathrm{H}}^{i-1}(\operatorname{lk}_{\Delta_\rho}\sigma; K)$$

where $\widetilde{\mathrm{H}}^*(\ ; K)$ denotes reduced simplicial cohomology with coefficients in K.

Proof. The map $\varphi\colon V \to V$ given by $\varphi(e_j) = \lambda_j^{-1} e_j$ for $j \in \rho$ and $\varphi(e_j) = e_j$ for $j \notin \rho$ extends to an isomorphism of K-algebras $\varphi\colon K\langle\Delta\rangle \to K\langle\Delta\rangle$, with $\varphi(v) = v_\rho$.

As a $K\langle\Delta_\rho\rangle$-module the algebra $K\langle\Delta\rangle$ decomposes as follows:

$$K\langle\Delta\rangle = \bigoplus_{\sigma\in\Delta, \sigma\subseteq[n]\setminus\rho} e_\sigma \cdot K\langle\Delta_\rho\rangle.$$

Now note that $e_\sigma K\langle\Delta_\rho\rangle \cong K\langle\operatorname{lk}_{\Delta_\rho}\sigma\rangle$, and that $(K\langle\operatorname{lk}_{\Delta_\rho}\sigma\rangle, v)$ is isomorphic to the augmented oriented cochain complex of $\operatorname{lk}_{\Delta_\rho}\sigma$ with values in K. $\square$

By a theorem of Hochster [12], $\rho \subseteq [n]$ is the support of a shift of the resolution of $k[\Delta]$ if and only if $\widetilde{\mathrm{H}}(\Delta_\rho; K) \neq 0$, so Theorem 4.2 and Proposition 4.3 yield

Corollary 4.4. *Let Δ be a simplicial complex with n vertices. For a subset $\sigma \subseteq [n]$ and a field K the following conditions are equivalent:*

(i) *There exists $\rho \subseteq [n]$ with $\sigma \subseteq \rho$ such that $\widetilde{\mathrm{H}}(\Delta_\rho; K) \neq 0$.*
(ii) *There exists $\tau \in \Delta$ with $\tau \cap \sigma = \varnothing$, such that $\widetilde{\mathrm{H}}(\operatorname{lk}_{\Delta_\sigma}\tau; K) \neq 0$.* $\square$

We single out a special case: For any simplicial complex Δ with $\widetilde{\mathrm{H}}^*(\Delta; k) \neq 0$ and any subset σ of the vertex set of Δ, there is a face τ of Δ such that $\widetilde{\mathrm{H}}(\operatorname{lk}_{\Delta_\tau}\sigma; K) \neq 0$.

Proof of Theorem 4.2. Let F be a minimal free resolution of S/I over S, let G be the minimal free resolution of E/J over E of Theorem 1.3, and let Y_ℓ be the basis of G_ℓ from Construction 1.1. A homogeneous K-basis of $\operatorname{Hom}_E(G_\ell, K) = \operatorname{Ext}_E^\ell(E/J, K)$ is given by $\{ \varkappa_f^a \mid \varkappa_f^a(y^{(a)}f) = 1 \text{ and } \varkappa_f^a(Y_\ell \setminus \{y^{(a)}f\}) = 0 \}$.

In the Cartan resolution C of K over E (cf. Example 2.5) set $1 = w^{(0)}$ and $w_j = w^{(\varepsilon_j)}$. Fixing a homomorphism $\varkappa_f^a\colon G_\ell \to K$, with $f \in B_i$ and $\deg(f) = b$, we note that a lifting of $\varkappa_f^a$ to a chain map $\widetilde{\varkappa}_f^a\colon G \to C$ can be started by

$$\big(\widetilde{\varkappa}_f^a\big)_\ell\big(y^{(a')}f'\big) = \begin{cases} 1 & \text{when } a = a' \text{ and } f = f'\,; \\ 0 & \text{otherwise}\,; \end{cases}$$

$$\big(\widetilde{\varkappa}_f^a\big)_{\ell+1}\big(y^{(a')}f'\big) = \begin{cases} (-1)^{|b|} w_j & \text{when } a = a' + \varepsilon_j\,,\ j \in \operatorname{supp}(b)\,,\ \text{and } f = f'\,; \\ (-1)^{|a|} w_j \lambda_{f'f} e_j^{-1} e_b^{-1} e_{b'} & \begin{array}{l}\text{when } a = a',\ j \in \operatorname{supp}(b' - b)\,, \\ \text{and } \theta(f') = \sum_{g\in B_i} \lambda_{f'g} x^{b'-c} g \\ \text{with } b' = \deg(f')\,,\ c = \deg(g)\,;\end{array} \\ 0 & \text{otherwise}\,. \end{cases}$$

These cases are disjoint because b' is squarefree, so by Construction 3.5 we have

$$\chi_j \varkappa_f^a = \begin{cases} (-1)^{|b|} \varkappa_f^{a+\varepsilon_j} & \text{for } j \in \operatorname{supp}(f)\,; \\ (-1)^{|a|} \sum_{f'\in B_{i+1}\colon b'=b+\varepsilon_j} \lambda_{f'f} \varkappa_{f'}^a & \text{for } j \in \operatorname{null}(f) = [n] \setminus \operatorname{supp}(f)\,. \end{cases}$$

Ordering the subsets of $[n]$ by inclusion, we set $B[0] = \varnothing$ and

$$B[p] = \{ f \in B \setminus B[p-1] \mid \operatorname{supp}(f) \text{ is maximal in } B \setminus B[p-1] \} \quad \text{for} \quad p \geq 1 .$$

The multiplication table shows that the K-span of $\{ \varkappa_f^a \mid \operatorname{supp}(f) \in \bigcup_{p \leq q} B[p] \}$ is a submodule $\mathcal{M}[q]$ of $\mathcal{M} = \operatorname{Ext}_E^*(M, K)$ over $\mathcal{S} = K[\chi_1, \dots, \chi_n]$, such that

$$\frac{\mathcal{M}[q]}{\mathcal{M}[q-1]} \cong \bigoplus_{f \in B[q]} \mathcal{S} \overline{\varkappa}_f^0 \qquad \text{and} \qquad \operatorname{Ann}_{\mathcal{S}}(\overline{\varkappa}_f^0) = \big(\operatorname{null}(f)\big) .$$

From the finite filtration $0 = \mathcal{M}[0] \subseteq \cdots \subseteq \mathcal{M}[n] = \mathcal{M}$ we get

$$\sqrt{\operatorname{Ann}_{\mathcal{S}} \mathcal{M}} = \sqrt{\bigcap_{q=1}^{n} \operatorname{Ann}_{\mathcal{S}} \frac{\mathcal{M}[q]}{\mathcal{M}[q-1]}} = \bigcap_{q=1}^{n} \sqrt{\operatorname{Ann}_{\mathcal{S}} \frac{\mathcal{M}[q]}{\mathcal{M}[q-1]}} = \bigcap_{f \in B} \big(\operatorname{null}(f)\big) .$$

The desired result now follows from Theorem 3.9. □

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