

AUTOMORPHISM SCHEME OF A FINITE FIELD EXTENSION

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ABSTRACT. Let $k \rightarrow K$ be a finite field extension and let us consider the automorphism scheme $\text{Aut}_k K$. We prove that $\text{Aut}_k K$ is a complete k -group, i.e., it has trivial centre and any automorphism is inner, except for separable extensions of degree 2 or 6. As a consequence, we obtain for finite field extensions K_1, K_2 of k , not being separable of degree 2 or 6, the following equivalence:

$$K_1 \simeq K_2 \Leftrightarrow \text{Aut}_k K_1 \simeq \text{Aut}_k K_2.$$

1. INTRODUCTION

The aim of this paper is to study the automorphism scheme of any finite field extension $k \rightarrow K$. In the separable case, the k -group $\text{Aut}_k K$ is quite simple; in fact, after a base change, the k -group $\text{Aut}_k K$ becomes isomorphic to the symmetric group S_n , where n denotes the degree of K over k . Hence, for finite separable extensions, the k -group $\text{Aut}_k K$ is finite and étale. Surprisingly, the facts are quite different for non-separable extensions. In such a case $\text{Aut}_k K$ is a large group (it has positive dimension) and it is not smooth.

In any case, it is not difficult to prove that $\text{Spec } K$ is a homogeneous k -scheme under the natural action of the k -group $\text{Aut}_k K$. This is a standard property of any classical transformation group. Hence, it seems natural to consider $\text{Aut}_k K$ as a new example of such groups (even if $\text{Aut}_k K$ is not smooth and has few rational points). Traditionally, the great interest of transformation groups is due to the fact that a transformation group encloses the essential structure of the geometric space in which it acts. The same holds for the group $\text{Aut}_k K$ and its geometric space $\text{Spec } K$; in fact, it is possible to recover the extension $k \rightarrow K$ from the group $\text{Aut}_k K$.

The main result of this paper states that $\text{Aut}_k K$ is a complete group, i.e., it has trivial centre and any automorphism is inner. Specifically:

Theorem 1.1. *Let $k \rightarrow K$ be a finite field extension. Then we have:*

1. $Z(\text{Aut}_k K) = \text{Id}$, except for a separable extension of degree 2.
2. The natural morphism defined by conjugation

$$\text{Aut}_k K \rightarrow \text{Aut}_{k\text{-gr}}(\text{Aut}_k K)$$

is an isomorphism of k -groups, except for separable extensions of degree 2 or 6.

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For separable extensions, the above theorem is a reformulation of the classical Hölder's theorem on the completeness of symmetric groups S_n . Hence, our result may be regarded as a broad extension of Hölder's theorem for non-separable extensions.

An interesting consequence of the above theorem is the following result, stating that the automorphism scheme fully determines the field extension.

Theorem 1.2. *Let K_1 and K_2 be finite extensions of a field k . Let us further assume that K_1 and K_2 are not separable extensions of degree 2 or 6 over k . Then we have*

$$K_1 \simeq K_2 \Leftrightarrow \text{Aut}_k K_1 \simeq \text{Aut}_k K_2.$$

In a forthcoming paper we shall provide an explicit construction of the given field extension $k \rightarrow K$ from the k -group $\text{Aut}_k K$.

Finally, let us remark that the group $\text{Aut}_k K$, as well as its subgroups, have been studied by several authors (Begueri [1], Shatz [15], Chase [2],[3], etc.) in order to extend Galois Theory to non-separable extensions.

2. PRELIMINARY SECTION

Let $\bar{k}$ be the algebraic closure of a field k of characteristic $p > 0$. Our starting point is the following theorem stating the local structure of finite field extensions (the term "local" refers to the f.p.f. topology).

Theorem 2.1. *Let $k \rightarrow K$ be a finite field extension. There exists an isomorphism of $\bar{k}$ -algebras*

$$K \otimes_k \bar{k} = \prod_{i=1}^m \bar{k}[x_1, \dots, x_n] / (x_1^{p^{e_1}}, \dots, x_n^{p^{e_n}})$$

where $e_1 \geq e_2 \geq \dots \geq e_n$.

In the purely inseparable extensions case, the above theorem is a direct consequence of a result of Pickert [11]. The general case is stated in Rasala [12], Chapter II, Theorem 6 and Lemma 4 (see also Pauer [10]). A more general result for finite generated algebras is obtained in Sancho [14].

The integer m is clearly the separability degree of K . The sequence $\varepsilon = \{e_1 \geq e_2 \geq \dots \geq e_n\}$ may be easily determined from the ranks of the K^{p^r} -modules $\Omega_{K^{p^r}/k}^1$; in fact, we have $e_i = \min \{r \in \mathbb{N} : \text{rank } \Omega_{K^{p^r}/k}^1 < i\}$. Hence, the sequence $\varepsilon = \{e_1 \geq e_2 \geq \dots \geq e_n\}$ is an invariant of K .

In this paper we study the automorphism scheme of finite field extensions and, in general, of finite algebras satisfying Theorem 2.1. Specifically, we consider the following class of finite k -algebras.

Definition 2.2. Let k be a ring of prime characteristic $p > 0$. A finite k -algebra A is said to be of type (m, ε) if there exists a faithfully flat morphism $k \rightarrow \bar{k}$ and an isomorphism of $\bar{k}$ -algebras

$$A \otimes_k \bar{k} = \prod_{i=1}^m \bar{k}[x_1, \dots, x_n] / (x_1^{p^{e_1}}, \dots, x_n^{p^{e_n}})$$

where $\varepsilon = \{e_1 \geq e_2 \geq \dots \geq e_n\}$.

Note that finite field extensions of type $(m, 0)$ are separable ones, while those of type $(1, \varepsilon)$ are purely inseparable.

Finite k -algebras with the splitting property stated in 2.1 may be characterized as finite k -algebras whose modules of r -jets $J_{A/k}^r$ are projective for $r \geq 0$ (see Sancho [14]). This characterization will not be used here.

Now we shall recall some concepts about the functor of points of a scheme that we shall use systematically in this paper.

Given a k -scheme X , we denote by $X^\cdot$ the corresponding functor of points defined on the category of k -schemes:

$$X^\cdot(S) = \text{Hom}_{k\text{-sch}}(S, X).$$

Any element of $X^\cdot(S)$, i.e., any morphism $x : S \rightarrow X$, will be said to be a point of X (parametrized by S). We shall not use the set theoretic concept of point, denoted by the notation $x \in X$. The statement “point x of X ” will be always used in the sense of the functor of points, i.e., it will denote a morphism $x : S \rightarrow X$ for some k -scheme S .

Notation. The composition of a point $x : S \rightarrow X$, with a morphism $\bar{S} \rightarrow S$ is a new point $\bar{S} \rightarrow S \xrightarrow{x} X$ which will also be denoted by x .

Given two points $x_1 : S_1 \rightarrow X$ and $x_2 : S_2 \rightarrow X$ we may always assume that both points have the same parameter space, since we may consider the compositions $S_1 \times_k S_2 \xrightarrow{\pi_1} S_1 \xrightarrow{x_1} X$ and $S_1 \times_k S_2 \xrightarrow{\pi_2} S_2 \xrightarrow{x_2} X$ where π_1, π_2 are the natural projections.

The identity morphism $Id : X \rightarrow X$ is said to be the general point of X .

The following elementary formula is essential:

$$\text{Hom}_{k\text{-sch}}(X, Y) = \text{Hom}_{\text{functor}}(X^\cdot, Y^\cdot).$$

According to this equality, to define a morphism of schemes is equivalent to defining a morphism between the corresponding functors of points (which is often easier).

Finally, recall that the functor of points of a k -scheme X is a sheaf in the f.p.p.f. topology (S.G.A. 3, IV 6.3.1. iii).

Let k be a Noetherian ring and let A be a flat finite k -algebra. The automorphism scheme of A is defined to be the k -group scheme $\text{Aut}_k A$ representing the functor of automorphisms of A , i.e., the functor of points of $\text{Aut}_k A$ is

$$\text{Hom}_{k\text{-sch}}(S, \text{Aut}_k A) = \text{Aut}_{\mathcal{O}_S\text{-alg}}(A \otimes_k \mathcal{O}_S)$$

for any k -scheme S .

The existence of the scheme $\text{Aut}_k A$ is a particular elementary case of the existence of the automorphism scheme for any flat projective morphism [7]. The scheme $\text{Aut}_k A$ may also be constructed as a closed subscheme of the general linear group $\text{Gl}(n, k) = \text{Aut}_{k\text{-lineal}}(A)$ where $n = \text{rank}_k A$. Moreover, this construction shows that $\text{Aut}_k A$ is an affine scheme. In this paper we shall only consider affine schemes of finite type over a Noetherian ring k .¹

¹The Noetherian hypothesis has been introduced for the sake of simplicity.

3. THE SEPARABLE CASE: REFORMULATION OF HÖLDER'S THEOREM

Let k be a ring and let A be a flat étale finite k -algebra of constant degree n . It is well known that there exists a faithfully flat base change $k \rightarrow \bar{k}$ such that

$$A \otimes_k \bar{k} = \bar{k} \times \cdots \times \bar{k}.$$

Since the automorphism scheme commutes with base changes, we have

$$(Aut_k A) \times_k \bar{k} = Aut_{\bar{k}}(A \otimes_k \bar{k}) = Aut_{\bar{k}}(\prod_{i=1}^n \bar{k}) = S_n.$$

Hence, the scheme $Aut_k A$ is locally isomorphic to the symmetric group S_n .

A classical result of Hölder [8] states that the symmetric group S_n has trivial centre (when $n \neq 2$) and that any automorphism of S_n is inner (when $n \neq 6$). As a direct consequence we obtain the following:

Theorem 3.1. *Let A be an étale finite k -algebra of degree n . Then*

- (a) *The k -group scheme $Aut_k A$ has trivial centre: $Z(Aut_k A) = \{Id\}$ (when $n \neq 2$).*
- (b) *The k -group scheme $Aut_k A$ is complete, i.e., the conjugation morphism*

$$Aut_k A \longrightarrow Aut_{k\text{-group}}(Aut_k A)$$

is an isomorphism (when $n \neq 2, 6$).

In the case $n = 2$ it is obvious that $Aut_k K = \mathbb{Z}/2\mathbb{Z}$. When $n = 6$, the corresponding result of Hölder gives us an exact sequence:

$$0 \rightarrow Aut_k K \longrightarrow Aut_{k\text{-gr}}(Aut_k K) \longrightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0.$$

4. THE PURELY INSEPARABLE CASE: THE GROUP S_ε IS COMPLETE

Now k will be a Noetherian ring of prime characteristic $p > 0$.

Given a sequence $\varepsilon = \{e_1 \geq \cdots \geq e_n\}$, we shall denote by S_ε the automorphism scheme of the “trivial” k -algebra of type $(1, \varepsilon)$

$$A = k[x_1, \dots, x_n]/(x_1^{p^{e_1}}, \dots, x_n^{p^{e_n}}).$$

By 2.1, any purely inseparable finite extension of fields $k \rightarrow K$ is locally isomorphic to A for a certain sequence ε . Therefore, the scheme $Aut_k K$ is a k -group locally isomorphic to S_ε .

In this section we shall determine the maximal tori of the k -group S_ε . This study will be necessary to prove that S_ε is complete and, hence, that automorphism schemes of purely inseparable field extensions are complete.

First let us determine S_ε . A basis for A as a free k -module is provided by the monomials $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ where $\alpha = (\alpha_1, \dots, \alpha_n)$ and $0 \leq \alpha_i < p^{e_i}$. Any automorphism τ of A is fully determined by $\tau(x_i)$,

$$\tau(x_i) = \sum_{\alpha} \lambda_{i,\alpha} x^\alpha.$$

The fact that τ is a morphism of k -algebras is expressed by the conditions $\tau(x_i)^{p^{e_i}} = 0$ or, equivalently,

$$\lambda_{i,\alpha}^{p^{e_i}} = 0 \text{ for any } \alpha < p^{e-e_i}$$

(the notation $\alpha < p^{e-e_i}$ means that $\alpha_j < p^{e_j-e_i}$ for every $1 \leq j \leq n$). The fact that τ is bijective is expressed by the condition of its determinant being invertible.

Such determinant is a polynomial in the coefficients $\lambda_{i,\alpha}$ that we shall denote by *det*. Hence we have:

Proposition 4.1. *The ring of coordinates of S_ε is*

$$\mathcal{O}_{S_\varepsilon} = (k[\lambda_{i,\alpha}]/I)_{\det}$$

where I is the ideal generated by the monomials $\lambda_{i,\alpha}^{p^{e_i}}$, with $\alpha < p^{e-e_i}$.

Remark 4.2. It follows directly from the above proposition that S_ε is a non-smooth k -group with geometrically connected fibres. The same holds for the automorphism scheme of any purely inseparable finite extension of a field.

The subgroup $T = G_m \times_k \cdots \times_k G_m$ of S_ε defined by the equations $\{\lambda_{i,\alpha} = 0, \alpha \neq (0, \dots, 1_i, \dots, 0)\}$ is said to be the standard torus of S_ε . The action $T \times_k \text{Spec } A \xrightarrow{*} \text{Spec } A$ transforms any two points $(t_1, \dots, t_n) : S \rightarrow T = (G_m)^n$, $(x_1, \dots, x_n) : S \rightarrow \text{Spec } A \subset \mathbb{A}^n$ into the point

$$(t_1, \dots, t_n) * (x_1, \dots, x_n) = (t_1 x_1, \dots, t_n x_n).$$

Proposition 4.3. *The centralizer of the standard torus T in S_ε coincides with T , i.e.,*

$$C(T) = T.$$

Proof. Let $t = (t_1, \dots, t_n)$ be the general point of T and let g be any point of $C(T)$. Write

$$g(x_i) = \sum_{\alpha} \mu_{i,\alpha} x^\alpha.$$

By definition of $C(T)$ we have $g \circ t = t \circ g$. Applying this equality to x_i we have

$$\sum_{\alpha} \mu_{i,\alpha} t_i x^\alpha = \sum_{\alpha} \mu_{i,\alpha} t^\alpha x^\alpha$$

so that $\mu_{i,\alpha}(t_i - t^\alpha) = 0$. Since $t_1, \dots, t_n$ are algebraically independent, we conclude that $\mu_{i,\alpha} = 0$ for any $\alpha \neq (0, \dots, 1_i, \dots, 0)$. Hence g is a point of T . $\square$

Theorem 4.4. *The centre of the k -group S_ε is trivial: $Z(S_\varepsilon) = \{Id\}$.*

Proof. It is clear that $Z(S_\varepsilon) \subseteq C(T) \stackrel{4.3}{=} T$, so that it is enough to prove that if a point $t = (t_1, \dots, t_n)$ of T belongs to the centre, then it is the identity. Let g be the general point of S_ε , defined by

$$g(x_i) = \sum_{\alpha} \lambda_{i,\alpha} x^\alpha$$

where $\lambda_{i,\alpha}^{p^{e_i}} = 0$ for any $\alpha < p^{e-e_i}$.

Applying the equality $g \circ t = t \circ g$ to x_i , we obtain

$$\sum_{\alpha} \lambda_{i,\alpha} t_i x^\alpha = \sum_{\alpha} \lambda_{i,\alpha} t^\alpha x^\alpha$$

so that $\lambda_{i,\alpha}(t_i - t^\alpha) = 0$. Since $\lambda_{i,\alpha}^{p^{e_i}} = 0$ (with $\alpha < p^{e-e_i}$) are the only relations between the variables $\lambda_{i,\alpha}$, it follows that $t_i = t^\alpha$ for any $\alpha < p^e$. In particular, taking $\alpha = (0, \dots, 0)$, we obtain that $t_i = 1$. $\square$

Let us recall the concept of homogeneous scheme. Consider the action of a k -group G on a k -scheme X . We say that X is homogeneous under the G -action if for any two points $x_1, x_2 \in X(S)$ there exists a faithfully flat morphism $\bar{S} \rightarrow S$ and a point $g \in G(\bar{S})$ such that $g * x_1 = x_2$.

Proposition 4.5. *Consider the trivial k -algebra $A = k[x_1, \dots, x_n]/(x_1^{p^{e_1}}, \dots, x_n^{p^{e_n}})$ of type $(1, \varepsilon)$. Then $\text{Spec } A$ is a homogeneous scheme under the natural action of the k -group $S_\varepsilon = \text{Aut}_k A$.*

Proof. Let us consider two (affine) points $\phi, \bar{\phi} : \text{Spec } B \rightarrow \text{Spec } A$ that will be defined by two k -algebra morphisms $\phi, \bar{\phi} : A \rightarrow B$. Then we have $\phi(x_i) = b_i$, $\bar{\phi}(x_i) = \bar{b}_i$, where $b_i^{p^{e_i}} = \bar{b}_i^{p^{e_i}} = 0$. It is easy to check that the automorphism g of $A_B = A \otimes_k B$ defined by $g(x_i) = x_i - b_i + \bar{b}_i$ satisfies $g * \phi = \bar{\phi}$. $\square$

Now, let us consider the action of a k -group G on a k -scheme X . Given a point $x : S \rightarrow X$, the isotropy subgroup of x is defined to be the subgroup H_x of $G_S = G \times_k S$ whose functor of points is

$$H_x(\bar{S}) = \{g \in G(\bar{S}) : g * x = x\}$$

for any S -scheme $\bar{S}$.

This subgroup is the fibre over x of the S -morphism $G_S \xrightarrow{*} X_S, g \mapsto g * x$. This fact proves the existence of H_x .

Let $x : \text{Spec } k \rightarrow \text{Spec } A$ be a point.

Corollary 4.6. *The quotient S_ε/H_x exists (in the f.f.p.f. topology) and is isomorphic to $\text{Spec } A$. Moreover, the natural morphism $S_\varepsilon \rightarrow S_\varepsilon/H_x = \text{Spec } A$ has a section.*

Proof. Since $\text{Spec } A$ is homogeneous, we may assume that x is the “origin”, i.e., the point defined by the ideal $\mathfrak{m} = (x_1, \dots, x_n)$ of A . The morphism $S'_\varepsilon \rightarrow \text{Spec } A$, $g \mapsto g * x$ is an epimorphism of sheaves by 4.5. It is clear that the quotient sheaf S'_ε/H'_x (associated to the presheaf quotient) is isomorphic to $\text{Spec } A$. Hence, S'_ε/H'_x is representable and $S_\varepsilon/H_x = \text{Spec } A$.

A section s of the morphism $S_\varepsilon \rightarrow \text{Spec } A, g \mapsto g * x$ may be defined as follows: Given a point $a = (a_1, \dots, a_n)$ of $\text{Spec } A$ (i.e., $(a_1, \dots, a_n)$ is the point defined by the ideal $(x_1 - a_1, \dots, x_n - a_n)$ of A), $s(a) = \tau_a$ is the point of S_ε defined by $\tau_a(x_i) = x_i + a_i$. $\square$

Let $G \times_k X \xrightarrow{*} X$ be an action of a k -group G on a k -scheme X . A point $x \in X(S)$ is said fixed by G when for any base change $\bar{S} \rightarrow S$ and any point $g \in G(\bar{S})$ we have $g * x = x$.

We shall denote by X^G the subscheme of X representing the functor of fixed points, i.e.,

$$(X^G)(S) = \{\text{points } x \in X(S) \text{ fixed by } G\}.$$

For the existence of the scheme of fixed points we refer to (S.G.A. 3; VIII 6.5d)

Corollary 4.7. *For any point $x : S \rightarrow \text{Spec } A$, we have*

$$S \stackrel{x}{=} (\text{Spec } A_S)^{H_x}.$$

Proof. Since $\text{Spec } A$ is homogeneous, we may assume that x is the “origin”, i.e., the point defined by the ideal $\mathfrak{m} = (x_1, \dots, x_n)$ of $A_S = \bar{k}[x_1, \dots, x_n]/(x_1^{p^{e_1}}, \dots, x_n^{p^{e_n}})$ (where $\bar{k}$ denotes the affine ring of S). In this case the standard torus T , which acts on $\text{Spec } A_S$ by the formula

$$(t_1, \dots, t_n) * (x_1, \dots, x_n) = (t_1 x_1, \dots, t_n x_n),$$

is contained in H_x . It is clear that the unique fixed point of T is the “origin”. Therefore

$$S \stackrel{x}{\subseteq} (\text{Spec } A_S)^{H_x} \subseteq (\text{Spec } A_S)^T = S$$

and the corollary follows. $\square$

Remark 4.8. Let us consider the functor of isotropy subgroups

$$F(S) = \{\text{Isotropy groups } H_x \text{ of points } x : S \rightarrow \text{Spec } A\}.$$

The above corollary shows that this functor is isomorphic to the functor of points of $\text{Spec } A$: any point x of $\text{Spec } A$ defines an isotropy subgroup H_x and, conversely, any subgroup H_x determines a unique point $(\text{Spec } A)^{H_x} \rightarrow \text{Spec } A$.

The action of S_ε on itself by conjugation, defines a natural action of S_ε on $T_{Id}S_\varepsilon$ (where Id is the unity section of S_ε). Let us denote the action of S_ε on $T_{Id}S_\varepsilon$ by $*$.

Corollary 4.9. H_x is the subgroup of S_ε which stabilizes the k -sub-bundle $T_{Id}H_x \subset T_{Id}S_\varepsilon$.

Proof. Since $\text{Spec } A$ is homogeneous, we may assume that x is the “origin”. Recall ([4] II §4 2.4) that $T_{Id}S_\varepsilon = T_{Id}\text{Aut}_k A$ is identified to $\text{Der}_k(A, A)$ and with this identification $T_{Id}H_x$ is equal to the set of derivations of A leaving $\mathfrak{m} = (x_1, \dots, x_n)$ stable, i.e., it is equal to the set of derivations of A vanishing at x . Then, given a point τ of S_ε , $\tau * (T_{Id}H_x)$ is the set of derivations of A vanishing at $\tau * x$. The derivations of A vanishing at x , do not vanish at any other point (for example, take $\{x_i \partial_{x_1}\}_{1 \leq i \leq n}$). Therefore, $\tau * (T_{Id}H_x) = T_{Id}H_x \Leftrightarrow \tau * x = x \Leftrightarrow \tau$ is a point of H_x . $\square$

Let $\mathcal{O}_{S_\varepsilon}$ be the affine ring of S_ε and let $\tilde{\mathfrak{m}}$ be the ideal corresponding to the unity section of S_ε . Let $G_{\tilde{\mathfrak{m}}}\mathcal{O}_{S_\varepsilon} = \bigoplus_{n=0}^{\infty} \tilde{\mathfrak{m}}^n / \tilde{\mathfrak{m}}^{n+1}$. It is clear from 4.1 that we have an isomorphism of graduate algebras

$$G_{\tilde{\mathfrak{m}}}\mathcal{O}_{S_\varepsilon} = k[\lambda_{i,\alpha}] / I^2$$

where I is the ideal generated by the monomials $\lambda_{i,\alpha}^{p^{e_i}}$ with $\alpha < p^{e-e_i}$. Given $\bar{f} \in \tilde{\mathfrak{m}}/\tilde{\mathfrak{m}}^2 \subset G_{\tilde{\mathfrak{m}}}\mathcal{O}_{S_\varepsilon}$, we say that the exponent of $\langle \bar{f} \rangle$ is n if $\bar{f}^{n-1} \neq 0$ and $\bar{f}^n = 0$.

Lemma 4.10. Assume that k is a field and write $e_1 = \dots = e_r > e_{r+1} \geq \dots \geq e_n$. Then $\langle \lambda_{i,0} \rangle$, $0 < i \leq r$, are the unique T -line k -sub-bundles of $\tilde{\mathfrak{m}}/\tilde{\mathfrak{m}}^2$ of exponent p^{e_1} .

Proof. Let $t = (t_1, \dots, t_n)$ be a general point of T . Remark that $t * \lambda_{i,\alpha} = t^{1_i - \alpha} \lambda_{i,\alpha}$. Then $\lambda_{i,0}$ is the unique T -line sub-bundle of $\tilde{\mathfrak{m}}/\tilde{\mathfrak{m}}^2$ where T acts by the character t_i , i.e., $t * \lambda_{i,0} = t_i \lambda_{i,0}$. If L is a sub-bundle in the conditions of the lemma, we have to prove that T acts over L by the character t_i , for some i such that $0 < i \leq r$.

²Instead of the term $\lambda_{i,(0,\dots,1_i,\dots,0)}$, we should write $\lambda_{i,(0,\dots,1_i,\dots,0)} - 1$, but we shall maintain the formula as we write it for the sake of simplicity.

It is clear that the exponent of $\lambda_{i,\alpha} \in \tilde{\mathfrak{m}}/\tilde{\mathfrak{m}}^2$ is p^{e_i} if $\alpha < p^{e-e_i}$, and it does not have a finite exponent if $\alpha \not< p^{e-e_i}$. Note that $\lambda_{i,\alpha}$ has exponent finite p^{e_1} if and only if $0 < i \leq r$ and $\alpha = 0$. Let $E_i = \{\text{elements of } \tilde{\mathfrak{m}}/\tilde{\mathfrak{m}}^2 \text{ of exponent } \leq e_i\}$. Obviously, E_i is a T -sub-bundle of $\tilde{\mathfrak{m}}/\tilde{\mathfrak{m}}^2$ and we have that $E_1/E_{r+1} = \langle \lambda_{i,0} \rangle_{0 < i \leq r}$. The natural morphism $L \rightarrow E_1/E_{r+1}$ is injective, therefore T acts over L by the character t_i , $0 < i \leq r$. $\square$

Lemma 4.11. *Let x be the “origin” of $\text{Spec } A$, and H_x the associated isotropy subgroup in $\text{Aut}_k A$ of x . Then the incident $T_{Id}H_x^\circ \subset \tilde{\mathfrak{m}}/\tilde{\mathfrak{m}}^2$ of $T_{Id}H_x$ verifies*

$$T_{Id}H_x^\circ = \langle \lambda_{1,0}, \dots, \lambda_{n,0} \rangle.$$

Proof. A point g , $g(x_i) = \sum_{\alpha} \lambda_{i,\alpha} x^\alpha$ of $\text{Aut}_k A$ is a point of H_x if and only if g stabilizes the ideal $(x_1, \dots, x_n)$ of A ; that is, if and only if $\lambda_{1,0} = \dots = \lambda_{n,0} = 0$. Hence, H_x is the subgroup of $\text{Aut}_k A$ defined by the ideal $(\lambda_{1,0}, \dots, \lambda_{n,0})$. The conclusion follows easily. $\square$

Theorem 4.12. *H_x is the unique flat and closed k -subgroup of S_ε , up to conjugation, such that*

- (1) *the quotient S_ε/H_x exists (in the f.f.p.f. topology) and is isomorphic to $\text{Spec } A$;*
- (2) *the natural morphism $S_\varepsilon \rightarrow S_\varepsilon/H_x$ has a section.*

Proof. H_x verifies these conditions by 4.6.

Now, let H' be a flat and closed subgroup of S_ε satisfying conditions (1) and (2).

a) Let us assume that k is a field. Remark that H' contains $(S_\varepsilon)_{\text{red}}$: The rational point $Id \cdot H'$ of $S_\varepsilon/H' \simeq \text{Spec } A$ is defined by a nilpotent ideal, then H' , which is the fibre of $Id \cdot H'$ by the morphism $S_\varepsilon \rightarrow S_\varepsilon/H'$, is defined by a nilpotent ideal of $\mathcal{O}_{S_\varepsilon}$. The conclusion follows.

If s is a section of the morphism $S_\varepsilon \rightarrow S_\varepsilon/H' = \text{Spec } A$, then we have an isomorphism $\text{Spec } A \times H' \simeq S_\varepsilon$ $((x, h') \mapsto s(x) \cdot h)$, where x is a point of $\text{Spec } A$ and h' is a point of H' .

Thus, via this isomorphism, the function $x_1 \in A = k[x_1, \dots, x_n]/(x_1^{p^{e_1}}, \dots, x_n^{p^{e_n}})$ defines a function $f \in \mathcal{O}_{S_\varepsilon}$ vanishing at H' , so that its class $\bar{f} \in \tilde{\mathfrak{m}}/\tilde{\mathfrak{m}}^2$ is an element of exponent p^{e_1} and belongs to the incident $T_{Id}H'^\circ$ of $T_{Id}H'$. Then, there exist elements of exponent p^{e_1} in $T_{Id}H'^\circ$. Note that the standard torus $T \subset (S_\varepsilon)_{\text{red}} \subset H'$ acts on $T_{Id}H'^\circ$. Since T transforms elements of exponent m to elements of exponent m and any T -module is a direct sum of submodules where T acts by means of characters ([4], II 2, 2.5), there exists a T -line sub-bundle in $T_{Id}H'^\circ$ of exponent p^{e_1} . By 4.10, we may assume that $\lambda_{1,0} \in T_{Id}H'^\circ$.

Now, let g be a point of $(S_\varepsilon)_{\text{red}} \subset H'$ defined by $g(x_1) = x_1 + x_i$ and $g(x_j) = x_j$, for any $j \neq 1$. Then

$$g * \lambda_{1,0} = \lambda_{1,0} + \lambda_{i,0} \in T_{Id}H'^\circ$$

Hence, we have $\lambda_{i,0} \in T_{Id}H'^\circ$. Then $T_{Id}H'^\circ = \langle \lambda_{1,0}, \dots, \lambda_{n,0} \rangle$ because $\dim_k T_{Id}H'^\circ = \dim_k T_{Id}^* \text{Spec } A = n$. By Lemma 4.11 $T_{Id}H_x^\circ = \langle \lambda_{1,0}, \dots, \lambda_{n,0} \rangle$. In conclusion, $T_{Id}H'^\circ = T_{Id}H_x^\circ$ and $T_{Id}H' = T_{Id}H_x$. By 4.9, $H' \subseteq H_x$ and, since both of them are subgroups of S_ε of the same index, we have $H' = H_x$.

b) Let assume that k is a ring. Let us see that $T \subset H'$, after conjugation if necessary.

Consider the action of T on S_ε/H' by translations. There exists a decomposition $\mathcal{O}_{S_\varepsilon/H'} = k \oplus I$, where I is the direct sum of the T -modules where T acts by means

of some non-trivial character, so that the product of two of these characters is non-trivial (in particular, I is an ideal): It suffices to prove these statements on fibres over k , then we may assume that k is a field. By a) S_ε/H' coincides with $S_\varepsilon/H_x = \text{Spec } A$ and we have that $A = k \oplus \mathfrak{m} = k \oplus (\bigoplus k \cdot x^\alpha)$, $0 < \alpha < p^e$, where the general point $t = (t_1, \dots, t_n)$ of T verifies that $t(x^\alpha) = t^\alpha x^\alpha$.

Then, T fixes the point $\text{Spec } k \hookrightarrow S_\varepsilon/H'$ defined by I , and T is included in a conjugated group of H' .

Let us now see that $H' = H_x$. Note that $\langle \lambda_{i,0} \rangle$ is the unique line sub-bundle of $\tilde{\mathfrak{m}}/\tilde{\mathfrak{m}}^2$ such that $t * \lambda_{i,0} = t_i \lambda_{i,0}$, then $T_{Id} H_x^\circ \subset \mathfrak{m}_{Id}/\mathfrak{m}_{Id}^2$ is the unique T -sub-bundle of $\tilde{\mathfrak{m}}/\tilde{\mathfrak{m}}^2$ such that on fibres over k is equal to $\langle \lambda_{1,0}, \dots, \lambda_{n,0} \rangle \subset \tilde{\mathfrak{m}}/\tilde{\mathfrak{m}}^2$. Therefore $T_{Id} H'^\circ = T_{Id} H_x^\circ$. Again as in a) we have that $H' \subseteq H_x$. Since S_ε/H' and S_ε/H_x are finite and flat schemes over k and the natural morphism $S_\varepsilon/H' \rightarrow S_\varepsilon/H_x$ is an isomorphism on fibres over k we have that $S_\varepsilon/H' = S_\varepsilon/H_x$ and $H' = H_x$. $\square$

Theorem 4.13. *The k -group S_ε is complete, i.e., the natural morphism defined by conjugation*

$$S_\varepsilon \longrightarrow \text{Aut}_{k\text{-gr}}(S_\varepsilon)$$

is an isomorphism of k -groups.

Proof. We have to prove that the natural morphism

$$S'_\varepsilon \longrightarrow \text{Aut}_{k\text{-gr}}(S'_\varepsilon)$$

is an isomorphism of sheaves for the f.p.p.f. topology.

The considered morphism is injective because $Z(S_\varepsilon) = \{Id\}$ (see 4.4).

Let us see that it is surjective.

Let τ be a point of $\text{Aut}_{k\text{-gr}}(S_\varepsilon)$. By 4.12, τ defines an automorphism of the functor of isotropy subgroups of S_ε . Now, this functor is isomorphic to the functor of points of $\text{Spec } A$ (see Remark after 4.7). We conclude that τ defines an automorphism σ of $\text{Spec } A$. By definition, we have

$$(*) \quad \tau(H_x) = H_{\sigma(x)},$$

and we have to prove that τ is the inner automorphism of S_ε defined by σ .

Let g be a point of $S_\varepsilon = \text{Aut}_k A$. For any point x of $\text{Spec } A$ we clearly have

$$(**) \quad g \cdot H_x \cdot g^{-1} = H_{g(x)}.$$

Applying τ to this equality and using $(*)$ we obtain

$$\tau(g) \cdot H_{\sigma(x)} \cdot \tau(g)^{-1} = H_{\sigma(g(x))}$$

and, by $(**)$, it follows that $\tau(g)(\sigma(x)) = \sigma(g(x))$; that is, $\tau(g) \circ \sigma = \sigma \circ g$ and we obtain that $\tau(g) = \sigma \circ g \circ \sigma^{-1}$. $\square$

5. THE GENERAL CASE

Let k be a Noetherian ring of prime characteristic $p > 0$ and let A be the trivial k -algebra of type (m, ε)

$$A = \prod_{i=1}^m k[x_1, \dots, x_n] / (x^{p^{\varepsilon_1}}, \dots, x^{p^{\varepsilon_n}}).$$

We denote by $S_{m,\varepsilon}$ the automorphism scheme of A .

5.1. Structure of the group $S_{m,\varepsilon}$. Let $e = e_1$ be the greatest exponent of A . It is obvious that $A^{p^e} = \prod^m k$ is the maximal étale subalgebra of A .

Since p -th powers commute with base changes (because the morphisms $A^{p^n} \rightarrow A$ are faithfully flat), there exists a restriction morphism

$$S_{m,\varepsilon}^\cdot = (Aut_k A)^\cdot \longrightarrow (Aut_k A^{p^e})^\cdot = S_m^\cdot.$$

This morphism has an obvious section $S_m^\cdot \hookrightarrow S_{m,\varepsilon}^\cdot$. On the other hand, the kernel of the restriction morphism is just the sheaf

$$F(S) = \{\tau \in (Aut_k A)^\cdot(S) : \tau(a \otimes 1) = a \otimes 1 \text{ for any } a \otimes 1 \in A^{p^e} \otimes_k \mathcal{O}_S\}.$$

It is easy to verify that $F = \prod^m S_\varepsilon^\cdot$, so that we obtain:

Proposition 5.1. *There exists an isomorphism of k -groups*

$$S_{m,\varepsilon} = \left(\prod^m S_\varepsilon \right) \rtimes S_m$$

where S_m acts on $\prod^m S_\varepsilon$ in the obvious way, permuting factors.

Corollary 5.2. *When $(m, \varepsilon) \neq (2, 0)$, we have*

$$Z(S_{m,\varepsilon}) = \{Id\}.$$

Proof. Let us consider an abstract group G with more than one element. It is an elementary exercise in group theory to verify that

$$Z\left(\left(\prod^m G\right) \rtimes S_m\right) = Z(G).$$

Therefore, if $\varepsilon \neq 0$, we have

$$Z(S_{m,\varepsilon}) \stackrel{5.1}{=} Z\left(\left(\prod^m S_\varepsilon\right) \rtimes S_m\right) = Z(S_\varepsilon) \stackrel{4.4}{=} \{Id\}.$$

If $\varepsilon = 0$, then $S_{m,\varepsilon} = S_m$ and the conclusion follows because $Z(S_m) = \{Id\}$ when $m \neq 2$ (see Rotman [13], 7.4). $\square$

5.2. The group $S_{m,\varepsilon}$ is complete. In this section we shall assume that $\text{Spec } k$ is connected.

Proposition 5.3. *The group S_ε is not a direct product of non-trivial subgroups.*

Proof. (1) First we assume that k is a field.

Let us consider a decomposition $S_\varepsilon = G_1 \times_k G_2$ and let τ be the general point of the standard torus T of S_ε , so that

$$\tau(x_i) = t_i x_i.$$

Then $\tau = g_1 \cdot g_2$ where g_1, g_2 are points of G_1, G_2 respectively. Since g_1 and g_2 commute, it follows that g_1 and g_2 commute with τ . It then follows that g_1, g_2 are points of $C(T) \stackrel{4.3}{=} T$ and we have

$$g_1(x_i) = \alpha_i x_i, \quad g_2(x_i) = \beta_i x_i$$

where α_i, β_i are elements of the ring $\mathcal{O}_T = k[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$ such that $\alpha_i \beta_i = t_i$.

For any index j , we consider the conjugate $\bar{g}_1$ of g_1 defined after a suitable base change by

$$\bar{g}_1(y_i) = \alpha_i y_i$$

where $y_i = x_i$ when $i \neq j$ and $y_j = x_j + a$, where a is a new variable that is independent of t_i satisfying $a^{p^{e_j}} = 0$. The point $\bar{g}_1$ belongs to G_1 because it is a conjugate of g_1 , hence it commutes with g_2 , so that

$$(\bar{g}_1 \cdot g_2)(x_j) = (g_2 \cdot \bar{g}_1)(x_j)$$

and we obtain

$$(\alpha_j - 1)(\beta_j - 1) = 0.$$

Since $\mathcal{O}_T$ is a domain, we conclude that $\alpha_j = 1$ (and $\beta_j = t_j$) or $\beta_j = 1$ (and $\alpha_j = t_j$).

Assume that $\alpha_1 = t_1$ (so that $\beta_1 = 1$). For any index $j \neq 1$ we consider the conjugate g'_1 of g_1 defined by

$$g'_1(z_i) = \alpha_i z_i$$

where $z_1 = x_1 + x_j$, and $z_i = x_i$ when $i \neq 1$. By the above argument, g'_1 commutes with g_2 , so that

$$(g'_1 \cdot g_2)(x_1) = (g_2 \cdot g'_1)(x_1)$$

and we obtain

$$t_1 - \alpha_j = t_1 \beta_j - t_j.$$

Of the two possible cases ($\alpha_j = t_j$, $\beta_j = 1$) and ($\alpha_j = 1$, $\beta_j = t_j$), only the first one satisfies the above condition. Therefore we obtain that $g_1 = \tau$. Since the general point of T is a point of G_1 , it follows that $T \subset G_1$. Now, G_2 commutes with G_1 and, in particular, it commutes with T . Hence $G_2 \subset C(T) = T \subset G_1$ so that $G_2 = \{Id\}$.

(2) Now let k be a ring with connected spectrum.

If $S_\varepsilon = G_1 \times_k G_2$, then we denote by I_{G_i} the ideal defining the subgroup G_i in S_ε and we denote by $\mathfrak{m}_{Id}$ the ideal defining the identity element $\text{Spec } k \hookrightarrow S_\varepsilon$. Let us consider the finitely generated $\mathcal{O}_{S_\varepsilon}$ -module $\mathfrak{m}_{Id}/I_{G_i}$. Note that $\mathfrak{m}_{Id}/I_{G_i} = 0 \Leftrightarrow G_i = \{Id\}$.

Considering the fibres of $S_\varepsilon \rightarrow \text{Spec } k$, the case (1) shows that

$$S_\varepsilon = \text{Sup}(\mathfrak{m}_{Id}/I_{G_1}) \cup \text{Sup}(\mathfrak{m}_{Id}/I_{G_2}),$$

$$\emptyset = \text{Sup}(\mathfrak{m}_{Id}/I_{G_1}) \cap \text{Sup}(\mathfrak{m}_{Id}/I_{G_2}),$$

and we obtain that $G_1 = \{Id\}$ or $G_2 = \{Id\}$ because S_ε is connected. □

Corollary 5.4. *Let $\varphi : \prod^m S_\varepsilon \rightarrow S_\varepsilon$ be a morphism of k -groups. If φ is surjective (as a morphism of sheaves for the f.p.p.f. topology), then φ factors through the natural projection onto some factor.*

Proof. Let $\phi : G_1 \times \cdots \times G_m \rightarrow G$ be a surjective morphism between abstract groups and let ϕ_i denote the restriction of ϕ to the subgroup G_i . If G has a trivial centre, it is easy to verify that

$$\ker \phi = \ker \phi_1 \times \cdots \times \ker \phi_m$$

and this result may be readily generalized to the k -groups. Hence we have

$$\ker \varphi = \ker \varphi_1 \times_k \cdots \times_k \ker \varphi_m$$

so that

$$S_\varepsilon = (\prod_{i=1}^m S_\varepsilon) / \ker \varphi = \prod_{i=1}^m (S_\varepsilon / \ker \varphi_i).$$

According to 5.3 only one factor $S_\varepsilon / \ker \varphi_i$ is $\neq 0$ and we may assume that $S_\varepsilon / \ker \varphi_1 \neq 0$. Then $\ker \varphi_i = S_\varepsilon$ for $2 \leq i \leq m$, i.e.,

$$\{Id\} \times S_\varepsilon \times \cdots \times S_\varepsilon \subset \ker \varphi$$

and $\varphi : \prod S_\varepsilon \rightarrow S_\varepsilon$ factors through the projection onto the first factor.

This argument uses the existence of quotient groups $S_\varepsilon / \ker \varphi_i$, which follows from this well-known result: If the product of a finite number of sheaves is representable, then every factor is representable. Applying this result to the product

$$\prod (S_\varepsilon / \ker \varphi_i) = \left(\prod S_\varepsilon \right) / \ker \varphi = S_\varepsilon$$

we obtain the existence of a k -group $S_\varepsilon / \ker \varphi_i$ representing the quotient sheaf $S_\varepsilon / \ker \varphi_i$. $\square$

Corollary 5.5. *We have*

$$\text{Aut}_{k\text{-gr}} \left(\prod_{i=1}^m S_\varepsilon \right) = \left(\prod_{i=1}^m S_\varepsilon \right) \rtimes S_m.$$

Proof. Let τ be a point of $\text{Aut}_{k\text{-gr}}(\prod^m S_\varepsilon)$. Applying the above corollary, it is easy to prove that τ permutes the factors of $\prod^m S_\varepsilon$. Therefore, composing τ with a convenient permutation, we may assume that τ takes each factor of $\prod^m S_\varepsilon$ into itself. The corollary follows, because S_ε is complete (4.13). $\square$

Theorem 5.6. *The k -group $S_{m,\varepsilon}$ is complete, i.e., the natural morphism induced by conjugation*

$$S_{m,\varepsilon} \longrightarrow \text{Aut}_{k\text{-gr}}(S_{m,\varepsilon})$$

is an isomorphism, except for the types $(m = 2, \varepsilon = 0)$ and $(m = 6, \varepsilon = 0)$.

Proof. If $\varepsilon = 0$, then $S_{m,\varepsilon} = S_m$ and Hölder's theorem proves this case.

Let us assume that $\varepsilon \neq 0$.

Let τ be an automorphism of $S_{m,\varepsilon} \stackrel{5.1}{=} (\prod^m S_\varepsilon) \rtimes S_m$. This automorphism induces an automorphism of the connected component of the identity $(S_{m,\varepsilon})^0 = \prod^m S_\varepsilon$. By 5.5, composing τ with an inner automorphism we may assume that τ induces the identity on $\prod^m S_\varepsilon$. Let us verify that τ is the identity on $S_{m,\varepsilon}$, then the theorem will follow.

It is enough to prove that τ is the identity on the subgroup S_m . First, it is easy to characterize points of S_m as the only points of $S_{m,\varepsilon}$ which coincide with a permutation of the factors $\prod^m S_\varepsilon$ when acting by conjugation. Now, we consider a point σ of S_m and $(g_1, \dots, g_n)$ of $\prod^m S_\varepsilon$. We have

$$(*) \quad \sigma \cdot (g_1, \dots, g_n) \cdot \sigma^{-1} = (g_{\sigma(1)}, \dots, g_{\sigma(n)}).$$

Applying τ to this equality, we obtain

$$(**) \quad \tau(\sigma) \cdot (g_1, \dots, g_n) \cdot \tau(\sigma)^{-1} = (g_{\sigma(1)}, \dots, g_{\sigma(n)})$$

because τ acts on $\prod^m S_\varepsilon$ by the identity. From the above mentioned characterization of S_m , it follows that $\tau(\sigma)$ is a point of S_m . Finally, comparing (*) and (**) we obtain that $\tau(\sigma) = \sigma$. $\square$

6. FINAL RESULTS

Theorem 6.1. *Let $k \rightarrow K$ be a finite field extension. Then*

- (a) $Z(\text{Aut}_k K) = \{Id\}$, *except for separable extensions of degree 2.*
- (b) $\text{Aut}_k K$ *is a complete k -group, i.e., the action by conjugation defines an isomorphism*

$$\text{Aut}_k K \longrightarrow \text{Aut}_{k\text{-gr}}(\text{Aut}_k K)$$

except for separable extensions of degree 2 or 6.

Proof. After a base change, we may assume by 2.1 that K is the trivial k -algebra of a certain type (m, ε) . In that case, $\text{Aut}_k K = S_{m, \varepsilon}$ and the conclusion follows from 5.2 and 5.6. $\square$

Remark 6.2. The proof of the above theorem holds for any finite ring morphism $k \rightarrow K$ of type $(2, 0)$ in case (a) and of a type different from $(2, 0)$ and $(6, 0)$ in case (b).

Lemma 6.3. *Let K_1 and K_2 be two finite k -algebras of type (m, ε) and (m', ε') respectively. If $\text{Aut}_k K_1 \simeq \text{Aut}_k K_2$, then $(m, \varepsilon) = (m', \varepsilon')$.*

Proof. We have to show that the type of any k -algebra K may be determined from the group $\text{Aut}_k K$. After a base change we may assume that K is the trivial k -algebra of a certain type (m, ε) , so that $\text{Aut}_k K = S_{m, \varepsilon}$.

The integer $m!$ (and therefore m) is determined by the degree of the quotient $S_{m, \varepsilon}/S_{m, \varepsilon}^0$ (where $S_{m, \varepsilon}^0$ denotes the connected component of the identity in $S_{m, \varepsilon}$) since, by 5.1,

$$S_{m, \varepsilon}/S_{m, \varepsilon}^0 = S_{m, \varepsilon}/(\prod^m S_\varepsilon) = S_m.$$

To determine the sequence ε , first we note that $(S_{m, \varepsilon}^0)^{S_m} = (\prod^m S_\varepsilon)^{S_m} = S_\varepsilon$. Then we must prove that the sequence ε is determined by S_ε . We denote by $G^{(p^r)}$ the p^r -th power of $G = S_\varepsilon$, i.e., $G^{(p^r)} = \text{Spec } \mathcal{O}_{S_\varepsilon}^{p^r}$. A straightforward calculation from the equations in 4.1 for S_ε gives us the following formula:

$$\text{rank } \Omega_{G^{(p^r-1)}/k}^1 = \text{rank } \Omega_{G^{(p^r)}/k}^1 \Leftrightarrow r \neq e_1, \dots, e_n.$$

This formula shows that the sequence $\varepsilon = (e_1, \dots, e_n)$ is fully determined by the group S_ε . $\square$

Theorem 6.4. *Let K_1 and K_2 be two finite k -algebras of a type different from $(2, 0)$ and $(6, 0)$. Then*

$$K_1 \simeq K_2 \Leftrightarrow \text{Aut}_k K_1 \simeq \text{Aut}_k K_2.$$

Proof. If $\text{Aut}_k K_1 \simeq \text{Aut}_k K_2$, by 6.3, both algebras have the same type, say (m, ε) . By definition, k -algebras of type (m, ε) are locally (in the sense of f.p.p.f. topology) isomorphic to the k -algebra

$$A = k[x_1, \dots, x_n]/(x_1^{p^{e_1}}, \dots, x_n^{p^{e_n}}).$$

These k -algebras are classified by the first cohomology “group” of automorphisms of A , i.e., by $H^1(k, S_{m,\varepsilon})$ (see [6], 4, or [9], II 8.1).

On the other hand, both groups $\text{Aut}_k K_1$ and $\text{Aut}_k K_2$ are locally isomorphic to $S_{m,\varepsilon}$. Moreover, k -groups locally isomorphic to $S_{m,\varepsilon}$ are classified by the first cohomology “group” $H^1(k, \text{Aut}_{k\text{-gr}}(S_{m,\varepsilon}))$.

By 5.6 we have $S_{m,\varepsilon} = \text{Aut}_{k\text{-gr}}(S_{m,\varepsilon})$ and the conclusion follows. $\square$

Corollary 6.5. *Let K_1 and K_2 be finite extensions of a field k , not being separable extensions of degree 2 or 6. Then*

$$K_1 \simeq K_2 \Leftrightarrow \text{Aut}_k K_1 \simeq \text{Aut}_k K_2.$$

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