TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 352, Number 3, Pages 1113–1129
S 0002-9947(99)02282-5
Article electronically published on July 7, 1999

THE SECOND BOUNDED COHOMOLOGY OF AN AMALGAMATED FREE PRODUCT OF GROUPS

KOJI FUJIWARA

Dedicated to Professor John Stallings for his 60th birthday

Abstract. We study the second bounded cohomology of an amalgamated free product of groups, and an HNN extension of a group. As an application, we show that a group with infinitely many ends has infinite dimensional second bounded cohomology.

1. Introduction

The bounded cohomology was defined by F.Trauber for groups and by Gromov for spaces. We review the definition of the bounded cohomology of a group G. Let

$$C_b^k(G;A) = \{f : G^k \to A \mid f \text{ has bounded range}\},$$

where $A = \mathbb{Z}$ or $\mathbb{R}$. The boundary $\delta : C_b^k(G;A) \to C_b^{k+1}(G;A)$ is given by

$$\begin{aligned}\delta f(g_0, \dots, g_k) &= f(g_1, \dots, g_k) + \sum_{i=1}^{k} (-1)^i f(g_0, \dots, g_{i-1}g_i, \dots, g_k) \\ &\quad + (-1)^{k+1} f(g_0, \dots, g_{k-1}).\end{aligned}$$

The cohomology of the complex $\{C_b^k(G;A), \delta\}$ is the *bounded cohomology* of G, denoted by $H_b^*(G;A)$. See [G], [I] as general references for the theory of the bounded cohomology.

For any group G, the first bounded cohomology $H_b^1(G;A)$ is trivial. If G is amenable, then $H_b^n(G;\mathbb{R})$ is trivial for all $n \geq 1$. The first example of a group with non-trivial second bounded cohomology was obtained by Brooks [B]. He showed that a free group of rank greater than 1 has infinite dimensional second bounded cohomology. Grigorchuk investigated the structure of the second bounded cohomology of free groups, torus knot groups and surface groups [Gr]. Yoshida [Y] and Soma [So1], [So2] studied the third bounded cohomology of surfaces and hyperbolic 3-manifolds. Epstein and the author showed that a non-trivial word-hyperbolic group has infinite dimensional second bounded cohomology [EF].

In order to state our results, we recall that l^1 denotes the Banach space of summable sequences of real numbers with the norm $||(x_i)|| = \sum_{i=1}^{\infty} |x_i|$. It is

Received by the editors February 20, 1996 and, in revised form, November 7, 1997.
1991 *Mathematics Subject Classification.* Primary 20F32; Secondary 55U99, 20E06.
Key words and phrases. Bounded cohomology, ends of groups.
Most of the work was done when the author visited MSRI supported in part by NSF grant DMS-9022140 and a JSPS grant. He is supported in part by The Inamori Foundation.

well known that the $\mathbb{R}$-vector space l^1 has dimension equal to the cardinal of the continuum.

In the case of an amalgamated free product of groups, we have the following.

Theorem 1.1. *Let $G = A *_C B$. If $|C\backslash A/C| \geq 3$ and $|B/C| \geq 2$, then there is an injective $\mathbb{R}$-linear map $\omega : l^1 \to H_b^2(G;\mathbb{R})$. In particular, the dimension of $H_b^2(G;\mathbb{R})$ as a vector space over $\mathbb{R}$ is the cardinal of the continuum.*

Corollary 1.1. *Let $G = A * B$ with $A \neq \{1\}, B \neq \{1\}$. If $G \neq \mathbb{Z}_2 * \mathbb{Z}_2$, then there is an injective $\mathbb{R}$-linear map $\omega : l^1 \to H_b^2(G;\mathbb{R})$. In particular, the dimension of $H_b^2(G;\mathbb{R})$ as a vector space over $\mathbb{R}$ is the cardinal of the continuum.*

Remark. (1) Corollary 1.1 generalizes the Brooks' result on free groups.

(2) Since $\mathbb{Z}_2 * \mathbb{Z}_2$ is amenable, $H_b^2(\mathbb{Z}_2 * \mathbb{Z}_2;\mathbb{R})$ is trivial.

Corollary 1.2. *Let $G = A *_C B$. If $|A| = \infty$, $|C| < \infty$, and $|B/C| \geq 2$, then there is an injective $\mathbb{R}$-linear map $\omega : l^1 \to H_b^2(G;\mathbb{R})$. In particular, the dimension of $H_b^2(G;\mathbb{R})$ as a vector space over $\mathbb{R}$ is the cardinal of the continuum.*

Corollary 1.3. *Let $G = A *_C B$. If A is abelian, $|A/C| \geq 3$, and $|B/C| \geq 2$, then there is an injective $\mathbb{R}$-linear map $\omega : l^1 \to H_b^2(G;\mathbb{R})$. In particular, the dimension of $H_b^2(G;\mathbb{R})$ as a vector space over $\mathbb{R}$ is the cardinal of the continuum.*

Example. $PSL_2(\mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$ and $SL_2(\mathbb{Z}) = \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ satisfy the assumption of Theorem 1.1. They are non-elementary word-hyperbolic groups too.

In the case of HNN extensions of groups, we obtain the following result.

Theorem 1.2. *Let $G = A*_{C,\varphi}$. If $|A/C| \geq 2$, $|A/\varphi(C)| \geq 2$, then there is an injective $\mathbb{R}$-linear map $\omega : l^1 \to H_b^2(G;\mathbb{R})$. In particular, the dimension of $H_b^2(G;\mathbb{R})$ as a vector space over $\mathbb{R}$ is the cardinal of the continuum.*

Due to the Stallings' structure theorem in [S] on a group with infinitely many ends, Theorems 1.1 and 1.2 imply the following.

Theorem 1.3. *If G is a finitely generated group with infinitely many ends, then there is an injective $\mathbb{R}$-linear map $\omega : l^1 \to H_b^2(G;\mathbb{R})$. In particular, the dimension of $H_b^2(G;\mathbb{R})$ as a vector space over $\mathbb{R}$ is the cardinal of the continuum.*

Theorems 1.1 and 1.2 sometimes enable us to decide the second bounded cohomologies of spaces which decompose along (codimension one) subspaces. For example if a 3-manifold M is Haken, with an incompressible embedded surface S, $\pi_1(M)$ decomposes along $\pi_1(S)$. In the case S is separating/non-separating, then Theorem 1.1/1.2 (respectively) may apply. The conditions on the numbers of (double) cosets in the theorems are not very restrictive, so these would be satisfied in this case, but the author does not know if it is always the case. In fact it is true that the second bounded cohomology of a compact, geometric (in the sense of Thurston) 3-manifold M is either trivial or infinite dimensional, which depends on the geometry of M [FO]. In the proof one applies Theorems 1.1 and 1.2 to the canonical decomposition of M (due to Jaco-Shalen and Johannson) along embedded 2-spheres and tori. One extra ingredient in the argument is that the second bounded cohomology of the fundamental group of a complete hyperbolic manifold with finite volume is infinite dimensional [F].

The author would like to thank J. Stallings for his useful suggestions. He also appreciates very insightful comments by the referee.

2. Quasi homomorphism and the counting function

Let G be a discrete group with a (finite or infinite) set of generators and $\Gamma(G)$ the Cayley graph. $\Gamma(G)$ is a path metric space with each edge of length 1. If the generating set is infinite, $\Gamma(G)$ is not locally compact. We only consider paths which start/end at vertices of $\Gamma(G)$, whose lengths are non-negative integers. Note that the distance between two vertices of $\Gamma(G)$ is well-defined and there always exist paths which achieve the distance (i.e. geodesics).

For a word $w = x_1 x_2 \dots x_n$ in these generators, define $|w| = n$. Let $\overline{w}$ be the element of G which is represented by the word w. Define $w^{-1} = x_n^{-1} \dots x_1^{-1}$. We sometimes identify a word w and the path starting at 1 and labeled by w in $\Gamma(G)$. For a path α labeled by w, define $|\alpha| = |w|$ and $\overline{\alpha} = \overline{w}$. For an element g in G, define $|g| = \inf_\alpha |\alpha|$, where α ranges over all the paths with $\overline{\alpha} = g$.

Let α be a finite path in $\Gamma(G)$. Define $|\alpha|_w$ to be the maximal number of times that w can be seen as a subword of α without overlapping. We define

$$c_w(\alpha) = \sup_{\alpha'}\{|\alpha'|_w - (|\alpha'| - |\overline{\alpha}|)\} = |\overline{\alpha}| - \inf_{\alpha'}(|\alpha'| - |\alpha'|_w),$$

where α' ranges over all the paths with the same starting point as α and the same finishing point. Note that the infimum in the above definition is always attained by some paths. If α' attains the infimum, we say that α' *realizes* c_w *at* α.

Lemma 2.1. *If α is a geodesic, then*

$$\frac{|\alpha|}{|w|} \geq c_w(\alpha) \geq |\alpha|_w.$$

Proof. Let α' realize c_w at α. Then, since $|\alpha'| - |\alpha'|_w \leq |\alpha| - |\alpha|_w$, we find

$$c_w(\alpha) = |\alpha| - (|\alpha'| - |\alpha'|_w) \geq |\alpha|_w.$$

To show the other inequality, note that $|\alpha'|_w \leq \frac{|\alpha'|}{|w|}$. This implies

$$|\alpha'| - |\alpha'|_w \geq |\alpha'| - \frac{|\alpha'|}{|w|} = \left(1 - \frac{1}{|w|}\right)|\alpha'| \geq \left(1 - \frac{1}{|w|}\right)|\alpha|.$$

Thus,

$$c_w(\alpha) = |\alpha| - (|\alpha'| - |\alpha'|_w) \leq \frac{|\alpha|}{|w|}.$$

□

For each g in G, we choose γ_g to be a path from 1 to g and set $c_w(g) = c_w(\gamma_g)$. Then $c_w(g)$ does not depend on the choice of γ_g. The following result is clear from the previous lemma.

Lemma 2.2. *For all $g \in G$,*

$$\frac{|g|}{|w|} \geq c_w(g).$$

Let $f \in C^1(G;\mathbb{R})$. If there exists a constant $D < \infty$ such that

$$|f(gh) - f(g) - f(h)| \leq D,$$

for all $g, h \in G$, then we say f is a *quasi-homomorphism* with *defect* D. Let f be a quasi-homomorphism with defect D. Then $|\delta f| \leq D$ and $\delta(\delta f) = 0$; thus $\delta f \in Z_b^2(G;\mathbb{R})$, which defines $[\delta f] \in H_b^2(G;\mathbb{R})$. Note that we always have $[\delta f] = 0$

in $H^2(G;\mathbb{R})$; however, we may have $[\delta f] \neq 0$ in $H_b^2(G;\mathbb{R})$ since f is not necessarily in $C_b^1(G;\mathbb{R})$. Now we define

$$h_w = c_w - c_{w^{-1}} \in C^1(G;\mathbb{Z}).$$

These 1-cochains are candidates for bounded quasi-homomorphisms (see Propositions 3.1 and 6.1).

3. Quasi homomorphisms on $A *_C B$

Let $G = A *_C B$ with $|A/C| \geq 2$, $|B/C| \geq 2$. Take the set $\{A \cup B\} \backslash \{1\}$ as a set of generators of G and denote its Cayley graph by $\Gamma(G)$.

If a word $w = x_1 \dots x_n$ satisfies $n = 1$ or $x_1, x_3, \dots \in A \backslash C$ (or $B \backslash C$) and $x_2, x_4, \dots \in B \backslash C$ (or $A \backslash C$, respectively), then we say w is *reduced*.

Lemma 3.1. *A word $w = x_1 \dots x_n$ is reduced if and only if it is a geodesic in $\Gamma(G)$.*

Proof. Assume w is not reduced; then there exists a subword $x_i x_{i+1}$ with $x_i, x_{i+1} \in A$ (or $x_i, x_{i+1} \in B$). Then $\overline{x_i x_{i+1}} \in A$ (or B, respectively); thus $|\overline{x_i x_{i+1}}| \leq 1$. Therefore $x_i x_{i+1}$ is not a geodesic; hence w is not a geodesic.

On the other hand, assume w is not a geodesic. To show that w is not reduced by contradiction, suppose w is reduced. Take a geodesic γ such that $\overline{w} = \overline{\gamma}$. Note that γ is reduced. Then $|w| = |\gamma|$ since reduced words representing a common element have the same length. Thus w is a geodesic. This is a contradiction. □

Lemma 3.2. *Let w be a word such that w^2 is reduced. Let α be a path. Then there is a geodesic which realizes c_w at α.*

Proof. Since w^2 is reduced, w is reduced. Let γ be a path which realizes c_w at α such that $|\gamma|_w$ is minimal among all the realizing paths at α. We claim that γ is a geodesic. Indeed, if $|\gamma|_w = 0$, then γ is a geodesic. Suppose $|\gamma|_w = n > 0$. Then γ is written such that

$$\gamma_1 w_1 \gamma_2 \dots w_n \gamma_{n+1},$$

where w_i is a copy of w and γ_i may be an empty word. First, to show that every γ_i is reduced by contradiction, suppose γ_I is not reduced. Replace γ_I by a reduced word γ'_I with $\overline{\gamma_I} = \overline{\gamma'_I}$; then we have a new path

$$\gamma' = \gamma_1 w_1 \gamma_2 \dots \gamma'_I \dots w_n \gamma_{n+1},$$

which satisfies $|\gamma'| < |\gamma|$, $|\gamma'|_w \geq |\gamma|_w$ and $\overline{\gamma} = \overline{\gamma'}$. This is impossible since γ is a realizing path. Thus every γ_i is reduced. Next, in order to show that γ is reduced by contradiction, suppose not. Since w^2 is reduced, there is a subword $w_i \gamma_{i+1} w_{i+1}$ of γ which is not reduced and γ_{i+1} is not empty. Since $w_i, w_{i+1}, \gamma_{i+1}$ are reduced, one of the following four cases occurs.
(i) The last letter of w_i and the initial letter of γ_{i+1} are in A.
(ii) The last letter of w_i and the initial letter of γ_{i+1} are in B.
(iii) The last letter of γ_{i+1} and the initial letter of w_{i+1} are in A.
(iv) The last letter of γ_{i+1} and the initial letter of w_{i+1} are in B.

Suppose (i) holds. Then

$$w_i = \dots b_1 a_1,\ \gamma_{i+1} = a_2 b_2 \dots,\ a_i \in A, b_i \in B.$$

We rewrite the subword $w_i \gamma_{i+1}$ in γ as

$$w_i \gamma_{i+1} = \dots b_1 a_1 a_2 b_2 \dots = \dots b_1 a' b_2 \dots,$$

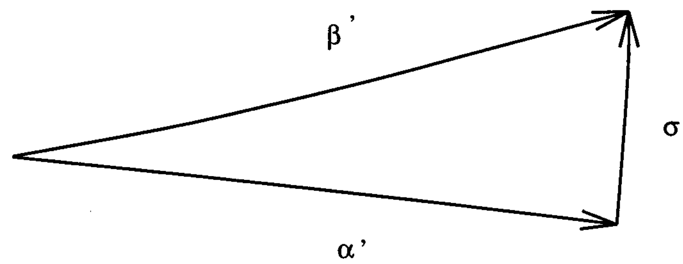

FIGURE 1. Geodesic triangle. This illustrates Lemma 3.3.

where $a' = a_1 a_2 \in A$. This gives a new word γ' with $|\gamma'| = |\gamma| - 1$, and $|\gamma'|_w \geq |\gamma|_w - 1$. Since $\overline{\gamma'} = \overline{\gamma}$ and $|\gamma| - |\gamma|_w \geq |\gamma'| - |\gamma'|_w$, we find γ' is another realizing path with $|\gamma'|_w < |\gamma|_w$. This contradicts the choice of γ. We showed that γ is reduced. By Lemma 3.1, γ is a geodesic. Similar argument applies to the other three cases. □

Lemma 3.3. *Suppose w^2 is reduced. Let α, β be paths starting at 1. Then we have*

$$|c_w(\alpha) - c_w(\beta)| \leq 2|\overline{\alpha^{-1}\beta}|.$$

Proof. Take realizing geodesics α' and β' of c_w at α and β respectively (see Figure 1). Then $c_w(\alpha) = |\alpha'|_w$, $c_w(\beta) = |\beta'|_w$. Take a geodesic σ with $\overline{\alpha^{-1}\beta} = \overline{\sigma}$. Since the path $\alpha'\sigma$ satisfies $\overline{\alpha'\sigma} = \overline{\beta'}$,

$$|\beta'| - |\beta'|_w \leq |\alpha'\sigma| - |\alpha'\sigma|_w.$$

Since $|\alpha'\sigma| = |\alpha'| + |\sigma|$ and $|\alpha'\sigma|_w \geq |\alpha'|_w + |\sigma|_w$,

$$|\beta'| - |\beta'|_w \leq |\alpha'| + |\sigma| - |\alpha'|_w - |\sigma|_w.$$

Thus

$$c_w(\beta) = |\beta'|_w \geq |\alpha'|_w + |\beta'| - |\alpha'| - |\sigma| + |\sigma|_w \geq c_w(\alpha) - 2|\sigma|,$$

since $|\alpha'|_w = c_w(\alpha)$, $|\beta'| - |\alpha'| \geq -|\sigma|$, and $|\sigma|_w \geq 0$. Similarly, $c_w(\alpha) \geq c_w(\beta) - 2|\sigma|$. We get $|c_w(\alpha) - c_w(\beta)| \leq 2|\sigma| = 2|\overline{\alpha^{-1}\beta}|$. □

Lemma 3.4. *Suppose w^2 is reduced. Let α, β be paths starting at 1. Then we have*

$$|h_w(\alpha) - h_w(\beta)| \leq 4|\overline{\alpha^{-1}\beta}|.$$

Proof. By definition $h_w = c_w - c_{w^{-1}}$. Apply Lemma 3.3. □

Lemma 3.5.

$$c_w(\alpha) = c_{w^{-1}}(\alpha^{-1}).$$

Proof. Clear from the definition of c_w. □

Lemma 3.6.

$$h_w(\alpha) = -h_w(\alpha^{-1}).$$

Proof. By definition, $h_w(\alpha^{-1}) = c_w(\alpha^{-1}) - c_{w^{-1}}(\alpha^{-1}) = c_{w^{-1}}(\alpha) - c_w(\alpha) = -h_w(\alpha)$. □

Lemma 3.7. *Suppose w^2 is reduced. Let α be a geodesic. If $\alpha = \alpha_1\alpha_2$, then*

$$|h_w(\alpha) - h_w(\alpha_1) - h_w(\alpha_2)| \leq 10.$$

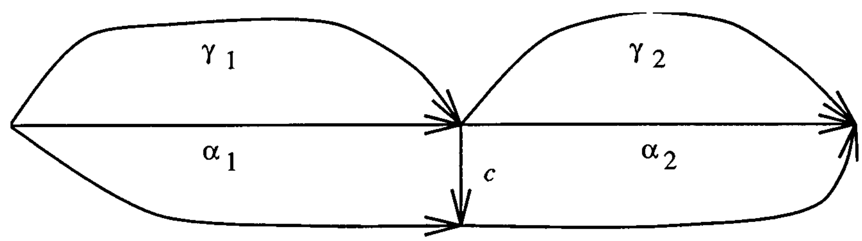

FIGURE 2. Dividing geodesics. This illustrates Lemma 3.7. Note $\alpha = \alpha_1\alpha_2$, $\gamma = \gamma_1\gamma_2$ and $\sigma = \sigma_1\sigma_2$.

Proof. We first show $|c_w(\alpha) - c_w(\alpha_1) - c_w(\alpha_2)| \le 5$. Let γ_1 and γ_2 be realizing geodesics of c_w at α_1 and α_2, respectively. We have $|\gamma_1|_w = c_w(\alpha_1)$, $|\gamma_2|_w = c_w(\alpha_2)$. Set $\gamma = \gamma_1\gamma_2$; then $\overline{\gamma} = \overline{\alpha}$. As $|\alpha| = |\alpha_1| + |\alpha_2| = |\gamma_1| + |\gamma_2| = |\gamma|$, γ is a geodesic. Since γ is a geodesic with $\overline{\gamma} = \overline{\alpha}$, we get $c_w(\alpha) \ge |\gamma|_w$. Thus

$$c_w(\alpha) \ge |\gamma|_w \ge |\gamma_1|_w + |\gamma_2|_w = c_w(\alpha_1) + c_w(\alpha_2).$$

On the other hand, take a realizing geodesic σ at α; then $c_w(\alpha) = |\sigma|_w$. Since α and σ are reduced, there exists subdivision of σ, $\sigma = \sigma_1\sigma_2$ such that $\overline{\alpha_1^{-1}\sigma_1} = \overline{\alpha_2\sigma_2^{-1}} = c$, for some $c \in C$ (see Figure 2). Since $|c| \le 1$, by Lemma 3.3 and 3.5, we get, for $i = 1, 2$,

$$|c_w(\alpha_i) - c_w(\sigma_i)| \le 2.$$

Since σ_1, σ_2 are geodesics, we see $|\sigma_i|_w \le c_w(\sigma_i)$ for $i = 1, 2$. Thus

$$c_w(\alpha) = |\sigma|_w \le |\sigma_1|_w + |\sigma_2|_w + 1 \le c_w(\sigma_1) + c_w(\sigma_2) + 1 \le c_w(\alpha_1) + c_w(\alpha_2) + 5.$$

We get $|c_w(\alpha) - c_w(\alpha_1) - c_w(\alpha_2)| \le 5$. Similarly, we have $|c_{w^{-1}}(\alpha) - c_{w^{-1}}(\alpha_1) - c_{w^{-1}}(\alpha_2)| \le 5$. Since $h_w = c_w - c_{w^{-1}}$, we obtain $|h_w(\alpha) - h_w(\alpha_1) - h_w(\alpha_2)| \le 10$. □

Proposition 3.1. *Let w be a word. Suppose w^2 is reduced. Then $h_w : G \to \mathbb{Z}$ is a quasi-homomorphism whose defect is uniformly bounded by 78; $|\delta h_w| \le 78$.*

Proof. Let x, y be elements in G. By definition $\delta h_w(x, y) = h_w(x) + h_w(y) - h_w(xy)$. Let α, β, γ be geodesics such that $\overline{\alpha} = x, \overline{\beta} = y, \overline{\gamma} = xy$. They are reduced paths. Since $\overline{\alpha\beta} = \overline{\gamma}$ and α, β, γ are reduced, there exist subdivisions of α, β, γ such that

$$\alpha = \alpha_1\alpha_2\alpha_3,\ \beta = \beta_1\beta_2\beta_3,\ \gamma = \gamma_1\gamma_2\gamma_3,$$

and that

$$\overline{\gamma_1^{-1}\alpha_1} = c_1,\ \overline{\alpha_3\beta_1} = c_2,\ \overline{\beta_3\gamma_3^{-1}} = c_3$$

for some $c_1, c_2, c_3 \in C$ and $\overline{\alpha_2}, \overline{\beta_2}, \overline{\gamma_2}$ are (simultaneously) in A or B (see Figure 3).

Since $\overline{\alpha_2}$ is in A or B, we find $c_w(\alpha_2), c_{w^{-1}}(\alpha_2) \le |\overline{\alpha_2}| \le 1$. Thus $|h_w(\alpha_2)| \le 2$. Similarly, $|h_w(\beta_2)|, |h_w(\gamma_2)| \le 2$.

Using Lemma 3.7 twice for $\alpha = ((\alpha_1\alpha_2)\alpha_3)$, we get

$$|h_w(\alpha) - h_w(\alpha_1) - h_w(\alpha_2) - h_w(\alpha_3)| \le 20;$$

thus $|h_w(\alpha) - h_w(\alpha_1) - h_w(\alpha_3)| \le 22$. Similarly,

$$|h_w(\beta) - h_w(\beta_1) - h_w(\beta_3)| \le 22,$$

$$|h_w(\gamma) - h_w(\gamma_1) - h_w(\gamma_3)| \le 22.$$

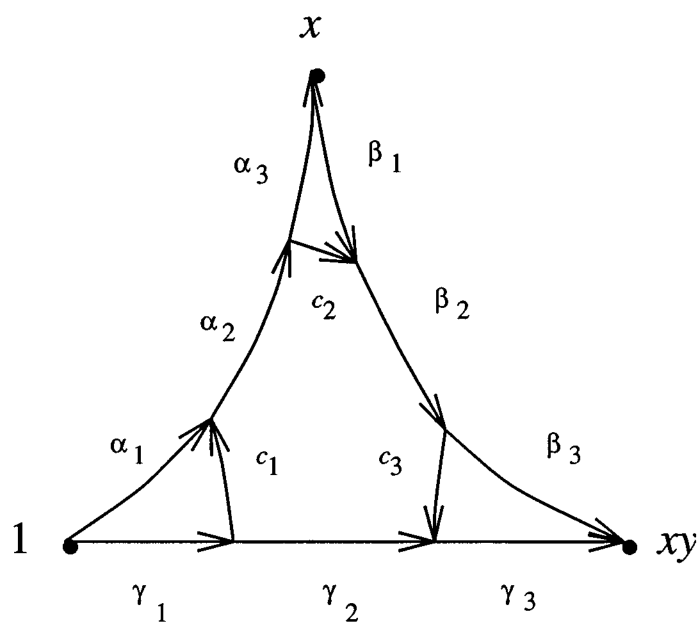

FIGURE 3. Dividing a geodesic triangle. This illustrates Proposition 3.1. Elements c_1, c_2, c_3 are in C.

Therefore, by Lemmas 3.4 and 3.6,

$$\begin{aligned} & |h_w(\alpha) + h_w(\beta) - h_w(\gamma)| \\ \leq \quad & |h_w(\alpha_1) + h_w(\alpha_3) + h_w(\beta_1) + h_w(\beta_3) - h_w(\gamma_1) - h_w(\gamma_3)| + 66 \\ \leq \quad & |h_w(\alpha_1) - h_w(\gamma_1)| + |h_w(\beta_1) - h_w(\alpha_3^{-1})| + |h_w(\gamma_3^{-1}) - h_w(\beta_3^{-1})| + 66 \\ \leq \quad & 4(|c_1| + |c_2| + |c_3|) + 66 \leq 78, \end{aligned}$$

since $|c_i| \leq 1$.

By definition, $h_w(\alpha) = h_w(x), h_w(\beta) = h_w(y)$ and $h_w(\gamma) = h_w(xy)$. This implies

$$|h_w(x) + h_w(y) - h_w(xy)| \leq 78.$$

□

4. Choice of words for $A *_C B$

The goal of this section is to prove the following proposition.

Proposition 4.1. *Let $G = A *_C B$ with $|C\backslash A/C| \geq 3$ and $|B/C| \geq 2$. Then there exist words w_i, $0 \leq i < \infty$, which satisfy the following properties.*
(1) For all $i \geq 0$ and all $n \geq 1$, $h_{w_i}(w_i^n) = n$.
(2) For all $j > i \geq 0$ and all $n \geq 1$, $h_{w_j}(w_i^n) = 0$.
(3) For all $i \geq 0$, $\overline{w_i} \in [G, G]$.
(4) For all $i \geq 0$, w_i^2 is reduced.
(5) $\lim_{i\to\infty} |w_i| = \infty$.

Let $G = A *_C B$ with $|C\backslash A/C| \geq 3$ and $|B/C| \geq 2$. Let w and w' be reduced paths which share the starting point and the finishing point as well. Let

$$w = a_1 b_1 \dots a_n b_n,$$

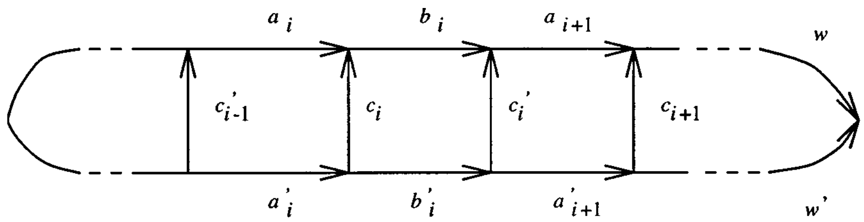

FIGURE 4. Reduced paths w and w' with common end points.

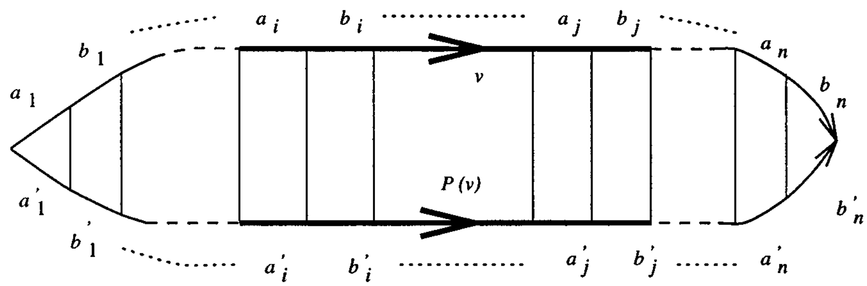

FIGURE 5. Definition of $P(v)$.

$$w' = a'_1 b'_1 \dots a'_m b'_m,$$

where a_1, b_n, a'_1, b'_m may be empty. Since $\overline{w} = \overline{w'}$ and w and w' are reduced, we get $n = m$ and there exist $c_1, \dots, c_n$ and $c'_0, c'_1, \dots, c'_n$ in C such that

$$c'_{i-1} a_i c_i^{-1} = a'_i, \; c_i b_i c_i'^{-1} = b'_i$$

for $1 \le i \le n$, where $c'_0 = c'_n = 1$. See Figure 4.

Let v be a subword of w,

$$v = a_i b_i \dots a_j b_j,$$

where a_i, b_j may be empty. We define a subword of w', denoted by $P(v)$ (see Figure 5), by

$$P(v) = a'_i b'_i \dots a'_j b'_j.$$

Let v' be a subword of w'. If v' is a subword of $P(v)$, we say v *covers* v'. If $v' = P(v)$, then we say v *faces* v'.

Since $|C \backslash A / C| \ge 3$, there exist elements $a_1, a_2 \in A$ such that $a_1, a_2 \in A \backslash C$ and $a_2 \notin C a_1 C$. Taking an element $b \in B \backslash C$, we define words $w_i, 0 \le i < \infty$, by

$$w_i = (a_1 b)^{10^i} (a_1^{-1} b^{-1})^{10^i} (a_2 b)^{10^i} (a_2^{-1} b^{-1})^{10^i}$$
$$(a_1 b)^{4 \cdot 10^i} (a_1^{-1} b^{-1})^{4 \cdot 10^i} (a_2 b)^{4 \cdot 10^i} (a_2^{-1} b^{-1})^{4 \cdot 10^i}.$$

We write the subword $(a_1 b)^{4 \cdot 10^i}$ by $w_i(1, +)$ and the subword $(a_1^{-1} b^{-1})^{4 \cdot 10^i}$ by $w_i(1, -)$.

Lemma 4.1. *The words $w_i, 0 \le i < \infty$, satisfy the following properties.*

(1) For all $i \ge 0$ and all $n \ge 1$, w_i^n is reduced.

(2) For all $i \ge 0$, $|w_i| = 40 \cdot 10^i$ and $|w_i(1, \pm)| = 8 \cdot 10^i$.

(3) For all $i \ge 0$, $\overline{w_i} \in [G, G]$.

Proof. (1) is clear from the way we chose a_1, a_2 and b. (2) is obvious. Note that $\overline{(a_ib)^p(a_i^{-1}b^{-1})^p} \in [G,G]$ for $i = 1, 2$ and all $p \geq 1$. This implies (3). □

Lemma 4.2. *For any pair of paths having a common starting point and a common finishing point, the following properties hold for their subpaths.*

(1) a_1 cannot face a_2.

(2) For all $i \geq 0$, w_i^2 cannot cover w_i^{-1}.

(3) For all $i < j$, $w_j(1,+)$ cannot cover w_i and $w_j(1,-)^{-1}$ cannot cover w_i.

(4) For all $k > 0$ and all $i < j$, w_i^k cannot cover w_j nor w_j^{-1}.

Proof. (1) If a_1 faces a_2 in some pair of paths, then there exist $c_1, c_2 \in C$ such that $c_1a_1c_2 = a_2$; thus $a_2 \in Ca_1C$, which contradicts our choice of a_1 and a_2.
(2) We show this claim for $i = 0$;

$$w_0 = a_1ba_1^{-1}b^{-1}a_2ba_2^{-1}b^{-1}(a_1b)^4(a_1^{-1}b^{-1})^4(a_2b)^4(a_2^{-1}b^{-1})^4.$$

We denote the order of the elements labeled by $a_1^{\pm 1}$ or $a_2^{\pm 1}$ in w_0 by W_0 in the following way:

$$W_0 = 1\overline{1}2\overline{2}1111\overline{1}\overline{1}\overline{1}\overline{1}2222\overline{2}\overline{2}\overline{2}\overline{2},$$

where the symbols $1, 2, \overline{1}$ and $\overline{2}$ are for a_1, a_2, a_1^{-1} and a_2^{-1}, respectively. We put

$$W_0^{-1} = 2222\overline{2}\overline{2}\overline{2}\overline{2}1111\overline{1}\overline{1}\overline{1}\overline{1}2\overline{2}1\overline{1}$$

to represent w_0^{-1}. To argue by contradiction, assume w_0^2 covers w_0^{-1}. Then W_0^2 covers W_0^{-1}. For each conceivable position for W_0^2 covering W_0^{-1}, one can find at least one of 1's, 2's, $\overline{1}$'s, or $\overline{2}$'s in W_0^{-1} facing some 2, 1, $\overline{2}$, $\overline{1}$, respectively, in W_0^2. See Figure 6. This implies that there is a pair of subpaths labelled by a_1 and a_2 which faces each other in w_0^2 and w_0^{-1}, which contradicts (1). We showed (2) for $i = 0$. A similar argument applies to w_i for each $i \geq 1$. (3) Observe that $w_j(1,+)$ consists of a_1 and b, while w_i contains a_2. If $w_j(1,+)$ covers w_i, then each a_2 in w_i faces some a_1 in $w_j(1,+)$, which contradicts (1). The same argument applies to $w_j(1,-)^{-1}$.
(4) Assume that w_i^k covers w_j for some $i < j$ and some $k > 0$. Then $w_j(1,+)$ covers some w_i of w_i^k since $2|w_i| \leq |w_j(1,+)|$, which contradicts (3). Assume that w_i^k covers w_j^{-1} for some $i < j$ and some $k > 0$. Then $w_j(1,-)^{-1}$ covers some w_i of w_i^k, which contradicts (3) as well. □

Lemma 4.3. *The words $w_i, 0 \leq i < \infty$, satisfy the following properties.*

(1) For all $n \geq 1$ and all $i \geq 0$, we have $c_{w_i}(w_i^n) = n$.

(2) For all $n \geq 1$ and all $i \geq 0$, we have $c_{w_i^{-1}}(w_i^n) = 0$.

(3) For all $j > i \geq 0$ and all $n \geq 1$, we have $c_{w_j}(w_i^n) = 0$.

(4) For all $j > i \geq 0$ and all $n \geq 1$, we have $c_{w_j^{-1}}(w_i^n) = 0$.

Proof. (1) Since w_i^n is reduced, w_i^n is a geodesic. We get $c_{w_i}(w_i^n) \geq |w_i^n|_{w_i} = n$. On the other hand, $c_{w_i}(w_i^n) \leq |\overline{w_i^n}|/|w_i| = n$. Thus $c_{w_i}(w_i^n) = n$.
(2) To show $c_{w_i^{-1}}(w_i^n) = 0$ by contradiction, assume $c_{w_i^{-1}}(w_i^n) > 0$. Let α be a realizing geodesic of $c_{w_i^{-1}}$ at w_i^n. Then $|\alpha|_{w_i^{-1}} > 0$. Fix a subword labeled by w_i^{-1} in α. Since w_i^n and α are reduced, there is a subword of w_i^n labeled by w_i^2 which covers the w_i^{-1} in α. But w_i^2 cannot cover w_i^{-1} by Lemma 4.2(2). We get a contradiction. See Figure 7.
(3) To show the claim by contradiction, assume $c_{w_j}(w_i^n) > 0$ for some $i < j$ and n. Take a realizing geodesic α of c_{w_j} at w_i^n. Then $|\alpha|_{w_j} > 0$. Fix some subword

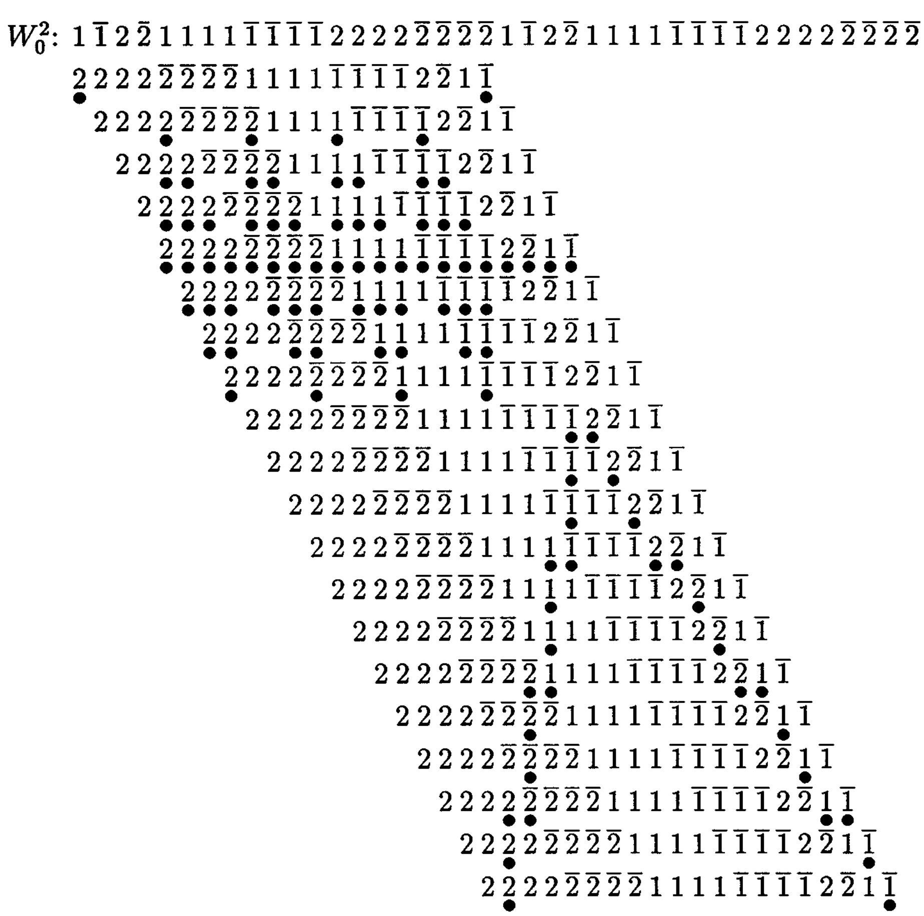

FIGURE 6. W_0^2 cannot cover W_0^{-1}. The first row is W_0^2. The second to the last rows describe all the possible positions for W_0^{-1}. We mark all the "illegal" pairs of $(1, 2)$ or $(\bar{1}, \bar{2})$ by $\bullet$. In each position for W_0^{-1}, we find at least one illegal pair, which is a contradiction.

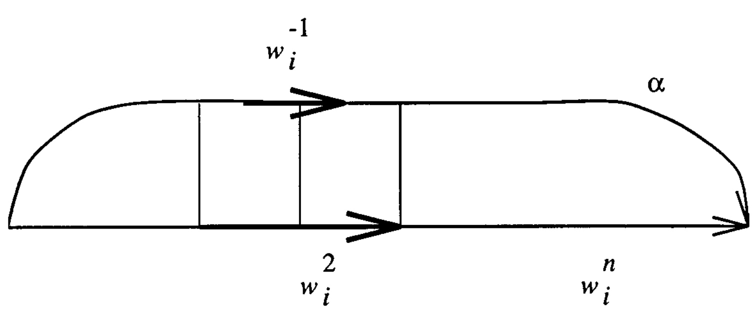

FIGURE 7. w_i^2 cannot cover w_i^{-1}.

labeled by w_j in α. Since α and w_i^n are reduced, there exists a subword labeled by w_i^k in w_i^n which covers the w_j in α. But w_i^k cannot cover w_j by Lemma 4.2(4). This is a contradiction.
(4) Similar to (3). □

We are in a position to prove Proposition 4.1.

Proof of Proposition 4.1. Let $w_i, 0 \leq i < \infty$, be the set of the words in Lemma 4.1 (and 4.2 and 4.3 as well). We show that they are desired words.

(1): Lemma 4.3(1) and (2) imply $h_{w_i}(w_i^n) = n$ since $h_{w_i} = c_{w_i} - c_{w_i^{-1}}$.

(2): By Lemma 4.3(3) and (4), we obtain $h_{w_j}(w_i^n) = 0$ for $i < j$.

(3): For a homomorphism $\phi : G \to \mathbb{R}$, we get $\phi(\overline{w_i}) = 0$ since $\overline{w_i} \in [G, G]$ by Lemma 4.1(3).

(4): By Lemma 4.1(1), w_i^2 is reduced for all $i \geq 0$.

(5): Clear from Lemma 4.1(2). □

5. Proofs of Theorem 1.1 and Corollaries 1.1, 1.2 and 1.3

Proof of Theorem 1.1. Let w_i, $0 \leq i < \infty$, be the words in Proposition 4.1. Note that h_{w_i} is an element of $C^1(G; \mathbb{Z})$. For each $g \in G$, there are only finitely many words w_i such that $h_{w_i}(g) \neq 0$. This follows from Lemma 2.2 and Proposition 4.1(5). Therefore, if $(a_i)_i \in l^1$, then $\sum_{i=1}^{\infty} a_i h_{w_i}$ is also well-defined as an element of $C^1(G; \mathbb{R})$ since this is in fact a finite sum for each $g \in G$. By the same reason, $\sum_{i=1}^{\infty} a_i \delta h_{w_i}$ is a well-defined cocycle, and the following equality holds.

$$\delta\left(\sum_{i=1}^{\infty} a_i h_{w_i}\right) = \sum_{i=1}^{\infty} a_i \delta h_{w_i}.$$

Further, since cocycles $\delta h_{w_i}, 0 \leq i < \infty$, have the same bound by Proposition 3.1, $\sum_i a_i \delta h_{w_i}$ is bounded. We get a real linear map

$$\omega : l^1 \to H_b^2(G; \mathbb{R}),$$

which sends $(a_i)_i$ to the cohomology class of $\sum_i a_i \delta h_{w_i}$. In order to show ω is injective, suppose $\omega((a_i)) = 0$. Then

$$\delta(\sum_{i=0}^{\infty} a_i h_{w_i}) = \delta b$$

for some $b \in C_b^1(G; \mathbb{R})$. This means

$$\sum_i a_i h_{w_i} - b = \phi,$$

for some homomorphism $\phi : G \to \mathbb{R}$. Applying this to $\overline{w_0^n} \in G$, we find

$$a_0 n - b(\overline{w_0^n}) = \phi(\overline{w_0^n}) = 0,$$

for all $n \geq 1$ by Proposition 4.1. Since b is bounded, $a_0 = 0$. Similarly, $a_i = 0$ for each $i \geq 1$. Thus ω is injective. It is well-known that the cardinality of the dimension of l^1 as a vector space is continuum. □

Proof of Corollary 1.1. Without loss of generality, we may assume $|A| \geq 3$. Since $C = \{1\}$, we have $|C \backslash A / C| = |A| \geq 3$. Apply Theorem 1.1. □

Proof of Corollary 1.2. Since $|C| < \infty$ and $|A| = \infty$, we have $|C \backslash A / C| = \infty$. Apply Theorem 1.1. □

Proof of Corollary 1.3. Since A is abelian, we have $|C\backslash A/C| = |A/C| \geq 3$. Apply Theorem 1.1. □

6. Quasi homomorphisms on $A*_{C,\varphi}$

Let $G = A*_{C,\varphi} = \langle A, t; c = t^{-1}\varphi(c)t \text{ for all } c \in C\rangle$. Suppose $|A/C| \geq 2$ and $|A/\varphi(C)| \geq 2$. Let g be

$$g = a_1 t^{n_1} a_2 t^{n_2} \dots a_I t^{n_I} a_{I+1},$$

with $a_i \in A$ and $n_i \neq 0$, where a_1, a_{I+1} may be empty. Suppose that for all i ($1 \leq i \leq I-1$), the following conditions (i) and (ii) are satisfied:
(i) If $n_i > 0$ and $n_{i+1} < 0$, then $a_{i+1} \notin C$.
(ii) If $n_i < 0$ and $n_{i+1} > 0$, then $a_{i+1} \notin \varphi(C)$.
Then we say g is *reduced*.

The following fact is known as Britton's lemma.

Lemma 6.1 (Britton, [LS]). *Suppose* $1 \leq I$. *If* g *is reduced, then* $g \neq 1$ *in* G.

As an application of Britton's lemma, we have the following lemma.

Lemma 6.2. *Let*

$$g = a_1 t^{n_1} a_2 t^{n_2} \dots a_I t^{n_I} a_{I+1}, \qquad a_i \in A,$$
$$h = b_1 t^{m_1} b_2 t^{m_2} \dots b_J t^{m_J} b_{J+1}, \qquad b_j \in A,$$

be reduced with $n_i = \pm 1$, $m_j = \pm 1$. *If* $g = h$ *in* G, *then*

$$I = J,\ n_1 = m_1, \dots, n_I = m_I,$$

and elements

$$a_{I+1}b_{I+1}^{-1},\ t^{n_I}a_{I+1}b_{I+1}^{-1}t^{-m_I},\ a_I t^{n_I} a_{I+1} b_{I+1}^{-1} t^{-m_I} b_I^{-1}, \dots$$
$$\dots, t^{n_1} a_2 t^{n_2} a_3 \dots a_{I+1} b_{I+1}^{-1} \dots b_3^{-1} t^{-m_2} b_2^{-1} t^{-m_1}$$

are in C *or* $\varphi(C)$, *in particular, in* A.

Proof. Without loss of generality, we may assume $J \leq I$. Since h is reduced, $h^{-1} = b_{J+1}^{-1} t^{-m_J} b_J^{-1} t^{-m_{J-1}} \dots b_1^{-1}$ is reduced as well. Set

$$d_{I+1} = a_{I+1} b_{J+1}^{-1};$$

then we have $d_{I+1} \in A$ and

$$gh^{-1} =_G (a_1 t^{n_1} a_2 t^{n_2} \dots a_I t^{n_I}) d_{I+1} (t^{-m_J} b_J^{-1} t^{-m_{J-1}} \dots b_1^{-1}),$$

where "$=_G$" means that the right- and left-hand sides of the equality represent a common element in G. Since $gh^{-1} =_G 1$, the word on the right-hand side of the above equality is not reduced by Britton's lemma. But, since $a_1 t^{n_1} a_2 t^{n_2} \dots a_I t^{n_I}$ and $t^{-m_J} b_J^{-1} t^{-m_{J-1}} \dots b_1^{-1}$ are reduced, it follows that

$$n_I = m_J = 1 \text{ (or } -1), \quad d_{I+1} \in C \text{ (or } \varphi(C), \text{ respectively)},$$

and $t^{n_I} d_{I+1} t^{-m_J} \in \varphi(C)$ (or C, respectively).

Set

$$c_I = t^{n_I} d_{I+1} t^{-m_J},\ d_I = a_I c_I b_J^{-1}.$$

Clearly $d_I \in A$ and

$$gh^{-1} =_G (a_1 t^{n_1} a_2 t^{n_2} \dots a_{I-1} t^{n_{I-1}}) d_I (t^{-m_{J-1}} b_{J-1}^{-1} t^{-m_{J-2}} \dots b_1^{-1}).$$

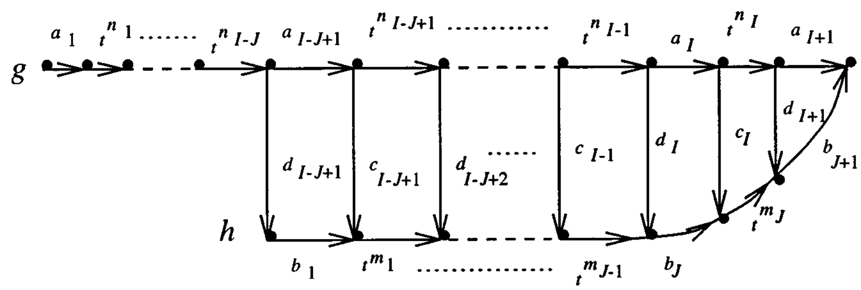

FIGURE 8. This illustrates Lemma 6.2. Elements d_i, c_i are in A for all i with $I-J+1 \le i \le I$. If $J < I$, then it contradicts Britton's lemma.

By Britton's lemma, the word on the right-hand side is not reduced. Thus

$$n_{I-1} = m_{J-1} = 1 \text{ (or } -1),$$
$$d_I \in C \text{ (or } \varphi(C), \text{ respectively)},$$

and

$$c_{I-1} = t^{n_{I-1}} d_I t^{-m_{J-1}} \in \varphi(C) \text{ (or } C, \text{ respectively)}.$$

We define elements $d_{I-1}, d_{I-2}, \ldots, d_{I-J+2}$ and $c_{I-1}, c_{I-2}, \ldots, c_{I-J+1}$ inductively by

$$c_i = t^{n_i} d_{i+1} t^{-m_{J-I+i}}, \quad d_i = a_i c_i b_{J-I+i}^{-1}.$$

See Figure 8. Using Britton's lemma repeatedly, we have

$$n_{I-J+j} = m_j, \qquad 1 \le j \le J,$$

and

$$d_{I+1}, d_I, \ldots, d_{I-J+2}, c_I, c_{I-1}, \ldots, c_{I-J+1} \in C \text{ or } \varphi(C).$$

Set

$$d_{I-J+1} = a_{I-J+1} c_{I-J+1} b_1^{-1}.$$

Clearly $d_{I-J+1} \in A$. To complete the proof, it suffices to show $J = I$. In order to show this by contradiction, suppose $J < I$. Then

$$gh^{-1} =_G a_1 t^{n_1} a_2 \ldots t^{n_{I-J-1}} a_{I-J} t^{n_{I-J}} d_{I-J+1}.$$

The word on the right-hand side is reduced. This contradicts Britton's lemma, since $gh^{-1} =_G 1$. We get $I = J$. □

We take $\{t\} \cup A \backslash \{1\}$ as a set of generators of G and write the Cayley graph of G for this set by Γ.

Lemma 6.3. *If α is a geodesic in Γ, then it is reduced.*

Proof. If a path α is not reduced, then we can make it shorter using a relation $tct^{-1} = \varphi(c)$; hence α is not a geodesic. □

Remark. Compare Lemma 6.3 with Lemma 3.1. A reduced path is not always a geodesic in this case. For example, the left-hand side of $\varphi(c)^{-1}tc = t$ for $c \in C \backslash \{1\}$ is reduced but not a geodesic.

Lemma 6.4. *Let α be a path and w a word. If w^2 is reduced, then there is a reduced path β which realizes c_w at α.*

Proof. Take a path β which realizes c_w at α such that $|\beta|_w$ is minimal among all the realizing paths. Then an argument similar to the proof of Lemma 3.2 shows β is reduced. □

Lemma 6.5. *Suppose w^2 is reduced. Let α, β be paths starting at 1. Then we have*

$$|c_w(\alpha) - c_w(\beta)| \leq 2|\overline{\alpha^{-1}\beta}|,$$

$$|h_w(\alpha) - h_w(\beta)| \leq 4|\overline{\alpha^{-1}\beta}|.$$

Proof. Similar to Lemmas 3.3 and 3.4. □

Lemma 6.6. *Suppose w^2 is reduced. Let α be a reduced path. If $\alpha = \alpha_1\alpha_2$, then*

$$|h_w(\alpha) - h_w(\alpha_1) - h_w(\alpha_2)| \leq 10.$$

Proof. Similar to Lemma 3.7. □

Proposition 6.1. *Suppose w^2 is reduced. Then we have*

$$|\delta h_w| \leq 78.$$

Proof. The outline is similar to the proof of Proposition 3.1. Let x, y be elements in G. It suffices to show $|h_w(xy) - h_w(x) - h_w(y)| \leq 78$. Take reduced paths α, β, γ with $\overline{\alpha} = x, \overline{\beta} = y, \overline{\gamma} = xy$. Since $\overline{\alpha\beta} = \overline{\gamma}$ and α, β, γ are reduced, by Lemmas 6.1 and 6.2, there exist subdivisions of α, β, γ such that

$$\alpha = \alpha_1\alpha_2\alpha_3,\ \beta = \beta_1\beta_2\beta_3,\ \gamma = \gamma_1\gamma_2\gamma_3,$$

and that $\overline{\gamma_1^{-1}\alpha_1} = c_1, \overline{\alpha_3\beta_1} = c_2, \overline{\beta_3\gamma_3^{-1}} = c_3$ for some $c_1, c_2, c_3 \in C \cup \varphi(C)$, and $\overline{\alpha_2}, \overline{\beta_2}, \overline{\gamma_2} \in A$. By Lemma 6.6,

$$|h_w(\alpha) - h_w(\alpha_1) - h_w(\alpha_2) - h_w(\alpha_3)| \leq 20.$$

Since $|h_w(\alpha_2)| \leq 2$, $|h_w(\alpha) - h_w(\alpha_1) - h_w(\alpha_3)| \leq 22$. Similarly,

$$|h_w(\beta) - h_w(\beta_1) - h_w(\beta_3)| \leq 22,\ |h_w(\gamma) - h_w(\gamma_1) - h_w(\gamma_3)| \leq 22.$$

By an argument similar to the proof of Proposition 3.1,

$$|h_w(x) + h_w(y) - h_w(xy)| = |h_w(\alpha) + h_w(\beta) - h_w(\gamma)| \leq 78.$$

□

7. Choice of words for $A*_{C,\varphi}$

Let $G = A*_{C,\varphi}$ with $|A/C| \geq 2$, $|A/\varphi(C)| \geq 2$.

Lemma 7.1. *Let w be*

$$w = t^{n_1}a_1t^{n_2}a_2 \dots t^{n_I}a_I,$$

such that $a_i \in A\backslash\{1\}$ and $n_i \in \mathbb{Z}\backslash\{0\}$ for all i with $1 \leq i \leq I$. We denote the set of the following conditions (1.1), … , (1.4) by Condition I and the set (2.1), … , (2.4) by Condition II.

(1.1) $0 < n_1, n_3, n_5, \dots$.
(1.2) $0 > n_2, n_4, n_6, \dots$.
(1.3) $a_1, a_3, a_5, \dots \notin C$.
(1.4) $a_2, a_4, a_6, \dots \notin \varphi(C)$.
(2.1) $0 > n_1, n_3, n_5, \dots$.
(2.2) $0 < n_2, n_4, n_6, \dots$.

(2.3) $a_1, a_3, a_5, \dots \notin \varphi(C)$.
(2.4) $a_2, a_4, a_6, \dots \notin C$.
If either the Condition I or II holds, then w is a geodesic in Γ.

Proof. Suppose Condition I holds. The other case is similar. Clearly w is reduced. Let γ be a geodesic with $\overline{w} = \overline{\gamma}$. Then by Lemma 6.3, γ is reduced. By Lemma 6.2, we have

$$\gamma = b_0 \tau_1 b_1 \tau_2 b_2 \dots \tau_I b_I,$$

such that $b_i \in A \backslash \{1\}$ for $1 \le i \le I-1$, b_0, b_I are in $A\backslash\{1\}$ or empty, and

$$\begin{aligned} \tau_i &= t b_{i,1} t b_{i,2} \dots t b_{i,n_i-2} t b_{i,n_i-1} t, \text{ if } i \text{ is odd,} \\ \tau_i &= t^{-1} b_{i,1} t^{-1} b_{i,2} \dots t^{-1} b_{i,n_i-2} t^{-1} b_{i,n_i-1} t^{-1}, \text{ if } i \text{ is even,} \end{aligned}$$

where $b_{i,j}$ are in $A\backslash\{1\}$ or empty for $1 \le i \le I$ and $1 \le j \le n_i - 1$.

We claim that b_I is not empty. To show this by contradiction, assume b_I is empty. Then

$$\begin{aligned} w\gamma^{-1} &= \dots t a_I t^{-1} \dots, \text{ if } I \text{ is odd,} \\ w\gamma^{-1} &= \dots t^{-1} a_I t \dots, \text{ if } I \text{ is even.} \end{aligned}$$

Since w and γ^{-1} are reduced, $w\gamma^{-1}$ is reduced by (1.3) if I is odd or by (1.4) if I is even. Then it follows from Britton's lemma that $\overline{w\gamma^{-1}} \neq 1$. This is a contradiction. We get b_I is not empty. Thus

$$|\gamma| \ge \sum_{i=1}^{I} |\tau_i| + I \ge \sum_{i=1}^{I} |n_i| + I = |w|.$$

We have the first inequality because $b_1, \dots, b_I$ are not empty, and the second one since the number of t's in τ_i is n_i for $1 \le i \le I$. Since γ is a geodesic, we have $|\gamma| = |w|$; hence w is a geodesic. □

Lemma 7.2. *Take some elements $g \in A\backslash C$ and $h \in A \backslash \varphi(C)$ and fix them. Let $w_i, 0 \le i < \infty$, be words such that*

$$w_i = t^{10^i} g t^{-10^i} h t^{10^i} g^{-1} t^{-10^i} h^{-1} t^{2\cdot 10^i} g t^{-2\cdot 10^i} h t^{3\cdot 10^i} g^{-1} t^{-3\cdot 10^i} h^{-1}.$$

Then the words w_i satisfy the following properties.
(1) For all $i \ge 0$ and all $n \ge 1$, we have $c_{w_i}(w_i^n) = n$.
(2) For all $i \ge 0$ and all $n \ge 1$, we have $c_{w_i^{-1}}(w_i^n) = 0$.
(3) For all $j > i \ge 0$ and all $n \ge 1$, we have $c_{w_j^{\pm 1}}(w_i^n) = 0$.
(4) For all $i \ge 0$, we have $\overline{w_i} \in [G, G]$.
(5) For all $i \ge 0$, we have $|w_i| = 8 + 14 \cdot 10^i$.

Proof. (1) By Lemma 7.1, w_i^n is a geodesic. Thus $c_{w_i}(w_i^n) = n$.

(2) In order to show $c_{w_0^{-1}}(w_0^n) = 0$ for all $n \ge 1$ by contradiction, suppose $c_{w_0^{-1}}(w_0^n) > 0$ for some n. By Lemma 6.4, take a reduced path α such that $\overline{\alpha} = \overline{w_0^n}$ and $|\alpha|_{w_0^{-1}} > 0$. We describe the order of t's and t^{-1}'s appearing in w_0 by W_0,

$$W_0 = + - + - + + - - + + + - - -,$$

where $+$ stands for t, $-$ for t^{-1}, and we disregard $g^{\pm 1}, h^{\pm 1}$ in w_0. Then W_0^n represents w_0^n. In the same fashion let Λ denote the order of t's and t^{-1}'s in α.

Applying Lemma 6.2 to α and w_0^n, we get $W_0^n = \Lambda$. Λ contains W_0^{-1} as a subpiece since $|\alpha|_{w_0^{-1}} > 0$, where

$$W_0^{-1} = + + + - - - + + - - + - + - \,.$$

But it is easy to check that W_0^n cannot contain W_0^{-1} as a subset. We get a contradiction. Hence $c_{w_0^{-1}}(w_0^n) = 0$. Similarly, we get $c_{w_i^{-1}}(w_i^n) = 0$ for each $i \geq 1$ and all $n \geq 1$.

(3) Let W_i, $1 \leq i < \infty$, be the order of t's and t^{-1}'s in w_i. To show $c_{w_j}(w_i^n) = 0$ for all $n \geq 1$ and all $j > i > 0$ by contradiction, assume $c_{w_j}(w_i^n) > 0$ for some $n \geq 1$ and some $j > i > 0$. Then, by our assumption, W_j must be contained in W_i^n as a subword. This is a contradiction since W_j has $3 \cdot 10^j$ consecutive t's, but W_i has at most $3 \cdot 10^i$ consecutive t's. Thus we get $c_{w_j}(w_i^n) = 0$. Similarly, $c_{w_j^{-1}}(w_i^n) = 0$.

(4) Set $T = t^{10^i}$. Then

$$w_i = [T, g][gh, [T, g^{-1}]\,][T, g^{-1}][g, h][T^2, g][gh, [T^3, g^{-1}]\,][T^3, g^{-1}][g, h];$$

thus $\overline{w_i} \in [G, G]$.

(5) This is clear. □

Proposition 7.1. *Let $G = A*_{C,\varphi}$ with $|A/C| \geq 2$, $|A/\varphi(C)| \geq 2$. There exist words w_i, $0 \leq i < \infty$, which satisfy the following properties.*
(1) For all $i \geq 0$ and all $n \geq 1$, we have $h_{w_i}(w_i^n) = n$.
(2) For all $j > i \geq 0$ and all $n \geq 1$, we have $h_{w_j}(w_i^n) = 0$.
(3) For all $i \geq 0$, we have $\overline{w_i} \in [G, G]$.
(4) For all $i \geq 0$, w_i^2 is reduced.
(5) $\lim_{i\to\infty} |w_i| = \infty$.

Proof. The words w_i, $0 \leq i < \infty$, in Lemma 7.2 clearly suffice. □

8. Proofs of Theorem 1.2 and Theorem 1.3

Proof of Theorem 1.2. Theorem 1.2 follows from Proposition 6.1 and 7.1 as Theorem 1.1 did from Proposition 3.1 and 4.1. □

Proof of Theorem 1.3. By Stallings' structure theorem [S], we know that G is either
(1) $A *_C B$ with $|C| < \infty$, $|A/C| \geq 3$ and $|B/C| \geq 2$,
or
(2) $A*_{C,\varphi}$ with $|C| < \infty$ and $|A/C| \geq 2$.

Suppose the condition(1) holds. If $|A| = \infty$ or $|B| = \infty$, then we have the conclusion by Corollary 1.2. If $|A| < \infty$ and $|B| < \infty$, then G is word-hyperbolic. Since G has infinitely many ends, it is non-elementary. It is known that the conclusion of Theorem 1.3 holds for a non-elementary word-hyperbolic group [EF]. Suppose the condition(2) holds. Then since $|A/C| \geq 2$ and $|C| < \infty$, we have $|A/\varphi(C)| \geq 2$ as well. Apply Theorem 1.2. □

References

[BaGh1] J. Barge, E. Ghys, Surfaces et cohomologie bornée, Invent. Math., 92, 1988, 509-526. MR **89e:**55015

[BaGh2] J. Barge, E. Ghys, Cocycles bornés et actions de groupes sur les arbres réels, in "Group Theory from a Geometric Viewpoint", World Sci. Pub., New Jersey, 1991, 617-622. MR **93f:**20033

[BaGh3] J. Barge, E. Ghys, Cocycles d'Euler et de Maslov, Math. Ann., 294, no 2, 1992, 235-265. MR **95b:**55021

[Bav] C. Bavard, Longueur stable des commutateurs, L'Enseignement Math., 37, 1991, 109-150. MR **92g:**20051

[B] R. Brooks, Some remarks on bounded cohomology, Ann. Math. Studies, 97, 1981, 53-63. MR **83a:**57038

[BS] R. Brooks, C. Series, Bounded cohomology for surface groups, Topology, 23, no 1, 1984, 29-36. MR **85c:**57009

[Br] K.S. Brown, "Cohomology of groups", Springer, New York, 1982. MR **83k:**20002

[EF] D.B.A. Epstein, K. Fujiwara, The second bounded cohomology of word-hyperbolic groups, Topology 36, 1997, 1275-1289. MR **98k:**20088

[F] K. Fujiwara, The second bounded cohomology of a group acting on a Gromov-hyperbolic space, Proc. London Math Soc. (3) 76, 1998, 70–94. MR **99c:**20072

[FO] K. Fujiwara, K. Ohshika, The second bounded cohomology of 3-manifolds, preprint, Univ Tokyo, 1997.

[Gh] E. Ghys, Groupes d'homéomorphismes du cercle et cohomologie bornée, Contemp. Math. 58, AMS, 1987, 81-106. MR **88m:**58024

[Gr] R.I. Grigorchuk, Some remarks on bounded cohomology, LMS Lecture Note 204, Cambridge Univ Press, Cambridge, 1995, 111-163. MR **96j:**20073

[G] M. Gromov, Volume and bounded cohomology, Publ. Math. IHES, 56, 1982, 5-99. MR **84h:**53053

[I] N.V. Ivanov, Foundations of the theory of bounded cohomology, Zap. Nauchn. Sem. Leningr. Otd. Mat. Inst., 143, 1985, 69-109. MR **87b:**53070

[LS] R.C. Lyndon, P.E. Schupp, "Combinatorial Group Theory", Berlin, Springer, 1977. MR **58:**28182

[MaMo] S. Matsumoto, S. Morita, Bounded cohomology of certain groups of homomorphisms, Proc. A.M.S., 94, 1985, 539-544. MR **87e:**55006

[Mi] Y. Mitsumatsu, Bounded cohomology and l^1-homology of surfaces, Topology, 23, 1984, 465-471. MR **86f:**57010

[So1] T. Soma, Bounded cohomology of closed surfaces, Topology 36, 1997, 1221-1246. MR **99a:**57011

[So2] T. Soma, Bounded cohomology and topologically tame Kleinian groups, Duke Math Jour. 88, 1997, no 2, 357-370. MR **98f:**57023

[S] J. Stallings, "Group theory and three-dimensional manifolds", Yale Univ. Press, New Haven, 1971. MR **54:**3705

[Y] T. Yoshida, On 3-dimensional bounded cohomology of surfaces, in "Homotopy Theory and Related Topics", Advanced Studies in Pure Math 9, Kinokuniya, Tokyo, 1986, 173-176. MR **88m:**57032

Department of Mathematics, Keio University, Yokohama, 223 Japan
Current address: Math Institute, Tohoku Univeristy, Sendai, 980-8578, Japan
E-mail address: `fujiwara@math.tohoku.ac.jp`