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# LOW-DIMENSIONAL LINEAR REPRESENTATIONS OF Aut $F_n$, $n \geq 3$

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Abstract. We classify all complex representations of Aut $F_n$, the automorphism group of the free group $F_n$ ($n \geq 3$), of dimension $\leq 2n-2$. Among those representations is a new representation of dimension $n+1$ which does not vanish on the group of inner automorphisms.

## Introduction

In this paper we study low-dimensional linear representations of $\Gamma_n = \text{Aut } F_n$ ($n \geq 3$), the automorphism group of the free group (low-dimensional representations of Aut $F_2$ were analyzed in [DP]). It is known (see Theorem 1.2) that any $n$-dimensional representation of $\text{Aut} F_n$ factors through the the canonical homomorphism $f\colon \Gamma_n \to GL_n(\mathbf{Z})$. We show that in dimension higher than $n$, the group $\Gamma_n$ acquires new representations. Namely, we will establish the existence of a homomorphism $g\colon \Gamma_n \to GL_n(\mathbf{Z}) \ltimes \mathbf{Z}^n$ which "lifts" $f$ and gives rise to an $(n+1)$-dimensional representation. The main result of the paper (Theorem 3.1) claims that this representation accounts in fact for all representations of $\Gamma_n$ of dimension $\leq 2n-2$ (in particular, any such representation factors through $g$). We also use the homomorphism $g$ to construct an infinite family of pairwise inequivalent representations of $\Gamma_n$ (Proposition 2.5).

We are grateful to the referee for helpful comments and especially for pointing out a number of relevant references we were not aware of.

## 1. Notations and preliminaries.

Throughout the paper, $E_m$ will denote the identity $m \times m$-matrix; for a partition $m = m_1 + \cdots + m_k$, $G(m_1, \ldots, m_k)$ will denote the product

$$GL_{m_1}(\mathbf{C}) \times \cdots \times GL_{m_k}(\mathbf{C})$$

diagonally embedded in $GL_m(\mathbf{C})$, $p_i(m_1, \ldots, m_k)\colon G(m_1, \ldots, m_k) \to GL_{m_i}(\mathbf{C})$ being the corresponding projection; $S_n$ is the symmetric group on $\{1, \ldots, n\}$. To comply with the classical tradition, $\Gamma_n$ and $S_n$ will be treated as groups of *right transformations* (in other words, the result of application of $f$ to $a$ will be written as $(a)f$, with the law of composition given by $(a)(fg) = ((a)f)g$.

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We will use the following elements of $\Gamma_n = \operatorname{Aut} F_n$:

$$\rho_{ij}\colon \begin{cases} x_i \to x_i x_j, \\ x_k \to x_k, \quad k \neq i, \end{cases} \qquad \lambda_{ij}\colon \begin{cases} x_i \to x_j x_i, \\ x_k \to x_k, \quad k \neq i, \end{cases}$$

$$\varepsilon_i\colon \begin{cases} x_i \to x_i^{-1}, \\ x_k \to x_k, \quad k \neq i, \end{cases} \qquad \begin{array}{l} \varepsilon_{ij} = \varepsilon_i \varepsilon_j; \\ \varepsilon = \varepsilon_1 \dots \varepsilon_n\colon x_i \to x_i^{-1} \quad \forall i, \end{array}$$

$$\forall \pi \in S_n, \quad \sigma_\pi\colon x_i \to x_{(i)\pi}.$$

It is known that these elements generate $\Gamma_n$ (cf. [MKS]). One easily checks the following identities:

$$\begin{gathered} \varepsilon_i^2 = \mathrm{id}_{F_n}, \quad \varepsilon_i \varepsilon_j = \varepsilon_j \varepsilon_i, \quad \sigma_\pi^{-1} \varepsilon_i \sigma_\pi = \varepsilon_{(i)\pi}, \\ \varepsilon_i^{-1} \rho_{ij} \varepsilon_i = \lambda_{ij}^{-1}, \quad \varepsilon_{ij}^{-1} \rho_{ij} \varepsilon_{ij} = \varepsilon^{-1} \rho_{ij} \varepsilon = \lambda_{ij} \quad \forall i, j, \ i \neq j. \end{gathered} \tag{1}$$

It follows that $H_n$, the subgroup generated by all $\varepsilon_i$, is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^n$; all $\sigma_\pi$, $\pi \in S_n$, form a subgroup $\Sigma_n$ which is a replica of $S_n$; and the subgroup $\Omega_n$, generated by all $\varepsilon_i$ and $\sigma_\pi$ together, is the semidirect product $\Sigma_n \ltimes H_n$.

A special feature that distinguishes the case $n \geq 3$ from the case $n = 2$ is the existence of the following commutator relations:

$$[\lambda_{ij}, \lambda_{jk}] = \lambda_{ik}, \qquad [\rho_{ij}, \rho_{jk}] = \rho_{ik} \tag{2}$$

($i, j$, and $k$ are all distinct), where $[x, y] = xyx^{-1}y^{-1}$.

Passing to abelianization, we obtain a (surjective) homomorphism

$$f\colon \Gamma_n \to \operatorname{Aut}(F_n/[F_n, F_n]) = GL_n(\mathbf{Z})$$

(matrix presentation is taken with respect to the basis of $F_n/[F_n, F_n] \simeq \mathbf{Z}^n$ made up of the images of $x_1, \dots, x_n$; for consistency, $\mathbf{Z}^n$ is treated as *n-rows* of integers, with *right* action of $GL_n(\mathbf{Z})$). Let $N = \operatorname{Ker} f$. Our argument makes essential use of the fact (see [LS], p.28) that $N$ as a normal subgroup of $\Gamma_n$ is generated by the element

$$\beta = \lambda_{n,n-1}^{-1} \rho_{n,n-1}.$$

Another element to be frequently used is

$$\gamma = \lambda_{n,n-1} \rho_{n,n-1}.$$

**Lemma 1.1.** *Let $\varphi\colon \Gamma_n \to G$ be a homomorphism of $\Gamma_n$ to some group $G$. If the restriction of $\varphi$ to $\Omega_n$ is not injective, then $\varphi$ factors through $f$.*

*Proof.* First, assume that the restriction $\varphi \mid H_n$ is not injective. It is easy to see that $H_n$ has only two proper subgroups normalized by $\Sigma_n$ : $\langle \varepsilon \rangle$ and $H' = \{h \in H_n \mid \det f(h) = 1\}$. If $\operatorname{Ker} f = \langle \varepsilon \rangle$, then (cf. (1))

$$\varphi(\lambda_{ij}) = \varphi(\varepsilon \rho_{ij} \varepsilon^{-1}) = \varphi(\rho_{ij}),$$

implying that $\varphi(\beta) = 1_G$, which in view of the fact quoted above implies that $\varphi(N) = \{1_G\}$; hence our claim. If $\operatorname{Ker}(\varphi \mid H_n) \supset H'$, we observe that $\varepsilon_{ij} \in H'$ ($i \neq j$), so the same argument applies with $\varepsilon$ replaced by $\varepsilon_{ij}$.

Now, suppose $z = (\sigma_\pi, h) \in \operatorname{Ker}(\varphi \mid \Omega_n)$ with $\pi$ nontrivial. If $i$ is such that $\pi(i) \neq i$, then the commutator $[z, \varepsilon_i] = \varepsilon_{i\pi(i)} \in \operatorname{Ker}(\varphi \mid H_n)$, reducing the argument to the previous case. □

Lemma 1.1 implies

**Theorem 1.2** ([DF], [R1]). *Every $n$-dimensional representation $\theta\colon \Gamma_n \to GL_n(\mathbf{C})$ factors through the canonical homomorphism $f\colon \Gamma_n \to GL_n(\mathbf{Z})$.*

*Proof.* It follows from Lemma 1.1 that $\theta(H_n) \subset GL_n(\mathbf{C})$ is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^n$, so after replacing $\theta$ by an equivalent representation we may assume that $\theta(\Sigma_n)$ contains representatives of all cosets of the Weyl group $\mathcal{W} = \mathcal{N}/\mathcal{D}_n$. Since $\varepsilon$ is fixed by $\Sigma_n$, and $-E_n$ is the only nontrivial element of $\theta(H_n)$ fixed by $\mathcal{W}$, we conclude that $\theta(\varepsilon) = -E_n$. But then $\theta(\beta) = \theta([\rho_{nn-1}, \varepsilon]) = E_n$, and our claim follows. □

*Remark* 1.3. As pointed out by the referee, Theorem 1.2 was first proved in [DF]. Being unaware of this result, the second-named author rediscovered it in [R1] (with the same proof which we reproduced above for the sake of completeness) in an attempt to see whether $\Gamma_n$ ($n \geq 3$) has finite representation type.

## 2. The homomorphism $g\colon \Gamma_n \to GL_n(\mathbf{Z}) \ltimes \mathbf{Z}^n$.

To avoid ambiguity, we note that since we are using the *right* action of $GL_n(\mathbf{Z})$ on $\mathbf{Z}^n$, the operation on the semidirect product $GL_n(\mathbf{Z}) \rtimes \mathbf{Z}^n$ is given by $(A, u)(B, v) = (AB, uB + v)$.

Let $v_1, \ldots, v_n$ be the standard basis of $\mathbf{Z}^n$.

**Proposition 2.1.** *For every $n \geq 2$, there exists a homomorphism*

$$g\colon \Gamma_n \to GL_n(\mathbf{Z}) \rtimes \mathbf{Z}^n$$

*such that*

$$g(\lambda_{ij}) = (f(\lambda_{ij}), v_j)\ ,\ g(\rho_{ij}) = (f(\rho_{ij}), -v_j),$$

$$g(\varepsilon_i) = (f(\varepsilon_i), 0)\ , g(\sigma_\pi) = (f(\sigma_\pi), 0).$$

*Proof.* For $n \geq 4$, $\Gamma_n$ is generated by the following four automorphisms:

$$P = \sigma_{(12)}, \quad Q = \sigma_{(12\ldots n)}, \quad S = \varepsilon_1, \quad U = \lambda_{12},$$

and admits the following presentation in terms of these generators (cf. [MKS], [N]):

$$P^2 = S^2 = Q^n = (QP)^{n-1} = 1, \tag{1}$$

$$[S, Q^{-1}PQ] = [S, QP] = [S, Q^{-1}SQ] = [P, Q^{-i}PQ^i] = 1, \quad (i = 2, 3, ..., [n/2]) \tag{2}$$

$$U^{-1}PUPSUSPS = (PSPU)^2 = (PQ^{-1}UQ)^2UQ^{-1}U^{-1}QU^{-1} = 1, \tag{3}$$

$$\begin{aligned} &[U, Q^{-2}PQ^2] = [U, QPQ^{-1}PQ] = [U, Q^{-2}SQ^2] = [U, Q^{-2}UQ^2] = 1, \\ &[U, SUS] = [U, PQ^{-1}SUSQP] = [U, PQ^{-1}PQPUPQ^{-1}PQP] = 1. \end{aligned} \tag{4}$$

Since the $GL_n$-components of $g(P), g(Q), g(S)$ and $g(U)$ as defined in the statement of the proposition coincide with $f(P)$, $f(Q), f(S)$ and $f(U)$ respectively, for the existence of a homomorphism $g\colon \Gamma_n \to GL_n(\mathbf{Z}) \ltimes \mathbf{Z}^n$ with such images of $P, Q, S$, and $U$, we need to verify only the equality of the $\mathbf{Z}^n$-components in the relations (1)–(4). For computations, we need to observe that $f(S) = \mathrm{diag}(-1, 1, \ldots, 1)$, $f(U) = E_{12}$, the elementary matrix with 1 as the (12)-entry, and $f(\sigma_\pi) = (\delta_{i(i)\pi})$, the permutation matrix. Note that since the $\mathbf{Z}^n$-components of $g(P), g(Q)$ and $g(S)$ are trivial, we only need to verify the relations involving $U$, i.e. (3) and (4).

The triviality of the $\mathbf{Z}^n$-component in all relations (3) is easily verified by direct computation. Namely, the $\mathbf{Z}^n$-component of $U^{-1}PUPSUSPS$ is

$$\begin{aligned}(-v_2)f(PUPSUSPS) + (v_2)f(PSUSPS) + (v_2)f(SPS)\\ = -v_2 + (v_1 + v_2) - v_1 = 0.\end{aligned}$$

The $\mathbf{Z}^n$-component of $(PSPU)^2$ is

$$(v_2)f(PSPU) + v_2 = -v_2 + v_2 = 0.$$

Finally, the $\mathbf{Z}^n$-component of $(PQ^{-1}UQ)^2UQ^{-1}U^{-1}QU^{-1}$ is

$$\begin{aligned}(2v_2)f(UQ^{-1}U^{-1}QU^{-1}) + (v_2)f(Q^{-1}U^{-1}QU^{-1}) - (v_2)f(QU^{-1}) - v_2\\ = 2v_3 + (v_2 - v_3) - v_3 - v_2 = 0.\end{aligned}$$

For verification of (4), we note the following commutator identity:

$$[(A, u), (B, v)] = ([A, B], u(B - E_n)A^{-1}B^{-1} + v(A^{-1} - E_n)B^{-1}).$$

It follows that the $\mathbf{Z}^n$-component of $[(A, u), (B, v)]$ is trivial if $uB = u$ and $vA = v$. In the first three commutators, the $\mathbf{Z}^n$-component of the second element is trivial, and it suffices to show that this element fixes $v_2$ (= $\mathbf{Z}^n$-component of $g(U)$). This is easily done by direct computation; for example,

$$(v_2)Q^{-2}PQ^2 = (v_n)PQ^2 = (v_n)Q^2 = v_2, \qquad \text{etc.}$$

For triviality of the $\mathbf{Z}^n$-component in the remaining four commutators, we need to show in addition that the $\mathbf{Z}^n$-component of this element is fixed by $U$, i.e. it doesn't involve $v_1$. The computations are routine, and we consider below only two out of four cases:

$$(v_2)f(Q^{-2}UQ^2) = (v_n)f(UQ^2) = (v_n)f(Q^2) = v_2,$$

and the $\mathbf{Z}^n$-component of $Q^{-2}UQ^2$ is $v_2f(Q^2) = v_4$;

$$(v_2)f(SUS) = (v_2)f(US) = (v_2)f(S) = v_2,$$

and the $\mathbf{Z}^n$-component of $SUS$ is $v_2f(Q^2) = v_4$ (in the remaining two commutators the $\mathbf{Z}^n$-components of the second element are $v_3$ and $v_4$, respectively).

Once we have established the existence of a homomorphism $g\colon \Gamma_n \to GL_n(\mathbf{Z}) \ltimes \mathbf{Z}^n$ which has the required images on $P, Q, S$, and $U$, the standard relations in $\Gamma_n$ show that the images under this $g$ of *all* elements $\varepsilon_i$, $\lambda_{ij}$, $\rho_{ij}$, and $\sigma_\pi$ coincide with those given in the statement of the proposition. Indeed, since the permutations (12) and $(12\ldots n)$ generate $S_n$, we have $g(\sigma_\pi) = (f(\sigma_\pi), 0)$ for all $\pi \in S_n$. The identity $\varepsilon_1\lambda_{12}\varepsilon_1^{-1} = \rho_{12}^{-1}$ implies that $g(\rho_{12}) = (f(\rho_{12}), -v_2)$, and then the identities

$$\sigma_\pi^{-1}\varepsilon_1\sigma_\pi = \varepsilon_{(1)\pi}, \qquad \sigma_\pi^{-1}\lambda_{12}\sigma_\pi = \lambda_{(1)\pi(2)\pi}, \qquad \sigma_\pi^{-1}\rho_{12}\sigma_\pi = \rho_{(1)\pi(2)\pi}$$

easily complete the verification that $g$ indeed has the prescribed images on all elements.

For the remaining two dimensions $n = 2$ and 3, one can argue similarly, using the known presentations of $\Gamma_2$ and $\Gamma_3$ (see [MKS]). However, there is a shorter argument based on the "functoriality" of $g = g_n$ with respect to $n$, by which we mean the following: for $m < n$, consider the embedding $\iota_n^m : \Gamma_m \to \Gamma_n$ obtained by letting $\Gamma_m$ act trivially on $x_{n-m+1}, \ldots, x_n$; then the following diagram is commutative:

$$\begin{array}{ccc} \Gamma_m & \xrightarrow{g_m} & GL_m(\mathbf{Z}) \ltimes \mathbf{Z}^m \\ {\scriptstyle \iota_n^m}\downarrow & & \downarrow{\scriptstyle \theta_n^m} \\ \Gamma_n & \xrightarrow{g_n} & GL_n(\mathbf{Z}) \ltimes \mathbf{Z}^n \end{array}$$

where $\theta_n^m$ identifies $\mathbf{Z}^m$ with $(\mathbf{Z}^m, 0, \ldots, 0) \subset \mathbf{Z}^n$, and embeds $GL_m(\mathbf{Z})$ in the left upper corner of $GL_n(\mathbf{Z})$. In fact, since $\theta_n^m$ is an embedding, the images under $g_n$ of the elements $\iota_n^m(\lambda_{ij})$, $\iota_n^m(\rho_{ij})$, $\iota_n^m(\varepsilon_i)$ $(1 \leq i, j \leq m)$ and $\iota_n^m(\sigma_\pi)$ $(\pi \in S_m)$ belong to $\operatorname{Im} \theta_n^m$, and $(\theta_n^m)^{-1}$ applied to these elements yields exactly the elements given in the statement of the proposition for $\Gamma_m$. The *existence* of $g_n$ implies the existence of $g_m$ for any $m < n$, completing the proof. $\square$

*Remark* 2.2. The referee suggested that it should be possible to obtain a different proof of Proposition 2.1 using the $GL_n(\mathbf{Z})$-isomorphism between $N/[N,N]$ and $\operatorname{Hom}_{\mathbf{Z}}(F_n/[F_n,F_n], F_n/F_n^{(3)})$, where $F_n^{(3)}$ is the third term of the lower central series of $F_n$ (see [F], pp. 426-428).

**Corollary 2.3.** *For the group of inner automorphisms* Int $F_n \subset \Gamma_n$, *one has*

$$g(\operatorname{Int} F_n) = (E_n, 2(n-1)\mathbf{Z}^n). \tag{5}$$

*In particular, $g$ doesn't vanish on* Int $F_n$.

*Proof.* Since

$$\operatorname{Int} x_j = \prod_{i \neq j} \lambda_{ij}^{-1} \cdot \prod_{i \neq j} \rho_{ij}^{-1},$$

(5) follows. $\square$

Note that the homomorphism $g$ is not surjective. This plays a crucial role in the proof of the following fact.

**Corollary 2.4.** *The extension*

$$1 \to N^{\mathrm{ab}} \to \Gamma_n/[N,N] \to GL_n(\mathbf{Z}) \to 1, \tag{6}$$

*and consequently also the extension*

$$1 \to N \to \Gamma_n \to GL_n(\mathbf{Z}) \to 1,$$

*do not split.*

*Proof.* If (6) were split, so would be the extension

$$1 \to g(N) \to g(\Gamma_n) \to GL_n(\mathbf{Z}) \to 1;$$

in particular, $g(\Gamma_n) \subset GL_n(\mathbf{Z}) \ltimes g(N)$. But $g(\beta) = (2v_n, E_n)$, so $g(N) = 2\mathbf{Z}^n$, and we see that $g(\lambda_{nn-1}) \notin GL_n(\mathbf{Z}) \ltimes g(N)$. $\square$

As yet another application of the existence of the homomorphism $g$, we will show that (contrary to some expectations) $\Gamma_n$ does not have finite representation type.

**Proposition 2.5.** *For any $n \geq 2$, there exists an infinite family $\{\rho_s\}$ of pairwise inequivalent representations $\rho_s : \Gamma_n \to GL_m(\mathbf{C})$ in some dimension $m = m(n)$. Thus, $\Gamma_n$ has infinite representation type for all $n \geq 2$.*

*Proof.* Since $\Gamma_2$ has the virtually free group $GL_2(\mathbf{Z})$ as a quotient, our assertion is trivial for $n = 2$. So, in what follows $n \geq 3$.

We will say that two representations $\rho_1, \rho_2 : H \to GL_m(\mathbf{C})$ of a certain group $H$ are *strongly inequivalent* if $\mathrm{Hom}_H\left((\mathbf{C}^m, \rho_1), (\mathbf{C}^m, \rho_2)\right) = 0$; in other words, there is no nonzero operator $T : \mathbf{C}^m \to \mathbf{C}^m$ such that $\rho_1(h)T = T\rho_2(h)$ for all $h \in H$. Our construction of the required family $\{\rho_s\}$ (which generalizes the construction described in [R2]) hinges on the existence of an infinite family of pairwise strongly inequivalent representation of the algebraic group $G_n = SL_n(\mathbf{C}) \ltimes \mathbf{C}^n$. Though this should be well-known to the specialists in representation theory, we have been unable to find a reference, and therefore include the formulation and proof of the required fact.

**Lemma 2.6.** *For any $n \geq 3$, there exists an $m = m(n)$ such that $G_n$ admits an infinite family of pairwise strongly inequivalent representations $\theta_s : G_n \to GL_m(\mathbf{C})$.*

*Proof.* We will need six pairwise inequivalent irreducible rational representations $\rho_1, ..., \rho_6$ of the algebraic group $SL_n(\mathbf{C})$ with the following property: for for every pair $(i, j)$ from the set

$$I = \{(1,4), (2,4), (2,5), (3,5), (3,6), (1,6)\},$$

the tensor product $\hat{\rho}_i \otimes \rho_j$ contains the standard representation of $SL_n(\mathbf{C})$ on $V = \mathbf{C}^n$, where for a representation $\rho$, $\hat{\rho}$ denotes the contragredient representation defined by $\hat{\rho}(g) = {}^t\rho(g)^{-1}$. Such representations (which exist for any $n \geq 3$) were kindly constructed for us by E. B. Vinberg, to whom we acknowledge our thanks. Namely, it follows from the formula for the multiplicities of irreducible components of the tensor products of two irreducible representations (cf. [OV], pp. 290, and Exercise 14 to §9 of Ch. VIII in [Bo]) that the representations with the following highest weights possess the required property:

$$\begin{array}{llll} \rho_1: & 3\varepsilon_1 + \varepsilon_2 + \varepsilon_3, & \rho_4: & 2\varepsilon_1 + \varepsilon_2 + \varepsilon_3, \\ \rho_2: & 2\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3, & \rho_5: & 2\varepsilon_1 + 2\varepsilon_3, \\ \rho_3: & 3\varepsilon_1 + 2\varepsilon_2, & \rho_6: & 3\varepsilon_1 + \varepsilon_2. \end{array}$$

Let $n_i = \dim \rho_i$ $(i = 1, ..., 6)$, and let $M_{ij}$ be the space of complex $n_i \times n_j$-matrices with the following action of $SL_n(\mathbf{C})$ :

$$A \cdot g = \rho_i(g)^{-1} A \rho_j(g),\ g \in SL_n(\mathbf{C}),$$

where in the right-hand side we have the usual product of matrices. Obviously, this representation of $SL_n(\mathbf{C})$ is equivalent to $\hat{\rho}_i \otimes \rho_j$. Let $m = n_1 + ... + n_6$. This partition of $m$ makes it sensible to talk about the $ij$-block (of size $n_i \times n_j$) of a matrix from $M_m(\mathbf{C})$, where $i, j = 1, ..., 6$. Let $V_{ij} \subset M_m(\mathbf{C})$ be the subspace of matrices in which all blocks, except $ij$, equal zero. Then $V_{ij}$ is invariant under $\mathrm{Ad}\rho$, where $\rho = \rho_1 \oplus ... \oplus \rho_6$, and the restriction $\mathrm{Ad}\rho|_{V_{ij}}$ defines an $SL_n(\mathbf{C})$-module isomorphic to $M_{ij}$, hence to $\hat{\rho}_i \otimes \rho_j$. Thus, according to our choice of $\rho_i$, for each pair $(i, j) \in I$ there exists an embedding of $SL_n(\mathbf{C})$-modules $\varphi_{ij} \colon V \to V_{ij}$. (As remarked by E. B. Vinberg, since V does not have multiple weights, it follows from the above mentioned formula for multiplicities that $\varphi_{ij}$ is actually unique up to a scalar.)

Let $W_{ij} = \varphi_{ij}(V)$. For any two pairs $(i_1, j_1)$, $(i_2, j_2) \in I$ we have $j_1 \neq i_2$ and therefore $W_{i_1,j_1} \cdot W_{i_2,j_2} = 0$ (the product is taken in $M_m(\mathbf{C})$). Since $W_{ij}^2 = 0$ for all $i \neq j$, we see that $W = \bigoplus_{(i,j) \in I} W_{ij}$ satisfies $W^2 = 0$.

Consider the family of linear maps $\varphi_s : V \to W$, $s \in \mathbf{C}$, defined by

$$\varphi_s(v) = s\varphi_{14}(v) + \sum_{\substack{(i,j)\in I\\ (i,j)\neq(1,4)}} \varphi_{ij}(v). \tag{7}$$

One easily checks that $\theta_s : G_n \to GL_m(\mathbf{C})$, given by

$$\theta_s(g, v) = \rho(g)(E_m + \varphi_s(v)),$$

is a family of rational representations of $G_n$, so it only remains to show that $\theta_s$ and $\theta_t$ are strongly inequivalent for $s, t \in \mathbf{C}$, $s \neq t$.

Suppose $T : \mathbf{C}^m \to \mathbf{C}^m$ intertwines $\rho_s$ and $\rho_t$. Then $T$ must commute with $\rho$, and therefore $T = \mathrm{diag}(\alpha_1 E_{n_1}, ..., \alpha_6 E_{n_6})$ for some $\alpha_1, ..., \alpha_6 \in \mathbf{C}$, as $\rho$ is the direct sum of the pairwise inequivalent irreducible representations $\rho_1, ..., \rho_6$. In view of (7) the condition $T\varphi_s = \varphi_t T$ implies that $\alpha_i = \alpha_j$ for all $i, j$ such that $(i, j) \in I$ and $(i, j) \neq (1, 4)$, and therefore $\alpha_1 = ... = \alpha_6$. So, if $T \neq 0$, then $T\varphi_s = \varphi_t T$ forces $s = t$, completing the proof. □

We now proceed with the proof of the proposition. Clearly, it suffices to construct an infinite family of pairwise inequivalent representations of $\Lambda_n = g(\Gamma_n)$. Note that $\Phi_n = \Lambda_n \cap (SL_n(\mathbf{Z}) \ltimes \mathbf{Z}^n)$ is a normal subgroup of $\Lambda_n$ of index 2; on the other hand, being of finite index in $SL_n(\mathbf{Z}) \ltimes \mathbf{Z}^n$, it is Zariski dense in $G_n$. It follows that on restricting $\theta_s$ to $\Phi_n$ ($\theta_s$ as in Lemma 2.6) we obtain a family of pairwise strongly inequivalent representations $\pi_s$ of $\Phi_n$. Fix $g \in \Lambda_n - \Phi_n$ and let $\tilde{\pi}_s(h) = \pi_s(ghg^{-1})$. We show that the family

$$\rho_s = \mathrm{Ind}_{\Phi_n}^{\Lambda_n}(\pi_s)$$

contains infinitely many inequivalent representations. Since representations $\tilde{\pi}_s$ are pairwise inequivalent, it suffices to establish the implication

$$\rho_s \simeq \rho_t \text{ for } s \neq t \;\Rightarrow\; \pi_s \simeq \tilde{\pi}_t, \tag{8}$$

where $\simeq$ denotes the equivalence of representations.

Obviously, the restriction $\rho_s|_{\Phi_n}$ is equivalent to $\pi_s \oplus \tilde{\pi}_s$, so $\rho_s \simeq \rho_t$ implies the existence of an isomorphism

$$T : \pi_s \oplus \tilde{\pi}_s \to \pi_t \oplus \tilde{\pi}_t.$$

However, since $\pi_s$ and $\pi_t$ are strongly inequivalent, the projection of $T \circ \pi_s$ to $\pi_t$ must be trivial, implying that $T$ yields the equivalence of $\pi_s$ and $\tilde{\pi}_t$, which proves (8) and completes the proof of Proposition 2.5. □

*Remark* 2.7. Note that for the family $\rho_s$ constructed in Proposition 2.5, $\rho_s(N)$ is abelian, so Theorem 2 in [R1], which claims that $\Gamma_n$ ($n \geq 3$) is $SS$-rigid with respect to the class of representations $\rho$ for which $\rho(N)$ is nilpotent of a fixed nilpotency class, cannot be upgraded to a statement about finiteness of representation type. On the other hand, representations $\rho_s$ are not completely reducible, so the question of whether or not $\Gamma_n$ is $SS$-rigid, i.e. has only finitely many inequivalent completely reducible representations in every dimension, remains open.

## 3. Statement of the Main Theorem.

We need to introduce some special representations of $\Gamma_m$. Let $\mu\colon \Gamma_n \to GL_n(\mathbf{C})$ be the representation obtained by composing $f\colon \Gamma_n \to GL_n(\mathbf{Z})$ with the embedding $GL_n(\mathbf{Z}) \subset GL_n(\mathbf{C})$. Let $\delta(x) = \det \mu(x)$; since for $n \geq 3$ the commutator subgroup $[\Gamma_n, \Gamma_n]$ coincides with $f^{-1}(SL_n(\mathbf{Z}))$ (this easily follows from the commutator relations), $\delta$ is the only nontrivial character of $\Gamma_n$.

Next, the homomorphism $g\colon \Gamma_n \to GL_n(\mathbf{Z}) \ltimes \mathbf{Z}^n$, $g(x) = (f(x), v(x))$, constructed in the previous section, gives rise to the following representation of $\Gamma_n$ in $GL_{n+1}(\mathbf{Z})$ :

$$x \mapsto \left(\begin{array}{c|c} f(x) & 0 \\ \hline v(x) & 1 \end{array}\right),$$

and we let $\nu\colon \Gamma_n \to GL_{n+1}(\mathbf{C})$ denote the composition of this representation with the embedding $GL_{n+1}(\mathbf{Z}) \subset GL_{n+1}(\mathbf{C})$. (Obviously, $\operatorname{Ker} \nu = \operatorname{Ker} g$; in particular, $\nu$ doesn't vanish on $\operatorname{Int} F_n$.)

**Theorem 3.1.** *Let $n \geq 3$, and let $\theta\colon \Gamma_n \to GL_m(\mathbf{C})$ be a representation of dimension $m \leq 2n-2$. Then either $\theta$ factors through $f$, or it is equivalent to a direct sum $\theta_1 \oplus \theta_2$, where $\theta_1$ is one of the following $(n+1)$-dimensional representations: $\nu, \delta\nu$, or their contragredient representations, and $\theta_2$ is a direct sum of 1-dimensional representations of $\Gamma_n$. In particular, $\theta$ always factors through $g$ and therefore vanishes on $[N, N]$.*

Here $\delta\nu$ denotes the $\delta$-twist of $\nu$ given by $\delta\nu(x) = \delta(x)\nu(x)$, and the contragredient representation for $\varphi\colon \Gamma_n \to GL_d(\mathbf{C})$ is $\tau\varphi$, where $\tau\colon g \mapsto {}^t g^{-1}$.

**Corollary 3.2.** *Let $n \geq 3$. Then $\Gamma_n$ has only finitely many inequivalent representations in any dimension $\leq 2n-2$.*

Indeed, it follows, for example, from the affirmative solution of the congruence subgroup problem for $SL_n(\mathbf{Z})$ (see [BMS]), or from Margulis's superrigidity (see [Ma]), that $GL_n(\mathbf{Z})$ has finitely many inequivalent representations in every dimension.

*Remark* 3.3. There is a general construction of representations of $\Gamma_n$ (see [BL]): one looks at the natural action of $\Gamma_n$ on the representation variety $R_m(F_n)$ for some $m$, which gives rise to an action of $\Gamma_n$ on the ring of regular functions $A = \mathbf{C}[R_m(F_n)]$; now, if $\mathfrak{m} \subset A$ is the maximal ideal associated with the trivial representation, one gets a family of finite-dimensional representations of $\Gamma_n$ on the quotients $\mathfrak{m}^i/\mathfrak{m}^j$ $(i < j)$. It would be interesting to find out if $\nu$ occurs as a subrepresentation in some such representation.

## 4. Restricting low-dimensional representations of $\Gamma_n$ to $\Omega_n$.

Let $\hat{H}_n$ be the group of characters of $H_n$, $\hat{\varepsilon}_i$ and let be defined by $\hat{\varepsilon}_i(\varepsilon_j) = \delta_{ij}$ (Kronecker's delta). We will need some specific representations of $\Omega_n$. Let $\psi\colon \Sigma_n \to \{\pm 1\}$ be the sign homomorphism, and $\chi_0 = \hat{\varepsilon}_1 + \ldots + \hat{\varepsilon}_n$. Then for $k, l \in \{0, 1\}$ we define a character of $\Omega_n$,

$$\eta_{k,l}(\sigma_\pi, h) = \psi(\sigma_\pi)^k \chi_0(h)^l$$

and the corresponding twisted $n$-dimensional representation

$$\mu_{k,l} = \eta_{k,l} \cdot \mu \mid_{\Omega_n},$$

where $\mu\colon \Gamma_n \to GL_n(\mathbf{C})$ is the standard representation introduced in the previous section.

**Proposition 4.1.** *Let $\theta\colon \Gamma_n \to GL_m(\mathbf{C})$ $(n \geq 3)$ be a representation of dimension $m \leq 2n-2$. Then $\theta(N) \neq \{E_m\}$ is possible only if the restriction $\theta \mid \Omega_n$ is equivalent to a representation of the form:*

$$\mu_{p,q} \oplus \left( \bigoplus_{k,l \in \{0,1\}} (\eta_{k,l})^{\alpha_{k,l}} \right)$$

*for some $p, q \in \{0,1\}$, and some integers $\alpha_{k,l}$ $(k,l \in \{0,1\})$ such that $\alpha_{00} + \cdots + \alpha_{11} = m - n$.*

*Proof.* Let $V = \mathbf{C}^m$, and let

$$V = V_{\chi_1} \oplus \cdots \oplus V_{\chi_r} \tag{1}$$

be the decomposition of $V$ as the direct sum of nonzero eigenspaces of $H_n$ corresponding to characters $\chi_i \in \hat{H}_n$ (observe that $\{\chi_1, \ldots, \chi_r\}$ is the union of some orbits of $\Sigma_n$ acting on $\hat{H}_n$). If $\chi \in \hat{H}_n$ is of the form

$$\chi = \hat{\varepsilon}_{i_1} + \cdots + \hat{\varepsilon}_{i_l},$$

then the orbit $\Sigma_n \cdot \chi$ consists of $\binom{n}{l}$ elements. It follows that if $n \geq 5$, $\mid \Sigma_n \cdot \chi \mid \geq 2n$ unless $l = 0, 1, n-1$, or $n$, i.e. $\chi = 0, \hat{\varepsilon}_i, \chi^0 - \hat{\varepsilon}_i$, or $\chi^0$ for some $i$, and therefore only such characters can appear in (1), as $\dim V < 2n$. On the other hand, if $n = 3$, then the characters of such form exhaust all characters, so only the case $n = 4$ will require special consideration. Furthermore, in view of Lemma 1.1, there should be a $\chi_i$ in (1) which is different from 0 and $\chi^0$. Then the orbit $\mathcal{O} = \Sigma_n \cdot \chi_i$ consists of $n$ elements, so $W = \bigoplus_{\chi \in \mathcal{O}} V_\chi$ has dimension $n \cdot \dim V_{\chi_i}$, so $\dim V_{\chi_i} = 1$. The stabilizer $\Sigma_n(\chi_i)$ is isomorphic to $S_{n-1}$, so its representation on $V_{\chi_i}$ is either the trivial or the sign representation, which easily implies that the representation of $\Omega_n$ on $W$ is equivalent to one of the $\mu_{p,q}$. Those $V_\chi$ in (1) which are not in $W$ correspond to $\chi = 0$ or $\chi^0$; hence they are $\Sigma_n$-invariant, and have dimension $\leq n-2$. But for $n \neq 4$, $S_n$ has only one nontrivial irreducible representation of dimension $\leq n-2$, viz. the 1-dimensional sign representation (see [JK], Theorem 2.4.10), and our assertion follows.

For $n = 4$, there is one orbit of $\Sigma_4$ on $\hat{H}_4$ consisting of 6 elements, viz. $\mathcal{O} = \Sigma_4 \cdot \chi$, $\chi = \hat{\varepsilon}_1 + \hat{\varepsilon}_2$, so we need to eliminate the possibility of this $\chi$ in (1) when $\dim V = 6$. In this case $\mathcal{O} = \{\hat{\varepsilon}_i + \hat{\varepsilon}_j \mid i \neq j\}$, which restricts to three distinct characters of the subgroup $\tilde{H} = \langle \varepsilon_1, \varepsilon_2 \rangle$, each of the corresponding eigenspaces being 2-dimensional. Let $\tilde{\Gamma}$ be the group Aut $F\langle x_3, x_4 \rangle$ isomorphically embedded in $\Gamma_4$ as acting identically on $x_1$ and $x_2$. Then $\tilde{H}$ and $\tilde{\Gamma}$ commute, so each of the eigenspaces is $\tilde{\Gamma}$-invariant. Since any 2-dimensional representation of Aut $F_2$ factors through the canonical homomorphism Aut$F_2 \to GL_2(\mathbf{Z})$ (Theorem 1.2), this implies that $\theta(\beta) = E_m$, and hence $\theta(N) = \{E_m\}$—a contradiction.

It remains to be shown that if $V_\chi$ from (1) doesn't occur in $W$ (so $\chi = 0$ or $\chi^0$, and $V_\chi$ is $\Sigma_4$-invariant), then the representation of $\Sigma_4$ on $V_\chi$ cannot be a 2-dimensional irreducible representation of $\Sigma_4$. It is known (see [FuH], p.19) that

$S_4$ has only one 2-dimensional irreducible representation $\tau$ which is obtained by composing the standard representation of $S_3$ with the isomorphism

$$S_4/\{1, (12)(34), (13)(24), (14)(23)\} \simeq S_3.$$

We need only the fact that $\det \tau(t) = -1$ for any transvection $t \in S_4$. So, let us suppose that

$$\theta \mid \Omega_n \simeq \mu_{p,q} \oplus \tau_l,$$

where $\tau_l(\sigma_\pi, h) = \chi^0(h)^l \tau(\pi)$, $l \in \{0, 1\}$. Let $x = (\sigma_{(12)}, \varepsilon_1)$. Then

$$\det \theta(x) = \det f(x) \det \tau((1,2)) = -1.$$

On the other hand, since $[\Gamma_n, \Gamma_n] = f^{-1}(SL_n(\mathbf{Z}))$ for $n \geq 3$, we have $x \in [\Gamma_n, \Gamma_n]$, and consequently $\det \theta(x) = 1$—a contradiction. □

## 5. Proof of Theorem 3.1.

Our proof of Theorem 3.1 is based on the description of the restriction of a representation $\theta\colon \Gamma_n \to GL_m(\mathbf{C})$ to $\Omega_n$ provided by Proposition 4.1, and information about how elements of $\Omega_n$ interact with $\rho_{ij}$ and $\lambda_{ij}$. We may (and we will) assume that

$$\theta \mid \Omega_n = \mu_{p,q} \oplus \bar{\theta},$$

where $\bar{\theta} = \bigoplus_{k,l \in \{0,1\}} (\eta_{k,l})^{\alpha_{k,l}}$ (notation as in §4) has dimension $l = m - n$, $0 \leq l \leq n-2$. We collect in the following statement some properties of $\mu_{p,q}$ and $\bar{\theta}$ that are immediate consequences of their description and will be used below.

**Lemma 5.1.** (i) *$\mu_{p,q}(\varepsilon_{ij}) = \mathrm{diag}(\alpha_1, \dots, \alpha_n)$, where $\alpha_i = \alpha_j = -1$ and $\alpha_k = 1$ for $k \neq i, j$.*

(ii) *$\bar{\theta}(\varepsilon_i) = \bar{\theta}(\varepsilon_j)$ for all $i, j$, so $\bar{\theta}(\varepsilon_{ij}) = E_l$ for all $i \neq j$.*

(iii) *Let $g \in \Gamma_n$ be such that $\theta(g) = \mathrm{diag}(A, B) \in G(n, l)$, and suppose that either $n = 3$, or $g$ commutes with $\varepsilon_i$ and $\sigma_{(jk)}$ for some $i$ and some $j \neq k$. Then $B$ commutes with $\bar{\theta}(\Omega_n)$.*

(Assertion (iii) is immediate if $n = 3$, since then $\bar{\theta}$ is at most one-dimensional; otherwise one needs to use (ii) and the fact that for any two transposition, the corresponding automorphisms have the same image under $\bar{\theta}$.)

Let $\Gamma'$ be the subgroup of $\Gamma_n$ generated by $\rho_{ij}$, $\lambda_{ij}$, $\varepsilon_i$ for $i, j \in \{n-1, n\}$, $i \neq j$, and $\sigma_{(n-1 n)}$, $\Delta = \langle \Gamma', H_n \rangle$.

**Lemma 5.2.** *If either $n \geq 4$, or $n = 3$ but $\theta$ doesn't factor through $f$, then $\theta(\Delta) \subset \mathcal{D}_{n-2} \times GL_{l+2}(\mathbf{C})$, where $\mathcal{D}_{n-2} \subset GL_{n-2}(\mathbf{C})$ is the diagonal torus.*

Proof. Let $H_{n-2} \subset H_n$ be the subgroup generated by $\varepsilon_i$ for $i \leq n-2$. Then $\Delta$ commutes elementwise with $H_{n-2}$, implying that $\theta(\Delta)$ is contained in the centralizer $C = C_{GL_m(\mathbf{C})}(\theta(H_{n-2}))$. First, we assume that $n \geq 4$. Since $\theta(\varepsilon_{ij}) = \mathrm{diag}(\alpha_1, \dots, \alpha_m)$ with $\alpha_i = \alpha_j = -1$ and all other $\alpha$'s equal to 1, we see that $C \subset G(n-2, l+2)$. Furthermore, since for $i \leq n-2$ we have $p_1(n-2, l+2)(\theta(\varepsilon_i)) = \pm\mathrm{diag}(\beta_1, \dots, \beta_{n-2})$, where $\beta_i = -1$ and all other $\beta$'s are 1, we get our claim.

If $n = 3$, we only need to consider the case $m = 4$. We have $\theta(\varepsilon_1) = \pm\mathrm{diag}(-1, 1, 1, \alpha)$, where $\alpha$ can be 1 or $-1$. If $\alpha = 1$, our claim is immediate. Otherwise, the subspaces $V_1, V_2 \subset \mathbf{C}^4$ spanned by the $1^{st}$ and $4^{th}$, and the $2^{nd}$ and

$3^{rd}$ basic vectors, respectively, are invariant under $\theta(\Delta)$. Since $\Gamma' \simeq \operatorname{Aut} F_2$, we conclude from Theorem 1.2 that the restriction of $\theta(\Gamma')$ to each of them factors through the canonical homomorphism $f'\colon \Gamma' \to GL_2(\mathbf{Z})$. This implies that $\theta(\beta) = E_4$, and therefore $\theta$ factors through $f$.

So, in terms of proving Theorem 3, we may (and will) assume henceforth the inclusion given in Lemma 5.2. Since $\beta = [\varepsilon_{nn-1}, \lambda_{nn-1}]$ and $\gamma = [\lambda_{nn-1}, \varepsilon_n]$, we obtain

$$p_1(n-2, l+2)(\theta(\beta)) = p_1(n-2, l+2)(\theta(\gamma)) = E_{n-2}.$$

We let $\theta'$ denote the representation of $\Delta$ obtained by composing $\theta$ with the projection $p_2(n-2, l+2)$. Then

$$\theta'(\varepsilon_{1,n}) = \operatorname{diag}(1, -1, E_l) \quad \text{and} \quad \theta'(\varepsilon_{n,n-1}) = \operatorname{diag}(-1, -1, E_l),$$

so the fact that $\gamma$ and $\varepsilon_{n,n-1}$ commute (which immediately follows from (1) in §1) implies that

$$\theta'(\gamma) = \operatorname{diag}(A, B)$$

with $A \in GL_2(\mathbf{C})$, and $B \in GL_l(\mathbf{C})$. Moreover, using the identity

$$(\gamma\varepsilon_{1,n})^2 = 1$$

(which is again an immediate consequence of (1) in §1), we conclude that

$$(\operatorname{diag}(1, -1)A)^2 = E_2 \quad \text{and} \quad B^2 = E_m. \tag{1}$$

If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then we obtain from (1) that

$$a^2 - bc = d^2 - bc = 1 \quad \text{and} \quad (a-d)b = (a-d)c = 0. \tag{2}$$

Case 1. $b \neq 0$ and $c \neq 0$. We claim that in this case $A$ cannot possibly have eigenvalues $\pm 1$. Indeed, we obtain from (2) that $a = d$ and $\det A = 1$. This means that if one of the eigenvalues is $\pm 1$, the other is the same. So, $\operatorname{tr} A = 2a = \pm 2$, which implies that $bc = 0$—a contradiction. Since the eigenvalues of $B$ are $\pm 1$, the fact that $\gamma$ and $\lambda_{nn-1}$ commute implies that $\theta'(\lambda_{nn-1}) \in G(2, l) \subset GL_{2+l}(\mathbf{C})$. Hence

$$\theta'(\beta) = \theta'([\varepsilon_{nn-1}, \lambda_{nn-1}]) = E_{l+2}.$$

So, $\theta(\beta) = E_m$, and $\theta$ factors through $f$.

Case 2. $b = c = 0$, i.e. $A$ is diagonal. We will show that in this case $\theta$ factors through $f$ as well. We claim that in addition to the obvious identity $\theta'(\gamma)^2 = E_{l+2}$, we also have

$$\theta'(\varepsilon_n \sigma_{(nn-1)} \gamma)^4 = E_{l+2}. \tag{3}$$

Indeed, the matrices $\theta'(\gamma)$, $\theta'(\sigma_{(nn-1)})$ and $\theta'(\varepsilon_n)$ belong to $G(2, l)$. Then (3) for their $2 \times 2$-blocks follows from the fact that these have the following shape:

$$\operatorname{diag}(\pm 1, \pm 1), \quad \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \operatorname{diag}(\pm 1, \pm 1).$$

To analyze their $l \times l$ blocks, we observe that since $\gamma$ commutes with $\sigma_{(12)}$ and $\varepsilon_1$ if $n \geq 4$, it follows from Lemma 5.1 that the $m \times m$ block of $\gamma$ commutes with those of $\sigma_{(nn-1)}$ and $\varepsilon_n$, and again (3) follows.

Now, the identity

$$(\sigma_{(nn-1)}\lambda_{nn-1})^{-1}\lambda_{nn-1}(\sigma_{(nn-1)}\lambda_{nn-1}) = \varepsilon_n\sigma_{(nn-1)}\gamma$$

implies that $\theta'(\lambda_{nn-1})^4 = E_{l+2}$, so

$$\theta'(\beta)^2 = \theta'(\lambda_{nn-1})^{-4}\theta'(\gamma)^2 = E_{l+2}.$$

Thus, $\theta'(\varepsilon_{nn-1}\beta\varepsilon_{nn-1}^{-1}) = \theta'(\beta)^{-1} = \theta'(\beta)$, and since $\theta'(\varepsilon_{nn-1}) = \mathrm{diag}(-E_2,\, E_l)$, we obtain

$$\theta'(\beta) = \mathrm{diag}(B_1, B_2) \in G(2,l).$$

Moreover, since $\beta$ commutes with $\varepsilon_n$, and $\varepsilon_n$ has $\pm\mathrm{diag}(1,-1)$ as its $2\times 2$ block, we see that $B_1$ is a diagonal matrix.

Next, we need the following easily verifiable identity:

$$\lambda_{n-1n}^{-1}\beta\lambda_{n-1n} = (\sigma_{(nn-1)}\beta)^2. \tag{4}$$

Since $\theta'(\beta)$ and $\theta'(\sigma_{(nn-1)})$ belong to $G(2,l)$, and the same argument as above shows that their $m\times m$ blocks commute, (4) implies that $\beta = \mathrm{diag}(B_1,\, B_2)$ is conjugate to

$$\mathrm{diag}\left(\left(\pm B_1\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\right)^2, B_2^2\right).$$

The eigenvalues of $\beta$ are $\pm 1$, and it follows from the conjugacy of these matrices that the multiplicity of $-1$ is $\leq 2$. Since $\det\theta(\beta) = 1$ ($\beta$ is a commutator), it can only be 0 or 2. If it is 0 or $B_1 = \pm E_2$, we obtain $\theta(\beta) = E_m$. So, it remains to eliminate the possibility of –1 occurring as an eigenvalue in each of $B_1$ and $B_2$ with multiplicity one.

Assume that this is the case. Then the matrix $B_1$ can be either $\mathrm{diag}(1,-1)$ or $\mathrm{diag}(-1,1)$, however by switching $x_{n-1}$ and $x_n$ (or alternatively by passing to the element $\beta' = \sigma_{(n-1n)}^{-1}\beta\sigma_{(n-1n)}$, which equally generates $N$), we may assume that $B_1 = \mathrm{diag}(1,-1)$. Furthermore, it follows from Lemma 5.1 that $B_2$ commutes with $\bar{\theta}(\Omega_n)$, and therefore there exists a matrix $C \in GL_l(\mathbf{C})$ that conjugates $B_2$ to $\mathrm{diag}(-1,1,\ldots,1)$ and preserves the shape of $\bar{\theta}$ (i.e. $C^{-1}\bar{\theta}C$ remains the direct sum of 1-dimensional representations of $\Omega_n$). So, by passing to an equivalent representation whose restriction to $\Omega_n$ has the same structure, we may assume that $\theta(\beta) = \mathrm{diag}(E_{n-1}, -E_2, E_{l-1})$.

Then, since $\lambda_{nn-1}$ commutes with $\beta$, $\theta'(\lambda_{nn-1})$ must be of the form

$$\begin{pmatrix} s & 0 & 0 & u \\ 0 & x & y & 0 \\ 0 & z & t & 0 \\ v & 0 & 0 & W \end{pmatrix}$$

where $s \in \mathbf{C}^*$, $u \in \mathbf{C}^{1\times(l-1)}$, $v \in \mathbf{C}^{(l-1)\times 1}$, and $W \in M_{l-1}(\mathbf{C})$. Now, using the identity

$$\varepsilon_{nn-1}^{-1}\lambda_{nn-1}\varepsilon_{nn-1} = \lambda_{nn-1}\beta \tag{5}$$

and the fact that $\theta'(\varepsilon_{nn-1}) = \mathrm{diag}(-E_2, E_m)$, one easily obtains that $u = 0$ and $v = 0$. In particular, $\theta(\lambda_{nn-1}) \in G(n+1, l-1)$. Since also $\theta(\Omega_n) \subset G(n+1, l-1)$, and $\Omega_n$ and $\lambda_{nn-1}$ together generate $\Gamma_n$, we see that $\theta(\Gamma_n) \subset G(n+1, l-1)$. Let $\theta^! \colon \Gamma_n \to GL_{n+1}$ be the composition of $\theta$ with the projection to $p_1(n+1, l-1)$.

Obviously, $\theta(\lambda_{nn-1}) \in G(n-2, 3, l-1)$, and letting $p = p_2(n-2, 3, l-1)$, we will have

$$p(\theta(\lambda_{nn-1})) = \begin{pmatrix} s & 0 & 0 \\ 0 & x & y \\ 0 & z & t \end{pmatrix}.$$

Since $p(\theta(\varepsilon_{nn-1})) = \mathrm{diag}(-1, 1, 1)$ and $p(\theta(\beta)) = \mathrm{diag}(1, -1, -1)$, the identity (5) implies that $x = t = 0$. In particular, $\theta^!(\lambda_{nn-1})$ belongs to the group of monomial matrices $\mathcal{M}_{n+1} \subset GL_{n+1}$. Since $\theta^!(\Omega_n) \subset \mathcal{M}_{n+1}$, we obtain $\theta^!(\Gamma_n) \subset \mathcal{M}_{n+1}$. If $\mathcal{M}_{n+1} \xrightarrow{\phi} S_{n+1}$ is the canonical homomorphism, then $\phi(\lambda_{nn-1}) = (n, n+1)$. Using $\pi = (n-1, n, n-2) \in S_n$, we obtain

$$\phi(\lambda_{n-2,n}) = \phi(\sigma_\pi^{-1} \lambda_{nn-1} \sigma_\pi) = (n-2, n+1).$$

Then the commutator identity (2) of §1 implies that $\phi(\lambda_{n-2,n-1}) = (n-2, n+1, n)$; on the other hand, $\phi(\lambda_{n-2,n-1}) = (n-2, n+1)$—a contradiction.

CASE 3. $b = 0, c \neq 0$. Then

$$A^2 = \begin{pmatrix} 1 & 0 \\ 2c & 1 \end{pmatrix}.$$

Since $\lambda_{nn-1}$ and $\gamma$ commute, we have

$$\theta'(\lambda_{nn-1}) = \left(\begin{array}{cc|c} x & 0 & 0 \\ y & x & u \\ \hline v & 0 & W \end{array}\right),$$

for some $x, y \in \mathbf{C}$, $u \in \mathbf{C}^{1\times m}$, $v \in \mathbf{C}^{m\times 1}$, $W \in M_m(\mathbf{C})$. The relation $\gamma = \varepsilon_{nn-1}^{-1} \lambda_{nn-1} \varepsilon_{nn-1} \lambda_{nn-1}$ implies that

$$x^2 = 1 \quad \text{and} \quad xu - uW = 0,$$

while the identity $\varepsilon_{1n-1}^{-1} \lambda_{nn-1} \varepsilon_{1n-1} = \lambda_{nn-1}^{-1}$ yields

$$xu + uW = 0.$$

So, $u = 0$.

Next, since $\lambda_{nn-1}$ commutes with $\sigma_{(12)}$, $W$ commutes with $p_2(n, l)(\theta(\sigma))$ for any $\sigma \in \Sigma_n$, to the effect that $\theta(\lambda_{ij})$ has the structure

$$\left(\begin{array}{c|c} * & 0 \\ \hline * & W \end{array}\right),$$

with the *same* $W$ for *all* $i \neq j$. Then the identity (2) in §1 implies that actually $W = E_l$. Since the elements $\lambda_{ij}$, $\varepsilon_i$, and $\sigma_\pi$ generate $\Gamma_n$, $\theta$ has the following block structure:

$$\theta(g) = \begin{pmatrix} \omega(g) & 0 \\ * & \eta(g) \end{pmatrix}, \quad g \in \Gamma_n,$$

where $\omega$ and $\eta$ have dimension $n$ and $l$, respectively. It follows from [R1] that $\omega$ coincides either with $\mu$ or with $(\det \mu)\mu$ (it cannot be $\tau\mu$ or $(\det \mu)\tau\mu$, as $\omega(\lambda_{nn-1})$ is a lower triangular matrix). Replacing $\theta$ by $(\det \mu)\theta$ if necessary, we may assume that $\omega = \mu$; in particular, $\omega(\lambda_{ij}) = E_{ij}$. We see that $\theta(\lambda_{nn-1})$ has the following shape:

$$\left(\begin{array}{c|c|c} E_{n-2} & 0 & 0 \\ \hline 0 & \begin{matrix} 1 & 0 \\ 1 & 1 \end{matrix} & 0 \\ \hline 0 & v \quad 0 & E_l \end{array}\right).$$

The fact that $\lambda_{nn-1}$ commutes with $\varepsilon_1$ and $\sigma_{(12)}$ easily implies that $v$ is fixed by $\eta(\Gamma_n)$, and therefore, after conjugation by a suitable matrix of the form $\mathrm{diag}(E_n, D)$, $D \in GL_l(\mathbf{C})$, which preserves the shape of $\eta$, we may assume that

$$v = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then, in particular, $\theta(\lambda_{nn-1}) \in G(n+1, l-1)$, and therefore $\theta(\Gamma_n) \subset G(n+1, l-1)$ as the images under $\theta$ of the elements $\varepsilon_i$ and $\sigma_\pi$, $\pi \in S_n$, also belong to $G(n+1, l-1)$, and together with $\lambda_{nn-1}$ these elements generate $\Gamma_n$. Moreover, on all these elements (including $\lambda_{nn-1}$) $p_1(n+1, l-1)(\theta)$ coincides with $\nu$. On the other hand, $p_2(n+1, l-1)(\theta)$ is $E_{l-1}$ on $\lambda_{ij}$ and is the direct some of 1-dimensional representations of $\Omega_n$. So, we finally are able to conclude that $\theta$ is equivalent to a representation of the form $\nu \oplus \varkappa$, where $\varkappa$ is a direct sum of 1-dimensional representations of $\Gamma_n$, as required.

Case 4. $b \neq 0, c = 0$. This case is immediately reduced to Case 3 by applying the automorphism $g \overset{\tau}{\mapsto} {}^t g^{-1}$ of $GL_n(\mathbf{C})$ (observe that $\theta(\Omega_n)$ lies in the orthogonal group, hence is fixed by $\tau$).

The proof of Theorem 3.1 is now complete.

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