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THE KO-THEORY OF TORIC MANIFOLDS

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Abstract. Toric manifolds, a topological generalization of smooth projective toric varieties, are determined by an n-dimensional simple convex polytope and a function from the set of codimension-one faces into the primitive vectors of an integer lattice. Their cohomology was determined by Davis and Januszkiewicz in 1991 and corresponds with the theorem of Danilov-Jurkiewicz in the toric variety case. Recently it has been shown by Buchstaber and Ray that they generate the complex cobordism ring. We use the Adams spectral sequence to compute the KO-theory of all toric manifolds and certain singular toric varieties.

1. Introduction

We take as our definition of toric manifold the construction of Davis and Januszkiewicz ([5], section 1.5). Let P^n be an n-dimensional, simple (at each vertex, n codimension-one faces meet), convex polytope. Set

$$\mathcal{F} = \{F_1, F_2, \dots , F_m\}$$

the set of codimension-one faces of P^n. The fact that P^n is simple implies that every codimension-l face F can be written uniquely as

$$F = F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_l}$$

where the F_{i_j} are codimension-one faces containing F. Let

$$\lambda : \mathcal{F} \to \mathbf{Z}^n$$

be a function into an n-dimensional integer lattice satisfying the condition that whenever $F = F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_l}$ then $\lambda(F_{i_1}), \lambda(F_{i_2}), \dots , \lambda(F_{i_l})$ span an l-dimensional submodule of $\mathbf{Z}^n$ which is a direct summand. Next, regarding $\mathbf{R}^n$ as the Lie algebra of $\mathbf{T}^n$, we see that λ associates to each codimension-l face F of P^n a rank-l subgroup $G_F \subset \mathbf{T}^n$. Finally, let $p \in P^n$ and $F(p)$ be the unique face with p in its relative interior. Define an equivalence relation $\sim$ on $\mathbf{T}^n \times P^n$ by $(g,p) \sim (h,q)$ if and only if $p = q$ and $g^{-1}h \in G_{F(p)} \cong \mathbf{T}^l$. Set

$$M^{2n}(\lambda) = \mathbf{T}^n \times P^n \big/ \sim .$$

$M^{2n}(\lambda)$ is a smooth, closed, connected, $2n$-dimensional manifold with a $\mathbf{T}^n$ action induced by left translation ([5], page 423). There is a projection

$$\pi : M^{2n}(\lambda) \to P^n$$

induced from the projection $\mathbf{T}^n \times P^n \to P^n$.

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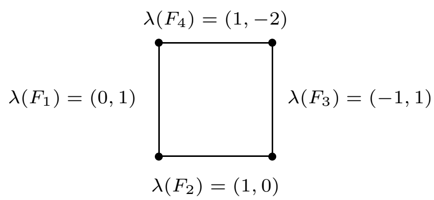
Following [5], we note that every toric manifold has this description, in particular, every smooth projective toric variety does too. Recently, Buchstaber and Ray [4] have shown that toric manifolds generate the complex cobordism ring.

Here is a simple example selected from the list in [5]. Let $n = 2$ and P^2 be a square. Here $\mathcal{F} = \{F_1, F_2, F_3, F_4\}$ consists of four codimension-one faces. Define $\lambda : \mathcal{F} \to \mathbf{Z}^2$ as in the diagram below.



yields $M^4(\lambda) \cong CP^2\#CP^2$.

Davis and Januszkiewicz point out that $CP^2\#CP^2$ is a toric manifold but does not have an almost complex structure and so cannot be a toric variety. Our main results are:

Theorem 1. *The Adams spectral sequence for the real connective KO-theory of the toric manifold $M^{2n}(\lambda)$ collapses.*

Corollary 2. *$KO^*M^{2n}(\lambda)$ is determined by the* $\bmod 2$ *cohomology ring of $M^{2n}(\lambda)$. In particular, the KO-theory depends only the values of $\lambda \bmod 2$.*

Our methods yield the additional result that the theorem remains true for certain singular toric varieties, of real dimension less than 12. We note that the K-theory of toric varieties has been computed by Robert Morelli in [8]

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2. Homology and Cohomology of $M^{2n}(\lambda)$

In order to compute the KO-theory of $M^{2n}(\lambda)$ we shall need the computation of its homology from [5]. To state their result we recall certain numbers defined in terms of the combinatorics of P^n. Let f_i be the number of faces of P^n of codimension $(i+1)$. Define numbers h_i by the equality of polynomials in t

$$(t-1)^n + \sum_{i=0}^{n-1} f_i (t-1)^{n-1-i} = \sum_{i=0}^{n} h_i t^{n-i}.$$

$(h_0, \dots, h_n)$ is called the *h-vector* of P^n. Notice $h_0 = h_n = 1$ and

$$\sum_{i=0}^{n} h_i = f_{n-1} = \text{the number of vertices of } P^n.$$

For each k-face F of P^n we have a connected $2k$-dimensional submanifold M_F of $M^{2n}(\lambda)$ defined by $M_F = \pi^{-1}(F)$.

Theorem 3 (M. Davis and T. Januszkiewicz [5])**.** *The group* $H_*(M^{2n}(\lambda);\mathbf{Z})$ *is independent of the function* λ*. Specifically,*

$$\begin{aligned} H_{2i+1}(M^{2n}(\lambda);\mathbf{Z}) &= 0, \\ H_{2i}(M^{2n}(\lambda);\mathbf{Z}) &= \text{free of rank } h_i. \end{aligned}$$

The group $H^{2l}(M^{2n}(\lambda);\mathbf{Z})$ *is generated by the Poincaré duals of classes of the form* $[M_F]$ *with* F *a face of codimension* l*. As a ring,* $H^*(M^{2n}(\lambda);\mathbf{Z})$ *is generated by the degree-two classes dual to* $[M_F]$ *with* F *a face of codimension one.* □

The ring structure of $H^*(M^{2n}(\lambda);\mathbf{Z})$ is determined from the Serre spectral sequence of the fibration

$$M^{2n}(\lambda) \to BP^n \to B\mathbf{T}^n$$

where BP^n denotes the Borel construction

$$BP^n = E\mathbf{T}^n \times_{\mathbf{T}^n} M^{2n}(\lambda).$$

Let $v_1, v_2, \ldots, v_m$ denote the degree-two generators of $H^*(M^{2n}(\lambda);\mathbf{Z})$, one for each codimension-one face of P^n. We need to define two ideals of relations in I and J.

Let K be the simplicial complex *dual* to P^n. That is, an $(n-1)$-dimensional simplicial complex with vertex set $\mathcal{F}$, the set of codimension-one faces of P^n. A set of $(k+1)$ elements in $\mathcal{F}$, $\{F_{i_0}, \ldots, F_{i_k}\}$ span a k-simplex in K if and only if $F_{i_0} \cap \cdots \cap F_{i_k} \neq \phi$. The ideal I is the homogenous ideal of relations generated by all square free monomials of the form $v_{i_1} \cdots v_{i_s}$, where $\{v_{i_1}, \ldots, v_{i_s}\}$ does not span a simplex in K.

The ideal J is defined in terms of the function λ. Let $\{e_1, \ldots, e_m\}$ be the standard basis of $\mathbf{Z}^m$. Then, identifying the codimension-one face F_i with e_i, we can regard

$$\lambda : \mathcal{F} \to \mathbf{Z}^n$$

as a linear map $\mathbf{Z}^m \to \mathbf{Z}^n$ given by an $m \times n$ matrix (λ_{ij}). In example 3 above, the linear map $\lambda : \mathbf{Z}^4 \to \mathbf{Z}^2$ is the matrix

$$\lambda = \begin{pmatrix} 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix}.$$

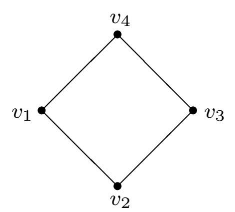
The ideal of relations J is determined by the system of equations

$$\begin{aligned} \lambda_{11}v_1 + \lambda_{12}v_2 + \ldots + \lambda_{1m}v_m &= 0 \\ \lambda_{21}v_1 + \lambda_{22}v_2 + \ldots + \lambda_{2m}v_m &= 0 \\ \vdots \qquad\qquad & \vdots \quad \vdots \\ \lambda_{ni}v_1 + \lambda_{n2}v_2 + \ldots + \lambda_{nm}v_m &= 0. \end{aligned}$$

Theorem 4 (M. Davis and T. Januszkiewicz [5])**.** *As rings*

$$H^*(M^{2n}(\lambda);\mathbf{Z}) = \mathbf{Z}[v_1, v_2, \ldots, v_m]/(I+J).$$

As an illustration, we compute $H^*(M^4(\lambda);\mathbf{Z})$ with $M^4(\lambda) \cong CP^2 \# CP^2$, the example from the introduction. The dual of P^2 is a one-dimensional simplicial complex K with vertices $\{v_1, v_2, v_3, v_4\}$.



$\{v_1, v_3\}$ does not span a simplex
$\{v_2, v_4\}$ does not span a simplex
so $I = \langle v_1v_3, v_2v_4\rangle \subset \mathbf{Z}[v_1, v_2, v_3, v_4]$

The relations J are read off from the matrix λ above

$$\left.\begin{array}{rcl} v_2 - v_3 + v_4 &=& 0 \\ v_1 + v_3 - 2v_4 &=& 0 \end{array}\right\} \Rightarrow \begin{array}{rcl} v_3 &=& v_2 + v_4, \\ v_1 &=& v_4 - v_2. \end{array}$$

Choosing generators $v_2, v_4 \in H^2(M^4(\lambda); \mathbf{Z})$ we get

$$\begin{array}{rclll} H^0(M^4(\lambda); \mathbf{Z}) &=& \mathbf{Z}, & & \\ H^2(M^4(\lambda); \mathbf{Z}) &=& \mathbf{Z} \oplus \mathbf{Z} & \langle v_2, v_4\rangle, & \\ H^4(M^4(\lambda); \mathbf{Z}) &=& \mathbf{Z} & \langle v_2^2 = v_4^2\rangle, & \\ H^i(M^4(\lambda); \mathbf{Z}) &=& 0 & i > 4, & v_{i_1}v_{i_2}v_{i_3} = 0, \quad i_j \in \{2, 4\}. \end{array}$$

3. The Action of the Steenrod Algebra

For our calculation, we require the structure of $H^*(M^{2n}(\lambda); \mathbf{Z}_2)$ as a module over the subalgebra $\mathcal{A}(1)$, generated by Sq^1 and Sq^2, of the mod 2 Steenrod algebra $\mathcal{A}$. Let S^0 denote the $\mathcal{A}(1)$ module consisting of a single class in dimension 0 and the trivial action of Sq^1 and Sq^2. Denote by $\mathcal{M}$ the $\mathcal{A}(1)$ module with a class x in dimension 0, a class y in dimension 2 and the action given by $Sq^2(x) = y$.

Lemma 5. *Let X be a space with $H^*(X; \mathbf{Z}_2)$ concentrated in even degrees. Then, as an $\mathcal{A}(1)$ module, $H^*(X; \mathbf{Z}_2)$ is isomorphic to a direct sum of suspended copies of S^0 and $\mathcal{M}$. Furthermore, the splitting is natural with respect to maps of spaces.*

Proof. The sequence

$$\to H^{2n-2}(X; \mathbf{Z}_2) \xrightarrow{Sq^2} H^{2n}(X; \mathbf{Z}_2) \to$$

is a chain complex since $Sq^2Sq^2 = Sq^3Sq^1 = 0$ because $H^*(X; \mathbf{Z}_2)$ is concentrated in even degrees. Its homology is defined to be the "Sq^2 homology of X" and is denoted

$$H_*(X; Sq^2).$$

Let $A_{2n} = \mathrm{Ker}\{Sq^2\colon H^{2n}(X; \mathbf{Z}_2) \to H^{2n+2}(X; \mathbf{Z}_2)\}$. Then $H^{2n}(X; \mathbf{Z}_2) \approx A_{2n} \oplus B_{2n}$ for some vector subspace B_{2n}. Define $C_{2n} \subseteq A_{2n}$ to be $\mathrm{Im}\{Sq^2 : H^{2n-2}(X; \mathbf{Z}_2) \to H^{2n}(X; \mathbf{Z}_2)\}$. Then $A_{2n} \approx C_{2n} \oplus D_{2n}$ for some vector subspace D_{2n}. Hence we have

$$H^{2n}(X; \mathbf{Z}_2) \approx C_{2n} \oplus D_{2n} \oplus B_{2n}$$

with $H_{2n}(X; Sq^2) \approx D_{2n}$ and $Sq^2 : B_{2n-2} \to C_{2n}$ an isomorphism. The lemma now follows since D_{2n} generates copies of suspensions of S^0 and $B_{2n}(\approx C_{2n+2})$ generates suspensions of $\mathcal{M}$. The naturality follows since $H_*(X; Sq^2)$ and C_* are natural. □

An algorithm allows us to determine the $\mathcal{A}(1)$ module structure of $H^*(X; \mathbf{Z}_2)$ explicitly. Let $\{u_{(2,1)}, u_{(2,2)}, \dots , u_{(2,s_2)}\}$ be a $\mathbf{Z}_2$ basis for $H^2(X; \mathbf{Z}_2)$. We construct a new basis $\{w_{(2,1)}, w_{(2,2)}, \ldots, w_{(2,s_2)}\}$ which will yield the decomposition above. Set $w_{(2,1)} = u_{(2,1)}$. If $Sq^2 u_{(2,2)} = Sq^2 w_{(2,1)}$ set $w_{(2,2)} = w_{(2,1)} + u_{(2,2)}$, else $w_{(2,2)} = u_{(2,2)}$. Suppose now that $w_{(2,t-1)}$ has been defined. If $Sq^2 u_{(2,t)}$ is linearly independent of $\{Sq^2 w_{(2,1)}, Sq^2 w_{(2,2)}, \ldots, Sq^2 w_{(2,t-1)}\}$ set $w_{(2,t)} = u_{(2,t)}$. Otherwise, if

$$Sq^2 u_{(2,t)} = Sq^2 w_{(2,i_1)} + Sq^2 w_{(2,i_2)} + \ldots + Sq^2 w_{(2,i_t)}$$

set $w_{(2,t)} = u_{(2,t)} + w_{(2,i_1)} + \ldots + w_{(2,i_t)}$. Next, reorder $\{w_{(2,1)}, w_{(2,2)}, \ldots, w_{(2,s_2)}\}$ so that $Sq^2 w_{(2,j)} = 0$ for $j = 1, \ldots, t_2$ and $Sq^2 w_{(2,j)} \neq 0$ for $j = t_2 + 1, \ldots, s_2$. Set $d_{(2,j)} = w_{(2,j)}$ for $j = 1, \ldots, t_2$ and $b_{(2,j)} = w_{(2,t_2+j)}$ for $j = 1, \ldots, s_2 - t_2$. So, in the notation above,

$$D_2 = \{d_{(2,1)}, d_{(2,2)}, \ldots, d_{(2,t_2)}\}$$

and

$$B_2 = \{b_{(2,1)}, b_{(2,2)}, \ldots, b_{(2,s_2-t_2)}\}.$$

Of course, $C_2 = \phi$ and $C_4 \approx B_2$. Now suppose that A_{2n-2}, B_{2n-2} and C_{2n-2} have been constructed. Set

$$C_{2n} = \{Sq^2 b_{(2n-2,1)}, Sq^2 b_{(2n-2,2)}, \ldots, Sq^2 b_{(2n-2,s_{2n-2}-t_{2n-2})}\} \approx B_{2n-2}.$$

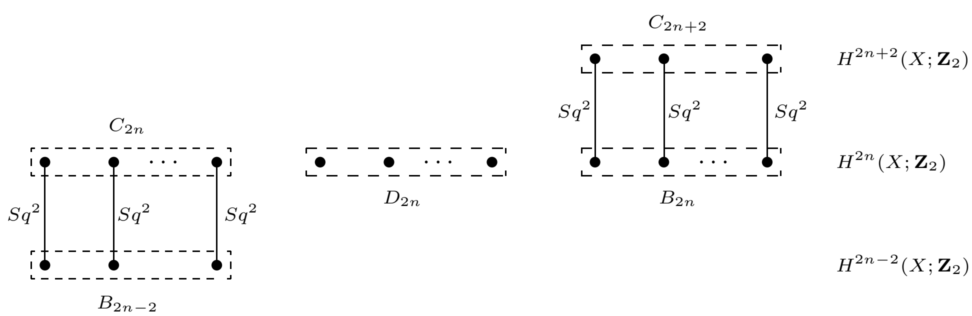
The elements of C_{2n} are linearly independent by the construction of B_{2n-2}. Choose *any* extension of C_{2n} to a basis of $N^{2n} = H^{2n}(X; \mathbf{Z}_2)$. Denote the basis by

$$C_{2n} \cup \{u_{(2n,1)}, u_{(2n,2)}, \ldots, u_{(2n,s_{2n})}\}.$$

Finally, repeat the process above on the set

$$\{u_{(2n,1)}, u_{(2n,2)}, \ldots, u_{(2n,s_{2n})}\}$$

to produce B_{2n} and D_{2n}. Diagrammatically, the $\mathcal{A}(1)$ module structure looks like

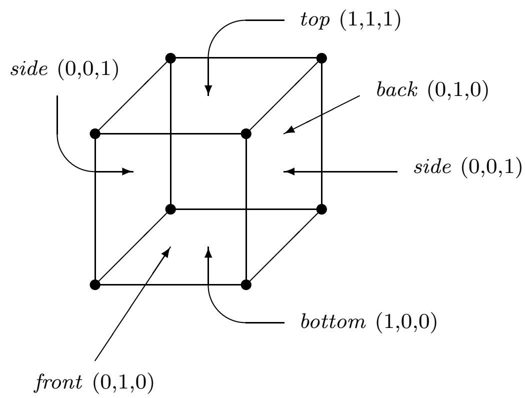


We conclude that the ring structure of $M^{2n}(\lambda)$ determines the $\mathcal{A}(1)$ module structure. *Notice that the $\mathcal{A}(1)$ module structure of $H^*(X; \mathbf{Z}_2)$ can depend only on the map λ mod 2.*

Example. Let P^3 be the three dimensional cube and the map

$$\lambda : \mathcal{F} \to \mathbf{Z}^3$$

(mod 2), be as in the diagram below.



Now

$$H^*(M^6(\lambda);\mathbf{Z}_2) = \mathbf{Z}[v_1, v_2, \ldots, v_6]/(I+J) \quad \mod 2.$$

For P^3 we have $f_0 = 6$, $f_1 = 12$ and $f_2 = 8$ from which it follows easily that $h_0 = 1$, $h_1 = 3$, $h_2 = 3$ and $h_3 = 1$ where h_i is the rank of $H^{2i}(M^6(\lambda);\mathbf{Z}_2)$. The simplicial complex K dual to P^3 is an octohedron with vertices $\{v_1, v_2, \ldots, v_6\}$. The ideal of relations I is generated by $v_1v_6 = 0$, $v_2v_4 = 0$ and $v_3v_5 = 0$. The ideal of relations J is determined by the matrix representation

$$\lambda = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

This gives $v_1 = v_6 = v_3 + v_5 = v_2 + v_4$. Choose as generators of $H^2(M^6(\lambda);\mathbf{Z}_2)$, $\{v_1, v_2, v_3\}$. The relations in $H^4(M^6(\lambda);\mathbf{Z}_2)$ become ${v_1}^2 = 0$, ${v_2}^2 = v_1v_2$ and ${v_3}^2 = v_1v_3$. In $H^6(M^6(\lambda);\mathbf{Z}_2)$ we have $v_1{v_2}^2 = v_1{v_3}^2 = v_3{v_1}^2 = {v_3}^3 = {v_2}^3 = 0$ and $v_3{v_2}^2 = v_2{v_3}^2 = v_1v_2v_3$. We conclude that as $\mathcal{A}(1)$ modules

$$H^*(M^6(\lambda);\mathbf{Z}_2) \;\cong\; \bigoplus_{j=0}^{3} \sum\nolimits^{2j} S^0 \oplus\; 2\sum\nolimits^{2} \mathcal{M}.$$

In the next section we show that this is sufficient to enable us to read off $KO_*(M^6(\lambda))$.

Problem. Given P^n and λ, find an algorithm which will determine the Sq^2 connections directly from the matrix representing λ, that is, *without* doing the algebra involved in solving the relations.

4. The Adams Spectral Sequence for *ko*-Homology

Let X be *any* space with $H^*(X;\mathbf{Z}_2)$ concentrated in even degrees. The (mod 2) Adams spectral sequence relevant for our calculation takes the form

$$E_2 \cong \mathrm{Ext}^{s,t}_{\mathcal{A}}(H^*(ko \wedge X),\mathbf{Z}_2) \cong \mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(H^*(X),\mathbf{Z}_2) \Longrightarrow ko_{t-s}X.$$

More details about this Adams spectral sequence can be found in, for example, [3]. At odd primes, in the case $X = M^n(\lambda)$, the Atiyah-Hirzebruch spectral sequence converging to ko_*X collapses for dimensional reasons and we can conclude easily that ko_*X has no odd torsion. In fact,

$$ko_*(M^n(\lambda)) \otimes \mathbf{Z}_{(p)} \;\cong\; H_*(M^n(\lambda); \mathbf{Z}_{(p)}) \otimes ko_*$$

where $\mathbf{Z}_{(p)}$ denotes the integers localized at p odd. So, a mod 2 calculation suffices for the whole ko-theory.

Lemma 5 tells us that as $\mathcal{A}(1)$ modules

$$H^*(X; \mathbf{Z}_2) \;\cong\; \bigoplus_{j=0}^{k} m_j \sum^{2j} S^0 \oplus \bigoplus_{j=0}^{l} n_j \sum^{2j} \mathcal{M}$$

where positive integers m_j and n_j denote the number of copies of each summand located in dimension $2j$. Then

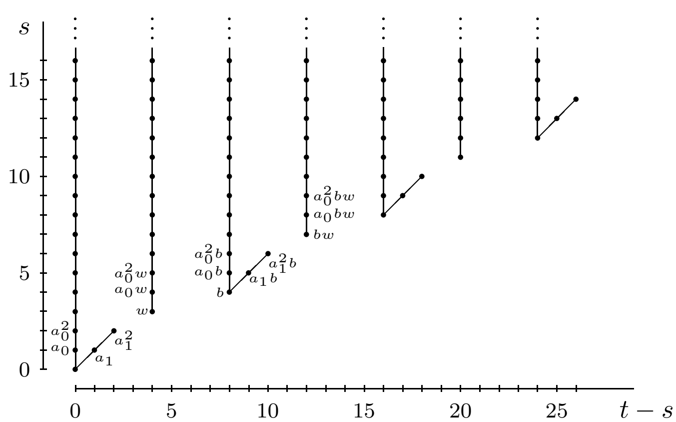
$$\begin{aligned}\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(H^*(X), \mathbf{Z}_2) \;\cong\; & \bigoplus_{j=0}^{k} m_j \cdot \mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(\textstyle\sum^{2j} S^0, \mathbf{Z}_2) \\ & \oplus \bigoplus_{j=0}^{l} n_j \cdot \mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(\textstyle\sum^{2j} \mathcal{M}, \mathbf{Z}_2)\end{aligned}$$

where the isomorphism is as $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(S^0, \mathbf{Z}_2)$ modules.

The bigraded algebra $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(S^0, \mathbf{Z}_2)$ is well known, [7].

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(S^0, \mathbf{Z}_2) \;\cong\; \mathbf{Z}_2[a_0, a_1, w, b]/(a_0a_1, a_1^3, a_1w, w^2 + a_0^2 b)$$

with $|a_0| = (0,1)$, $|a_1| = (1,1)$, $|w| = (4,3)$ and $|b| = (8,4)$, where $|x| = (t-s, s)$ specifies the geometric degree $t-s$ and the Adams filtration s. It's most easily represented by the picture following. The vertical line segments indicate multiplication by a_0 and the sloping line segments, multiplication by a_1.



$\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(S^0, \mathbf{Z}_2)$

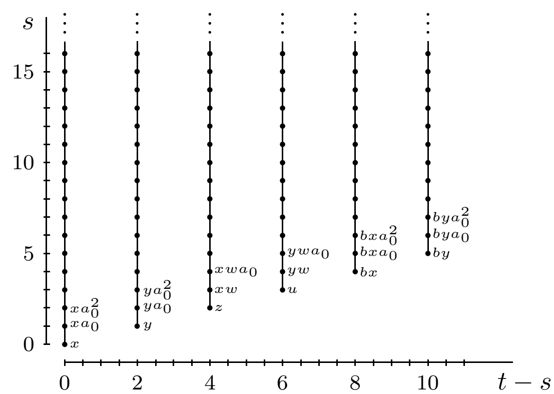
The vertical multiplication by a_0 yields *multiplication-by-two* extensions at E_∞. The vertical towers in this diagram produce copies of $\mathbf{Z}_{(2)}$, the integers localized at 2, in ko_*S^0. The other classes yield copies of $\mathbf{Z}_2$. The class b represents the

Bott periodicity operator. Embedded in this picture then is ko_* the coefficients of ko-theory.

$$ko_*S^0 \cong \mathbf{Z}_{(2)} \oplus \sum^1 \mathbf{Z}_2 \oplus \sum^2 \mathbf{Z}_2 \oplus \sum^4 \mathbf{Z}_{(2)} \oplus \sum^8 \mathbf{Z}_{(2)} \oplus \sum^9 \mathbf{Z}_2 \oplus \dots .$$

$\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(\mathcal{M}, \mathbf{Z}_2)$ is computed easily from $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(S^0, \mathbf{Z}_2)$ and the cofibration sequence associated to $\mathcal{M}$. As a module over $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(S^0, \mathbf{Z}_2)$, $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(\mathcal{M}, \mathbf{Z}_2)$ has generators x, y, z, u with $|x| = (0,0)$, $|y| = (2,1)$, $|z| = (4,2)$ and $|u| = (6,3)$ and relations

$$a_1x = a_1y = a_1z = a_1u = 0,\ a_0z = wx,\ a_0u = wy,\ wz = a_0bx,\ wu = a_0by$$



$\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(\mathcal{M}, \mathbf{Z}_2)$

Since $\sum^2 \mathcal{M} \simeq H^*(CP^2, \mathbf{Z}_2)$ and noting that no differentials are possible in the spectral sequence, we can read off the connective ko-homology of the complex projective plane

$$ko_*CP^2 \cong \sum^2 \mathbf{Z}_{(2)} \oplus \sum^4 \mathbf{Z}_{(2)} \oplus \sum^6 \mathbf{Z}_{(2)} \oplus \sum^8 \mathbf{Z}_{(2)} \oplus \dots .$$

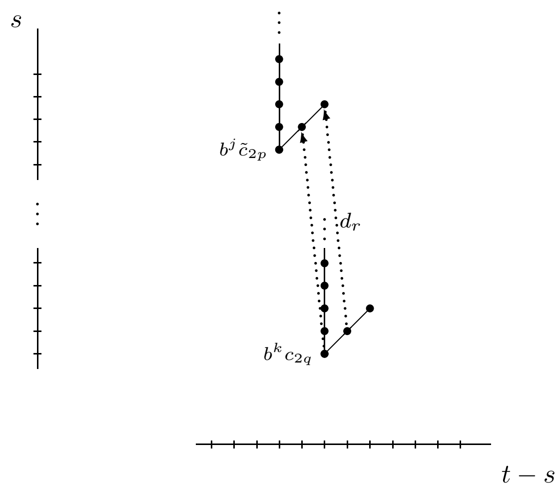
The decomposition above of $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(H^*(X), \mathbf{Z}_2)$ implies that its diagram is obtained by superimposing shifted copies of the diagrams for $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(S^0, \mathbf{Z}_2)$ and $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(\mathcal{M}, \mathbf{Z}_2)$. Dimensional considerations and the fact that d_r is a derivation with respect to the action of $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(S^0, \mathbf{Z}_2)$ allow us to conclude that one type of non-zero differential

$$d_r : E_r^{s,t} \longrightarrow E_r^{s+r,t+r-1}$$

is possible in the spectral sequence. It occurs on a copy of $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(S^0, \mathbf{Z}_2)$ as in the diagram below. In the diagram we have identified the generator

$$c_{2j} \in \mathrm{Ext}^{0,2j}_{\mathcal{A}(1)}(H^*(X), \mathbf{Z}_2)$$

of an $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(\sum^{2j} S^0, \mathbf{Z}_2)$ summand, with the dual of $c_{2j} \in C_{2j} \subseteq H^{2j}(X; \mathbf{Z}_2)$. The class $\tilde{c}_{2p}$ represents some linear combination of classes in $\mathrm{Ext}^{0,2p}_{\mathcal{A}(1)}(H^*(X), \mathbf{Z}_2)$.



A Differential in the Adams Spectral Sequence for ko_*X

Important Remark. Since b has $(t-s,s)$ bidegree $(8,4)$, this differential *cannot* occur in the Adams Spectral Sequence for a toric manifold or toric variety, of dimension less than 12, with mod 2 cohomology concentrated in even degrees. Consequently, the spectral sequence collapses without any further analysis and theorem 1 holds for such spaces.

We shall use the fact that a toric manifold is a manifold to prove that there can be no non-zero differentials in the spectral sequence. Choose q minimal so that for some r, we have $d_r(b^k c_{2q}) \neq 0$. Next, choose the smallest such r so that for some k, we have $d_r(b^k c_{2q}) \neq 0$. The derivation property of d_r with respect to multiplication by the periodicity operator b, implies then that $d_r(c_{2q}) \neq 0$ and so we can assume that $k=0$.

We restrict now to the case $X = M^{2n}(\lambda)$ a toric manifold of dimension $2n$. Consider *all* $2q$ dimensional submanifolds M_{F_i} of $M^{2n}(\lambda)$ corresponding to q-faces F_i. The inclusions

$$M_{F_i} \hookrightarrow M^{2n}(\lambda)$$

induce maps of Adams Spectral Sequences and in particular, maps

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(H^*(M_{F_i}),\mathbf{Z}_2) \longrightarrow \mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(H^*(M^{2n}(\lambda)),\mathbf{Z}_2).$$

In each $\mathrm{Ext}^{0,2q}_{\mathcal{A}(1)}(H^*(M_{F_i}),\mathbf{Z}_2)$ there is a unique class corresponding to the fundamental class $[M_{F_i}]$. Theorem 3 tells us that c_{2q} is a linear combination of the images of the classes $[M_{F_i}]$. Because $d_r(c_{2q}) \neq 0$, the naturality of the Adams Spectral Sequence implies that $d_r([M_{F_i}]) \neq 0$ for some i. In other words, a q-face $F = F_i$ of P^n must exist with a non-zero differential in the Adams Spectral Sequence for $ko_*(M_F)$ supported on the top class of filtration zero. We shall use the result following to show that this cannot be the case for the manifold M_F and so complete the proof of theorem 1

Theorem 6. *Let M be an orientable manifold of dimension n Then M is a spin manifold if the top dimensional cohomology class is not in the image of Sq^2.*

Proof. Let $v \in H^*(M)$ be the total Wu class of M. It satisfies the property that $Sq(v) = w$ where Sq is the total Steenrod operation and w is the total Stiefel-Whitney class. Since M is orientable we have $v_2 = w_2$ where w_2 is the second

Stiefel-Whitney class. The Wu formula for M, ([6], page 261), is

$$\langle a \cup v, [M] \rangle \;=\; \langle Sq(a), [M] \rangle$$

for any $a \in H^*(M)$. In particular, for any class $x \in H^{n-2}(M)$, we have

$$\langle x \cup w_2, [M] \rangle \;=\; \langle x \cup v_2, [M] \rangle \;=\; \langle Sq^2(x), [M] \rangle.$$

So, if $Sq^2(x) = 0$ for all x we must have $w_2 = 0$ by Poincaré duality and so M is a spin manifold. □

Corollary 7. *There are no non-zero differentials in the Adams Spectral Sequence for $ko_*(M_F)$ supported on the top class in filtration zero.*

Proof. Suppose such a differential did exist. Then the $\mathcal{A}(1)$ module $H^*(M_F), \mathbf{Z}_2)$ must contain a summand S^0 in the top dimension $2q$. In particular, the top class in $H^{2q}((M_F), \mathbf{Z}_2)$ is not in the image of Sq^2 and so M_F must be spin manifold. This implies, ([2]), that M_F is *orientable* with respect to ko_*. We can now apply Poincaré-Lefschetz duality, ([9], page 39(a)), to conclude that as a ko_* module, $ko_*(M_F)$ must contain a summand, free on a single generator in $ko_{2q}(M_F)$ dual to the single summand on the generator in $ko^0(M_F)$. This contradicts the existence of the differential. □

The fact that the Adams spectral sequence collapses leaves us with possible group extension problems before we can read off the group $ko_*(M^n(\lambda))$. Fortunately, in our case these are not difficult. As mentioned earlier, the vertical multiplication by a_0 yields *multiplication-by-two* extensions at E_∞. All other classes in the spectral sequence are products of a_1. Vertical extensions across copies of $ko_*(S^0)$, of $\mathbf{Z}_2$ groups to groups of higher torsion, cannot occur because products of a_1 yield elements of order two in *ko*-theory.

We conclude that, if as $\mathcal{A}(1)$ modules

$$H^*(M^n(\lambda); \mathbf{Z}_2) \;\cong\; \bigoplus_{j=0}^{k} m_j \textstyle\sum^{2j} S^0 \oplus \bigoplus_{j=0}^{l} n_j \sum^{2j} M,$$

then

$$ko_*(M^n(\lambda)) \;\cong\; \bigoplus_{j=0}^{k} m_j \textstyle\sum^{2j} ko_* S^0 \oplus \bigoplus_{j=0}^{l} n_j \sum^{2j} ko_* M$$

where the graded groups ko_*S^0 and ko_*M are described above.

Our calculation shows that multiplication by the Bott element b is a monomorphism in E_∞ and hence in $ko_*(M^n(\lambda))$. So, we can invert b to get the periodic KO-homology of $M^n(\lambda)$.

$$KO_*(M^n(\lambda)) \;\cong\; \bigoplus_{j=0}^{k} m_j \textstyle\sum^{2j} KO_* S^0 \oplus \bigoplus_{j=0}^{l} n_j \sum^{2j} KO_* M$$

where

$$KO_*S^0 \;\cong\; \ldots \oplus \textstyle\sum^{-6} \mathbf{Z}_2 \oplus \sum^{-4} \mathbf{Z} \oplus \mathbf{Z} \oplus \sum^{1} \mathbf{Z}_2 \oplus \sum^{2} \mathbf{Z}_2 \oplus \sum^{4} \mathbf{Z} \oplus \ldots$$

and

$$KO_*M \;\cong\; \ldots \oplus \textstyle\sum^{-4} \mathbf{Z} \oplus \sum^{-2} \mathbf{Z} \oplus \mathbf{Z} \oplus \sum^{2} \mathbf{Z} \oplus \sum^{4} \mathbf{Z} \oplus \sum^{6} \mathbf{Z} \oplus \ldots.$$

5. The *KO*-cohomology of Toric Manifolds

We employ the universal coefficient exact sequence following to compute the KO-cohomology from the KO-homology.

Theorem 8 (D. W. Anderson, [1], theorem 2.4)**.** *Let X be a CW-complex. For all n, there is a natural exact sequence*

$$0 \to \lim{}^1 KO^{m-1}(X) \to \mathrm{Ext}_{\mathbf{Z}}(KSp_{m-1}(X), \mathbf{Z}) \\ \to \lim{}^0 KO^m(X) \to \mathrm{Hom}_{\mathbf{Z}}(KSp_m(X), \mathbf{Z}) \to 0$$

where these limits are over the filtration of X by finite subcomplexes.

In our case, $X = M^n(\lambda)$ is a finite complex and we are left with the sequence

$$0 \to \mathrm{Ext}_{\mathbf{Z}}(KSp_{m-1}M^n(\lambda), \mathbf{Z}) \to KO^m M^n(\lambda) \\ \to \mathrm{Hom}_{\mathbf{Z}}(KSp_m M^n(\lambda), \mathbf{Z}) \to 0$$

Bott periodicity implies $KSp_m M^n(\lambda) \cong KO_{m-4}M^n(\lambda)$. Combining this with the results of the previous section, namely, that the groups $KO_*M^n(\lambda)$ are direct sums of copies of $\mathbf{Z}$ and $\mathbf{Z}_2$, we see that the short exact sequence splits. Explicitly, if $KO_m M^n(\lambda) \cong \alpha_m \cdot \mathbf{Z} \oplus \beta_m \cdot \mathbf{Z}_2$, for integers α_m and β_m, then, as groups

$$KO^m M^n(\lambda) \cong \alpha_{m-4} \cdot \mathbf{Z} \oplus \beta_{m-5} \cdot \mathbf{Z}_2.$$

We conclude with a remark about the module structure. Let $DM^n(\lambda)$ denotes the S-dual of $M^n(\lambda)$. If

$$H^*(M^n(\lambda); \mathbf{Z}_2) \cong \bigoplus_{j=0}^{k} m_j \sum^{2j} S^0 \oplus \bigoplus_{j=0}^{l} n_j \sum^{2j} M,$$

then by duality

$$H^*(DM^n(\lambda); \mathbf{Z}_2) \cong \bigoplus_{j=0}^{k} m_j \sum^{-2j} S^0 \oplus \bigoplus_{j=0}^{l} n_j \sum^{2j-2} M.$$

So, except for dimension shifts. the Adams spectral sequence for $ko_*DM^n(\lambda)$ looks much as it did for $ko_*M^n(\lambda)$ We cannot use the same arguments however to conclude that the spectral sequence collapses. Instead, we now know the groups $KO^m M^n(\lambda)$ and so we can use a rank argument to conclude that all differentials must be zero. This allows us to read off $ko_*DM^n(\lambda)$ as a ko_*S^0 module because we know the ko_*S^0 module structure of ko_*M. Again, the Bott element b acts as a monomorphism and we can conclude the KO_*S^0 module structure of $KO_*DM^n(\lambda)$ and so of $KO^*M^n(\lambda)$

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