CHARACTERIZATIONS OF SPECTRA
WITH $\mathcal{U}$–INJECTIVE COHOMOLOGY WHICH SATISFY
THE BROWN–GITLER PROPERTY

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ABSTRACT. We work in the stable homotopy category of p–complete connective spectra having mod p homology of finite type. $H^*(X)$ means cohomology with $\mathbf{Z}/p$ coefficients, and is a left module over the Steenrod algebra $\mathcal{A}$.

A spectrum Z is called *spacelike* if it is a wedge summand of a suspension spectrum, and a spectrum X *satisfies the Brown–Gitler property* if the natural map $[X, Z] \to \mathrm{Hom}_{\mathcal{A}}(H^*(Z), H^*(X))$ is onto, for all spacelike Z.

It is known that there exist spectra $T(n)$ satisfying the Brown–Gitler property, and with $H^*(T(n))$ isomorphic to the injective envelope of $H^*(S^n)$ in the category $\mathcal{U}$ of unstable $\mathcal{A}$–modules.

Call a spectrum X *standard* if it is a wedge of spectra of the form $L \wedge T(n)$, where L is a stable wedge summand of the classifying space of some elementary abelian p–group. Such spectra have $\mathcal{U}$–injective cohomology, and all $\mathcal{U}$–injectives appear in this way.

Working directly with the two properties of $T(n)$ stated above, we clarify and extend earlier work by many people on Brown–Gitler spectra. Our main theorem is that, if X is a spectrum with $\mathcal{U}$–injective cohomology, the following conditions are equivalent:

(A) there exist a spectrum Y whose cohomology is a reduced $\mathcal{U}$–injective and a map $X \to Y$ that is epic in cohomology, (B) there exist a spacelike spectrum Z and a map $X \to Z$ that is epic in cohomology, (C) $\epsilon : \Sigma^\infty \Omega^\infty X \to X$ is monic in cohomology, (D) X satisfies the Brown–Gitler property, (E) X is spacelike, (F) X is standard. ($M \in \mathcal{U}$ is *reduced* if it has no nontrivial submodule which is a suspension.)

As an application, we prove that the Snaith summands of $\Omega^2 S^3$ are Brown–Gitler spectra–a new result for the most interesting summands at odd primes. Another application combines the theorem with the second author's work on the Whitehead conjecture.

Of independent interest, enroute to proving that (B) implies (C), we prove that the homology suspension has the following property: if an n–connected space X admits a map to an n–fold suspension that is monic in mod p homology, then $\epsilon : \Sigma^n \Omega^n X \to X$ is onto in mod p homology.

1. INTRODUCTION

In this paper, we will be working in the stable homotopy category of p–complete connective spectra having mod p homology of finite type, where the prime p will be clear from the context. $H_*(X)$ and $H^*(X)$ will mean homology and cohomology with $\mathbf{Z}/p$ coefficients, and the latter is a left module over the Steenrod algebra $\mathcal{A}$.

The properties of such spectra that we will be studying are inspired by the theory of Brown–Gitler spectra. In the early 1970's, E.Brown and S.Gitler [BG] constructed certain 2–primary finite C.W. spectra $T(n)$ with striking mapping properties with respect to suspension spectra. They were inspired by geometric immersion problems, and their techniques involved elaborate resolutions. Soon after, M.Mahowald [M] noticed that stable wedge summands of $\Omega^2 S^3$ had the same cohomology as the S-duals of the family $T(n)$, and he used his spectra to construct an infinite family of 2–primary elements in the stable homotopy groups of spheres.

Odd primary versions of the Brown–Gitler spectra were constructed by R.Cohen [C1] and P.Goerss [G1]. Working at the prime 2, Brown and F.Peterson connected the Brown–Gitler spectra to Mahowald's spectra in [BP], and this was partially done at odd primes by Cohen [C1]. Various papers characterizing Brown–Gitler spectra ensued (see e.g [BC] and [C2]).

Our original intention here was to revisit these old results, clarifying the key steps, and completing the story at odd primes. However, we have found ourselves doing considerably more, by taking into account algebraic results from the 1980's on the injectives in the category of unstable $\mathcal{A}$–modules (beginning with [Ca], and culminating in [LS]).

To state our main theorem we need to introduce some notation and terminology.

An $\mathcal{A}$–module M is *unstable* if, when $p = 2$, $Sq^i : M_n \to M_{n+i}$ is zero for all $i > n$, and when p is odd, $\beta^e P^i : M_n \to M_{n+2(p-1)i+e}$ is zero for all $2i + e > n$, $e = 0, 1$. The category of unstable modules is denoted $\mathcal{U}$, and an unstable $\mathcal{A}$–module will be called a $\mathcal{U}$–*injective* if it is an injective object in this abelian category. $M \in \mathcal{U}$ is called *reduced* if it has no nontrivial submodule which is a suspension (of an unstable module).

Reprising a definition from [K2], we call a spectrum X *spacelike* if it is a wedge summand of a suspension spectrum. Note that if X is spacelike, then $H^*(X) \in \mathcal{U}$.

We say that a spectrum X *satisfies the Brown–Gitler property* if the natural map

$$[X, Z] \to \operatorname{Hom}_{\mathcal{A}}(H^*(Z), H^*(X))$$

is onto, for all spacelike Z.

Using this terminology, the spectra $T(n)$, defined for $n \geq 0$, satisfy two properties:

- $H^*(T(n)) \simeq J(n)$, the $\mathcal{U}$–injective satisfying

$$\operatorname{Hom}_{\mathcal{U}}(M, J(n)) \simeq \operatorname{Hom}_{\mathbf{Z}/p}(M_n, \mathbf{Z}/p).$$

- $T(n)$ satisfies the Brown–Gitler property.

In what follows, these two properties will be *all* we will need to know about the $T(n)$: how they were constructed is not needed. The papers cited above do construct such spectra, but we recommend the paper of Goerss, J.Lannes and F.Morel [GLM] for the shortest and cleanest construction.

Another class of spectra having $\mathcal{U}$–injective cohomology is the family of spectra which are stable wedge summands of classifying spaces of elementary abelian p–groups. Let $\{L_\lambda\}_{\lambda \in \Lambda}$ be a set of spectra representing each of the homotopy types of indecomposable spectra occurring in this way[1]. (See [HaK] for natural algebraic indexing sets Λ.) Then the classification theorem of J.Lannes and L.Schwartz [LS]

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[1]Included in this set is the sphere spectrum $\Sigma^\infty S^0$.

says that every $\mathcal{U}$–injective is uniquely the direct sum of modules of the form

$$H^*(L_\lambda) \otimes J(n).$$

We say that a spectrum X is *standard* if it is a wedge of spectra of the form $L_\lambda \wedge T(n)$ (and still has homology of finite type).

Our main theorem now goes as follows.

Theorem 1.1. *Suppose X is a spectrum with $\mathcal{U}$–injective cohomology. Then the following conditions are equivalent.*

(A) *There exists a spectrum Y whose cohomology is a reduced $\mathcal{U}$–injective, together with a map $f : X \to Y$ that is epic in cohomology.*

(B) *There exists a spacelike spectrum Z, together with a map $g : X \to Z$ that is epic in cohomology.*

(C) *The evaluation map $\epsilon : \Sigma^\infty \Omega^\infty X \to X$ is monic in cohomology.*

(D) *X satisfies the Brown–Gitler property.*

(E) *X is spacelike.*

(F) *X is standard.*

Remarks 1.2. (1) The implication that (F) implies (E) is the theorem that the spectra $T(n)$ are spacelike. This was first proved by Goerss [G1] and Lannes [L], and one can read this off of the results in [GLM].

(2) If $H^*(X)$ is already reduced, then condition (A) is automatically satisfied[2], and we get a uniqueness theorem: if $H^*(X) \simeq H^*(Y)$ is a reduced $\mathcal{U}$–injective, then $X \simeq Y$. In contrast, there often exist "fake" $T(n)$, e.g. if $p = 2$ and $n \geq 6$, there exists $X(n)$ with $H^*(X(n)) \simeq H^*(T(n))$, but $X(n) \not\simeq T(n)$. (Thanks are due to Paul Goerss and Mark Mahowald for informing us of such examples.)

(3) It would be interesting to know if a spectrum satisfying both conditions (D) and (E) must necessarily have $\mathcal{U}$–injective cohomology. If so, then the theorem would classify such spectra.

We now describe two consequences of our theorem, aimed at applications.

For the first, recall that by Yoneda's lemma, given $a \in \mathcal{A}_d$, the natural transformation $a \cdot : M_n \to M_{n+d}$ will induce a map of $\mathcal{A}$–modules

$$\cdot a : J(n + d) \to J(n).$$

Corollary 1.3. *Suppose $X(n)$ are spectra such that $H^*(X(n)) \simeq J(n)$. Then $X(n) \simeq T(n)$ if either of the following two conditions hold.*

(A) *If $p = 2$, there exist maps $X(n) \to X(2n)$ realizing $\cdot Sq^n : J(2n) \to J(n)$. If p is odd, there exist maps $X(2i+e) \to X(2pi+2e)$ realizing $\cdot \beta^e P^i : J(2pi+2e) \to J(2i + e)$.*

(B) *There exist maps $\Psi : X(i+j) \to X(i) \wedge X(j)$ which are nonzero in cohomology in dimension $i + j$, and maps $g_j : X(2p^j) \to \Sigma^\infty B\mathbf{Z}/p$ which are nonzero in cohomology in dimension 1.*

In §5, we will show that each of the conditions in this corollary easily implies that the corresponding condition in the theorem is true. Our application will be to deduce that Snaith wedge summands of $\Omega^2 S^3$ can be identified with Brown–Gitler

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[2]This situation, in which $H^*(X)$ is a reduced $\mathcal{U}$–injective, is much easier than the general situation. One can easily deduce that such a spectrum satisfies conditions (D), (E), and (F) just using the classification of reduced $\mathcal{U}$–injectives, together with Proposition 1.5 below. This bypasses most of our work.

spectra (Theorem 5.1). At odd primes, this is a new result for the most interesting family of summands.

Our other consequence combines the main theorem with the work of the second author on the Whitehead Conjecture [K1, KP]. Let $SP^{p^j}(S^0)$ denote the p^jth symmetric power of the sphere spectrum, and, following [MP], let $SP^{p^j}_\Delta(S^0)$ denote the cofiber of the "diagonal" map $SP^{p^{j-1}}(S^0) \to SP^{p^j}(S^0)$.

Corollary 1.4.

1. *Suppose given spectra $Y(j)$, $j \geq 0$, and maps*
$$Y(0) \to Y(1) \to Y(2) \to \cdots \to H\mathbf{Z}/p$$
   realizing the admissible sequence length filtration of $H^*(H\mathbf{Z}/p) = \mathcal{A}$. Then $Y(j) \simeq SP^{p^j}_\Delta(S^0)$ for all j.
2. *Suppose given spectra $Y(j)$, $j \geq 0$, and maps*
$$Y(0) \to Y(1) \to Y(2) \to \cdots \to H\mathbf{Z}_p$$
   *realizing the admissible sequence length filtration of $H^*(H\mathbf{Z}_p) = \mathcal{A}/\mathcal{A}\beta$. Then $Y(j) \simeq SP^{p^j}(S^0)$ for all j.*

This will be discussed in §6. As an application, we can immediately prove one of the conjectures in the second author's recent preprint [K3].

We end this introduction by discussing the structure of the proof of Theorem 1.1. The diagram indicates the implications proved in the proof, with (1), ..., (8) indicating the order in which implications will be proved. For example, step (3) will be the implication $(B) \Rightarrow (C)$. (Step (6) is the implication $(D) \Rightarrow (F)$.)
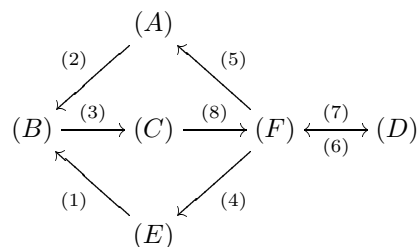
Implications (1) and (2) are proved independently of the other implications. The other six are related as follows:
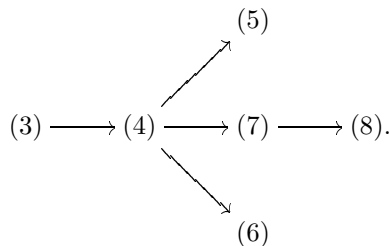
For example, to deduce (4), we will need to use (special cases of) (3).

Implication (1) is obvious.

Using the Adams spectral sequence, implication (2) will follow easily from the following algebraic proposition, which is a generalization to all primes of [LZ2, Prop.5.4.7.1] (a 2–primary result).

Proposition 1.5. *If M is an unstable $\mathcal{A}$–module, and N is a reduced $\mathcal{U}$–injective, then*

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(M,N) = 0, \ \text{for all } t-s < 0.$$

In §2, we will sketch two proofs: one based on Lambda algebra techniques, in the spirit of [BG, M, BC], and one based on studies of derived functors of destablization, in the spirit of [Si, LZ2, Z, G2].

Implication (3) will follow quite easily from the following curious proposition about the homology suspension, which should be of independent interest.

Proposition 1.6.

1. *Let X be an n–connected space with $H_*(X)$ of finite type. Suppose there exists a map $X \to \Sigma^n Z$ which is monic in homology. Then $\epsilon : \Sigma^n \Omega^n X \to X$ is epic in homology.*
2. *Let X be a 0–connected spectrum with $H_*(X)$ of finite type. Suppose there exists a map $X \to \Sigma^\infty Z$ which is monic in homology. Then $\epsilon : \Sigma^\infty \Omega^\infty X \to X$ is epic in homology.*

This will be proved in §3. We will sketch two proofs: one based on recent work of G. Arone [A] studying certain Goodwillie towers, and one based on the more classic Eilenberg–Moore spectral sequence. Implications (4) through (8) are then proved in §4 using, to a greater or lesser degree, algebraic information about the $\mathcal{U}$–injectives, together with the systematic use of mapping telescopes to topologically realize various inverse limits of $J(n)$'s.

Hearty thanks are due to Paul Goerss. In particular, over the last couple of years he has strongly hinted to us that something like Proposition 1.6 should be true, and our discussions in both §2 and §3 borrow heavily from his published work [G1, G2]. Some of the results in this paper appeared in the first author's thesis [H].

2. Ext of $\mathcal{U}$–injectives

We begin this section by proving implication (2) ((A) implies (B) in Theorem 1.1), assuming the algebraic result Proposition 1.5. So we are assuming that there exists a spectrum Y whose cohomology is a reduced $\mathcal{U}$–injective, together with a map $f : X \to Y$ that is epic in cohomology, and we need to show that there exists a spacelike spectrum Z, together with a map $g : X \to Z$ that is epic in cohomology.

By the classification of $\mathcal{U}$–injectives [LS], any reduced injective is a direct sum of $H^*(L_\lambda)$'s. Thus there exists a spacelike Z with $H^*(Y) \simeq H^*(Z)$, as $\mathcal{A}$–modules. Using the Adams spectral sequence and Proposition 1.5, this isomorphism can be realized by a map $h : Y \to Z$, and we let $g = h \circ f : X \to Z$.

We now turn to proofs of Proposition 1.5. It is useful to begin with the following observation (made by many people before us, see e.g. [G2]). Let

$$\Omega^\infty : \mathcal{A}\text{–modules} \ \to \mathcal{U}$$

be left adjoint to the inclusion, and let

$$\Omega_s^\infty : \mathcal{A}\text{–modules} \ \to \mathcal{U}$$

be the sth associated left derived functor. Then one formally deduces

Lemma 2.1. *If M is any $\mathcal{A}$–module, and N is a $\mathcal{U}$–injective, then there is a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{A}}(\Omega_s^{\infty}(\Sigma^{-t}M), N) \simeq \mathrm{Ext}_{\mathcal{A}}^{s,t}(M, N).$$

As a consequence of this, we can reformulate the statement of Proposition 1.5.

Corollary 2.2. *The following statements are equivalent.*

 (i) *If M is an unstable $\mathcal{A}$–module, and N is a reduced $\mathcal{U}$–injective, then*

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(M, N) = 0, \ \text{for all } t - s < 0.$$

 (ii) *If M is a suspension of an unstable $\mathcal{A}$–module, and N is a reduced $\mathcal{U}$–injective, then*

$$\mathrm{Hom}_{\mathcal{A}}(\Omega_s^{\infty}(\Sigma^{-s}M), N) = 0.$$

Following convention (see e.g. [S, p.26]), given an $\mathcal{A}$–module M, define

$$P_0 : M \to M$$

to be

$$\beta^e P^i : M_{2i+e} \to M_{2pi+2e}$$

on grading $2i + e$, $e = 0, 1$. This definition makes sense for $p = 2$ if we adopt the convention that $P^i = Sq^{2i}$. Then we note that an unstable module M is the suspension of another unstable module if and only if P_0 is identically zero. (For $p = 2$, recall that $\beta = Sq^1$ and $Sq^1 Sq^{2i} = Sq^{2i+1}$.)

With this definition, Lemma 2.1 has the following consequence.

Corollary 2.3. *The following statements are equivalent.*

 (i) *If M is an unstable $\mathcal{A}$–module, then the map*

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(M, J(2pi + 2e)) \xrightarrow{\cdot P_0} \mathrm{Ext}_{\mathcal{A}}^{s,t}(M, J(2i + e))$$

 is identically zero, for all $t - s < 0$, $i \geq 0$, $e = 0, 1$.
 (ii) *If M is a suspension of an unstable $\mathcal{A}$–module, then $\Omega_s^{\infty}(\Sigma^{-s}M)$ is again such a suspension.*

As statement (ii) of Corollary 2.3 obviously implies statement (ii) of Corollary 2.2, it follows that Proposition 1.5 will be proved if we prove either of the two equivalent statements in Corollary 2.3. Proofs of each of these turn out to "almost" be in the literature.

When $p = 2$, statement (ii) is part of [LZ2, Theorem 1.5]. The odd prime analogue is Theorem 2.5 of the unpublished thesis [Z]. These authors were strongly influenced by Singer's work (as in [Si]), as was Goerss in [G2], where, in §5, the author essentially gives another proof of our statement (ii) in the $p = 2$ case.

Statement (i) of Corollary 2.3 is closer to "classical" work on Brown–Gitler modules. Rewritten using duality of $\mathcal{A}$–modules, [M, Lemma 5.6] and [BP, Lemma 4.5] are special cases of this, when $p = 2$, as is [C1, Corollary III.3.7], when p is odd. When $p = 2$, [BC, Lemma 2.3(i)] proves statement (i) for all M of the form $H^*(Z)$, where Z is a space.

For completeness, and since [Z] is perhaps not readily available, we end this section by sketching a proof of the following lemma, which immediately implies statement (i).

Lemma 2.4. *In the category of $\mathcal{A}$–modules, there exist injective resolutions,*

$$0 \to J(n) \to I_0(n) \to I_1(n) \to I_2(n) \to \dots ,$$

and chain maps under $\cdot P_0 : J(2pi + 2e) \to J(2i + e),$

$$\gamma_{2i+e} : I_*(2pi + 2e) \to I_*(2i + e),$$

with the following property: if M is an unstable $\mathcal{A}$–module,

$$\gamma_{2i+e} : \mathrm{Hom}_{\mathcal{A}}(M, \Sigma^t I_s(2pi + 2e)) \to \mathrm{Hom}_{\mathcal{A}}(M, \Sigma^t I_s(2i + e))$$

is zero for all $t - s < 0$.

Let $\bar{\Lambda}$ be the extended Lambda algebra as in [BG, §2] and [C1, §I.1]. This is an associative, bigraded algebra over $\mathbf{Z}/p$ with generators λ_i, when $p = 2$, and λ_i and μ_i when $p > 2$, subject to the appropriate Adem relations. To consolidate the notation for all primes p, for $j \geq 0$ and $\epsilon = 0, 1$, let

$$\nu_{j,\epsilon} = \begin{cases} \lambda_{j-1} & \text{if } \epsilon = 0 \text{ and } p > 2, \\ \mu_{j-1} & \text{if } \epsilon = 1 \text{ and } p > 2, \\ \lambda_{2j+\epsilon-1} & \text{if } p = 2. \end{cases}$$

With this definition, $\bar{\Lambda}$ is generated by the $\nu_{j,\epsilon}$, and $|\nu_{j,\epsilon}| = 2(p-1)j + \epsilon - 1$. There are certain admissible sequences $I = (i_1, \epsilon_1, i_2, \epsilon_2, \dots, i_m, \epsilon_m)$ such that the words $\nu_I = \nu_{i_1,\epsilon_1} \nu_{i_2,\epsilon_2} \cdots \nu_{i_m,\epsilon_m}$ form a $\mathbf{Z}/p$-basis for $\bar{\Lambda}$. Let $\{\nu^I\}$ be the dual basis.

For $n \geq 0$, define a left ideal L_n of $\bar{\Lambda}$ by

$$L_n = \bar{\Lambda}\{\nu_{i,\epsilon} : 2pi + 2\epsilon \leq n\}.$$

L_n is spanned by the admissible words it contains. (Compare with [BG, Lemma 2.4] and [C1, Lemma I.1.4].) We define $\Lambda(n)$ to be $\bar{\Lambda}/L_n$, and $\Lambda^s(n)$ to be the part consisting of words of length s.

Given an $\mathcal{A}$–module M, let $M^{\#}$ denote its $\mathbf{Z}/p$-dual viewed as a right $\mathcal{A}$-module. From [BC, §4] (when $p = 2$), and [C1, §I.1 and §III.3] (when p is odd), one gleans the following. There exist injective resolutions,

$$0 \longrightarrow J(n) \longrightarrow I_0(n) \xrightarrow{\;d\;} I_1(n) \xrightarrow{\;d\;} I_2(n) \xrightarrow{\;d\;} \dots ,$$

and chain maps under $\cdot P_0 : J(2pi + 2e) \to J(2i + e),$

$$\gamma : I_*(2pi + 2e) \to I_*(2i + e),$$

with the following properties:

- $\mathrm{Hom}_{\mathcal{A}}(M, I_s(n)) \simeq (\Lambda^s(n) \otimes M^{\#})_{n-s}$, naturally in M.
- Under this natural isomorphism,

$$d : (\Lambda^s(n) \otimes M^{\#})_{n-s} \to (\Lambda^{s+1}(n) \otimes M^{\#})_{n-s-1}$$

  is given by the formula

$$d(v \otimes u) = \sum_{j,\epsilon} \nu_{j,\epsilon} v \otimes u \beta^{\epsilon} P^j.$$

- Under this natural isomorphism,

$$\gamma : (\Lambda^s(2pi + 2e) \otimes M^{\#})_t \to (\Lambda^s(2i + e) \otimes M^{\#})_{t-2(p-1)i-e}$$

  has the property that $\gamma(v \otimes u)$ is the sum of terms of the form

$$\nu^{K,i,e}(\nu_{j,\epsilon} v)\nu_K \otimes u \beta^{\epsilon} P^j.$$

Now we need the following lemma, a variation on [BG, Proof of Lemma 2.8] and [C1, Lemma I.1.8].

Lemma 2.5. $\nu_{j,\epsilon}v$ *can be written as a sum of admissible basis elements* $\nu_{K,k,\kappa}$ *with*

$$|\nu_{k,\kappa}| \leq \left(\frac{p-1}{p}\right)|v| + |\nu_{j,\epsilon}|.$$

Assuming these results, we can now prove Lemma 2.4.

Proof of Lemma 2.4. Let M be an unstable $\mathcal{A}$-module.

We must show that

$$\gamma : (\Lambda^s(2pi + 2e) \otimes M^{\#})_t \to (\Lambda^s(2i + e) \otimes M^{\#})_{t-2i(p-1)-e}$$

is zero for $t < 2pi + 2e$.

From the above, we see that if $\gamma(v \otimes u) \neq 0$, then there exists a pair (j, ϵ) such that simultaneously

$$|v| \geq \left(\frac{p}{p-1}\right)(|\nu_{i,e}| - |\nu_{j,\epsilon}|)$$

and

$$u\beta^{\epsilon}P^j \neq 0.$$

Since M is unstable, $u\beta^{\epsilon}P^j \neq 0$ implies that

$$|u| \geq 2pj + 2\epsilon.$$

Adding these reveals that the integer $|u| + |v|$ is greater than $2pi + 2e - 1$, as needed. □

3. The homology suspension

In this section, we prove Proposition 1.6, and use this to deduce key implication (3) in our proof of Theorem 1.1. We work backwards through this program.

Proof of implication (3): condition (B) implies condition (C). We are given $f : X \to Z$ such that Z is spacelike and f is monic in homology, and we wish to show that the evaluation map $\epsilon : \Sigma^{\infty}\Omega^{\infty}X \to X$ is epic in homology. This would follow immediately from Proposition 1.6(2), if X was 0–connected. If it is not, we will argue that $X \simeq X_0 \vee S(X)$ where X_0 is 0–connected and $S(X)$ is a wedge of (p–completed) $\Sigma^{\infty}S^0$'s, and we will be done by the 0–connected case.

To show that we have such a splitting, first note that, since Z is spacelike, there certainly *is* an analogous splitting $Z \simeq Z_0 \vee S(Z)$. Now consider the map

$$f_* : \pi_0(X) \to \pi_0(Z).$$

This will be a homomorphism between finitely generated modules over the p–adic integers $\mathbf{Z}_p$. The module $\pi_0(Z)$ will be free, and, by the mod p Hurewicz theorem, $f \otimes \mathbf{Z}/p$ is monic. We can apply the following lemma.

Lemma 3.1. *Let* $f : A \to B$ *be a homomorphism between finitely generated* $\mathbf{Z}_p$–*modules. If* B *is free and* $f \otimes \mathbf{Z}/p$ *is monic, then* f *is split monic.*

Now we can find $S(X)$, a wedge of $\Sigma^{\infty}S^0$'s, and $i : S(X) \to X$ inducing an isomorphism on π_0. Then the composite

$$S(X) \xrightarrow{\quad i \quad} X \xrightarrow{\quad f \quad} Z \to S(Z) \xrightarrow{\quad r \quad} S(X)$$

will be an equivalence, if r is chosen to realize a splitting of $f_* : \pi_0(X) \to \pi_0(Z)$, and we conclude that there is a decomposition $X \simeq X_0 \vee S(X)$, as claimed. □

Proof that Proposition 1.6(1) implies Proposition 1.6(2). We are given a map $X \to \Sigma^\infty Z$ which is monic in homology, and we wish to conclude that $\epsilon : \Sigma^\infty \Omega^\infty X \to X$ is epic in homology. We need a lemma.

Lemma 3.2. *There exist finite spectra*

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset \bigcup_n X_n \simeq X$$

with $H_i(X_n) \simeq H_i(X)$ for $i \leq n$, and with $H_i(X_n) = 0$ for $i > n$.

This lemma is easily proved: inductively, X_n is obtained from X_{n-1} by attaching cells of dimension n and $n+1$. More precisely, X_n will be the cofiber of a map $M \to X_{n-1}$, where M is a wedge of S^{n-1}'s and $S^{n-1} \cup_{p^r} D^n$'s so that $\pi_{n-1}(M) \simeq \pi_n(X/X_{n-1})$.

With X_n chosen as in this lemma, the composite $X_n \to X \to \Sigma^\infty Z$ is still monic in homology. Since X_n is a finite spectrum, this map desuspends: for some large N, it is $\Sigma^{-N}\Sigma^\infty$ applied to an unstable map $X_n' \to \Sigma^N Z$. By Proposition 1.6(1), we conclude that $\Sigma^N \Omega^N X_n' \to X_n'$ is epic in homology, and thus that $\Sigma^\infty \Omega^N X_n' \to X_n$ is epic in homology. But this map factors through $\epsilon : \Sigma^\infty \Omega^\infty X_n \to X_n$, so we conclude that this map is also epic in homology.

Taking the colimit as n goes to infinity, then, finally shows that $\epsilon : \Sigma^\infty \Omega^\infty X \to X$ is also epic in homology. □

We now give two proofs of Proposition 1.6(1). The first is more elementary than the second, but longer. Both are ultimately consequences of the well known calculation of $H_*(\Omega^n \Sigma^n Z)$ [CLM].

Recall that we are given an n–connected space X of finite type, together with a map $f : X \to \Sigma^n Z$ which is monic in homology, and we wish to show that $\epsilon : \Sigma^n \Omega^n X \to X$ is epic in homology. We begin by noting that we can assume Z is connected: if not, one lets Z' be the wedge of the path components of Z, $p : Z \to Z'$ the evident quotient map, and then replaces f by $(\Sigma^n p) \circ f : X \to Z'$.

The first proof involves iterated use of the Eilenberg–Moore spectral sequence, and goes as follows.

Let $\mathcal{C}$ be the category of augmented, connected, coassociative, cocommutative, finite type coalgebras over $\mathbf{Z}/p$, and let $\mathcal{H}$ be the category of connected finite type Hopf algebras over $\mathbf{Z}/p$. It is useful to say that $C \in \mathcal{C}$ is a *primitive* coalgebra if all its positive degree elements are primitive. Given $C \in \mathcal{C}$, we let ΣC denote the primitive coalgebra obtained from C by suspending the positive degree part.

With this terminology, and letting $W_0 C = C$, what we need to know about operations in the homology of iterated loopspaces is summarized by the next proposition.

Proposition 3.3. *For each $n \geq 1$, there are a functor*

$$W_n : \mathcal{C} \longrightarrow \mathcal{H}$$

and natural maps of Hopf algebras

$$\Theta_n(X) : W_n H_*(\Omega^n X) \longrightarrow H_*(\Omega^n X),$$

defined for n–connected spaces of finite type, satisfying the following properties.

(i) If $s : H_*(Z) \to H_*(\Omega^n \Sigma^n Z)$ is any map of coalgebras that is also an additive section of $\epsilon_* : H_*(\Omega^n \Sigma^n Z) \to H_*(Z)$, then the composite

$$W_n H_*(Z) \xrightarrow{W_n s} W_n H_*(\Omega^n \Sigma^n Z) \xrightarrow{\Theta_n(\Sigma^n Z)} H_*(\Omega^n \Sigma^n Z)$$

will be an isomorphism of Hopf algebras.
(ii) W_n preserves monomorphisms.
(iii) If C is a primitive coalgebra, then $W_n C$ is primitively generated.
(iv) $\mathrm{Cotor}^{W_{n-1}(\Sigma C)}(\mathbf{Z}/p, \mathbf{Z}/p)$ is isomorphic to $W_n C$, as graded vector spaces.

Proof. This proposition can be read off of the first three sections of F. Cohen's contribution in [CLM]. Our functor W_n is what is called W_{n-1} in [CLM] (with lots of structure forgotten).

Property (i) is then well known if s is chosen to be the natural section $\eta_* : H_*(Z) \to H_*(\Omega^n \Sigma^n Z)$. Any other section will differ from this one by terms of "higher weight", and property (i) follows.

Properties (ii) and (iii) are clear. Verification of property (iv) amounts to a counting argument: $W_{n-1} \Sigma C$ will be an explicit tensor product of monogenic Hopf algebras[3], so additively Cotor can be explicitly computed. (Compare with [G1, Proof of Corollary 4.5].) □

We also need the following standard properties of the Eilenberg–Moore spectral sequence.

Proposition 3.4. *There is a 2nd quadrant spectral sequence $\{E^r_{p,q}(X)\}$, defined naturally for all simply connected X, converging to $H_{p+q}(\Omega X)$, with $E^2_{p,q}(X) = \mathrm{Cotor}^{H_*(X)}_{p,q}(\mathbf{Z}/p, \mathbf{Z}/p)$. Furthermore, $\epsilon_* : H_*(\Sigma \Omega X) \to H_*(X)$ can be identified with the composite*

$$H_{*-1}(\Omega X) \longrightarrow E^\infty_{-1,*}(X) \hookrightarrow E^2_{-1,*}(X) = PH_*(X) \hookrightarrow H_*(X).$$

Here PC denotes the primitives in a coalgebra C. All these properties are either proved or referenced in [Mc].

Turning (finally!) to the proof of Proposition 1.6(1), recall that we are given a map $f : X \to \Sigma^n Z$ that is monic in homology. Consider the following statements:
A(k) $(\Omega^k f)_* : H_*(\Omega^k X) \to H_*(\Omega^k \Sigma^n Z)$ is split monic as a map of coalgebras.
B(k) $W_k(\Sigma^{-k} H_*(X)) \simeq H_*(\Omega^k X)$ as Hopf algebras.
C(k) $\epsilon : \Sigma^k \Omega^k X \to X$ is epic in homology.
(In B(k), $\Sigma^{-k} H_*(X)$ denotes the obvious primitive coalgebra.) By induction on k, we will prove A(k) for $0 \leq k \leq n-1$, B(k) for $0 \leq k \leq n-1$, and C(k) for $0 \leq k \leq n$. B(0) and C(0) trivially hold. We are given that A(0) is true, and ultimately wish to know that C(n) is true.

For the inductive step, we assume that A($k-1$), B($k-1$), and C($k-1$) are all valid. By properties (i), (iii), and (iv) of Proposition 3.3, the Eilenberg–Moore spectral sequence for computing $H_*(\Omega^k \Sigma^n Z)$ from the coalgebra $H_*(\Omega^{k-1} \Sigma^n Z)$ collapses at E^2. By A($k-1$), $(\Omega^{k-1} f)_* : E^2(\Omega^{k-1} X) \to E^2(\Omega^{k-1} \Sigma^n Z)$ is monic, so the spectral sequence for computing $H_*(\Omega^k X)$ also collapses. From this collapsing, we deduce various things.

First, the collapsing implies that

$$(\Omega^k f)_* : H_*(\Omega^k X) \longrightarrow H_*(\Omega^k \Sigma^n Z)$$

---

[3]This is not quite accurate when $n = 1$ or 2, but these cases are easy to understand.

is a monic map of Hopf algebras. This is a weak form of A(k).

Second, the collapsing, together with B($k-1$), implies that

$$W_k(\Sigma^{-k}H_*(X)) \simeq H_*(\Omega^k X)$$

as graded vector spaces. This is a weak form of B(k).

Third, the collapsing implies that $\epsilon_* : H_*(\Sigma\Omega^k X) \to PH_*(\Omega^{k-1}X)$ is epic. By B($k-1$), we also know that $H_*(\Omega^{k-1}X)$ is primitively generated. Since the map $\epsilon_* : H_*(\Omega^{k-1}X) \to \Sigma^{1-k}H_*(X)$ factors though the projection onto the indecomposables $QH_*(\Omega^{k-1}X)$, and is known to be epic by C($k-1$), we conclude that $\epsilon_* : PH_*(\Omega^{k-1}X) \to \Sigma^{1-k}H_*(X)$ is epic. Thus C(k) is true.

If $k < n$, we continue the argument. Since $k < n$, $H_*(\Omega^k\Sigma^n Z)$ will be primitively generated. A subHopf algebra of a primitively generated Hopf algebra is again primitively generated [MM, Proposition 6.13]; thus, using our weak form of A(k), we can conclude that $H_*(\Omega^k X)$ is primitively generated also. Reasoning as above, we conclude that $\epsilon_* : PH_*(\Omega^k X) \to \Sigma^{-k}H_*(X)$ is epic. Thus we can choose maps $s' : \Sigma^{-k}H_*(X) \to PH_*(\Omega^k X)$ and $s : \Sigma^{-k}H_*(\Sigma^n Z) \to PH_*(\Omega^k\Sigma^n Z)$ to be compatible sections of the two horizontal maps in the commutative diagram

$$
\begin{array}{ccc}
PH_*(\Omega^k X) & \xrightarrow{\ \epsilon_*\ } & \Sigma^{-k}H_*(X) \\
{\scriptstyle P(\Omega^k f)_*}\downarrow & & {\scriptstyle f_*}\downarrow \\
PH_*(\Omega^k\Sigma^n Z) & \xrightarrow{\ \epsilon_*\ } & \Sigma^{-k}H_*(\Sigma^n Z).
\end{array}
$$

These then induce a commutative diagram of Hopf algebras:

$$
\begin{array}{ccc}
W_k(\Sigma^{-k}H_*(X)) & \xrightarrow{\ \Theta_k(X)\circ W_k(s')\ } & H_*(\Omega^k X) \\
{\scriptstyle W_k(f_*)}\downarrow & & {\scriptstyle (\Omega^k f)_*}\downarrow \\
W_k(\Sigma^{-k}H_*(\Sigma^n Z)) & \xrightarrow{\ \Theta_k(\Sigma^n Z)\circ W_k(s)\ } & H_*(\Omega^k\Sigma^n Z).
\end{array}
$$

In this diagram, the bottom map is an isomorphism by property (i) of Proposition 3.3, and the left vertical map is split monic, since f_* is. Using the weak form of B(k) already established, we conclude that the top map is an isomorphism, establishing B(k), and that the right vertical map is a split monomorphism, establishing A(k).

This completes the inductive step in our first proof of Proposition 1.6(1).

Our second proof relies on recent work of Arone, inspired by T. Goodwillie's "calculus of functors". In [A], the author proves that, for any finite complex K, there is a natural tower of spectra

$$\cdots \longrightarrow P_3^K(X) \longrightarrow P_2^K(X) \longrightarrow P_1^K(X) \longrightarrow P_0^K(X) \simeq *$$

strongly converging to $\Sigma^\infty \mathrm{Map}_*(K, X)$ if X is dim K connected. The spectrum

$$\text{fiber of } \{P_s^K(X) \longrightarrow P_{s-1}^K(X)\}$$

naturally identifies with the spectrum

$$F(K^{[s]}/\Delta^s(K), X^{[s]})_{h\Sigma_s},$$

where $\Delta^s(K) \subset K^{[s]}$ denotes the fat diagonal, $F(\ ,\ )$ denotes the function spectrum, and E_{hG} denotes the homotopy orbits of a spectrum with G action. Furthermore, the projection

$$\Sigma^\infty \operatorname{Map}_*(K, X) \longrightarrow P_1^K(X) \simeq F(K, X)$$

is the natural "linearization" map.

When we specialize to $K = S^n$ and apply $H_*(\ )$, one gets a second quadrant spectral sequence $\{E_{s,t}^r(X)\}$ converging to $H_{s+t}(\Omega^n X)$ with

$$E_{-s,t}^1(X) \simeq H_{t-s}(F(S^{sn}/\Delta^s(S^n), X^{[s]})_{h\Sigma_s}),$$

and with the edge map

$$H_*(\Omega^n X) \longrightarrow E_{-1,*+1}^1(X) = \Sigma^{-n} H_*(X)$$

agreeing with the homology suspension.

By a direct application of equivariant S–duality, there is a natural equivalence

$$F(S^{sn}/\Delta^s(S^n), X^{[s]})_{h\Sigma_s} \simeq D_{n,s}(\Sigma^{-n} X).$$

Here $D_{n,s}(E) = F(\mathbf{R}^n, s)_+ \wedge_{\Sigma_s} E^{[s]}$, where $F(\mathbf{R}^n, s)$ is the configuration space of ordered s–tuples of distinct points in $\mathbf{R}^n$, Y_+ denotes a space Y with a disjoint basepoint, and the symmetric group Σ_s acts in the obvious way on both $F(\mathbf{R}^n, s)$ and $E^{[s]}$. (See [BMMS] for such constructions on spectra, together with related homology operations.) Thus we get

$$E^1(X) = W_n(\Sigma^{-n} H_*(X))$$

with W_n the functor appearing in Proposition 3.3 above. (The second grading on $E^1(X)$ corresponds to the "weight" grading of W_n.)

In particular the spectral sequence preserves monomorphisms on the E^1 level, and $E^1(\Sigma^n Z) \simeq E^\infty(\Sigma^n Z)$. Thus given a map $X \to \Sigma^n Z$ that is monic in homology, $E^1(X) \simeq E^\infty(X)$ also, and so the homology suspension is epic, as needed.

4. Proofs of steps (4) through (8)

In this section, we complete the proof of Theorem 1.1. Implications are numbered as in the introduction.

4.1. **Proof of implication (4): condition (F) implies condition (E).** As remarked in the introduction, this implication is just the theorem that $T(n)$ is spacelike for all n.

Lemma 4.1. *For each n, there exist a space $Z(n)$ and an $\mathcal{A}$–module epimorphism $H^*(Z(n)) \to J(n)$.*

Proof. The natural operation of taking the tensor product of two unstable $\mathcal{A}$–modules induces a well known product on the bigraded vector space $J(*)_*$. This product interacts with the $\mathcal{A}$–module structure via the Cartan formula.

When $p = 2$, let $f_j : H^*(B\mathbf{Z}/2) \to J(2^j)$ be the unique nonzero $\mathcal{A}$–module map. Note that this map is nonzero in dimension 1, as there is an obvious chain of Steenrod operations linking the nonzero class in $H^1(B\mathbf{Z}/2)$ to that in $H^{2^j}(B\mathbf{Z}/2)$. Let

$$Tens(\bigoplus_{j \geq 0} H^*(B\mathbf{Z}/2)) \longrightarrow J(*)_*$$

be the associative algebra extension of the sum of these maps.

The explicit computation (as in [Mi] or [S, Theorem 2.4.7])

$$J(*)_* = \mathbf{Z}/2[x_j \mid j \geq 0],$$

where x_j is in $J(2^j)_1$, and $Sq^1(x_j) = x_{j-1}^2$, then shows that this map is onto. We conclude that the putative space $Z(n)$ can be taken to be an appropriate wedge of smash products of $B\mathbf{Z}/2$'s.

When p is odd, there is a similar proof, starting from basic maps $f_j : H^*(B\mathbf{Z}/p) \to J(2p^j)$ (together with $H^*(S^1) \simeq J(1)$), and using the calculation

$$J(*)_* = (T\langle e, x_j \mid j \geq 0\rangle \otimes \mathbf{Z}/p[y_j \mid j \geq 0])/I,$$

with $e \in J(1)_1$, $x_j \in J(2p^j)_1$, $y_j \in J(2p^j)_2$, $\beta x_j = y_j$, and $P^1(y_j) = y_{j-1}^p$, and with the ideal I generated by the relations $ex_j = -x_je$, $x_ix_j = -x_jx_i$, and $e^2 = y_0$. (See [Mi] or [S, Theorem 2.4.8].)   □

Armed with the lemma, we prove implication (4). The Brown–Gitler property of $T(n)$ implies that we can realize an $\mathcal{A}$–module epimorphism $H^*(Z(n)) \to J(n)$, as in the lemma. Thus there exists a map $f : T(n) \to \Sigma^\infty Z(n)$ that is epic in cohomology. By implication (3), proved in the last section, we conclude that $\epsilon : \Sigma^\infty \Omega^\infty T(n) \to T(n)$ is monic in cohomology. Again using the fact that $T(n)$ satisfies the Brown–Gitler property, there exists a map $r : T(n) \to \Sigma^\infty \Omega^\infty T(n)$ realizing any $\mathcal{A}$–module splitting of the (necessarily split) monomorphism ϵ^*. The composite $\epsilon \circ r : T(n) \to T(n)$ will then be an equivalence, proving that $T(n)$ is a summand of the suspension spectrum $\Sigma^\infty \Omega^\infty T(n)$.

Remark 4.2. When $p = 2$, a little variant on our proof of this lemma reproves an old result of W.Massey related to Brown and Gitler's original investigations. In 1960, Massey [Mas] calculated which Stiefel–Whitney classes vanish on the stable normal bundle of all closed n–dimensional manifolds. His result can be interpreted as follows. Given an n–manifold M, the fundamental class $[M] \in H_n(M)$ can be viewed as a map of $\mathcal{A}$–modules

$$[M] : H^*(M) \longrightarrow J(n).$$

Using duality and Wu's formula, Massey's theorem is precisely the statement that the sum of such maps is onto all of $J(n)$. To recover this from our argument, one just needs to replace the "generating" maps $f_j : H^*(B\mathbf{Z}/2) \to J(2^j)$ by

$$[RP^{2^j}] : H^*(RP^{2^j}) \longrightarrow J(2^j),$$

and note that fundamental classes behave well with respect to products. The M's needed for Massey's theorem can then be taken to all be of the form $RP^{2^{j_1}} \times \cdots \times RP^{2^{j_r}}$, where $2^{j_1} + \cdots + 2^{j_r} = n$ (and in the lemma, we can let $Z(n)$ be the disjoint union of such manifolds).

4.2. **Proof of implication (5): condition (F) implies condition (A).** In proving this implication, we can clearly assume that $X = L_\lambda \wedge T(n)$.

As a consequence of implication (4), we know that $T(n)$ both satisfies the Brown–Gitler property and is spacelike. It follows that any map of $\mathcal{A}$–modules $J(m) \to J(n)$ can be realized by a map $T(n) \to T(m)$.

In particular we can realize the family of $\mathcal{A}$–module maps $\cdot P_0 : J(2pi + 2e) \to J(2i + e)$ defined in §2. If $n = 2i + e$, let $\tilde{T}(n)$ denote the resulting telescope:

$$\tilde{T}(n) = \text{hocolim } \{T(2i + e) \longrightarrow T(2pi + 2e) \longrightarrow T(2p^2i + 2pe)$$
$$\longrightarrow T(2p^3i + 2p^2e) \longrightarrow \dots \}.$$

Let $i : T(n) \to \tilde{T}(n)$ be the inclusion, and let $Y = L_\lambda \wedge \tilde{T}(n)$. Then $H^*(Y)$ is a reduced $\mathcal{U}$–injective, and $1 \wedge i : X \to Y$ is epic in cohomology.

4.3. Proof of implication (6): condition (D) implies condition (F).
Here we assume that X satisfies the Brown–Gitler property, and we wish to prove it is standard.

By the classification of the $\mathcal{U}$–injectives, there is a unique standard spectrum Y with $H^*(Y) \simeq H^*(X)$ as $\mathcal{A}$–modules. By implication (4), we know that Y is spacelike. Thus this $\mathcal{A}$–module isomorphism can be realized by an equivalence $X \simeq Y$.

4.4. Proof of implication (7): condition (F) implies condition (D).
We need to show that standard spectra satisfy the Brown–Gitler property.

Following [Ca, LZ1], define unstable $\mathcal{A}$–modules $K(n)$ as inverse limits: with $n = 2i + e$,

$$K(n) = \lim \{ J(2i + e) \xleftarrow{P_0} J(2pi + 2e) \xleftarrow{P_0} J(2p^2 i + 2pe) \xleftarrow{P_0} \dots \}.$$

These same references (e.g. [LZ1, p.310]) show that, given m and n, there exist $a_j \in A$ and n_j such that

$$K(m) \otimes J(n) \simeq \lim_j (J(n_j), \cdot a_j).$$

It follows quite formally [LZ1, Lemme 4.3] that $K(m) \otimes J(n)$ is a $\mathcal{U}$–injective.

From the classification of $\mathcal{U}$–injectives we learn that there exist standard spectra $T(m, n)$ such that

- $H^*(T(m, n)) \simeq K(m) \otimes J(n)$,
- each $L_\lambda \wedge T(n)$ is a wedge summand of some $T(m, n)$.

It thus suffices to show that $T(m, n)$ satisfies the Brown–Gitler property.

As noted above, implication (4) implies that we can realize the maps $\cdot a_j : J(n_{j+1}) \to J(n_j)$ by maps $T(n_j) \to T(n_{j+1})$, and we let $\tilde{T}(m, n)$ denote the resulting telescope:

$$\tilde{T}(m, n) = \mathrm{hocolim} \{ T(n_0) \longrightarrow T(n_1) \longrightarrow T(n_2) \longrightarrow T(n_3) \longrightarrow \dots \}.$$

Let Z be any spacelike spectrum (with, as usual, cohomology of finite type). In the commutative diagram

$$
\begin{array}{ccc}
[\tilde{T}(m, n), Z] & \longrightarrow & \lim_j [T(n_j), Z] \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{A}}(H^*(Z), H^*(\tilde{T}(m, n))) & \longrightarrow & \lim_j \mathrm{Hom}_{\mathcal{A}}(H^*(Z), H^*(T(n_j)))
\end{array}
$$

the horizontal maps are isomorphisms[4] , and the right vertical map is the inverse limit of continuous epimorphisms between finitely generated $\mathbf{Z}_p$–modules, thus is also an epimorphism. We conclude that the left vertical map is also epic, and thus that $\tilde{T}(m, n)$ satisfies the Brown–Gitler property.

Using implication (4) again, we know that $T(m, n)$ is spacelike. Thus we can realize the $\mathcal{A}$–module isomorphism $H^*(T(m, n)) \simeq H^*(\tilde{T}(m, n))$ by an equivalence $\tilde{T}(m, n) \simeq T(m, n)$, so that $T(m, n)$ also satisfies the Brown–Gitler property.

---

[4]The top map is epic (which is all we care about) by the Milnor telescope sequence, and the $\lim^1$ term making up the kernel is 0 because the relevant inverse system is an inverse system of finitely generated $\mathbf{Z}_p$–modules.

4.5. **Proof of implication (8): condition (C) implies condition (F).** We are given that $\epsilon : \Sigma^\infty \Omega^\infty X \to X$ is monic in cohomology, and we wish to conclude that X is standard.

By the classification of the $\mathcal{U}$–injectives, there exists a standard spectrum Y with $H^*(X) \simeq H^*(Y)$ as $\mathcal{A}$–modules. As $H^*(X)$ is $\mathcal{U}$–injective, the unstable $\mathcal{A}$–module monomorphism $\epsilon^* : H^*(X) \to H^*(\Sigma^\infty \Omega^\infty X)$ is *split* monic. By implication (7), we know that Y satisfies the Brown–Gitler property; thus we can realize a splitting of ϵ^* by a map $r : Y \to \Sigma^\infty \Omega^\infty X$. Then the composite

$$ Y \xrightarrow{\ r\ } \Sigma^\infty \Omega^\infty X \xrightarrow{\ \epsilon\ } X $$

will be an equivalence, so that X is standard.

5. Brown–Gitler spectra and $\Omega^2 S^3$

We begin this section with a proof of Corollary 1.3.

Proof of Corollary 1.3. Suppose condition (A) of the corollary holds, i.e. we have spectra $X(n)$ with $H^*(X(n)) \simeq J(n)$, and maps $X(2i + e) \to X(2pi + e)$ realizing $\cdot P_0 : J(2pi + 2e) \to J(2i + e)$. With $n = 2i + e$, let $Y(n)$ denote the resulting mapping telescope:

$$ Y(n) = \text{hocolim } \{ X(2i + e) \to X(2pi + 2e) \to X(2p^2 i + 2pe) \to \ldots \}. $$

Then the inclusion $X(n) \to Y(n)$ realizes the epimorphism $K(n) \to J(n)$. As $K(n)$ is a reduced $\mathcal{U}$–injective (essentially by definition!), we see that condition (A) of Theorem 1.1 holds, and we conclude that $X(n) \simeq T(n)$.

Suppose condition (B) of the corollary holds, i.e. we have spectra $X(n)$ with $H^*(X(n)) \simeq J(n)$, maps $\Psi : X(i + j) \to X(i) \wedge X(j)$ which are nonzero in cohomology in dimension $i + j$, and maps $g_j : X(2p^j) \to \Sigma^\infty B\mathbf{Z}/p$ which are nonzero in cohomology in dimension 1. In cohomology, the maps Ψ will induce the product on the $\mathcal{A}$–algebra $J(*)_*$, and the maps g_j will induce the maps f_j appearing in the proof of Lemma 4.1 (and we note that $X(1) \simeq S^1$ is clear). We can thus realize the algebra in the proof of Lemma 4.1, and we end with maps

$$ X(n) \longrightarrow \Sigma^\infty Z(n), $$

with $Z(n)$ as in that lemma, which are epic in cohomology. Thus condition (B) of Theorem 1.1 holds, and we conclude that $X(n) \simeq T(n)$. $\qquad\square$

Our main application goes as follows. With $D_{2,k}(Z)$ as in §3, define $X(n)$, for $n \geq 0$, by S–duality:

$$ X(2i + e) = \Sigma^{2pi+2e} \text{ Dual } D_{2,pi+e}(S^1) $$

for all $i \geq 0$, and $e = 0, 1$, as usual.

We remind the reader that these spectra are thus related to $\Omega^2 S^3$. Recall that, if Z is path connected, the Milgram–May model [May] for $\Omega^2 \Sigma^2 Z$ comes equipped with a natural filtration, and if $F_k(\Omega^2 \Sigma^2 Z) \subset \Omega^2 \Sigma^2 Z$ denotes the kth stage of this, $D_{2,k}(Z) = F_k(\Omega^2 \Sigma^2 Z)/F_{k-1}(\Omega^2 \Sigma^2 Z)$. Furthermore, stably this filtration splits, and there is a stable decomposition [Sn]:

$$ \Sigma^\infty \Omega^2 \Sigma^2 Z \simeq \bigvee_{k>0} \Sigma^\infty D_{2,k}(Z). $$

Thus our spectra $X(n)$ are appropriately suspended S–duals of the Snaith summands of $\Omega^2 S^3$ (and *all* the Snaith summands are accounted for, as homology calculations reveal that $D_{2,k}(S^1) \simeq *$ unless k has the form $pi + e$ with $e = 0$ or 1).

As observed by Mahowald [M], when $p = 2$, and by R. Cohen [C1, Theorem II.b.], when p is odd, there are $\mathcal{A}$–module isomorphisms $H^*(X(n)) \simeq J(n)$. Below we will show that our family satisfies condition (A) of Corollary 1.3, and thus we can conclude:

Theorem 5.1. $X(n) \simeq T(n)$.

Remark 5.2. When $p = 2$, this theorem was first proved in [BP], where the authors also start with the geometric input that condition (A) of Corollary 1.3 holds. Comparing our proofs, one finds that they use Brown and Gitler's explicit Posnikov tower presentation for the S–duals of the spectra $T(n)$, while we don't. Our extra ingredient is Proposition 1.6 (so that condition (B) implies condition (C) in Theorem 1.1), a statement independent of the theory of Brown–Gitler spectra.

At odd primes p, homology calculations and an easy geometric argument show that $X(n) \simeq \Sigma X(n-1)$ unless n has the form $2pi$ or $2pi + 2$. Since $T(n)$ is both spacelike and satisfies the Brown–Gitler property, these same homology calculations show that a similar statement holds for the family $T(n)$. Translated into our notation, R. Cohen in [C1] constructed the family $T(2pi + 2)$, and proved that $T(2pi + 2) \simeq X(2pi + 2)$. The family $T(2pi)$ was only constructed later by Goerss [G1], and our result that $T(2pi) \simeq X(2pi)$ is new. Note that this family includes the examples $X(2p^j)$ that seem to be key in constructing interesting maps in stable homotopy (see [HuK]).

To prove that the family satisfies condition (A) of Corollary 1.3, one shows more. There are the so–called "Mahowald exact sequences".

Lemma 5.3. [S, Proposition 2.2.3] *There are exact sequences of $\mathcal{A}$–modules*

$$0 \longrightarrow \Sigma J(2pi + 2e - 1) \longrightarrow J(2pi + 2e) \xrightarrow{\cdot P_0} J(2i + e) \longrightarrow 0.$$

We note that the first maps in these sequences are (up to nonzero scalar) the unique nontrivial $\mathcal{A}$–module maps between these modules, and it follows that the Mahowald exact sequences can be characterized as the only chain complexes of $\mathcal{A}$–modules having the form

$$0 \longrightarrow \Sigma J(2pi + 2e - 1) \xrightarrow{\beta_*} J(2pi + 2e) \xrightarrow{\alpha_*} J(2i + e) \longrightarrow 0,$$

with both α_* and β_* nonzero.

Now we have

Proposition 5.4. *The Mahowald exact sequences are realized by cofibration sequences*

$$X(2i + e) \longrightarrow X(2pi + 2e) \longrightarrow \Sigma X(2pi + 2e - 1).$$

When $p = 2$, this was proved by Cohen, Mahowald, and Milgram: this is one of the two results in the note [CMM]. In this rest of this section, we will sketch how the (slightly more delicate) odd prime analogue of their argument goes, establishing the proposition in general.

Using that $X(2pi + 2i - 1) \simeq \Sigma X(2pi + 2i - 2)$, and letting $k = pi + e$, unraveling our definitions reveals that we are trying to construct maps

$$\Sigma^{2p-2} D_{2,pk-p}(S^1) \longrightarrow D_{2,pk}(S^1) \longrightarrow D_{2,k}(S^1)$$

with null homotopic composite, and with both maps nonzero in homology.

We will need the other (prime free) result from [CMM].

Lemma 5.5. $D_{2,k}(\Sigma^2 Z) \simeq \Sigma^{2k} D_{2,k}(Z)$.

Using this, it suffices to show that, given a fixed k, there exists an r, together with maps

$$\Sigma^{2pr-2} D_{2,pk-p}(S^{2r-1}) \xrightarrow{\ \beta\ } D_{2,pk}(S^{2r-1}) \xrightarrow{\ \alpha\ } D_{2,k}(S^{2pr-1})$$

with $\alpha \circ \beta \simeq *$, and with α_* and β_* both nonzero.

Fixing r for the moment, it is convenient to let $F_l = F_l(\Omega^2 S^{2r+1})$, and to let $\bar F_l = F_l(\Omega^2 S^{2pr+1})$. Using an $(l-1)$ dimensional Σ_l–equivariant C.W. model for $F(\mathbf{R}^2, l)$ (e.g. as constructed by Fox and Neuwirth [FN]), one gets that F_l is $(2rl-1)$ dimensional, and that $\Omega^2 S^{2pr+1}$ is obtained from $\bar F_l$ by attaching cells of dimension at least $(2pr-1)(l+1)$. In particular, if we let F_p' be the $(2pr-2)$ skeleton of F_p, we will have $F_{p-1} \subset F_p'$.

The loopspace structure on $\Omega^2 \Sigma^2 Z$ induces pairings

$$\mu : F_i(\Omega^2 \Sigma^2 Z) \times F_j(\Omega^2 \Sigma^2 Z) \longrightarrow F_{i+j}(\Omega^2 \Sigma^2 Z).$$

Noting that

$$\Sigma^{2pr-2} D_{2,pk-p}(S^{2r-1}) \simeq (F_p'/F_{p-1}) \wedge (F_{pk-p}/F_{pk-p-1}),$$

we let $\beta : \Sigma^{2pr-2} D_{2,pk-p}(S^{2r-1}) \to D_{2,pk}(S^{2r-1})$ be the map induced by the composite $F_p' \times F_{pk-p} \subset F_p \times F_{pk-p} \xrightarrow{\mu} F_{pk}$.

Let $J : \Omega S^{2r+1} \to \Omega S^{2pr+1}$ be the pth James–Hopf invariant. If $r \geq k/2$, the dimension calculations above show that, for all $l \leq k$, $F_{pl} \subset \Omega^2 S^{2r+1} \xrightarrow{\Omega J} \Omega^2 S^{2pr+1}$ will factor, uniquely up to homotopy, through $\bar F_l$, and, similarly, ΩJ restricted to F_{pl-1} will factor through $\bar F_{l-1}$. Then we let $\alpha = \alpha_k$, where $\alpha_l : D_{2,pl}(S^{2r-1}) \to D_{2,l}(S^{2pr-1})$ is the map induced by $F_{pl} \xrightarrow{\Omega J} \bar F_l$.

Since ΩJ is a loop map, we get a commutative diagram

$$
\begin{array}{ccc}
F_p \times F_{pk-p} & \xrightarrow{\ \Omega J \times \Omega J\ } & \bar F_1 \times \bar F_{k-1} \\
\mu \downarrow & & \mu \downarrow \\
F_{pk} & \xrightarrow{\ \Omega J\ } & \bar F_k,
\end{array}
$$

and thus a commutative diagram

$$
\begin{array}{ccc}
D_{2,p}(S^{2r-1}) \wedge D_{2,pk-p}(S^{2r-1}) & \xrightarrow{\ \alpha_1 \wedge \alpha_{k-1}\ } & D_{2,1}(S^{2pr-1}) \wedge D_{2,k-1}(S^{2pr-1}) \\
\mu \downarrow & & \mu \downarrow \\
D_{2,pk}(S^{2r-1}) & \xrightarrow{\ \alpha\ } & D_{2,k}(S^{2pr-1}).
\end{array}
$$

Since $\alpha_1 : D_{2,p}(S^{2r-1}) \to D_{2,1}(S^{2pr-1}) \simeq S^{2pr-1}$ restricted to $S^{2pr-2} \simeq F_p'/F_{p-1}$ is null homotopic, we conclude that $\alpha \circ \beta$ is also null.

Now we need to use the calculation that

$$H_*(\Omega^2 S^{2r+1}) \simeq \Lambda(a_j \mid j \geq 0) \otimes \mathbf{Z}/p[b_j \mid j \geq 1]$$

as a filtered algebra, with $a_j \in H_{2p^j r-1}(F_{p^j})$, and $b_j \in H_{2p^j r-2}(F_{p^j})$. It follows that β_* corresponds to multiplication by b_1, and is thus nonzero.

To see that α_* is nonzero, it suffices to show that $(\Omega J)_*$ is onto. We need to know a bit more about the homology generators. a_0 is, of course, the bottom class, then a_{j+1} is obtained from a_j by applying the appropriate Dyer–Lashof operation, and finally the homology Bockstein applied to a_j is b_j. Now consider the commutative diagram

$$
\begin{array}{ccc}
QH_*(\Omega^2 S^{2r+1}) & \xrightarrow{(\Omega J)_*} & QH_*(\Omega^2 S^{2pr+1}) \\
\epsilon_* \downarrow & & \epsilon_* \downarrow \\
H_{*+1}(\Omega S^{2r+1}) & \xrightarrow{J_*} & H_{*+1}(\Omega S^{2pr+1}).
\end{array}
$$

It is well known, and easy to prove, that the bottom map is an epimorphism. Using that the homology suspension commutes with Dyer–Lashof operations, one learns that both vertical maps are isomorphisms when $*$ has the form $2p^j r - 1$. Thus the top map is onto in those degrees. Since the homology Bockstein applied to a_j is b_j, the top map is onto in *all* degrees, and our proof is finished.

6. Symmetric products of spheres

We begin this section with a proof of Corollary 1.4.

Proof of Corollary 1.4. This is an easy corollary of work on the Whitehead conjecture, combined with Theorem 1.1 in the "easy" reduced $\mathcal{U}$–injective case.

We will prove part (2) of the corollary. The proof of part (1) is similar.

From [K1, KP], as interpreted in [K2], one has the following theorem.

Theorem 6.1. *Suppose given spectra $Y(j)$, $j \geq 0$, and maps*

$$
Y(0) \longrightarrow Y(1) \longrightarrow Y(2) \longrightarrow \ldots \longrightarrow H\mathbf{Z}_p
$$

realizing the admissible sequence length filtration of $H^*(H\mathbf{Z}_p) = \mathcal{A}/\mathcal{A}\beta$.

If $\Sigma^{-j} Y(j)/Y(j-1)$ is spacelike, then this filtration of $H\mathbf{Z}_p$ is equivalent to

$$
S^0 \longrightarrow SP^p(S^0) \longrightarrow SP^{p^2}(S^0) \longrightarrow \ldots \longrightarrow H\mathbf{Z}_p.
$$

Now suppose we have a sequence of spectra

$$
Y(0) \longrightarrow Y(1) \longrightarrow Y(2) \longrightarrow \ldots \longrightarrow H\mathbf{Z}_p
$$

realizing the admissible sequence length filtration of $H^*(H\mathbf{Z}_p) = \mathcal{A}/\mathcal{A}\beta$. We assert that the geometric hypothesis of this last theorem automatically holds! To see this, first note that there is an $\mathcal{A}$–module isomorphism

$$
H^*(\Sigma^{-j} Y(j)/Y(j-1)) \simeq H^*(\Sigma^{-j} SP^{p^j}(S^0)/SP^{p^{j-1}}(S^0)).
$$

Now we note that $H^*(\Sigma^{-j} SP^{p^j}(S^0)/SP^{p^{j-1}}(S^0))$ was proved to be the cohomology of a reduced $\mathcal{U}$–injective enroute to proving the Whitehead conjecture: it is the cohomology of the "Steinberg module" summand of $B(\mathbf{Z}/p)_+^j$ [MP]. We can thus conclude that $\Sigma^{-j} Y(j)/Y(j-1)$ is spacelike, by the implication (A) $\Rightarrow$ (E) of Theorem 1.1. $\qquad\square$

As an application of this corollary, we can prove a conjecture from [K3]. Let $p = 2$. In [K3], one of us defined a bigraded family of spectra $X(N,n)$, for $N \geq 0$ and $n \geq 0$, by letting $X(N,n)$ be the S-dual of $D_{N+1,n}(S^{-N})$. With this definition, $X(1,n) \simeq X(n)$, where $X(n)$ as in Theorem 5.1 (and thus $X(1,n) \simeq T(n)$). Furthermore, evaluation on loopspaces was used to construct maps

$$
\delta : X(N,n) \longrightarrow \Sigma^{-1} X(N+1,n),
$$

and Hopf invariant techniques were used to construct maps

$$\Phi : X(N, n) \longrightarrow X(N, 2n),$$

generalizing the maps $X(n) \to X(2n)$ appearing in Proposition 5.4.

These families of maps were shown to be compatible: δ and Φ commute up to homotopy. Then $X(\infty, n)$ was defined to be

$$X(\infty, n) = \text{hocolim } \{ \ X(0, n) \ \xrightarrow{\ \delta\ } \ \cdots \ \xrightarrow{\ \delta\ } \ \Sigma^{-N} X(N, n) \ \xrightarrow{\ \delta\ } \ \cdots \},$$

and it was proved [K3, Theorem 1.5] that the sequence

$$X(\infty, 1) \ \xrightarrow{\ \Phi\ } \ X(\infty, 2) \ \xrightarrow{\ \Phi\ } \ X(\infty, 4) \ \xrightarrow{\ \Phi\ } \ \ldots$$

realizes the length filtration of $\mathcal{A}$.

By Corollary 1.4, we conclude

Theorem 6.2. $X(\infty, 2^j) \simeq SP_{\Delta}^{2^j}(S^0)$.

This proves [K3, Conjecture 1.6.].

References

[A]   G. Arone, *A generalization of Snaith–type filtration*, preprint, 1997.

[BC]   E. H. Brown and R. L. Cohen, *The Adams Spectral Sequence of $\Omega^2 S^3$ and Brown-Gitler Spectra*, Annals of Mathematics Studies **113** (1987), 101–125. MR **89d:**55020

[BG]   E. H. Brown and S. Gitler, *A Spectrum whose Cohomology is a Certain Cyclic Module over the Steenrod Algebra*, Topology **12** (1973), 283–295. MR **52:**11893

[BP]   E. H. Brown and F. P. Peterson, *On the Stable Decomposition of $\Omega^2 S^{r+2}$*, Transactions of the American Mathematical Society **243** (1978), 287–298. MR **58:**18424

[BMMS] R. R. Bruner, J. P. May, J. E. McClure, and M. Steinberger, *H_∞ Ring spectra and their applications*, Springer Lecture Notes in Mathematics **1176**, Springer, 1986. MR **88e:**55001

[Ca]   G. Carlsson, *G.B.Segal's Burnside ring conjecture for $(\mathbf{Z}/2)^k$*, Topology **22**(1983), 83–103.

[CLM]   F. R. Cohen, T. J. Lada, and J. P. May, *The Homology of Iterated Loop Spaces*, Springer Lecture Notes in Mathematics **533**, Springer, 1976. MR **55:**9096

[CMM]   F. R. Cohen, M. Mahowald, and R. J. Milgram, *The Stable Decomposition of the Double Loop Space of a Sphere*, A. M. S. Proc. Symp. Pure Math. **32** (1978), 225–228. MR **80j:**55009

[C1]   R. L. Cohen, *Odd Primary Infinite Families in Stable Homotopy Theory*, Memoirs of the A. M. S. **242** (1981). MR **82d:**55011

[C2]   R. L. Cohen, *Representations of Brown–Gitler spectra*, Springer L. N. Math. **788** (1980), 399–417. MR **81k:**55011

[FN]   R. Fox and L. Neuwirth, *The braid groups*, Math. Scand. **10** (1962), 119–126. MR **27:**742

[G1]   P.G.Goerss, *A direct construction for the duals of Brown-Gitler spectra*, Ind. U. Math. J. **34** (1985), 733–751. MR **87g:**55005

[G2]   P.G.Goerss, *Unstable projectives and stable Ext: with applications*, Proc. London Math. Soc. **53** (1986), 539–561. MR **88d:**55011

[GLM]   P. Goerss, J. Lannes, and F. Morel, *Hopf Algebras, Witt Vectors, and Brown-Gitler Spectra*, A.M.S. Cont. Math. **146** (1993), 111–128. MR **95a:**55007

[HaK]   J. C. Harris and N. J. Kuhn, *Stable decompositions of classifying spaces of finite abelian p–groups*, Math. Proc. Camb. Phil. Soc. **103** (1988), 427–449. MR **89d:**55021

[H]   D.J.Hunter, *Stable homotopy groups of spheres and Brown–Gitler spectra*, Ph.D. dissertation, University of Virginia, 1997.

[HuK]   D.J.Hunter and N.J.Kuhn, *Mahowaldean families of elements in stable homotopy groups revisited*, Math. Proc. Camb. Phil. Soc., to appear.

[K1]   N. J. Kuhn, *A Kahn–Priddy sequence and a conjecture of G. W. Whitehead*, Math. Proc. Camb. Phil. Soc. **92** (1982), 467–483. MR **85f:**55007b

[K2]   N. J. Kuhn, *Spacelike resolutions of spectra*, A. M. S. Cont. Math. Series **19** (1983), 153–165. MR **85d:**55009

[K3]   N. J. Kuhn, *New relationships among loopspaces, symmetric products, and Eilenberg Mac Lane space*, preprint, 1998.

[KP]   N. J. Kuhn and S. B. Priddy, *The transfer and Whitehead's conjecture*, Math. Proc. Camb. Phil. Soc. **98** (1985), 459–480. MR **87g:**55030

[L]   J.Lannes, *Sur le n–dual du n–ème spectre de Brown–Gitler*, Math.Zeit. **199**(1988), 29–42. MR **89h:**55020

[LS]   J.Lannes and L.Schwartz, *Sur la structure des $\mathcal{A}$–modules instables injectifs*, Topology **28**(1989), 153–169. MR **90h:**55027

[LZ1]   J.Lannes and S.Zarati, *Sur les $\mathcal{U}$–injectifs*, Ann.Scient.Ec.Norm.Sup. **19**(1986), 1–31. MR **89a:**55032

[LZ2]   J.Lannes and S.Zarati, *Sur les foncteurs dérivés de la déstabilisation*, Math. Zeit. **194**(1987), 25–59. MR **88j:**55014

[M]   M. Mahowald, *A New Infinite Family in $_2\pi_*^S$*, Topology **16** (1977), 249–256. MR **56:**3838

[Mas]   W. S. Massey, *On the Stiefel–Whitney classes of a manifold*, Amer. J. Math. **82** (1960), 92–102. MR **22:**1918

[May]   J. P. May, *The Geometry of Iterated Loop Spaces*, Springer Lecture Notes in Mathematics **271**, Springer, 1972. MR **54:**8623b

[Mc]   J. McCleary, *User's guide to spectral sequences*, Math. Lecture Series **12**, Publish or Perish Inc., 1985. MR **87f:**55014

[Mi]   H. Miller, *The Sullivan Conjecture on Maps from Classifying Spaces*, Annals of Mathematics **120** (1984), 39–87. MR **85i:**55012

[MM]   J. W. Milnor and J. C. Moore, *On the structure of Hopf algebras*, Annals of Mathematics **81** (1965), 211–264. MR **30:**4259

[MP]   S. A. Mitchell and S. B. Priddy, *Stable splittings derived from the Steinberg module*, Topology **22** (1983), 285–298. MR **85f:**55005

[S]   L. Schwartz, *Unstable modules over the Steenrod algebra and Sullivan's fixed point conjecture,* Chicago Lectures in Math., Unversity of Chicago Press, 1994. MR **95d:**55017

[Si]   W. M. Singer, *Iterated loop functors and the homology of the Steenrod algebra II: A chain complex for $\Omega_s^k M$*, J. Pure App. Math. **16** (1980), 85–97. MR **81b:**55040

[Sn]   V. P. Snaith, *A Stable Decomposition for $\Omega^2\Sigma^2 X$*, Journal of the London Mathematical Society **2** (1974), 577–583. MR **49:**3918

[Z]   S. Zarati, *Dérivés du foncteur de déstabilisation en caractéristique impaire et applications*, Thése de doctorat, Université Paris–Sud (Orsay), 1984.

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