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LIVŠIC THEOREMS FOR HYPERBOLIC FLOWS

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ABSTRACT. We consider Hölder cocycle equations with values in certain Lie groups over a hyperbolic flow. We extend Livšic's results that measurable solutions to such equations must, in fact, be Hölder continuous.

1. INTRODUCTION

Let ϕ^t be a hyperbolic flow on a set Λ. The precise definition of this will be given in §2, but hyperbolic flows include, for example, transitive Anosov flows and, more generally, Axiom A flows restricted to a basic set. Let G be a Lie group. A continuous map $\mathcal{F} : \Lambda \times \mathbb{R} \to G$ is called a *cocycle* if

$$\mathcal{F}(x, t+s) = \mathcal{F}(\phi^t x, s)\mathcal{F}(x,t)$$

for all $x \in \Lambda$, $t, s \in \mathbb{R}$. Two cocycles $\mathcal{F}, \mathcal{G}$ are said to be *cohomologous* if there exists a map $u : \Lambda \to G$ such that

$$\mathcal{F}(x,t) = u(\phi^t x)\mathcal{G}(x,t)u(x)^{-1}. \tag{1.1}$$

(If u is only assumed to be measurable, then (1.1) is assumed to hold a.e. with respect to a suitable ϕ^t-invariant measure.) We call u satisfying (1.1) a *cobounding function.* If $\mathcal{F}$ satisfies

$$\mathcal{F}(x,t) = u(\phi^t x)u(x)^{-1}, \tag{1.2}$$

then we call $\mathcal{F}$ a *coboundary.* Observe that when G is abelian, (1.1) reduces to (1.2).

One can obtain real and circle-valued cocycles by taking $f : \Lambda \to \mathbb{R}$ and defining

$$\begin{aligned} \mathcal{F}(x,t) &= \int_0^t f(\phi^u x)\, du, \\ \mathcal{F}(x,t) &= \exp\left(2\pi i \int_0^t f(\phi^u x)\, du\right), \end{aligned}$$

respectively. More generally, a cocycle determines and is determined by the Lie algebra valued function $f(x) = \lim_{t\to 0}(\exp^{-1} \mathcal{F}(x,t))/t$.

Three questions can be asked:

1. What conditions on $\mathcal{F}$ ensure that $\mathcal{F}$ is a coboundary?

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2. If $\mathcal{F}$ is a Hölder continuous cocycle (see §4 for the definition) and there is a measurable solution u to (1.1) (holding a.e. with respect to a Hölder equilibrium state), is the cobounding function Hölder continuous with the same exponent as $\mathcal{F}$?
3. If $\mathcal{F}$ possesses a higher degree of regularity (C^r for some $1 \le r \le \infty$ or $r = \omega$), is the same true of a continuous solution to (1.1)?

One can also formulate cocycle equations for a hyperbolic diffeomorphism ϕ. Let G be a Lie group and let $f, g : \Lambda \to G$ be Hölder continuous. We say that f, g are *cohomologous* if, for some $u : \Lambda \to G$,

$$f(x) = u(\phi x) g(x) u(x)^{-1}. \tag{1.3}$$

Again, f is called a *coboundary* if it is cohomologous to the trivial cocycle $g(x) \equiv e$, the group identity. We call results answering the above questions *Livšic theorems* after the celebrated work in [Li1], [Li2]. In this note, we shall mostly be interested in answering question 2 in the affirmative, but we shall also find it useful to give a brief discussion of results related to questions 1 and 3. Observe that questions 2 and 3 can be combined. However, since they require different methods and since we have nothing new to add to the third question, we prefer to keep them separate.

Let us first address question 1. Let $\mathcal{F}$ be a Hölder cocycle. Clearly, if u is a Hölder continuous solution to the coboundary equation (1.2), then $\mathcal{F}(x, T) = e$ whenever $\phi^T x = x$. This condition is also sufficient for the existence of a Hölder continuous cobounding function. We call a result of this form a *Livšic periodic data criterion*. Such criteria continue to hold for any compact Lie group (see [KH] for a highly readable account), for finite-dimensional Lie groups [Li2] and, for certain classes of cocycles $\mathcal{F}$, for some infinite-dimensional Lie groups [Li2]. For the general cocycle equation (1.1), there is a partial analogue of the periodic data criterion for compact Lie groups [Pa2].

Livšic also considered question 2 for real-valued cocycles. In [Li2] he proved that if ϕ^t is a C^2 transitive Anosov flow or diffeomorphism preserving a smooth invariant measure, then any measurable solution u to the real-valued cocycle equation

$$\int_0^t f(\phi^u x)\, du = u(\phi^t x) - u(x) \text{ a.e.}$$

where f is Hölder continuous must be equal almost everywhere to a Hölder continuous function u' for which the equation holds everywhere. We call results of this kind—where measurable solutions imply continuous or Hölder solutions—*Livšic regularity theorems*.

Recently, Nicol and Pollicott [NP] observed that Livšic's method for Anosov diffeomorphisms can be extended to coboundary equations taking values in an arbitrary matrix Lie group. The method does not extend to the general cocycle equation (1.1). Indeed, the regularity theorem can fail for non-compact non-connected Lie groups [Wa1].

In §4 we shall prove the following. Let G be a compact Lie group. Suppose ϕ^t is a C^1 hyperbolic flow equipped with a Hölder equilibrium state m. Let $\mathcal{F}$, $\mathcal{G}$ be Hölder continuous cocycles and suppose they satisfy (1.1) a.e. $[m]$ for a measurable function $u : \Lambda \to G$. Then u is equal a.e. to a Hölder continuous function for which (1.1) holds everywhere. A precise statement of this is given in §4. In §5 we show how one may deduce the real-valued regularity theorem from the circle-valued case.

To prove these results we first consider cocycle equations over shifts of finite type and their suspensions. By the well-known technique of Markov sections (which we sketch below), a hyperbolic flow is seen to be a factor of a suspension of a shift of finite type in a particularly nice way. This allows us to first solve the measurable cocycle equation in the symbolic cover and then push solutions down to the hyperbolic flow. We then obtain a continuous solution to the cocycle equation. For the case of a coboundary (equation (1.2)), this is sufficient: a Livšic periodic data criterion then implies the required degree of Hölder regularity on the solution. In the general case (equation (1.1)), a small further argument is necessary. The method of pushing down solutions that we use here had its origins in [PP2]. Here, hyperbolic diffeomorphisms were considered and the regularity theorem for cocycle equations of the form (1.3) was proved. Markov partitions were used to reduce the problem to the case of a shift of finite type; this had already been considered in [Pa1]. A summary of the results in [Pa1] is given in §3 before we prove regularity results for suspensions of a shift of finite type.

Building on earlier work of Livšic, Guillemin and Kazhdan, Hurder and Katok (amongst others), Niţică and Török considered in [NT1], [NT2] question 3. In §6 we link our results with those in [NT2].

2. Preliminaries

There is a standard (albeit non-canonical) way of relating a hyperbolic flow to a shift of finite type via Markov sections. This approach goes back to Bowen [Bo] in the hyperbolic case and Ratner [Ra] in the Anosov case. This technique has been well used in the study of hyperbolic dynamics (see [Bo, BR, PP1] et al.), and we summarise the definitions and main ideas below.

Let $S = \{1, \dots, k\}$ be a finite set. Let A be an aperiodic $k \times k$ matrix with entries 0 or 1. Define the *subshift of finite type* by

$$\Sigma = \Sigma_A = \{(x_j)_{j=-\infty}^{\infty} \in S^{\mathbb{Z}} \mid A_{x_j, x_{j+1}} = 1 \text{ for all } j \in \mathbb{Z}\}.$$

Then Σ is invariant under the shift map $\sigma : \Sigma \to \Sigma$ defined by $(\sigma x)_j = x_{j+1}$.

Let $n(x, y) = \sup\{n \mid x_j = y_j, |j| \le n\}$. For each $\theta \in (0, 1)$ define a metric on Σ by $d^\theta(x, y) = \theta^{n(x,y)}$. For a metric space Y, let $F_\theta(\Sigma, Y)$ denote the space of functions $\Sigma \to Y$ that are Lipschitz continuous with respect to d^θ. We shall often abuse this definition and speak of functions in $F_\theta(\Sigma, Y)$ as being Hölder of exponent θ.

For a continuous $f : \Sigma \to \mathbb{R}$ define the *pressure* of f to be

$$P(f) = \sup\{h_\mu(\sigma) + \mu(f) \mid \mu \text{ is a } \sigma\text{-invariant probability}\}. \tag{2.1}$$

If $f \in F_\theta(\Sigma, \mathbb{R})$, then this supremum is attained by a unique σ-invariant probability m. We call m the *equilibrium state* of f. One can check that F_θ-equilibrium states are ergodic and non-zero on non-empty open sets.

Let $r \in F_\theta(\Sigma, \mathbb{R})$ be strictly positive. Define

$$\widetilde{\Sigma_r} = \{(x, t) \in \Sigma \times \mathbb{R} \mid x \in \Sigma, 0 \le t \le r(x)\}/\sim$$

where $\sim$ identifies $(x, r(x))$ with $(\sigma x, 0)$. This is a compact metrisable space. We shall often denote a point in $\widetilde{\Sigma_r}$ by $\tilde{x}$ and write $\tilde{x} = (x, s)$ for $s \in [0, r(x))$. Define the *symbolic flow* σ_r^t on $\widetilde{\Sigma_r}$ by $\sigma_r^t(x, s) = (x, s + t)$ for small t and then extend to all t by using the identifications.

The pressure of a continuous function $f : \widetilde{\Sigma_r} \to \mathbb{R}$ can be defined in analogy with (2.1) (the entropy of a flow is defined to be the entropy of its time-one map). When f satisfies $\mathcal{I}f(x) := \int_0^{r(x)} f\sigma_r^u(x,0)\,du \in F_\theta(\Sigma, \mathbb{R})$ for some $\theta \in (0,1)$ the supremum is again attained by a unique σ_r^t-invariant probability m which we call the *equilibrium state* of f. One can show, [PP1], for example, that $m = \mu \times l/\mu(r)$ where μ is the equilibrium state of $\mathcal{I}f - P(f)r$ and l is a Lebesgue measure on the fibres $[0, r(x)]$. Clearly, m is ergodic (as μ is).

Let M be a compact C^∞ Riemannian manifold and let $\Lambda \subset M$. Let $C^\alpha(\Lambda, Y)$ denote the space of Hölder continuous functions $\Lambda \to Y$ with exponent $\alpha \in (0,1)$. Let $|f|_\theta$ denote the least possible Hölder constant and let $\|f\|_\theta = |f|_\theta + |f|_\infty$.

Let ϕ^t be a C^1 flow on M. A ϕ^t-invariant closed set $\Lambda \subset M$ is called a *basic set* and ϕ^t restricted to Λ is called a *hyperbolic flow* if the following conditions hold:

1. the tangent bundle $T_\Lambda M$ of M restricted to Λ has a continuous splitting into a Whitney sum $E^s \oplus E^u \oplus E^0$ of $d\phi^t$-invariant sub-bundles for which there exist constants $C > 0$, $0 < \lambda < 1$ such that
$$\|d\phi^t(v)\| \le C\lambda^t \|v\| \text{ for } t \ge 0, \text{ if } v \in E_x^s,$$
$$\|d\phi^{-t}(v)\| \le C\lambda^t \|v\| \text{ for } t \ge 0, \text{ if } v \in E_x^u,$$
and E^0 is one-dimensional and tangential to the flow direction;
2. the periodic points of ϕ^t in Λ are dense in Λ;
3. Λ contains a dense orbit;
4. there exists an open neighbourhood U of Λ in M such that
$$\Lambda = \bigcap_{t=-\infty}^{\infty} \phi^t U;$$
5. Λ contains no fixed points and is larger than a single orbit.

When $\Lambda = M$ we call ϕ^t a *(transitive) Anosov flow.* The definition of a hyperbolic diffeomorphism is exactly the same, except that there is no centre sub-bundle E^0 in 1.

Again, we can use an analogue of (2.1) to define the pressure of a continuous function on Λ. If f is Hölder continuous, then this supremum is achieved by a unique ϕ^t-invariant probability m, the *equilibrium state* of f [BR]. Moreover, m is positive on non-empty open sets and is ergodic [BR].

We now gather together some well-known facts about the existence and properties of stable and unstable manifolds for hyperbolic flows. For details, see for example [Sh] and the references cited therein.

For $\varepsilon > 0$ sufficiently small and $x \in \Lambda$, define the (local, strong) *stable* and *unstable manifolds*[1] by
$$W_\varepsilon^s(x) = \left\{ y \in M \;\middle|\; \begin{array}{l} d(\phi^t x, \phi^t y) \le \varepsilon \text{ for all } t \ge 0, \\ d(\phi^t x, \phi^t y) \to 0 \text{ as } t \to \infty \end{array} \right\},$$
$$W_\varepsilon^u(x) = \left\{ y \in M \;\middle|\; \begin{array}{l} d(\phi^{-t} x, \phi^{-t} y) \le \varepsilon \text{ for all } t \ge 0, \\ d(\phi^{-t} x, \phi^{-t} y) \to 0 \text{ as } t \to \infty \end{array} \right\}.$$

[1]Some authors use the notation $W_\varepsilon^{ss}(x)$ for our $W_\varepsilon^s(x)$ and use $W_\varepsilon^s(x)$ for the *weak* local stable manifold. Since we shall only refer to weak stable manifolds in passing, we use the simpler notation. Similar remarks hold for the unstable manifolds.

Then, provided ε is small enough, these are C^1 embedded discs in M and $T_x W^{s,u}_\varepsilon(x) = E^{s,u}_x$.

For $x \in \Lambda$ we have the following estimates:

$$\text{(2.2)} \qquad \begin{array}{l} \text{if } y \in W^s_\varepsilon(x), \text{ then } d(\phi^t x, \phi^t y) \le C\lambda^t d(x,y) \text{ for all } t \ge 0; \\ \text{if } y \in W^u_\varepsilon(x), \text{ then } d(\phi^{-t} x, \phi^{-t} y) \le C\lambda^t d(x,y) \text{ for all } t \ge 0. \end{array}$$

We will need the fact that Λ admits a *local product structure* (or *canonical coordinates*). Let $\varepsilon > 0$ be small. Then there exists $\delta > 0$ such that the following holds. If $x, y \in \Lambda$ and $d(x,y) \le \delta$, then there exists a unique $\nu(x,y) \in \mathbb{R}$ such that

1. $W^s_\varepsilon(\phi^{\nu(x,y)}x) \cap W^u_\varepsilon(y) \neq \emptyset$,
2. $|\nu(x,y)| < \varepsilon$.

Moreover, the set $W^s_\varepsilon(\phi^{\nu(x,y)}x) \cap W^u_\varepsilon(y)$ consists of a single point and this point lies in Λ. Denote this point by $\langle x, y\rangle$. Notice that for any $x \in \Lambda$, $\nu(x,x) = 0$ and $\langle x, x\rangle = x$. The maps $\nu : \Delta_\delta \to \mathbb{R}$, $\langle\cdot,\cdot\rangle : \Delta_\delta \to \Lambda$ on $\Delta_\delta := \{(x,y) \in \Lambda \times \Lambda \mid d(x,y) \le \delta\}$ are continuous (and so uniformly continuous as Δ_δ is compact).

One relates a hyperbolic flow to a symbolic flow by constructing the return map on a family of suitably nice local cross-sections, called a *Markov section*. The details of this construction are well-known [Bo], [BR] although somewhat complicated; we summarise them below.

A C^1-embedded codimension 1 closed disc $D \subset M$ is called a *local section* if for some $\xi > 0$ the map $D \times [-\xi, \xi] \to M : (x,t) \mapsto \phi^t x$ defines a homeomorphism onto its image $U_\xi(D)$, a neighbourhood of D in M (this is just the requirement that the flow is transverse to D). The map $\pi_D : U_\xi(D) \to D$ that projects a point in $U_\xi(D)$ along its orbit to D is differentiable. If $x, y \in \Lambda \cap D$, then $\langle x, y\rangle$ is defined and, provided ε is small enough, lies in $\Lambda \cap U_\xi(D)$. By the remarks above, $\pi_D\langle\cdot,\cdot\rangle$, when defined, is continuous.

Let D be a local section with small diameter. A subset $R \subset \operatorname{Int}(\Lambda \cap D)$ is called a *rectangle* if the following hold:

1. R is equal to the closure of its interior when viewed as a subset of $\Lambda \cap D$; in particular, ∂R is nowhere dense in $\Lambda \cap D$;
2. the diameter of R is small enough so that $\pi_D\langle\cdot,\cdot\rangle : R \times R \to \Lambda \cap D$ is defined; moreover, if $x, y \in R$, then $\pi_D\langle x, y\rangle \in R$.

Suppose $\mathcal{R} = \{R_1, \ldots, R_k\}$ is a finite collection of rectangles such that the corresponding D_i are pairwise disjoint. Define

$$\begin{aligned} \Gamma(\mathcal{R}) &= R_1 \cup \cdots \cup R_k, \\ \Gamma^\circ(\mathcal{R}) &= \operatorname{Int} R_1 \cup \cdots \cup \operatorname{Int} R_k, \\ \partial\mathcal{R} &= \partial R_1 \cup \cdots \cup \partial R_k, \end{aligned}$$

(here, interiors and boundaries are as subsets of $\Lambda \cap D_i$). Suppose

$$\Lambda = \bigcup_{t=0}^{T} \phi^{-t}\Gamma(\mathcal{R})$$

for some small $T > 0$. Then each orbit of ϕ^t crosses $\Gamma(\mathcal{R})$ infinitely often in both forward and backward time. For $x \in \Gamma(\mathcal{R})$, let $0 < \tau(x) \le T$ be the first return time of x. This induces a return map $H : \Gamma(\mathcal{R}) \to \Gamma(\mathcal{R})$ defined by $H(x) = \phi^{\tau(x)}(x)$. In general, the maps τ and H are not continuous.

In [Bo], [BR], it is shown how to construct a collection $\mathcal{R} = \{R_1, \ldots, R_k\}$ of rectangles satisfying all of the above conditions together with an additional Markov condition. This is then used to construct an aperiodic shift of finite type $\Sigma \subset \{1,\dots,k\}^{\mathbb{Z}}$ (with transition matrix A given by $A_{i,j}=1$ precisely when $H(\operatorname{Int} R_i)\cap \operatorname{Int} R_j \neq \emptyset$) such that the map $\pi : \Sigma \to \Gamma(\mathcal{R})$ defined by

$$\pi\left((x_j)_{j\in\mathbb{Z}}\right) = \bigcap_{j\in\mathbb{Z}} H^{-j} R_{x_j}$$

satisfies the following properties:

1. $\pi \in F_\theta(\Sigma,\Lambda)$ for some $0<\theta<1$,
2. $\pi\sigma = H\pi$,
3. π is bounded-to-one,
4. if $H^n x \notin \partial\mathcal{R}$ for all $n\in\mathbb{Z}$, then $\pi^{-1}\{x\}$ is a singleton.

Define, for $\pi(x)\in\Gamma(\mathcal{R})$ with orbit disjoint from $\partial\mathcal{R}$, $r(x)$ to be the first return time for x. Then it can be shown that r extends to a strictly positive $F_\theta(\Sigma,\mathbb{R})$-function for some θ. This allows us to construct a symbolic flow σ_r^t on $\widetilde{\Sigma_r}$ and a semi-conjugacy $\tilde{\pi}$ from σ_r^t to ϕ^t that is invertible on orbits that do not meet the boundary. In summary:

Theorem 2.1 ([Bo], [BR])**.** *There exists a strictly positive $r\in F_\theta(\Sigma,\mathbb{R})$ such that the map $\tilde{\pi} : \widetilde{\Sigma_r} \to \Lambda$ defined by $\tilde{\pi}(x,t) = \phi^t x$ satisfies:*

1. *$\tilde{\pi}\sigma_r^t = \phi^t\tilde{\pi}$;*
2. *$\tilde{\pi}$ is continuous;*
3. *$\tilde{\pi}$ is bounded-to-one;*
4. *if the orbit of x is disjoint from $\bigcup_{n=-\infty}^{\infty} H^n(\partial\mathcal{R})$, then $\tilde{\pi}^{-1}\{x\}$ is a singleton.*

If $f:\Lambda\to\mathbb{R}$ is Hölder continuous (of exponent α, say), then the equilibrium state of f is given by $m\tilde{\pi}^{-1}$ where m is the equilibrium state for $f\tilde{\pi}$ (which is well-defined since $\mathcal{I}(f\tilde{\pi})\in F_{\theta^\alpha}(\Sigma,\mathbb{R})$).

Remark 2.2. In fact, $\tilde{\pi}$ is Hölder continuous (in the sense of [Wa2]) with respect to the metric d_r^θ considered in [Wa2].

Remark 2.3. Observe that $\tilde{\pi}^{-1}\{x\}$ is a singleton if and only if $\tilde{\pi}^{-1}\{\phi^t x\}$ is a singleton for all $t\in\mathbb{R}$.

It will be important to understand how ϕ^t acts on the boundary of $\mathcal{R}$.

Lemma 2.4 ([Bo])**.** *The boundary $\partial\mathcal{R}$ of $\mathcal{R}$ can be decomposed $\partial\mathcal{R} = \partial^s\mathcal{R}\cup\partial^u\mathcal{R}$ such that $H(\partial^s\mathcal{R})\subset\partial^s\mathcal{R}$ and $H^{-1}(\partial^u\mathcal{R})\subset\partial^u\mathcal{R}$. Moreover, if we define*

$$\begin{aligned}\Delta^s\mathcal{R} &= \bigcup_{0\le t\le T}\phi^t(\partial^s\mathcal{R}),\\ \Delta^u\mathcal{R} &= \bigcup_{0\le t\le T}\phi^{-t}(\partial^u\mathcal{R}),\end{aligned}$$

then $\phi^t(\Delta^s\mathcal{R})\subset\Delta^s\mathcal{R}$, $\phi^{-t}(\Delta^u\mathcal{R})\subset\Delta^u\mathcal{R}$ for all $t\ge 0$.

Remark 2.5. Observe that $\Delta^u\mathcal{R}\cap\Gamma^\circ(\mathcal{R})=\emptyset$. If $z\in\Delta^u\mathcal{R}\cap\Gamma^\circ(\mathcal{R})$, then $\phi^t z\in\partial^u\mathcal{R}$ for some $t\ge 0$. Hence, $z = H^{-n}(\phi^t z)\in\partial^u\mathcal{R}\cap\Gamma^\circ(\mathcal{R})$ for some $n>0$, a contradiction. Similarly, $\Delta^s\mathcal{R}\cap\Gamma^\circ(\mathcal{R})=\emptyset$. However, for $t\ge 0$, $\phi^t(\partial^u\mathcal{R})$ can (and does) intersect $\Gamma^\circ(\mathcal{R})$, and similarly for the backwards iterates of $\partial^s\mathcal{R}$.

Remark 2.6. Since $\phi^t(\partial\mathcal{R})$ intersects $\Gamma^\circ(\mathcal{R})$ only countably many times and $\partial\mathcal{R}$ is nowhere dense, it follows from Theorem 2.1.4 and Remark 2.3 that $\tilde{\pi}^{-1}\{x\}$ is a singleton for a dense G_δ in Λ.

Let us call an orbit in Λ *good* if it never intersects $\partial\mathcal{R}$. The following is used in [PP2] in the case of a hyperbolic diffeomorphism.

Proposition 2.7. *There is a dense set of good periodic orbits in Λ that only intersect $\Gamma(\mathcal{R})$ in $\Gamma^\circ(\mathcal{R})$. In particular, if γ is such a periodic orbit and $x \in \gamma$, then there is a unique $\tilde{x} \in \widetilde{\Sigma_r}$ such that $\tilde{\pi}(\tilde{x}) = x$.*

Proof. Let γ be a periodic orbit in Λ and let $x \in \gamma \cap \Gamma(\mathcal{R})$. Then $H^n x = x$ for some $n > 0$. Suppose $x \in \partial\mathcal{R}$. If $x \in \partial^s\mathcal{R}$, then $Hx, \dots, H^{n-1}x \in \partial^s\mathcal{R}$ by Lemma 2.4. If $x \in \partial^u\mathcal{R}$, then $H^{-1}x = H^{n-1}x, \dots, H^{-(n-1)}x = Hx \in \partial^u\mathcal{R}$. In either case, $\{x, Hx, \dots, H^{n-1}x\} \subset \partial\mathcal{R}$. Hence, if $\gamma \cap \partial\mathcal{R} \neq \emptyset$, then γ only intersects $\Gamma(\mathcal{R})$ in $\partial\mathcal{R}$. Since $\partial\mathcal{R}$ is nowhere dense and the periodic orbits are dense in Λ, the claim follows. □

3. Shifts of finite type and their suspensions

In [PP2] the following is proved.

Proposition 3.1. *Let G be a compact Lie group and let $f, g \in F_\theta(\Sigma, G)$. Suppose $u : \Sigma \to G$ is measurable and satisfies $f = u\sigma \cdot g \cdot u^{-1}$ a.e. $[m]$ for some Hölder equilibrium state m. Then there exists $u' \in F_\theta(\Sigma, G)$ such that $u' = u$ a.e. and $f = u'\sigma \cdot g \cdot {u'}^{-1}$ everywhere.*

Remark 3.2. In fact, in [PP2] it is only proved that $u' \in F_{\beta^{\frac{1}{2}}}(G)$ where β denotes the maximum of θ and the Hölder exponent of the equilibrium state. To see that $u' \in F_\theta(G)$, we argue as follows. Let $x \in \Sigma$ and let $W^s_\varepsilon(x) = \{y \in \Sigma \mid y_i = x_i,\ i \geq 0\}$ denote the local stable 'manifold' through x. For $z \in W^s_\varepsilon(x)$ define $F^s(z, x) = \lim_{n\to\infty} f^n(z)^{-1} f^n(x)$ ($f^n = f\phi^{n-1} \cdots f\phi \cdot f$) and similarly define $G^s(z, x)$. Then one can easily check that $\rho(F^s(z, x), e) \leq \text{Const.}\, d^\theta(z, x)$ and $F^s(z, x) = u'(z) G^s(z, x) u'(x)^{-1}$ (ρ is an invariant metric on G). It follows that u' is Hölder continuous of exponent θ along stable 'manifolds'. The same is true for unstable 'manifolds'. It is then easy to see that $u' \in F_\theta(G)$. We shall use this kind of argument many times in the sequel.

We can now prove a regularity theorem for symbolic flows. Let $\mathcal{F} : \widetilde{\Sigma_r} \times \mathbb{R} \to G$ be a continuous, compact Lie group valued cocycle. Define

$$\mathcal{IF} : \Sigma \to G : x \mapsto \mathcal{F}((x, 0), r(x)).$$

(In the real-valued case we write $\mathcal{I}f(x) = \int_0^{r(x)} f\sigma_r^u(x, 0)\, du$ and similarly for the circle-valued case.)

Theorem 3.3. *Let $\mathcal{F}$ be a continuous cocycle and suppose $\mathcal{IF} \in F_\theta(\Sigma, G)$ for some $\theta \in (0, 1)$. Let m be a Hölder equilibrium state for σ_r^t. Suppose $\tilde{u} : \widetilde{\Sigma_r} \to G$ is a measurable solution to the cocycle equation*

$$\mathcal{F}(\tilde{x}, t) = \tilde{u}\sigma_r^t(\tilde{x})\mathcal{G}(\tilde{x}, t)\tilde{u}(\tilde{x})^{-1} \ \text{a.e. } [m]. \tag{3.1}$$

Then there exists a unique continuous $\tilde{u}' : \widetilde{\Sigma_r} \to G$ such that $\tilde{u}' = \tilde{u}$ a.e. and

$$\mathcal{F}(\tilde{x}, t) = \tilde{u}'\sigma_r^t(\tilde{x})\mathcal{G}(\tilde{x}, t)\tilde{u}'(\tilde{x})^{-1}$$

everywhere.

Proof. First note (c.f. [CFS, p. 13]) that we may take the set of measure zero on which (3.1) fails to hold to be independent of t.

Let $\tilde{x} = (x, s)$ and set $t = r(x)$ in (3.1) to obtain

$$\mathcal{F}((x,s), r(x)) = \tilde{u}\sigma_r^{r(x)}(x,s)\mathcal{G}((x,s), r(x))\tilde{u}(x,s)^{-1} \text{ a.e. } [m].$$

From the discussion in §2, $m = \mu \times l/\mu(r)$ for some Hölder equilibrium state μ for σ. By Fubini's Theorem, we can choose a 'good' small $s_0 \geq 0$ such that

$$\mathcal{F}((x,s_0), r(x)) = \tilde{u}\sigma_r^{r(x)}(x,s_0)\mathcal{G}((x,s_0), r(x))\tilde{u}(x,s_0)^{-1} \text{ a.e. } x. \tag{3.2}$$

Since $\mathcal{F}$ is a cocycle, one can easily check that

$$\mathcal{F}((x,s_0), r(x)) = \mathcal{F}((\sigma x, 0), s_0)\mathcal{F}((x,0), r(x))\mathcal{F}((x,0), s_0)^{-1},$$

and similarly for $\mathcal{G}$. Substituting this expression into (3.2) we obtain

$$\begin{aligned} \mathcal{IF}(x) &= \mathcal{F}((\sigma x, 0), s_0)^{-1}\tilde{u}(\sigma x, s_0)\mathcal{G}((\sigma x, 0), s_0) \\ &\quad \cdot \mathcal{IG}(x)\mathcal{G}((x,0), s_0)^{-1}\tilde{u}(x, s_0)^{-1}\mathcal{F}((x,0), s_0) \text{ a.e. } x. \end{aligned}$$

Let $v(x) = \mathcal{F}((x,0), s_0)^{-1}\tilde{u}(x, s_0)\mathcal{G}((x,0), s_0) : \Sigma \to G$. This is measurable and

$$\mathcal{IF}(x) = v\sigma(x)\mathcal{IG}(x)v(x)^{-1}$$

a.e. with respect to a Hölder equilibrium state for σ. By Proposition 3.1, there exists $v' : \Sigma \to G \in F_\theta(G)$ such that

$$\begin{aligned} v' &= v \text{ a.e.} \\ \mathcal{IF}(x) &= v'\sigma(x)\mathcal{IG}(x)v'(x)^{-1} \text{ everywhere.} \end{aligned} \tag{3.3}$$

Define $\tilde{u}'$ on $\Sigma \times \{s_0\}$ by $\tilde{u}'(x, s_0) = \mathcal{F}((x,0), s_0)v'(x)\mathcal{G}((x,0), s_0)^{-1}$. Then $\tilde{u}' = \tilde{u}$ a.e. on $\Sigma \times \{s_0\}$. Define $\tilde{u}'$ on $\widetilde{\Sigma_r}$ by

$$\tilde{u}'(x,s) = \mathcal{F}((x,0), s)\mathcal{F}((x,0), s_0)^{-1}\tilde{u}'(x, s_0)\mathcal{G}((x,0), s_0)\mathcal{G}((x,0), s)^{-1}.$$

We will show $\tilde{u}'$ is the required solution.

From (3.3) it follows that

$$\begin{aligned} \mathcal{IF}(x) &= \mathcal{F}((\sigma x, 0), s_0)^{-1}\tilde{u}'(\sigma x, s_0)\mathcal{G}((\sigma x, 0), s_0)\mathcal{IG}(x) \\ &\quad \cdot \mathcal{G}((x,0), s_0)^{-1}\tilde{u}'(x, s_0)^{-1}\mathcal{F}((x,0), s_0) \end{aligned}$$

for all $x \in \Sigma$. Then

$$\begin{aligned} \tilde{u}'(x, r(x)) &= \mathcal{IF}(x)\mathcal{F}((x,0), s_0)^{-1}\tilde{u}'(x, s_0)\mathcal{G}((x,0), s_0)\mathcal{IG}(x)^{-1} \\ &= \mathcal{F}((\sigma x, 0), s_0)^{-1}\tilde{u}'(\sigma x, s_0)\mathcal{G}((\sigma x, 0), s_0) \\ &= \tilde{u}'(\sigma x, 0). \end{aligned}$$

Hence $\tilde{u}'$ is a well-defined continuous function on $\widetilde{\Sigma_r}$.

To see that $\tilde{u}'$ is a solution to the cocycle equation, observe that

$$\begin{aligned} &\tilde{u}\sigma_r^t(x,s) \\ &= \mathcal{F}((x,0), s+t)\mathcal{F}((x,0), s_0)^{-1}\tilde{u}'(x, s_0)\mathcal{G}((x,0), s_0)\mathcal{G}((x,0), s+t)^{-1} \\ &= \mathcal{F}((x,s), t)\mathcal{F}((x,0), s)\mathcal{F}((x,0), s_0)^{-1}\tilde{u}'(x, s_0) \\ &\quad \cdot \mathcal{G}((x,0), s_0)\mathcal{G}((x,0), s_0)^{-1}\mathcal{G}((x,s), t)^{-1} \\ &= \mathcal{F}((x,s), t)\tilde{u}'(x,s)\mathcal{G}((x,s), t)^{-1} \end{aligned}$$

everywhere.

Finally, we show that $\tilde{u}' = \tilde{u}$ a.e. Recall that $\tilde{u}'(x, s_0) = \tilde{u}(x, s_0)$ a.e. on $\Sigma \times \{s_0\}$. For any $s \in \mathbb{R}$ we have

$$\begin{aligned}
\tilde{u}'(x, s + s_0) &= \tilde{u}'\sigma_r^s(x, s_0) \\
&= \mathcal{F}((x, s_0), s)\tilde{u}'(x, s_0)\mathcal{G}((x, s_0), s)^{-1} \\
&= \mathcal{F}((x, s_0), s)\tilde{u}(x, s_0)\mathcal{G}((x, s_0), s)^{-1} \text{ a.e.} \\
&= \tilde{u}(x, s + s_0) \text{ a.e.,}
\end{aligned}$$

proving the theorem. □

4. Livšic regularity theorems for hyperbolic flows

In this section we prove the regularity theorem for cocycles taking values in a compact connected Lie group over a hyperbolic flow ϕ^t.

Let G be such a group and let ρ be a left-right invariant Riemannian metric. Let $\mathcal{F}$ be a cocycle taking values in G. We say $\mathcal{F}$ is Hölder continuous of exponent $\alpha \in (0, 1)$ if the map

$$x \mapsto \lim_{t \to 0} \frac{1}{t} \exp^{-1} \mathcal{F}(x, t)$$

with values in the Lie algebra of G is Hölder continuous of exponent α. Equivalently, if we view G as a closed subgroup of a unitary group (as we may do), then we require the map

$$x \mapsto f(x) = \lim_{t \to 0} \frac{\mathcal{F}(x, t) - I}{t}$$

to be Hölder (I denotes the identity matrix).

Lemma 4.1. *Let $\mathcal{F}$ be a Hölder cocycle. Then*

1. $\rho(\mathcal{F}(x, t), \mathcal{F}(y, t)) \le \int_0^t \|f(\phi^u x) - f(\phi^u y)\| \, du$;
2. $\rho(\mathcal{F}(x, t), e) \le \int_0^t \|f(\phi^u x)\| \, du$.

Proof. Note that $\gamma : [0, t] \to G : u \mapsto \mathcal{F}(y, u)^{-1}\mathcal{F}(x, u)$ is a smooth curve from e to $\mathcal{F}(y, t)^{-1}\mathcal{F}(x, t)$. Then an easy calculation shows that

$$\gamma'(u) = \mathcal{F}(y, u)^{-1}(f(\phi^u x) - f(\phi^u y))\mathcal{F}(x, u)$$

whence

$$\|\gamma'(u)\| = \|f(\phi^u x) - f(\phi^u y)\|$$

as $\mathcal{F}$ is unitary-valued. The claim follows from the definition of the Riemannian metric and the left-right invariance of ρ. The second part is similar. □

Corollary 4.2. *For $z \in W_\varepsilon^s(x)$,*

$$F^s(z, x) = \lim_{t \to \infty} \mathcal{F}(z, t)^{-1}\mathcal{F}(x, t)$$

exists and satisfies $\rho(F^s(z, x), e) \le \mathrm{Const.}\, d(z, x)^\alpha$ where α is the Hölder exponent of $\mathcal{F}$ and the constant is independent of x, z.

Proof. Fix $t \geq 0$ and let $s > 0$. Then

$$\begin{aligned} &\rho(\mathcal{F}(z,t+s)^{-1}\mathcal{F}(x,t+s), \mathcal{F}(z,t)^{-1}\mathcal{F}(x,t)) \\ &\quad = \rho(\mathcal{F}(\phi^t z, s), \mathcal{F}(\phi^t x, s)) \\ &\quad \leq \|f\|_\alpha \int_0^t d(\phi^u(\phi^t z), \phi^u(\phi^t x))^\alpha \, du \\ &\quad \leq \text{Const.}\, d(\phi^t z, \phi^t x)^\alpha \\ &\quad \leq \text{Const.}\, d(z,x)^\alpha \lambda^{\alpha t} \end{aligned}$$

where the constant is independent of x, z and t. So, $\lim_{t\to\infty} \mathcal{F}(z,t)^{-1}\mathcal{F}(x,t)$ exists. Setting $t = 0$ in the above gives $\rho(F^s(z,x), e) \leq \text{Const.}\, d(z,x)^\alpha$. □

We can also define for $z \in W^u_\varepsilon(y)$, $F^u(z,y) = \lim_{t\to-\infty} \mathcal{F}(z,t)^{-1}\mathcal{F}(y,t)$. Again, $\rho(F^u(z,y), e) \leq \text{Const.}\, d(z,y)^\alpha$.

We can now prove our main result.

Theorem 4.3. *Let ϕ^t be a C^1 hyperbolic flow on a basic set Λ equipped with an equilibrium state corresponding to a Hölder continuous function. Let $\mathcal{F}$ and $\mathcal{G}$ be Hölder cocycles of exponent α. Suppose $u : \Lambda \to G$ is a measurable solution to*

$$\mathcal{F}(x,t) = u(\phi^t x)\mathcal{G}(x,t)u(x)^{-1} \text{ a.e. } [m]. \tag{4.1}$$

Then there exists a Hölder $u' \in C^\alpha(G)$ such that $u = u'$ a.e. and

$$\mathcal{F}(x,t) = u'(\phi^t x)\mathcal{G}(x,t)u'(x)^{-1}$$

everywhere.

Proof. Let $\mathcal{F}$, $\mathcal{G}$ and u be as in the statement of the theorem. Let $\tilde{\pi} : \widetilde{\Sigma_r} \to \Lambda$ be as in §2. Let $\tilde{u} = u \circ \tilde{\pi}$, $\tilde{\mathcal{F}}(\tilde{x},t) = \mathcal{F}(\tilde{\pi}\tilde{x},t)$, $\tilde{\mathcal{G}}(\tilde{x},t) = \mathcal{G}(\tilde{\pi}\tilde{x},t)$ and $\tilde{m} = m \circ \tilde{\pi}^{-1}$. Then $\tilde{u}$ is measurable, $\tilde{m}$ is a Hölder equilibrium state for σ^t_r and

$$\tilde{\mathcal{F}}(\tilde{x},t) = \tilde{u}\sigma^t_r(\tilde{x})\tilde{\mathcal{G}}(\tilde{x},t)\tilde{u}(\tilde{x})^{-1} \text{ a.e. } [\tilde{m}].$$

Moreover, for $x, y \in \Sigma$, we have

$$\begin{aligned} &\rho(\mathcal{I}\tilde{\mathcal{F}}(x), \mathcal{I}\tilde{\mathcal{F}}(y)) \\ &\quad = \rho(\mathcal{F}(\tilde{\pi}(x,0), r(x)), \mathcal{F}(\tilde{\pi}(y,0), r(y))) \\ &\quad \leq \rho(\mathcal{F}(\tilde{\pi}(x,0), r(x)), \mathcal{F}(\tilde{\pi}(x,0), r(y))) \\ &\qquad + \rho(\mathcal{F}(\tilde{\pi}(x,0), r(y)), \mathcal{F}(\tilde{\pi}(y,0), r(y))) \\ &\quad = \rho(\mathcal{F}(\phi^{r(y)}\tilde{\pi}(x,0), r(x) - r(y)), e) + \rho(\mathcal{F}(\tilde{\pi}(x,0), r(y)), \mathcal{F}(\tilde{\pi}(y,0), r(y))) \\ &\quad \leq \|f\|_\alpha \|r\|_\theta d^\theta(x,y) + \|f\|_\alpha \int_0^{r(y)} d(\phi^u\tilde{\pi}(x,0), \phi^u\tilde{\pi}(y,0))^\alpha \, du \\ &\quad \leq \text{Const.}\, d^\theta(x,y) + \text{Const.}\, d^{\theta^\alpha}(x,y) \end{aligned}$$

by Lemma 4.1, where the constants are independent of x, y. Hence $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$ satisfy the hypotheses of Theorem 3.3. By Theorem 3.3, $\tilde{u} = \tilde{u}'$ a.e. for some continuous $\tilde{u}' : \widetilde{\Sigma_r} \to G$ and

$$\tilde{\mathcal{F}}(\tilde{x},t) = \tilde{u}'\sigma^t_r(\tilde{x})\tilde{\mathcal{G}}(\tilde{x},t)\tilde{u}'(\tilde{x})^{-1}$$

everywhere. We shall show that $\tilde{u}' = u' \circ \pi$ for some continuous $u' : \Lambda \to G$. It then follows that (4.1) holds everywhere when u is replaced by u'.

Let γ be a 'good' periodic orbit. Then for each $x \in \gamma$, there exists a unique $\tilde{x} \in \widetilde{\Sigma_r}$ such that $\tilde{\pi}\tilde{x} = x$. If we define u' on γ by $u'(x) = \tilde{u}'(\tilde{x})$, then (4.1) holds everywhere on the dense set of 'good' periodic orbits. It remains to show that u' can be extended to Λ.

Let γ_1 and γ_2 be two 'good' periodic orbits with least periods τ_1, τ_2, respectively. Let $x \in \gamma_1$, $y \in \gamma_2$ be sufficiently close so that $z = \langle x, y\rangle = W^s_\varepsilon(\phi^{\nu(x,y)}x) \cap W^u_\varepsilon(y)$ is defined. We claim that the orbit of z is 'good'; it will then follow from Theorem 2.1 that $z = \tilde\pi\tilde z$ for a unique $\tilde z \in \widetilde{\Sigma_r}$.

Since $z \in W^u_\varepsilon(y)$, we have $d(\phi^{-t}z, \phi^{-t}y) \to 0$ as $t \to \infty$. Choose $t_y > 0$ such that $y' = \phi^{-t_y}y \in \operatorname{Int} R \subset \Gamma^\circ(\mathcal{R})$ for some $R \in \mathcal{R}$. Then $d(\phi^{-n\tau_2 - t_y}z, y') \to 0$ as $n \to \infty$. Thus, for all sufficiently large n, $\phi^{-n\tau_2-t_y}z \in U_\xi(D)$. Since π_D is continuous, $\pi_D\phi^{-n\tau_2-t_y}z \to \pi_D y' = y'$. It follows that $\pi_D\phi^{-n\tau_2-t_y}z \notin \partial\mathcal{R}$ if n is sufficiently large. Hence there exists a sequence of times $t_n^- \downarrow -\infty$ such that $\phi^{t_n^-}z \notin \partial\mathcal{R}$. Similarly, by replacing y by $\phi^{\nu(x,y)}x$ and reversing time, we can construct a sequence of times $t_n^+ \uparrow \infty$ such that $\phi^{t_n^+}z \notin \partial\mathcal{R}$. Suppose for a contradiction that, for some t, $\phi^t z \cap \partial\mathcal{R} \neq \emptyset$. If $\phi^t z \in \partial^s\mathcal{R}$, then, by Lemma 2.4, the forward orbit (under H) of z would remain in $\partial^s\mathcal{R}$, contradicting $\phi^{t_n^+}z \notin \partial\mathcal{R}$. Similarly, $\phi^t z \notin \partial^u\mathcal{R}$. By Theorem 2.1.4, the orbit of z is 'good'.

Define $u'(z) = \tilde u'(\tilde z)$ and extend u' along the orbit of z. Then (4.1) holds for u' along the orbit of z.

We have

$$\begin{aligned} &\mathcal{F}(z,t)^{-1}\mathcal{F}(\phi^{\nu(x,y)}x, t) \\ (4.2) \qquad &= \; u'(z)\mathcal{G}(z,t)^{-1}u'(\phi^t z)^{-1}u'(\phi^{t+\nu(x,y)}x)\mathcal{G}(\phi^{\nu(x,y)}x, t)u'(\phi^{\nu(x,y)}x)^{-1}. \end{aligned}$$

Now $\lim_{t\to\infty} u'(\phi^t z)^{-1}u'(\phi^{t+\nu(x,y)}x) = \lim_{t\to\infty} \tilde u'\sigma_r^t(\tilde\pi^{-1}z)^{-1}\tilde u'\sigma_r^{t+\nu(x,y)}(\tilde\pi^{-1}x) = 0$, as $\tilde\pi^{-1}z \in W^s_\varepsilon(\sigma_r^{\nu(x,y)}\tilde\pi^{-1}x)$ and $\tilde u'$ is continuous. Hence letting $t \to \infty$ in (4.2) we see that

$$F^s(z, \phi^{\nu(x,y)}x) = u'(z)G^s(z, \phi^{\nu(x,y)}x)u'(\phi^{\nu(x,y)}x)^{-1}.$$

Similarly,

$$F^u(z,y) = u'(z)G^u(z,y)u'(y)^{-1}.$$

Hence,

$$u'(\phi^{\nu(x,y)}x) = F^s(z,\phi^{\nu(x,y)}x)^{-1}F^u(z,y)u'(y)G^u(z,y)^{-1}G^s(z,\phi^{\nu(x,y)}x).$$

By the left-right invariance of ρ and Corollary 4.2 we have

$$\begin{aligned} \rho(u'(\phi^{\nu(x,y)}x), u'(y)) \quad &\leq \quad \rho(F^s(z,\phi^{\nu(x,y)}x), e) + \rho(F^u(z,y), e) \\ &\qquad + \rho(G^s(z,\phi^{\nu(x,y)}x), e) + \rho(G^u(z,y), e) \\ &\leq \quad \text{Const.}\,(d(z,\phi^{\nu(x,y)}x)^\alpha + d(z,y)^\alpha) \end{aligned}$$

for some constant independent of x, y and z.

Hence,

$$\begin{aligned} \rho(u(x), u(y)) \quad &\leq \quad \rho(\tilde u'\tilde\pi^{-1}(x), \tilde u'\sigma_r^{\nu(x,y)}\tilde\pi^{-1}(x)) \\ (4.3) \qquad &\qquad + \text{Const.}\,(d(z,\phi^{\nu(x,y)}x)^\alpha + d(z,y)^\alpha). \end{aligned}$$

Since both $\tilde u$ and ν are uniformly continuous, it follows that u is uniformly continuous on the dense set of good periodic points. Therefore, u' extends uniquely to a continuous function $u' : \Lambda \to G$.

Since $\tilde\pi$ is one-to-one a.e., $u' = u$ a.e. Moreover, u' solves (4.1) everywhere.

It remains to check that u' has the same degree of Hölder regularity as the cocycles $\mathcal{F}$, $\mathcal{G}$. In the case of a coboundary ($\mathcal{F}(x,t) = u(\phi^t x)u(x)^{-1}$) this is clear: a periodic data criterion gives the required degree of regularity on u. For the general case, we argue as follows. From the estimates above, it follows that if $z \in W^s_\varepsilon(x)$, then

$$F^s(z,x) = u'(z)G^s(z,x)u'(x)^{-1}.$$

Hence

$$\begin{aligned}\rho(u'(x), u'(z)) &\leq \rho(F^s(z,x), e) + \rho(G^s(z,x), e) \\ &\leq \text{Const.}\, d(z,x)^\alpha\end{aligned}$$

so that u' is Hölder continuous along stable manifolds. Similarly, u' is Hölder continuous along unstable manifolds. Along orbits it is easy to see, using flow-box co-ordinates, that for sufficiently small t,

$$\begin{aligned}\rho(u'(\phi^t x), u'(x)) &\leq \rho(\mathcal{F}(x,t), e) + \rho(\mathcal{G}(x,t), e) \\ &\leq \text{Const.}\, |t|^\alpha \\ &\leq \text{Const.}\, d(x, \phi^t x)^\alpha,\end{aligned}$$

for some constant independent of x and t. Hence u' is uniformly Hölder continuous along the leaves of three foliations whose leaves are uniformly transverse, continuously varying Lipschitz submanifolds. Proposition 19.1.1 of [KH] then shows that u' is Hölder on Λ. □

Remark 4.4. The above proof resembles the proof of the corresponding result for hyperbolic diffeomorphisms given in [PP2], taking into account the extra complications that arise from flows. For a hyperbolic diffeomorphism, the local product structure is given by $\langle x, y\rangle = W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$ so there is no term $\nu(x,y)$ in (4.3). In particular, we can deduce that u' is Hölder continuous from (4.3) without any extra argument.

5. More general groups

We consider some classes of non-compact Lie groups.

5.1. **Abelian Lie groups.** We show how to deduce the regularity theorem for real-valued cocycles from the circle-valued case. For notational convenience, we only consider diffeomorphisms.

Theorem 5.1. *Let ϕ be a hyperbolic diffeomorphism on Λ. Let m be an equilibrium state corresponding to a Hölder continuous function. Suppose $f : \Lambda \to \mathbb{R} \in C^\alpha(\Lambda, \mathbb{R})$ and u is a measurable solution to $f = u\phi - u$ a.e. $[m]$. Then there exists $u' \in C^\alpha(\Lambda, \mathbb{R})$ such that $u = u'$ a.e. and $f = u'\phi - u'$ everywhere.*

Proof. Let $\mu \in \mathbb{R}$. Then

$$\exp 2\pi i\mu f = \frac{\exp 2\pi i\mu u\phi}{\exp 2\pi i\mu u}.$$

By the regularity theorem for circle-valued cocycles, there exists a continuous map $u_\mu : \Lambda \to K$ such that

$$\exp 2\pi i\mu f = u_\mu\phi/u_\mu$$

everywhere. Suppose $\phi^n x = x$. Then $\exp 2\pi i\mu f^n(x) = 1$ ($f^n = \sum_{i=0}^{n-1} f\phi^i$). Hence, $\mu f^n(x)$ is an integer for each $\mu \in \mathbb{R}$. It follows that $f^n(x) = 0$ whenever $\phi^n x = x$. By the periodic data criterion for real-valued cocycles, $f = u'\phi - u'$ for some $u' \in C^\alpha(\Lambda, \mathbb{R})$. By ergodicity, we may choose u' so that $u = u'$ a.e. □

Remark 5.2. Hence we have the regularity theorem for cocycles taking values in an arbitrary connected abelian Lie group.

Remark 5.3. Livšic proved Theorem 5.1 in [Li2] for C^2 Anosov diffeomorphisms and flows preserving a smooth measure.

Remark 5.4. If u is assumed to be essentially bounded, then it is easy to deduce from the periodic data criterion that u must be Hölder continuous [Li1].

5.2. **Semi-direct products.** For notational simplicity, we only consider the case of a hyperbolic diffeomorphism.

Theorem 5.5. *Let G_1, G_2 be Lie groups and suppose that G_1 acts smoothly on G_2 by automorphisms. Let ϕ be a hyperbolic diffeomorphism and suppose the regularity theorem holds for coboundaries taking values in G_1 and G_2. Then the regularity theorem continues to hold for coboundaries taking values in the semi-direct product $G_1 \ltimes G_2$.*

Proof. Let $\alpha : G_1 \to \mathrm{Aut}(G_2)$ be the G_1-action on G_2. Then the group structure on $G_1 \ltimes G_2$ is given by $(g_1, g_2)(g_1', g_2') = (g_1 g_1', g_2 \cdot \alpha_{g_1} g_2')$.

Let $f = (f_1, f_2) : \Lambda \to G_1 \ltimes G_2$ be Hölder continuous and suppose $u = (u_1, u_2)$ is a measurable function such that

$$(f_1, f_2) = (u_1\phi, u_2\phi) \cdot (u_1, u_2)^{-1} \text{ a.e.} \tag{5.1}$$

Then (5.1) is equivalent to

$$f_1 = u_1\phi \cdot u_1^{-1} \text{ a.e.} \tag{5.2}$$

$$f_2 = u_2\phi \cdot \alpha_{u_1\phi \cdot u_1^{-1}} u_2^{-1} \text{ a.e.} \tag{5.3}$$

Since the regularity theorem holds for G_1, there exists a Hölder $u_1' : \Lambda \to G_1$ such that $u_1 = u_1'$ a.e. and $f_1 = u_1'\phi \cdot u_1'$ everywhere. Then $\alpha_{(u_1'\phi)^{-1}} f_2$ is Hölder continuous and

$$\alpha_{(u_1'\phi)^{-1}} f_2 = \alpha_{(u_1'\phi)^{-1}} u_2\phi \cdot \alpha_{u_1'^{-1}} u_2^{-1} \text{ a.e.} \tag{5.4}$$

Now $x \mapsto \alpha_{u_1'(x)^{-1}} u_2(x)$ is measurable and, since the regularity theorem holds for G_2, it is equal a.e. to a Hölder continuous function for which (5.4) holds everywhere. Hence, $u_2 = u_2'$ a.e. for some Hölder continuous u_2' for which (5.3) holds everywhere. □

Remark 5.6. So the regularity theorem holds for coboundaries taking values in subgroups of $O(d) \ltimes \mathbb{R}^d$, the 'orthogonal affine' transformations of $\mathbb{R}^d$. For more general Lie subgroups of $Gl(n)$ and C^2 Anosov diffeomorphisms, the regularity theorem for coboundaries continues to hold, provided the cobounding function and its inverse are assumed to be essentially bounded [NP].

Remark 5.7. For the general cocycle equation (1.1) the regularity theorem may fail for (non-compact) soluble Lie groups and semi-direct products [Wa1].

6. Higher regularity results

Let us now briefly address question 3 in the Introduction. We have already seen that if $\mathcal{F}, \mathcal{G}$ are Hölder cocycles such that $\mathcal{F}(x,t) = u(\phi^t x)\mathcal{G}(x,t)u(x)^{-1}$ a.e., then u has a continuous version for which this equation holds everywhere.

Now let us suppose that $\mathcal{F}$ and $\mathcal{G}$ possess a higher degree of regularity. Let $1 \leq r \leq \infty$ or $r = \omega$ and let ϕ^t be a C^r Anosov flow on a manifold M. We say that $\mathcal{F}$ is a C^r cocycle if the map $\mathcal{F} : M \times \mathbb{R} \to G$ is C^r.

In [Li1], [Li2], Livšic considered the C^1 case for abelian Lie groups (see also [KH, §19.2] for the details of this). He also considered $C^{r+\alpha}$, C^∞ and C^ω versions in some special cases. The C^∞ case was completely solved in [LMM] (it had previously been considered for geodesic flows on negative curved surfaces in [GK]) and since then two other proofs have been given [Jo], [HK]. In [Ll2], de la Llave used the method of [HK] to deduce the C^ω case. Building on earlier work in [NT1] for coboundaries, Niţică and Török proved in [NT2] the regularity theorem for cocycle equations taking values in any matrix group and in the diffeomorphism group of any compact manifold. Combining the results of [NT2] with Theorems 4.3 and 5.1, we have

Theorem 6.1. *Let ϕ^t be a C^r transitive Anosov flow on a compact manifold M ($1 \leq r \leq \infty$ or $r = \omega$). Let G be a compact Lie group or an abelian Lie group and let $\mathcal{F}, \mathcal{G} : M \times \mathbb{R} \to G$ be two C^r cocycles. Let m be the equilibrium state corresponding to a Hölder continuous function. Suppose $u : M \to G$ is a measurable solution to*

$$\mathcal{F}(x,t) = u(\phi^t x)\mathcal{G}(x,t)u(x)^{-1} \text{ a.e. } [m].$$

Then there exists a continuous $u' : M \to G$ such that $u = u'$ a.e. and $\mathcal{F}(x,t) = u'(\phi^t x)\mathcal{G}(x,t)u'(x)^{-1}$ everywhere. Moreover, u' is $C^{r-1+\alpha}$ for any $\alpha \in [0,1)$. (We define $r - 1 + \alpha = r$ for $r = 1, \infty, \omega$.)

Counterexamples [Ll1], [NT2] show that, in general, we cannot assume that u' is C^r.

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