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RANDOM INTERSECTIONS OF THICK CANTOR SETS

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ABSTRACT. Let C_1, C_2 be Cantor sets embedded in the real line, and let τ_1, τ_2 be their respective thicknesses. If $\tau_1\tau_2 > 1$, then it is well known that the difference set $C_1 - C_2$ is a disjoint union of closed intervals. B. Williams showed that for some $t \in \operatorname{int}(C_1 - C_2)$, it may be that $C_1 \cap (C_2 + t)$ is as small as a single point. However, the author previously showed that generically, the other extreme is true; $C_1 \cap (C_2 + t)$ contains a Cantor set for all t in a generic subset of $C_1 - C_2$. This paper shows that small intersections of thick Cantor sets are also rare in the sense of Lebesgue measure; if $\tau_1\tau_2 > 1$, then $C_1 \cap (C_2 + t)$ contains a Cantor set for almost all t in $C_1 - C_2$.

If C_1, C_2 are Cantor sets embedded in the real line, then their difference set is

$$C_1 - C_2 \equiv \{\, x - y \mid x \in C_1 \text{ and } y \in C_2 \,\}.$$

The difference set has another, more dynamical, definition as

$$C_1 - C_2 = \{\, t \mid C_1 \cap (C_2 + t) \neq \emptyset \,\},$$

where $C_2 + t = \{\, x + t \mid x \in C_2 \,\}$ is the translation of C_2 by the amount t. There are two reasons to say that the second definition is dynamical. First, it gives a dynamic way of visualizing the difference set; if we think of C_1 as being fixed in the real line and think of C_2 as sliding across C_1 with unit speed, then $C_1 - C_2$ can be thought of as giving those times when the moving copy of C_2 intersects C_1. Second, it has become a tool for studying dynamical systems. One Cantor set sliding over another one comes up in various studies of homoclinic phenomena, such as infinitely many sinks, [N1], antimonotonicity, [KKY], and Ω-explosions, [PT1]; for an elementary explanation of this, see [GH, pp. 331–342] or [R, pp. 110–115]. This has led to a number of problems and results of the following form: Given conditions on the sizes of C_1 and C_2, what can be said of the sizes of either $C_1 - C_2$, or $C_1 \cap (C_2 + t)$ for $t \in C_1 - C_2$. A wide variety of notions of size have been used, such as cardinality, topology, measure, Hausdorff dimension, limit capacity, and thickness; see for example [HKY], [KP], [MO], [PT2], [PS], [S], and [W]. In this paper we will be concerned with the thickness of C_1 and C_2, and our conclusion will be about the topology of $C_1 \cap (C_2 + t)$ for almost every $t \in C_1 - C_2$.

It is not hard to show that the difference set of two Cantor sets C_1, C_2 is always a compact, perfect set. So the simplest structure that we can expect $C_1 - C_2$ to have is the disjoint union of closed intervals. There is a condition we can put on C_1 and C_2 that will guarantee this; if τ_1, τ_2 are the thicknesses of C_1, C_2, and

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if $\tau_1\tau_2 > 1$, then $C_1 - C_2$ is a disjoint union of closed intervals. What about the size of $C_1 \cap (C_2 + t)$ for $t \in C_1 - C_2$? In [W] it was shown that even when $\tau_1\tau_2 > 1$, it is possible that $C_1 \cap (C_2 + t)$ can be as small as a single point for some $t \in \operatorname{int}(C_1 - C_2)$. But in [K1, Chapter 3], it was shown that this is exceptional, at least in the sense of category, and that in fact the other extreme is the case; if $\tau_1\tau_2 > 1$, then $C_1 \cap (C_2 + t)$ contains a Cantor set for all t in a generic subset of $C_1 - C_2$. Our main result in this paper is to prove a similar result for Lebesgue measure.

Theorem 1. *Let C_1, C_2 be Cantor sets embedded in the real line and let τ_1, τ_2 be their respective thicknesses. If $\tau_1\tau_2 > 1$, then $C_1 \cap (C_2 + t)$ contains a Cantor set for almost all $t \in C_1 - C_2$.*

It is worth mentioning here that, in [W], [HKY], and [K1], conditions are given on τ_1 and τ_2 so that $C_1 \cap (C_2 + t)$ contains a Cantor set for *all* $t \in \operatorname{int}(C_1 - C_2)$.

Before proving Theorem 1, let us look at the definition of thickness and see how it is used. If C is a Cantor set embedded in the real line, then the complement of C is a disjoint union of open intervals. We call the components of the complement of C the *gaps* of C. Let $\{U_n\}_{n=1}^{\infty}$ be an ordering of the bounded gaps of C by decreasing length, so $|U_{n+1}| \le |U_n|$, where $|U|$ denotes the Lebesgue measure of U. Let I_1 denote the smallest closed interval containing C. For $n > 1$, let $I_n = I_1 \setminus (\bigcup_{i=1}^{n-1} U_i)$. Note that I_n has n components. Let A_n denote the component of I_n that contains U_n. Let L_n and R_n denote the left and right components of $A_n \setminus U_n$. Then the *thickness* τ of C is defined by

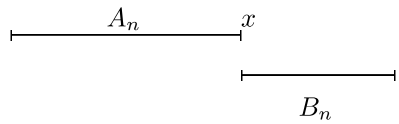
$$\tau(C) \equiv \inf_n \left\{ \min \left\{ \frac{|L_n|}{|U_n|}, \frac{|R_n|}{|U_n|} \right\} \right\}.$$

This definition of thickness is from [W]; in both [W] and [K1, pp. 15–16] it is shown that (i) this definition does not depend on the choice of an ordering for the gaps of C in the case when $|U_{n+1}| = |U_n|$ for some n, and (ii) this definition is equivalent to the usual definition of thickness (e.g., [N2, pp. 99–100]).

Thickness gives us a way of measuring the size of Cantor sets embedded in the real line. The larger the thickness, the "bigger" the Cantor set. So for example, as a consequence of the next lemma the condition $\tau_1\tau_2 > 1$ implies that C_1 and C_2 are big enough that their difference set is large in the sense that $C_1 - C_2$ is a disjoint union of closed intervals.

Lemma 2. *Let C_1, C_2 be Cantor sets embedded in the real line, with thicknesses τ_1, τ_2. If $\tau_1\tau_2 > 1$ and neither C_1 nor C_2 is contained in a gap of the other, then $C_1 \cap C_2 \neq \emptyset$.*

This lemma is often referred to as the Gap Lemma, [PT2, p. 63]. There is a slightly stronger version of the Gap Lemma that uses the notion of an overlapped point in the intersection of two Cantor sets. This is a simple, but useful, definition from [K1, pp. 17–18]. Suppose that $x \in C_1 \cap C_2$. Let $\{U_n\}_{n=1}^{\infty}$ and $\{V_n\}_{n=1}^{\infty}$ denote the bounded gaps, and let I_1, J_1 denote the convex hulls, of C_1 and C_2. Let A_n and B_n denote the components of $I_1 \setminus (\bigcup_{i=1}^{n-1} U_i)$ and $J_1 \setminus (\bigcup_{i=1}^{n-1} V_i)$, respectively, that contain x. Then x is an *overlapped point* from $C_1 \cap C_2$ if $A_n \cap B_n$ has nonempty interior for all n. To put this another way, if $x \in C_1 \cap C_2$, then x is *not* an overlapped point if and only if there is an n such that $A_n \cap B_n = \{x\}$, i.e., A_n and B_n look like the following picture.

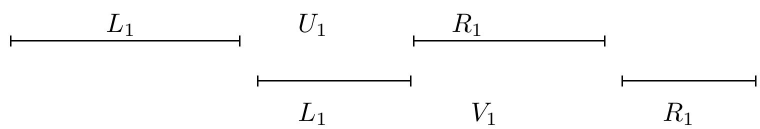


Now we can state the slightly stronger version of the Gap Lemma.

Lemma 3. *Let C_1, C_2 be Cantor sets embedded in the real line, with thicknesses τ_1, τ_2. If $\tau_1\tau_2 > 1$ and neither C_1 nor C_2 is contained in the closure of a gap of the other, then $C_1 \cap C_2$ contains an overlapped point.*

This version of the Gap Lemma implies that $C_1 - C_2$ is a disjoint union of closed intervals, and that $C_1 \cap (C_2 + t)$ contains an overlapped point for all $t \in \operatorname{int}(C_1 - C_2)$. It is not hard to see that $C_1 \cap (C_2 + t)$ contains only non-overlapped points when t is a boundary point of $C_1 - C_2$. We say that Cantor sets C_1 and C_2 are *interweaved* if neither C_1 nor C_2 is contained in the closure of a gap of the other.

Here is a sketch of the proof of the Gap Lemma. Let $\{U_n\}_{n=1}^{\infty}$ and $\{V_n\}_{n=1}^{\infty}$ denote the bounded gaps, and let I_1, J_1 denote the convex hulls, of C_1 and C_2, respectively. The key idea is that, since $\tau_1\tau_2 > 1$, we cannot have the following picture of $I_1 \setminus U_1$ and $J_1 \setminus V_1$.



So it must be that the intersection of $I_1 \setminus U_1$ and $J_1 \setminus V_1$ has nonempty interior. A careful induction argument, based on the above idea, gives that the intersection of $I_1 \setminus (\bigcup_{i=1}^{n} U_i)$ and $J_1 \setminus (\bigcup_{i=1}^{n} V_i)$ has nonempty interior for all $n > 1$; this implies that $C_1 \cap C_2$ contains an overlapped point. Notice that if the hypothesis $\tau_1\tau_2 > 1$ is replaced with $\tau_1\tau_2 \geq 1$, then we can still conclude that $C_1 \cap C_2 \neq \emptyset$, but we cannot conclude that $C_1 \cap C_2$ contains an overlapped point.

If C is a Cantor set embedded in the real line, then the components of each I_n are called the *bridges* of C; if B is any bridge of C, then $B \cap C$ is called a *segment* of C. Clearly any segment of C is also a Cantor set. As a consequence of the definition of thickness, we have the following simple lemma, [K1, p. 16], which will allow us to apply the Gap Lemma "locally."

Lemma 4. *Let C be a Cantor set embedded in the real line with thicknesses τ. If C' is any segment of C, then the thickness of C' is greater than or equal to τ.*

The main result we need in order to prove Theorem 1 is the following lemma, which at first glance seems to be only slightly stronger than the Gap Lemma.

Lemma 5. *Let C_1, C_2 be Cantor sets embedded in the real line, with thicknesses τ_1, τ_2. If $\tau_1\tau_2 > 1$, then $C_1 \cap (C_2 + t)$ contains at least two overlapped points for almost all $t \in C_1 - C_2$.*

Before proving this lemma, let us see how it is used to prove Theorem 1.

Proof of Theorem 1. Let $\{C_{1,n}\}_{n=1}^{\infty}$ be any ordering of all the segments of C_1, and let $\{C_{2,n}\}_{n=1}^{\infty}$ be any ordering of all the segments of C_2. Then, by Lemmas 4 and

5, for any i and j there is a set $E_{ij} \subset C_{1,i} - C_{2,j}$ of measure zero, such that $C_{1,i} \cap (C_{2,j} + t)$ contains at least two overlapped points for all $t \in (C_{1,i} - C_{2,j}) \setminus E_{ij}$. Let $E \equiv \bigcup_{i,j} E_{ij}$. So then E has measure zero, and $E \subset C_1 - C_2$.

Using terminology from [K1, p. 20], if $t \in (C_1 - C_2) \setminus E$, then $C_1 \cap (C_2 + t)$ has no *isolated overlapped points*. An overlapped point is isolated if there is a neighborhood of it which contains no other overlapped points. In [K1, pp. 20–21] it is shown that if the intersection of two Cantor sets does not contain isolated overlapped points, then the intersection must contain a Cantor set. But here we will sketch a proof that if $t \in (C_1 - C_2) \setminus E$, then $C_1 \cap (C_2 + t)$ contains a Cantor set.

Let $t \in (C_1 - C_2) \setminus E$. Then $C_1 \cap (C_2 + t)$ contains at least two overlapped points, so let x, y be distinct overlapped points in $C_1 \cap (C_2 + t)$. Choose integers i_1, j_1 large enough so that x and y are in distinct components of I_{i_1} and $J_{j_1} + t$. Let $K_1 = K_{1,1} \cup K_{1,2}$ denote the two components of I_{i_1} that contain x and y, and let $L_1 = L_{1,1} \cup L_{1,2}$ denote the two components of $J_{j_1} + t$ that contain x and y. Since $t \in (C_1 - C_2) \setminus E$, $K_{1,1} \cap L_{1,1}$ contains at least two overlapped points from $C_1 \cap (C_2 + t)$, and so does $K_{1,2} \cap L_{1,2}$. Now choose integers $i_2 > i_1$ and $j_2 > j_1$ large enough so that these four overlapped points are in distinct components of I_{i_2} and $J_{j_2} + t$, and let $K_2 = \bigcup_{\nu=1}^{4} K_{2,\nu}$ and $L_2 = \bigcup_{\nu=1}^{4} L_{2,\nu}$ denote these components. In general, suppose we are given integers i_n and j_n, and 2^n distinct components $K_n = \bigcup_{\nu=1}^{2^n} K_{n,\nu}$ from I_{i_n}, and 2^n distinct components $L_n = \bigcup_{\nu=1}^{2^n} L_{n,\nu}$ from $J_{j_n} + t$, such that each of $K_{n,\nu} \cap L_{n,\nu}$ contains an overlapped point from $C_1 \cap (C_2 + t)$. Then, since $t \in (C_1 - C_2) \setminus E$, each of $K_{n,\nu} \cap L_{n,\nu}$ actually contains two overlapped points from $C_1 \cap (C_2 + t)$. So we can choose integers $i_{n+1} > i_n$ and $j_{n+1} > j_n$ large enough so that these 2^{n+1} overlapped points are contained in 2^{n+1} distinct components $K_{n+1} = \bigcup_{\nu=1}^{2^{n+1}} K_{n+1,\nu}$ from $I_{i_{(n+1)}}$, and $L_{n+1} = \bigcup_{\nu=1}^{2^{n+1}} L_{n+1,\nu}$ from $J_{j_{(n+1)}} + t$. So for every $n \geq 1$, the set $K_n \cap L_n$ has 2^n components, $(K_n \cap L_n) \subset (I_{i_n} \cap J_{j_n})$, and $(K_{n+1} \cap L_{n+1}) \subset (K_n \cap L_n)$. Finally, the set $\bigcap_{n=1}^{\infty} (K_n \cap L_n)$ is a Cantor set contained in $C_1 \cap (C_2 + t)$.

Now we shall begin working on the proof of Lemma 5. For Cantor sets C_1, C_2 with thicknesses τ_1, τ_2, and $\tau_1 \tau_2 > 1$, let

$$\mathcal{O} \equiv \{\, t \mid C_1 \cap (C_2 + t) \text{ contains exactly one overlapped point} \,\},$$

and

$$\mathcal{T} \equiv \{\, t \mid C_1 \cap (C_2 + t) \text{ contains two or more overlapped points} \,\}.$$

Notice that $\mathcal{O} \cap \mathcal{T} = \emptyset$, and $\mathcal{O} \cup \mathcal{T} = C_1 - C_2$ up to a set of measure zero (in fact $(C_1 - C_2) \setminus (\mathcal{O} \cup \mathcal{T})$ is a countable set). To prove Lemma 5, we need to show that $\mathcal{O}$ has measure zero. To do this, it helps to make a distinction between three kinds of overlapped points. Suppose that $x \in C_1 \cap C_2$ is an overlapped point. Let A_n and B_n denote the components of $I_1 \setminus (\bigcup_{i=1}^{n-1} U_i)$ and $J_1 \setminus (\bigcup_{i=1}^{n-1} V_i)$, that contain x (where $\{U_n\}_{n=1}^{\infty}$ and $\{V_n\}_{n=1}^{\infty}$ denote the bounded gaps, and I_1, J_1 denote the convex hulls, of C_1, C_2). Then x is an *overlapped point of the first, second, or third kind*, respectively, if one of the following three conditions holds, respectively;

1. $x \in \operatorname{int}(A_n)$ and $x \in \operatorname{int}(B_n)$ for all n,
2. $x \in \operatorname{int}(A_n)$ for all n and there is an n such that x is an endpoint of B_n, or $x \in \operatorname{int}(B_n)$ for all n and there is an n such that x is an endpoint of A_n,
3. there is an n such that x is an endpoint of both A_n and B_n, and $A_n \cap B_n \neq \{x\}$.

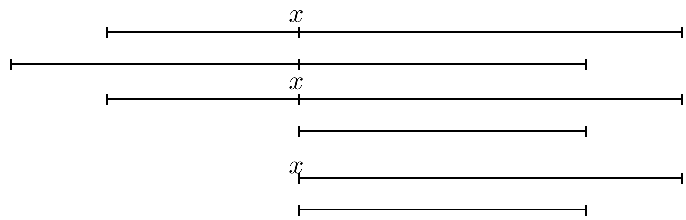


FIGURE 1. Overlapped points of the first, second, and third kind.

Figure 1 gives an idea of what the three different kinds of overlapped points look like with respect to the bridges A_n and B_n. For specific examples of Cantor sets whose intersection contains a single overlapped point of either the first or third kind, see [K3] and [K4].

If $t \in \mathcal{O}$, then $C_1 \cap (C_2 + t)$ contains only one overlapped point; so we can partition $\mathcal{O}$ into three subsets according to whether $C_1 \cap (C_2 + t)$ contains an overlapped point of the first, second or third kind. There are only a countable number of $t \in C_1 - C_2$ for which $C_1 \cap (C_2 + t)$ can have an overlapped point of the third kind (since there are only a countable number of "endpoints" in C_1 or C_2), so the part of $\mathcal{O}$ for which $C_1 \cap (C_2 + t)$ contains an overlapped point of the third kind has measure zero. So we need to concentrate on the part of $\mathcal{O}$ for which $C_1 \cap (C_2 + t)$ contains an overlapped point of the first or second kind. Define

$$\mathcal{O}' \equiv \{\, t \in \mathcal{O} \mid \text{ the overlapped point in } C_1 \cap (C_2 + t) \text{ is not of the third kind} \,\}.$$

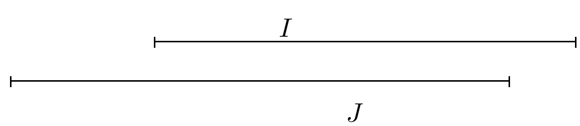
We need to show that $\mathcal{O}'$ has measure zero. Our proof is by contradiction; we assume that $\mathcal{O}'$ has positive measure, but then show that no point of $\mathcal{O}'$ is a density point. The main part of the proof is the next lemma; it gives a lower bound on the density of $\mathcal{T}$ in a neighborhood of any point $t \in \mathcal{O}'$.

We need two more definitions. Let us say that two bounded, closed, intervals are *linked* if each one contains exactly one boundary point of the other; see [PT2, pp. 63–64]. We say that two Cantor sets embedded in the real line are *linked Cantor sets* if their convex hulls are linked. Notice that linked Cantor sets are interweaved.

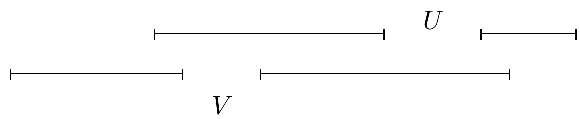
Lemma 6. *Let C_1, C_2 be linked Cantor sets, with thicknesses τ_1, τ_2, such that $C_1 \cap C_2$ contains a single overlapped point which is of the first or second kind. If $\tau_1 \tau_2 > 1$, then there is a constant $\epsilon = \epsilon(\tau_1, \tau_2) > 0$, which only depends on τ_1 and τ_2, and a neighborhood (a, b) of 0, such that*

$$\frac{|\mathcal{T} \cap (a, b)|}{b - a} \geq \epsilon.$$

Proof. Let I, J denote the convex hulls of C_1, C_2. We are assuming that I and J are linked so they are positioned, relative to each other, something like the following.



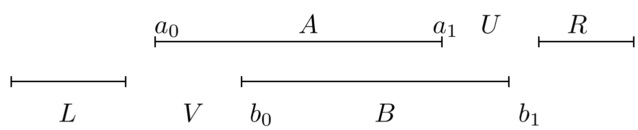
Let U denote the longest gap of C_1 which intersects with J, and let V denote the longest gap of C_2 which intersects with I. Now we make the following claim: Either the closure of U contains an endpoint of J, or the closure of V contains an endpoint of I. To prove the claim, suppose it is not true; suppose that the closure of U does not contain an endpoint of J, and the closure of V does not contain an endpoint of I. So U and V might be positioned, relative to each other, something like the following picture.



But then C_1 and C_2 have (at least) two pairs of linked segments, so by Lemma 4 and the Gap Lemma, $C_1 \cap C_2$ contains at least two overlapped points, which is a contradiction, which proves the claim.

Now we have two cases to consider. The first case is when both the closure of U contains an endpoint of J, and the closure of V contains an endpoint of I. The second case is when either the closure of U does not contain an endpoint of J, or the closure of V does not contain an endpoint of I.

Case 1. In this case $I \setminus U$ and $J \setminus V$ are positioned, relative to each other, as in the following picture.



Notice that we have two linked bridges, which are denoted by A and B (the intervals A and B cannot have a common endpoint, since it would have to be either an overlapped point of the third kind or a nonoverlapped point, contradicting in either case one of our hypotheses). The two nonlinked bridges are denoted by L and R. Let $A = [a_0, a_1]$ and $B = [b_0, b_1]$. Let $c \equiv a_1 - b_1 < 0$, and let $d \equiv a_1 - b_0 > 0$ (notice that $d - c = |B|$). Now (c, d) is a neighborhood of 0, and (c, d) has been chosen so that the segments $A \cap C_1$ and $(B \cap C_2) + t$ are interweaved for all $t \in (c, d)$. To prove this, notice that if $|B| \leq |A|$, then A and $B + t$ are in fact linked for all $t \in (c, d)$. On the other hand, if $|B| > |A|$, then for $t \in (a_0 - b_0, d)$, A and $B + t$ are linked, but for $t \in (c, a_0 - b_0]$, we have $A \subset B + t$. However, when $t \in (c, a_0 - b_0]$, C_1 and $C_2 + t$ are linked, so in order that $C_1 \cap (C_2 + t) \neq \emptyset$, it must be that $A \cap C_1$ and $(B \cap C_2) + t$ are interweaved. So for all $t \in (c, d)$, we know that $C_1 \cap (C_2 + t)$ contains at least one overlapped point in $A \cap (B + t)$.

When $t = d$, the segments $A \cap C_1$ and $(B \cap C_2) + d$ are no longer interweaved, but C_1 and $C_2 + d$ are, so $C_1 \cap (C_2 + d)$ still contains at least one overlapped point. So when $t = d$, it must be that at least one of the originally nonlinked intervals R and $L + d$ intersects with either A or $B + d$. There are eight possible "geometries" of $I \setminus U$ and $(J \setminus V) + d$, depending on how either R intersects with $B + d$, or $L + d$ intersects with A; they are listed in Figure 2. For each configuration, we want to show that there is an neighborhood $(a, b) \subset (c, d)$ of 0 such that the density of $\mathcal{T}$ in (a, b) has a lower bound that only depends on τ_1 and τ_2.

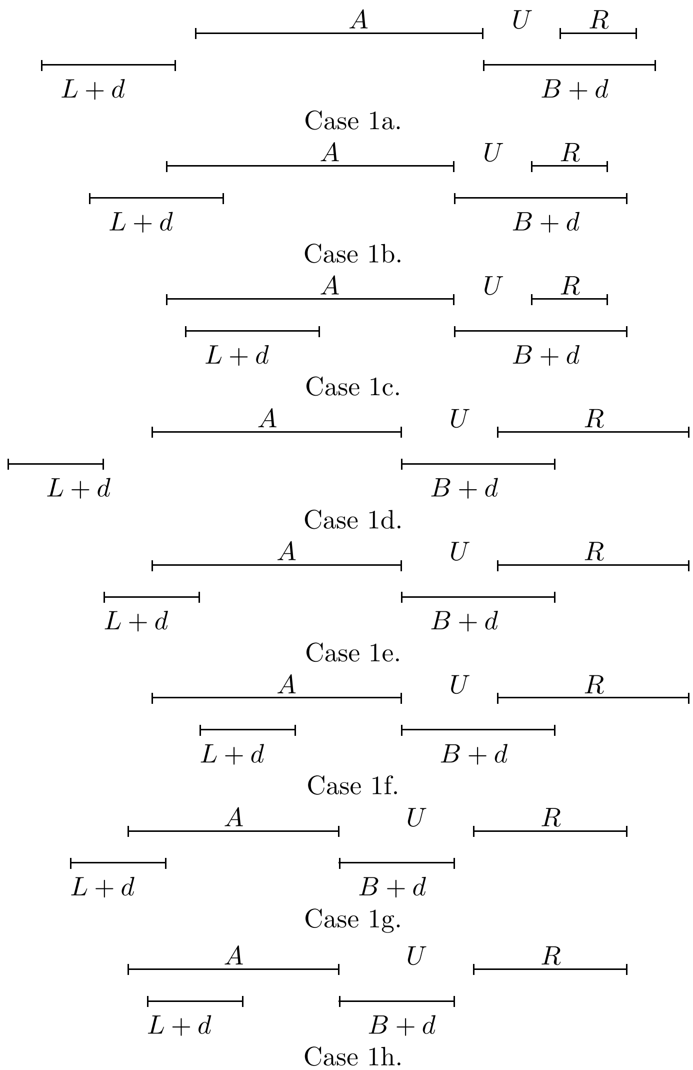
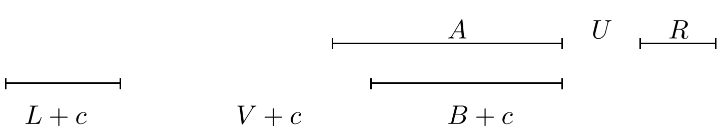
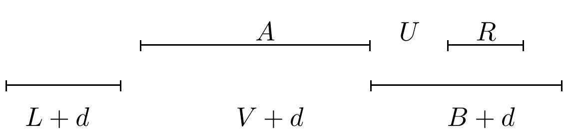


FIGURE 2. All the subcases of Case 1.

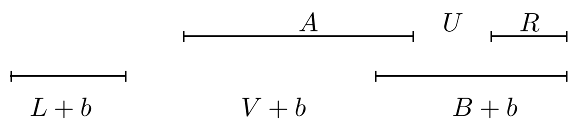
Case 1a. In this case, when $t = c$, we get the following picture of $I \setminus U$ and $(J \setminus V) + c$.



And when $t = d$, we get the following picture of $I \setminus U$ and $(J \setminus V) + d$.



For all $t \in (c,d)$, the segments in A and $B+t$ are interweaved. The intervals R and $B+t$ start out nonintersecting, then they are linked, then they become nonlinked but intersecting. By the Gap Lemma, the interweaved segments in A, $B+t$, and the linked pair R, $B+t$ each guarantee us an overlapped point. However, the segments contained in the nonlinked but still intersecting pair A, $B+t$ need not be interweaved. So we restrict t to avoid this situation. Let $a \equiv c$, and let $b \equiv (a_1+|U|+|R|)-b_1>0$. When $t=b$, we get the following picture of $I \setminus U$ and $(J \setminus V)+b$.



Now we can give a lower bound, for this case, on the density of those t in (a,b) for which $C_1 \cap (C_2+t)$ contains at least two overlapped points. Notice that $b-a = ((a_1+|U|+|R|)-b_1)-(a_1-b_1)=|U|+|R|$. Then

$$\frac{|\,(a,b)\cap\mathcal{T}\,|}{b-a} \geq \frac{|R|}{|U|+|R|} = \frac{1}{1+\frac{1}{|R|/|U|}} \geq \frac{1}{1+1/\tau_1} = \frac{\tau_1}{1+\tau_1}.$$

Case 1b. This case is handled the same as Case 1a, since the interval $L+t$ was not used in that case, and everything else is the same.

Case 1c. Again, this case is the same as Case 1a.

Case 1d. In this case, let $a \equiv c$ and $b \equiv d$, so $b-a=|B|$. Then

$$\begin{aligned}\frac{|\,(a,b)\cap\mathcal{T}\,|}{b-a} \geq \frac{|B|-|U|}{|B|} &= 1-\frac{1}{(|B|/|V|)(|V|/|U|)}\\ &\geq 1-\frac{1}{(|B|/|V|)(|A|/|U|)} \qquad \text{(since } |A|\leq|V|)\\ &\geq 1-\frac{1}{\tau_1\tau_2}>0.\end{aligned}$$

Case 1e. This is the most complicated case, and we handle it a bit differently. Let $b \equiv d$, $a' \equiv a_0-b_0>0$, and $a'' \equiv a_1-b_1>0$. Notice that $b-a'=|A|$, $b-a''=|B|$, and that $A\cap C_1$ and $(B\cap C_2)+t$ are interweaved for all t in either (a',b) or (a'',b). The density of $\mathcal{T}$ in (a',b) is bounded from below by

$$\frac{|\,(a',b)\cap\mathcal{T}\,|}{b-a'} \geq \frac{|A|-|V|}{|A|} = 1-\frac{|V|}{|A|},$$

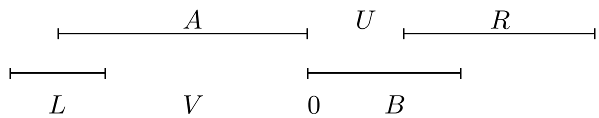
and density of $\mathcal{T}$ in (a'',b) is bounded from below by

$$\frac{|\,(a'',b)\cap\mathcal{T}\,|}{b-a''} \geq \frac{|B|-|U|}{|B|} = 1-\frac{|U|}{|B|}.$$

Since $|V|$ can be arbitrarily close to $|A|$, or $|U|$ can be arbitrarily close to $|B|$, we cannot say anything more about these last two estimates other than they are greater than zero. However, since $\tau_1\tau_2>1$, we cannot have both $|V|$ arbitrarily close to $|A|$, and $|U|$ arbitrarily close to $|B|$; as the lengths of A and V get close to each other, the lengths of U and V must be bounded away from each other, and vice versa. So there is a trade off between the density of $\mathcal{T}$ in the intervals (a',b)

and (a'', b); as one of the densities decreases, the other one must increase. We will analyze this trade off by introducing a rescaling of the Cantor set C_2.

To simplify the notation, make a couple of simple changes of variable so that $d = 0$ and $a_1 = b_0 = 0$. Case 1e then looks like the following picture:



where now $A = [-|A|, 0]$, $B = [0, |B|]$, $(a', b) = (-|A|, 0)$, and $(a'', b) = (-|B|, 0)$.

We shall apply a linear "rescaling" transformation

$$T(x) = \lambda x \qquad \text{with } \frac{|U|}{|B|} < \lambda < \frac{|A|}{|V|},$$

to the Cantor set C_2, and then compute the density of $\mathcal{T}(C_1, \lambda C_2)$ in each of the intervals (a', b) and $(\lambda a'', b)$. (We do not need to consider $\lambda \geq |A|/|V|$ and $\lambda \leq |U|/|B|$, since these are covered by Cases 1a or 1d, and Cases 1g or 1h.)

A lower bound for the density of $\mathcal{T}(C_1, \lambda C_2)$ in the interval (a', b) is given by

$$\frac{|\,(a', b) \cap \mathcal{T}(C_1, \lambda C_2)\,|}{b - a'} \geq \frac{|A| - \lambda |V|}{|A|} = 1 - \lambda \frac{|V|}{|A|},$$

and a lower bound for the density of $\mathcal{T}(C_1, \lambda C_2)$ in the interval $(\lambda a'', b)$ is given by

$$\frac{|\,(a'', b) \cap \mathcal{T}(C_1, \lambda C_2)\,|}{b - a''} \geq \frac{\lambda |B| - |U|}{\lambda |B|} = 1 - \frac{1}{\lambda} \frac{|U|}{|B|}.$$

What we want now is

$$\min_{|U|/|B| < \lambda < |A|/|V|} \left\{ \max \left\{ 1 - \lambda \frac{|V|}{|A|}, 1 - \frac{1}{\lambda} \frac{|U|}{|B|} \right\} \right\}.$$

Since $1 - (\lambda |V|/|A|)$ decreases and $1 - (|U|/\lambda |B|)$ increases with λ, it suffices to solve for λ so that $1 - (\lambda |V|/|A|) = 1 - (|U|/\lambda |B|)$. This is solved by

$$\lambda = \sqrt{\frac{|A||U|}{|B||V|}}.$$

If we plug this value of λ into our previous lower bounds, we get

$$\begin{aligned} \max \left\{ \frac{|\,(a', b) \cap \mathcal{T}\,|}{b - a'}, \frac{|\,(a'', b) \cap \mathcal{T}\,|}{b - a''} \right\} &\geq 1 - \frac{|V|}{|A|} \sqrt{\frac{|A||U|}{|B||V|}} \\ &= 1 - \left(\frac{|A|}{|U|} \frac{|B|}{|V|} \right)^{-1/2} \geq 1 - \frac{1}{\sqrt{\tau_1 \tau_2}} > 0. \end{aligned}$$

This is our lower bound for the density of $\mathcal{T}$ in one of the intervals (a', b) or (a'', b), though we cannot say which one.

Case 1f. This case is the same as Case 1b, if we reverse the roles of C_1 and C_2.

Case 1g. This case is the same as Case 1d, if we reverse the roles of C_1 and C_2.

Case 1h. This case is the same as Case 1a, if we reverse the roles of C_1 and C_2.

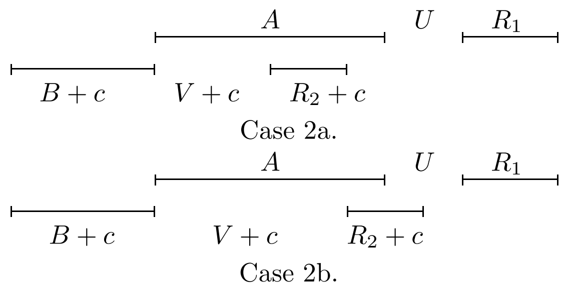
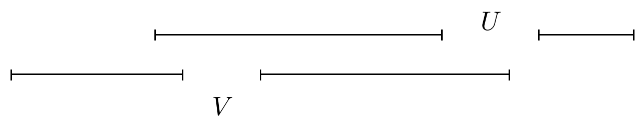
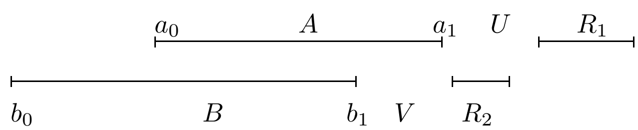


FIGURE 3. The two subcases for Case 2.

Case 2. Suppose that the closure of U contains an endpoint of J, but the closure of V does not contain an endpoint of I. So we might have $I \setminus U$ and $J \setminus V$ positioned, relative to each other, as in the following picture.



However, in order that C_1 and C_2 not have two pairs of linked segments, V must contain an endpoint of U. Thus, we in fact have U and V positioned as in the following picture.



Notice that we have two linked bridges, which are denoted by A and B, and two nonlinked bridges, which are denoted by R_1 and R_2. Let $A = [a_0, a_1]$ and $B = [b_0, b_1]$. Let $c \equiv a_0 - b_1 < 0$, and let $d \equiv \min\{\, a_0 - b_0, a_1 - b_1 \,\} > 0$. Notice that $d - c = |B|$ if $|B| \leq |A|$, and $d - c = |A|$ if $|A| < |B|$, and in either case $d - c \leq |A|$. So (c, d) is a neighborhood of 0, and (c, d) has been chosen so that the intervals A and $B + t$ are linked for all $t \in (c, d)$. So for all $t \in (c, d)$, we know that $C_1 \cap (C_2 + t)$ contains at least one overlapped point in $A \cap (B + t)$.

When $t = c$, A and $B + c$ are no longer linked, but C_1 and $C_2 + c$ are linked, so $C_1 \cap (C_2 + c)$ contains at least one overlapped point. So when $t = c$, it must be that the interval $R_2 + c$ intersects with A. There are two possible "geometries" of $I \setminus U$ and $(J \setminus V) + c$, depending on how $R_2 + c$ intersects with A; see Figure 3.

Case 2a. Let $a \equiv a_1 - (b_1 + |V| + |R_2|)$, so $c < a < 0$, and let $b \equiv d$. Notice that if $|A| < |B|$, then $b - a = (a_1 - b_1) - (a_1 - (b_1 + |V| + |R_2|)) = |V| + |R_2|$, and if $|B| \leq |A|$, then

$$
\begin{aligned}
b - a &= (a_0 - b_0) - (a_1 - (b_1 + |V| + |R_2|)) \\
&= |B| + |V| + |R_2| - |A| \\
&\leq |A| + |V| + |R_2| - |A| \qquad \text{(since } |B| \leq |A|\text{)} \\
&= |V| + |R_2|.
\end{aligned}
$$

In either case, a lower bound on the density of $\mathcal{T}$ in (a,b) is given by

$$\frac{|\,(a,b)\cap\mathcal{T}\,|}{b-a}\geq\frac{|R_2|}{|V|+|R_2|}=\frac{1}{1+\frac{1}{|R_2|/|V|}}\geq\frac{1}{1+1/\tau_2}=\frac{\tau_2}{1+\tau_2}.$$

Case 2b. Notice that, by using both the fact that $|R_2|/|U|\leq 1$ and the definition of thickness, we have

$$\frac{|A|}{|V|}\geq\frac{|R_2|}{|V|}\frac{|A|}{|U|}\geq\tau_1\tau_2. \tag{1}$$

Now let $a\equiv c$, and $b\equiv d$, so $b-a=d-c\leq|A|$. Using inequality (1), a lower bound on the density of $\mathcal{T}$ in (a,b) is given by

$$\frac{|\,(a,b)\cap\mathcal{T}\,|}{b-a}\geq\frac{|A|-|V|}{|A|}=1-\frac{|V|}{|A|}\geq 1-\frac{1}{\tau_1\tau_2}.$$

This concludes Case 2b, and also Case 2. Now that we have analyzed all the possible cases, let

$$\epsilon_1=\frac{\tau_1}{1+\tau_1},\quad\epsilon_2=\frac{\tau_2}{1+\tau_2},\quad\epsilon_3=1-\frac{1}{\tau_1\tau_2},\quad\epsilon_4=1-\frac{1}{\sqrt{\tau_1\tau_2}},$$

and let $\epsilon\equiv\min\{\,\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\,\}>0$. Then ϵ only depends on τ_1 and τ_2.

Lemma 7. *Let C_1, C_2 be linked Cantor sets, with thicknesses τ_1, τ_2, such that $C_1\cap C_2$ contains a single overlapped point which is of the first or second kind. If $\tau_1\tau_2>1$, then there is a constant $\epsilon=\epsilon(\tau_1,\tau_2)>0$, which only depends on τ_1 and τ_2, and neighborhoods (a_n,b_n) of 0 with $\lim_{n\to\infty}b_n-a_n=0$, such that for all n*

$$\frac{|\mathcal{T}\cap(a_n,b_n)|}{b_n-a_n}\geq\epsilon.$$

Proof. In both Cases 1 and 2 of Lemma 6, after we removed the open intervals U and V from the closed intervals I and J, we were left with a pair of linked bridges which were denoted by A and B. The segments of C_1 and C_2 contained in A and B satisfy the hypotheses of Lemma 6. So we can apply Lemma 6 to these new Cantor sets, and get new linked bridges A_2, B_2, and another open neighborhood (a_2,b_2) of zero where the density of $\mathcal{T}$ is bounded from below by ϵ.

By induction, given linked Cantor sets $C_1\cap A_n$ and $C_2\cap B_n$, we can apply Lemma 6 to get linked bridges A_{n+1} and B_{n+1}, and an open neighborhood (a_{n+1},b_{n+1}) of zero where the density of $\mathcal{T}$ is bounded from below by ϵ. Since τ_1, τ_2 are lower bounds on the thicknesses of $C_1\cap A_n$, $C_2\cap B_n$, and ϵ depends only on τ_1 and τ_2, the same value of ϵ works for all n.

To show that $\lim_{n\to\infty}b_n-a_n=0$, it suffices to show that $|A_n|\to 0$ and $|B_n|\to 0$ as $n\to\infty$, since $(a_n,b_n)\subset A_n-B_n$ (recall that A_n and B_n+t are interweaved for all $t\in(a_n,b_n)$). But $\{A_n\}_{n=1}^{\infty}$ is a sequence of bridges from C_1 that each contain the overlapped point x, so it must be that $|A_n|\to 0$, since C_1 is a Cantor set; similarly for the B_n.

Now we can give the proof of Lemma 5.

Proof of Lemma 5. We need to show that $\mathcal{O}'$ has measure zero. Suppose that it has positive measure. By the Lebesgue density theorem, [WZ, pp. 107–109],

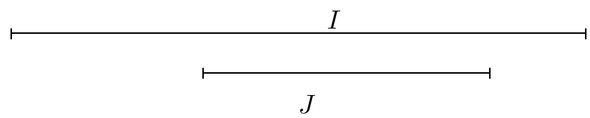
$$\lim_{n\to\infty}\frac{|\mathcal{O}'\cap(a_n,b_n)|}{b_n-a_n}=1,$$

for almost all t in $\mathcal{O}'$, where $\{ (a_n, b_n) \}_{n=1}^{\infty}$ is any sequence of intervals that *shrink regularly to* t. (The intervals (a_n, b_n) shrink regularly to t if (i) $\lim_{n\to\infty} b_n - a_n = 0$, (ii) if D_n is the smallest disk centered at t containing (a_n, b_n), then there is a constant k independent of n such that $|D_n| \leq k(b_n - a_n)$.)

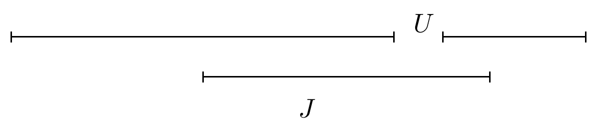
Suppose that $t_0 \in \mathcal{O}'$ is a density point. By a simple change of variable, we can assume that $t_0 = 0$. Let I, J denote the smallest closed interval containing C_1, C_2.

Claim. *Without loss of generality, we can assume that I and J are linked.*

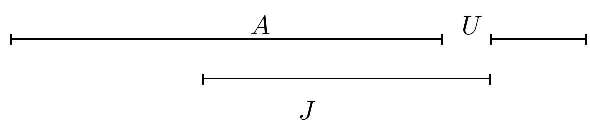
Proof. To prove this claim, first notice that I and J cannot have a common endpoint; for if they did, the common endpoint would have to be either an overlapped point of the third kind, or a nonoverlapped point, which contradicts our assumption that $0 \in \mathcal{O}'$. Since $I \cap J \neq \emptyset$ and I, J cannot have a common endpoint, it must be that either they are linked, in which case we are done, or one of I or J is contained in the interior of the other. Suppose that J is contained in the interior of I, so I and J are positioned, relative to each other, as in the following picture.



Let U be the longest gap of C_1 that intersects with J. So $I \setminus U$ and J might be positioned, relative to each other, as in the following picture.



But in order that C_1 and C_2 not have two linked segments, and hence two overlapped points in $C_1 \cap C_2$, it must be that U contains an endpoint of J, i.e., $I \setminus U$ and J are in fact positioned, relative to each other, as in the following picture.



The interval to the left of U, which is denoted by A, is linked with J. The segment $C_1 \cap A$ has thickness at least τ_1, and $(C_1 \cap A) \cap C_2$ contains a single overlapped point, which is still of the first or second kind. So, without loss of generality, we can replace C_1 with $C_1 \cap A$, and also I with A, and then I and J are linked.

So C_1 and C_2 are linked Cantor sets such that $0 \in \mathcal{O}'$, and their thicknesses satisfy $\tau_1 \tau_2 > 1$. By Lemma 7, we have neighborhoods (a_n, b_n) of 0 with $\lim_{n\to\infty} b_n - a_n = 0$, such that for all n

$$\frac{|\mathcal{T} \cap (a_n, b_n)|}{b_n - a_n} \geq \epsilon,$$

for some constant $\epsilon > 0$ which is independent of n. Since $0 \in (a_n, b_n)$ for all n, the intervals (a_n, b_n) shrink regularly to 0 (let $k = 2$ in the definition of shrink regularly). Since 0 is a density point of $\mathcal{O}'$, we can choose an n so that

$$\frac{|\mathcal{O}' \cap (a_n, b_n)|}{b_n - a_n} > 1 - \epsilon.$$

Since $\mathcal{T}$ and $\mathcal{O}'$ are disjoint, these last two inequalities contradict each other, so it must be that $\mathcal{O}'$ has measure zero.

For some intuition on what $\mathcal{O}'$ can look like see [K2], where the structure of $\mathcal{O}'$ is examined in detail using symbolic dynamics for the special case where $C_1 = C_2$ is a middle-α Cantor set with $\alpha \leq 1/3$.

We end this paper with a couple of conjectures. Since the proofs of both the Gap Lemma and Theorem 1 are essentially renormalization arguments, and since renormalization often leads to critical phenomena, we can conjecture that the condition $\tau_1\tau_2 = 1$ on thicknesses is some kind of critical boundary for difference sets of Cantor sets. Since $\tau_1\tau_2 > 1$ implies both that $C_1 - C_2$ is a union of intervals and that $C_1 \cap (C_2 + t)$ contains a Cantor set for almost all $t \in C_1 - C_2$, we can conjecture the following phenomena for the condition $\tau_1\tau_2 < 1$.

Conjecture 1. For any positive real numbers τ_1 and τ_2 with $\tau_1\tau_2 < 1$, there exist Cantor sets C_1, C_2 with thicknesses τ_1, τ_2 such that $C_1 - C_2$ does not contain any intervals (and hence it is a Cantor set).

Conjecture 2. For any positive real numbers τ_1 and τ_2 with $\tau_1\tau_2 < 1$, there exist Cantor sets C_1, C_2 with thicknesses τ_1, τ_2 such that $C_1 \cap (C_2 + t)$ does not contain a Cantor set for almost all real numbers t.

Notice that neither of these conjectures implies the other.

For any $\alpha \in (0, 1)$, let C_α denote the middle-α Cantor set in the interval $[0, 1]$. Since a middle-α Cantor set will minimize Hausdorff dimension among all Cantor sets of a given thickness ([PT2, pp. 77–78] and [K1, p. 23]) it would seem reasonable to expect them to be good candidates for solving the above conjectures. So we can make the following more specific conjectures.

Conjecture $1'$. For any real numbers $\alpha_1, \alpha_2 \in (0, 1)$ with $\alpha_1 + 3\alpha_1\alpha_2 + \alpha_2 > 1$, there exists a real number $\lambda > 0$ such that $C_{\alpha_1} - \lambda C_{\alpha_2}$ does not contain any intervals.

Conjecture $2'$. For any real numbers $\alpha_1, \alpha_2 \in (0, 1)$ with $\alpha_1 + 3\alpha_1\alpha_2 + \alpha_2 > 1$, there exists a real number $\lambda > 0$ such that $C_{\alpha_1} \cap (\lambda C_{\alpha_2} + t)$ does not contain a Cantor set for almost all real numbers t.

These conjectures are related to Problem 7 from [PT2, p. 151]. These conjectures are very easy to prove when $\tau_1 = \tau_2$; see [K3].

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