TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 352, Number 3, Pages 1247–1276
S 0002-9947(99)02516-7
Article electronically published on October 6, 1999

GAUGE INVARIANT EIGENVALUE PROBLEMS IN $\mathbb{R}^2$ AND IN $\mathbb{R}^2_+$

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ABSTRACT. This paper is devoted to the study of the eigenvalue problems for the Ginzburg-Landau operator in the entire plane $\mathbb{R}^2$ and in the half plane $\mathbb{R}^2_+$. The estimates for the eigenvalues are obtained and the existence of the associate eigenfunctions is proved when *curl A* is a non-zero constant. These results are very useful for estimating the first eigenvalue of the Ginzburg-Landau operator with a gauge-invariant boundary condition in a bounded domain, which is closely related to estimates of the upper critical field in the theory of superconductivity.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

In this paper we shall study the eigenvalue problems of the *Ginzburg-Landau operator* $-\nabla_A^2$ in the entire plane $\mathbb{R}^2$ and in the half plane $\mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$, with various type boundary conditions, where A is a real vector field, and the Ginzburg-Landau operator associated with A is defined by

$$-\nabla_A^2\psi = -\nabla_A \cdot (\nabla_A\psi) = -\Delta\psi + i[2A \cdot \nabla\psi + \psi \; div \; A] + |A|^2\psi.$$

Here $\nabla_A\psi = \nabla\psi - i\psi A$, $i = \sqrt{-1}$. Note that for a vector field $A = (A_1, \; A_2)$, $curl \; A = \partial_1 A_2 - \partial_2 A_1$, $curl^2 A = (\partial_2 curl \; A, -\partial_1 curl \; A)$, where $\partial_j = \frac{\partial}{\partial x_j}$ is the partial differential operator in the j-th coordinate.

First we consider the following eigenvalue problem:

$$-\nabla_A^2\psi = \alpha\psi \quad \text{in } \mathbb{R}^2. \tag{1.1}$$

Denote $\mathcal{W}(\mathbb{R}^2) = W^{1,2}_{loc}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, and define $\alpha(A)$ by

$$\alpha(A) = \inf_{\psi \in \mathcal{W}(\mathbb{R}^2)} \frac{\int_{\mathbb{R}^2} |\nabla_A\psi|^2 dx}{\int_{\mathbb{R}^2} |\psi|^2 dx}. \tag{1.2}$$

Generally, for all $\alpha < \alpha(A)$, equation (1.1) has no nontrivial bounded solution. If equation (1.1) has a nontrivial bounded solution ψ when $\alpha = \alpha(A)$, we say that $\alpha(A)$ is the *first eigenvalue* of (1.1), and the nontrivial bounded solution ψ is called a bounded eigenfunction. Furthermore, if $\alpha(A)$ is achieved in $\mathcal{W}(\mathbb{R}^2)$, then we call $\alpha(A)$ the *minimal eigenvalue* of equation (1.1). In this case the associated eigenfunctions are the minimizers of $\alpha(A)$.

Received by the editors November 1, 1996 and, in revised form, December 18, 1997.
1991 *Mathematics Subject Classification.* Primary 82D55.
Key words and phrases. Superconductivity, Ginzburg-Landau operator, eigenvalue.

Before stating our main results we should mention that the Ginzburg-Landau operator has the so-called *gauge-invariance* properties:

$$\nabla_{A+\nabla\chi}(e^{i\chi}\psi) = e^{i\chi}\nabla_A\psi, \qquad \nabla^2_{A+\nabla\chi}(e^{i\chi}\psi) = e^{i\chi}\nabla^2_A\psi$$

for every real smooth function χ. Hence, the number $\alpha(A)$ defined by (1.2) is invariant under the *gauge transformation* $A \to A+\nabla\chi$ for any real smooth function χ, that is, $\alpha(A) = \alpha(A+\nabla\chi)$, and the associated eigenfunction ψ, if it exists, is transformed to $e^{i\chi}\psi$. Note that *curl* A is invariant under the gauge transformation $A \to A + \nabla\chi$.

The gauge-invariance plays a very important role in the study of equations involving the Ginzburg-Landau operator. Because of the gauge-invariance, $\alpha(A)$ strongly depends on *curl* A, as shown by our first result (which will be proved in Section 3).

Theorem 1. *For every vector field* $A \in C^2(\mathbb{R}^2)$,

$$\lim_{h\to\infty} \frac{\alpha(hA)}{|h|} = \inf_{x\in\mathbb{R}^2} |curl\ A(x)|.$$

A special case is that *curl* $A = h$, a nonzero constant. Set

$$\omega(x) = \frac{1}{2}(-x_2, x_1). \tag{1.3}$$

Note that *curl* $\omega = 1$ and *div* $\omega = 0$. Then, *curl* $(A - h\omega) = 0$; therefore, there exists a real function χ such that $A = h\omega + \nabla\chi$. By the gauge invariance we have $\alpha(A) = \alpha(h\omega)$. For simplicity we denote $\alpha(h\omega)$ by $\alpha(h)$. So we consider the following problem:

$$-\nabla^2_{h\omega}\psi = \alpha\psi \quad \text{in } \mathbb{R}^2. \tag{1.4}$$

Denote

$$\alpha(h) = \inf_{\psi\in\mathcal{W}(\mathbb{R}^2)} \frac{\int_{\mathbb{R}^2} |\nabla_{h\omega}\psi|^2 dx}{\int_{\mathbb{R}^2} |\psi|^2 dx}, \tag{1.5}$$

where h is a non-zero real number, ψ is a complex-valued function, and ω is given by (1.3).

The next result gives the eigenvalues and the eigenfunctions for equation (1.4).

Theorem 2. *For each* $h \neq 0$, *the eigenvalues of* (1.4) *are* $\alpha_n(h) = (2n+1)|h|$, *for* $n = 0, 1, 2, \cdots$. *Especially,* $\alpha(h) = \alpha_0(h) = |h|$, *and the associated* L^2 *eigenfunctions are given by*

$$\exp(-|h|r^2/4)f(x) \ \ \textit{if } h > 0, \qquad \exp(-|h|r^2/4)\overline{f(x)} \ \ \textit{if } h < 0,$$

where $r = |x|$ *and* f *is any entire function (that is, a complex analytic function on the entire plane) such that* $\exp(-|h|r^2/4)f(x) \in L^2(\mathbb{R}^2)$.

Theorem 2 implies that the eigenfunctions associated with the minimal eigenvalue can only have isolated zeros. Theorem 2 will be proved in Sections 2, 3, 4. In Section 4 we will also give an integral representation of the L^2 eigenfunctions. We shall also see that for $\alpha = (2n+1)|h|$, $n = 0, 1, 2, \cdots$, equation (1.4) has bounded eigenfunctions which do not belong to $L^2(\mathbb{R}^2)$, and for all $\alpha < \alpha(h)$, equation (1.4) has no nontrivial bounded solution. Hence, $\alpha(h) = |h|$ is both the minimal eigenvalue and the first eigenvalue of equation (1.4), and the associated eigenspace in $L^2(\mathbb{R}^2)$ is of infinite dimension.

The problem in the half plane $\mathbb{R}^2_+$ is more complicated. Consider the following eigenvalue problem:

$$\begin{cases} -\nabla^2_\omega \psi = \beta\psi \quad \text{in } \mathbb{R}^2_+, \\ (\nabla_\omega \psi)\cdot\nu + \gamma\psi = 0 \quad \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{1.6}$$

where $\nu(x) = (0, -1)$ is the outer normal to $\mathbb{R}^2_+$, and $\gamma \geq 0$ is a constant. In this case we define

$$\beta_\gamma = \inf_{\psi\in\mathcal{W}(\mathbb{R}^2_+)} \frac{\int_{\mathbb{R}^2_+} |\nabla_\omega\psi|^2 dx + \gamma \int_{\partial\mathbb{R}^2_+} |\psi|^2 dx_1}{\int_{\mathbb{R}^2_+} |\psi|^2 dx}, \tag{1.7}$$

where $\mathcal{W}(\mathbb{R}^2_+) = W^{1,2}_{loc}(\mathbb{R}^2_+) \cap L^2(\mathbb{R}^2_+)$. Obviously,

$$\inf_{\psi\in\mathcal{W}(\mathbb{R}^2_+)} \frac{\int_{\mathbb{R}^2_+} |\nabla_{h\omega}\psi|^2 dx + \gamma|h| \int_{\partial\mathbb{R}^2_+} |\psi|^2 dx_1}{\int_{\mathbb{R}^2_+} |\psi|^2 dx} = |h|\beta_\gamma.$$

Theorem 3. *For all $\gamma \geq 0$, $0 < \beta_\gamma < 1$ and β_γ not achieved in $\mathcal{W}(\mathbb{R}^2_+)$. equation* (1.6) *has no nontrivial bounded solution for $\beta < \beta_\gamma$ and has exactly one linearly independent bounded solution for $\beta = \beta_\gamma$. Moreover, for $\gamma = 0$,*

$$\beta_0 < 1 - \frac{1}{\sqrt{2e\pi}}. \tag{1.8}$$

Theorem 3 is a short version of Theorem 5.3 in Section 5. Theorem 3 implies that β_γ is not a minimal eigenvalue of equation (1.6) in $\mathcal{W}(\mathbb{R}^2_+)$, but the first eigenvalue of equation (1.6) in $L^\infty(\mathbb{R}^2_+)$ with the eigenspace of one dimension.

Letting $\gamma \to +\infty$ in equation (1.6), one would expect that a convergent subsequence of the bounded eigenfunctions of equation (1.6) converges to a solution of the following *Dirichlet problem*:

$$\begin{cases} -\nabla^2_\omega \psi = \lambda\psi \quad \text{in } \mathbb{R}^2_+, \\ \psi = 0 \quad \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{1.9}$$

Define

$$\lambda_0 = \inf_{\psi\in\mathcal{W}_0(\mathbb{R}^2_+)} \frac{\int_{\mathbb{R}^2_+} |\nabla_\omega\psi|^2 dx}{\int_{\mathbb{R}^2_+} |\psi|^2 dx}, \tag{1.10}$$

where $\mathcal{W}_0(\mathbb{R}^2_+) = \{\psi \in \mathcal{W}(\mathbb{R}^2_+) : \psi = 0 \text{ on } \partial\mathbb{R}^2_+\}$. It surprises us that

Theorem 4. $\lambda_0 = 1$ *and is not achieved. Equation* (1.9) *has no nontrivial bounded solution for $\lambda \leq \lambda_0$ and has at least one linearly independent bounded solution for $\lambda > \lambda_0$.*

For the proof see Section 6. Theorem 4 says that the eigenvalue problem (1.9) has neither a minimal eigenvalue (in $\mathcal{W}_0(\mathbb{R}^2_+)$) nor a first eigenvalue (in $L^\infty(\mathbb{R}^2_+)$)!

The motivation to study such eigenvalue problems is to estimate the minimal magnitude of the applied magnetic field, called *the upper critical field*, or the *third critical field* H_{C_3}, at which superconductors lose superconductivity, see [SG], [dG]. Although there are some analyses concerning the estimates for H_{C_3}, to our best knowledge, no rigorous mathematical proof for general domains has been given.

Let the applied magnetic field be $H_a = \sigma H_0$. We estimate the minimal σ, say σ^*, such that under the applied magnetic field $\sigma^* H_0$ superconductivity disappears. Choose a vector field F so that *curl* $F = H_0$. In [LP3] we found out that for a type 2 superconductor with large Ginzburg-Landau parameter, the value of σ^* is close to the number σ_* for which $\mu(\sigma_*\kappa F) = \kappa^2$, where $\mu = \mu(bF)$ is the minimal eigenvalue of

$$\begin{cases} -\nabla_{bF}^2\psi = \mu\psi & \text{in } \Omega, \\ (\nabla_{bF}\psi)\cdot\nu + \gamma\psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.11}$$

This observation leads us to consider the eigenvalue problem (1.11) with large value of b. In [LP2] we obtained the asymptotic estimate for $\mu(bF)$ as $b \to \infty$ by using the results in this paper.

This paper is organized as follows. The eigenvalue problems in the entire plane $\mathbb{R}^2$ will be studied in Sections 2, 3, 4 by various methods. In Section 2 we use variational methods to study equation (1.4). In Section 3 we classify all the minimizers of $\alpha(h)$ by using complex variables. In Section 4 we use the Fourier transform to find all the eigenvalues of equation (1.4), and obtain integral representation of eigenfunctions. Sections 5, 6 are devoted to the study of eigenvalue problems in the half plane $\mathbb{R}^2_+$. Some basic lemmas used in previous sections will be given in Section 7.

We should mention that there are many recent works on the mathematical theory of superconductivity, see [BBH], [CHO], [DGP], [JT], [L], [LP1] and the references therein.

The authors would like to thank the referee for many useful comments and for telling us of the recent work, *Stable nucleation for the Ginzburg-Landau system with an applied magnetic field*, done by P. Bauman, D. Phillips and Q. Tang,* where the existence of solutions to the Ginzburg-Landau equation on a disc when the applied magnetic field is a large constant was studied by using of the bifurcation method. They also studied an ODE eigenvalue problem

$$\begin{cases} y'' = (x^2 - \beta y)y, & x > z, \\ y'(z) = 0, \end{cases}$$

which is closely related to our analysis in Section 7 (see (7.1)). However, the methods we used in this paper are different to theirs.

2. Variational Approach for Eigenvalue Problems in $\mathbb{R}^2$

In this section we use variational method to study the eigenvalue problem (1.4) in $\mathbb{R}^2$. Let $\alpha(h)$ be defined by (1.5). We have

Lemma 2.1. $\alpha(h) = |h|\alpha(1)$.

Proof. For every $\psi \in \mathcal{W}(\mathbb{R}^2)$ we set $\phi(x) = \overline{\psi(x)}$. Then $|\nabla_{-\omega}\psi| = |\nabla_\omega\phi|$. Hence $\alpha(-1) = \alpha(1)$.

When $h > 0$, for every $\psi \in \mathcal{W}(\mathbb{R}^2)$ we set $\psi_h(x) = \psi(\sqrt{h}x)$. Then

$$\frac{\int_{\mathbb{R}^2} |\nabla_{h\omega}\psi_h|^2 dx}{\int_{\mathbb{R}^2} |\psi_h|^2 dx} = h\frac{\int_{\mathbb{R}^2} |\nabla_\omega\psi|^2 dx}{\int_{\mathbb{R}^2} |\psi|^2 dx},$$

which yields $\alpha(h) = h\alpha(1)$ for $h > 0$. □

By Lemma 2.1, We only need to discuss the problem

$$-\nabla_\omega^2\psi = \alpha\psi \quad \text{in } \mathbb{R}^2. \tag{2.1}$$

*Since published in Arch. Rational Mech. Anal. **142** (1998), 1–43; MR **99g:**58040.

Denote

$$\alpha(1) = \inf_{\psi\in\mathcal{W}(\mathbb{R}^2)} \frac{\int_{\mathbb{R}^2} |\nabla_\omega \psi|^2 dx}{\int_{\mathbb{R}^2} |\psi|^2 dx}. \tag{2.2}$$

We shall use the polar coordinates (r, θ).

Theorem 2.2. *For equation* (2.1), *the minimal eigenvalue* $\alpha(1) = 1$. *For each* $k \ge 0$,

$$r^k \exp(-\frac{r^2}{4} + ik\theta)$$

is an L^2 *eigenfunction. Moreover, for any entire function* $f(x) \not\equiv 0$, $\exp(-r^2/4)f(x)$ *is an eigenfunction.*

Proof. Obviously, $\exp(-r^2/4)f$ is an eigenfunction associated with the eigenvalue 1, where $f \not\equiv 0$ is any entire function. Thus, $\alpha(1) \le 1$. We need to show that $\alpha(1) \ge 1$. Let $B_n = \{x \in \mathbb{R}^2 : |x| < n\}$ and

$$\alpha_n = \inf_{\psi\in W^{1,2}(B_n)} \frac{\int_{B_n} |\nabla_\omega \psi|^2 dx}{\int_{B_n} |\psi|^2 dx}.$$

Clearly, $\alpha_n \to \alpha(1)$ as $n \to +\infty$. We shall show that $\alpha_n \ge 1$ for all n. Let ϕ be a smooth function with expansion

$$\phi = \sum_{k=-\infty}^{+\infty} u_k(r) e^{ik\theta},$$

where $u_k(r)$ are radial functions. We compute

$$\begin{aligned}\int_{B_n} |\nabla_\omega \phi|^2 dx &= \sum_{k=-\infty}^{+\infty} 2\pi \int_0^n \{|u_k'|^2 + (\frac{k}{r} - \frac{r}{2})^2 |u_k|^2\} r dr \\ &\ge \sum_{k=-\infty}^{+\infty} 2\pi \alpha_n(k) \int_0^n |u_k|^2 r dr,\end{aligned}$$

where

$$\begin{aligned}\alpha_n(k) &= \inf_{u\in C^1[0,n]} \frac{\int_0^n \{|u'|^2 + (\frac{k}{r} - \frac{r}{2})^2 |u|^2\} r dr}{\int_0^n |u|^2 r dr} \\ &\ge \inf_{u\in C^1[0,\infty)} \frac{\int_0^{+\infty} \{|u'|^2 + (\frac{k}{r} - \frac{r}{2})^2 |u|^2\} r dr}{\int_0^{+\infty} |u|^2 r dr}.\end{aligned}$$

If $k \ge 0$, for every real smooth function $u(r)$ we have

$$\begin{aligned}&\int_0^{+\infty} \{|u'|^2 + (\frac{k}{r} - \frac{r}{2})^2 |u|^2\} r dr \\ &\ge 2\int_0^{+\infty} u'u(\frac{k}{r} - \frac{r}{2}) r dr = \int_0^{+\infty} |u|^2 r dr.\end{aligned}$$

The equality holds for $u = r^k \exp(-r^2/4)$. Hence, $\alpha_n(k) \ge 1$ and $\to 1$ as $n \to \infty$.

If $k < 0$ we have

$$(\frac{k}{r} - \frac{r}{2})^2 = (\frac{|k|}{r} - \frac{r}{2})^2 + 2|k|.$$

Hence, $\alpha_n(k) \ge 2|k| + 1$ and $\to 2|k| + 1$ as $n \to \infty$.

Therefore, $\alpha_n \geq 1$ for all n, which yields $\alpha(1) \geq 1$. □

Proposition 2.3. *For every $\alpha < 1$, equation* (2.1) *has no nontrivial bounded solution.*

In Section 5 we will prove a similar result for the Neumann eigenvalue problem (1.6). So we omit the proof here.

As mentioned in Section 1, for $\alpha = (2n+1)$, $n = 0, 1, 2, \cdots$, equation (2.1) has a bounded eigenfunction which does not belong to $L^2(\mathbb{R}^2)$ (see Section 4).

3. Complex Variable Approach for Eigenvalue Problems in $\mathbb{R}^2$

In this section we use complex variables to study equation (1.1) and prove Theorem 1, and we classify all the L^2 eigenfunctions of equation (1.4) associated with the minimal eigenvalue $\alpha(h)$. Throughout this section we always assume that the vector field $A \in C^2(\mathbb{R}^2)$. If *curl* $A \equiv h$, a nonzero constant, then $A = h\omega + \nabla\chi$ for some real function χ. Denote the eigenspace of $-\nabla_A^2$ in $L^2(\mathbb{R}^2)$ associated with the minimal eigenvalue $\alpha(A)$ by $\mathcal{L}(A)$. Then

$$\mathcal{L}(A) = e^{i\chi}\mathcal{L}(h\omega) = \{e^{i\chi}\phi : \ \phi \in \mathcal{L}(h\omega)\}.$$

Moreover,

$$\begin{aligned}&\phi \in \mathcal{L}(h\omega) \text{ iff } \ \overline{\phi} \in \mathcal{L}(-h\omega),\\ &\phi \in \mathcal{L}(\omega) \text{ iff } \ \phi_h(x) = \phi(\sqrt{h}x) \in \mathcal{L}(h\omega) \text{ for every } \ h > 0.\end{aligned}$$

Hence, it is enough to classify the L^2 eigenfunctions of $-\nabla_\omega^2$ associated with the minimal eigenvalue $\alpha(1)$

Theorem 3.1. *$\psi \in \mathcal{L}(\omega)$ if and only if*

$$\psi(x) = e^{-r^2/4} f(x), \tag{3.1}$$

where $r = |x|$ and $f \not\equiv 0$ is any entire function such that $f(x)e^{-r^2/4} \in L^2(\mathbb{R}^2)$.

Remark 3.1. Theorem 3.1 implies that $\mathcal{L}(\omega)$ is an infinite dimensional subspace of $L^2(\mathbb{R}^2)$ spanned by $r^k \exp(-\frac{r^2}{4} + ik\theta)$, $k = 0, \ 1, \ 2, \ \cdots$.

To prove Theorem 3.1, we use complex variables. Denote $z = x_1 + ix_2$, $\overline{z} = x_1 - ix_2$, $\partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$, $\partial_{\overline{z}} = \overline{\partial_z} = \frac{1}{2}(\partial_1 + i\partial_2)$. For a vector field $A = (A_1, A_2)$, we denote $a = \frac{1}{2}(A_1 - iA_2)$, $\overline{a} = \frac{1}{2}(A_1 + iA_2)$, $\partial_a = \partial_z - ia$, $\partial_{\overline{a}} = \partial_{\overline{z}} - i\overline{a}$.

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^2$ be a smooth domain, ν the unit outer normal vector to $\partial\Omega$ and τ the unit tangential vector such that $\{\nu, \tau\}$ is positively oriented. Let $\psi \in L^2(\Omega)$ be the solution of the equation*

$$-\nabla_A^2 \psi = f \qquad \text{in } \Omega. \tag{3.2}$$

Then,

$$\begin{aligned}&\int_\Omega \left\{4|\partial_{\overline{a}}\psi|^2 + (curl\ A)|\psi|^2 - f\bar{\psi}\right\} dx\\ &= \int_{\partial\Omega} \left\{\bar{\psi}(\nabla_A\psi)\cdot\nu + |\psi|^2 A\cdot\tau + i\bar{\psi}\frac{\partial\psi}{\partial\tau}\right\} ds\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}(3.4)\qquad &\int_\Omega \left\{4|\partial_a\psi|^2 - (curl\ A)|\psi|^2 - f\bar{\psi}\right\} dx \\ &= \int_{\partial\Omega} \left\{\bar{\psi}(\nabla_A\psi)\cdot\nu - |\psi|^2 A\cdot\tau - i\bar{\psi}\frac{\partial\psi}{\partial\tau}\right\} ds.\end{aligned}$$

Proof. Rewrite equation (3.2) as

$$-4\partial_a\partial_{\overline{a}}\psi + (curl\ A)\psi = f. \tag{3.5}$$

Multiplying (3.5) by $\overline{\psi}$ and integrating over Ω, we get (3.3). We can also write equation (3.2) as

$$-4\partial_{\overline{a}}\partial_a\psi - (curl\ A)\psi = f, \tag{3.6}$$

and (3.4) follows. ◻

Applying Lemma 3.2 to equation (1.1), we have

Lemma 3.3. *Assume that ψ is a minimizer of* (1.2). *Then we have*

$$\int_{\mathbb{R}^2} \left\{4|\partial_{\overline{a}}\psi|^2 + (curl\ A)|\psi|^2\right\} dx = \alpha(A)\int_{\mathbb{R}^2}|\psi|^2 dx \tag{3.7}$$

and

$$\int_{\mathbb{R}^2} \left\{4|\partial_a\psi|^2 - (curl\ A)|\psi|^2\right\} dx = \alpha(A)\int_{\mathbb{R}^2}|\psi|^2 dx. \tag{3.8}$$

As a consequence we get

Corollary 3.4. *For every vector field A,*

$$\alpha(A) \geq \inf_{x\in\mathbb{R}^2}|curl\ A(x)|. \tag{3.9}$$

Proof. Let $H(x) = curl\ A(x)$ and $m = \inf|H(x)|$. If $m = 0$, then (3.9) is obvious. If $m > 0$, then $H(x)$ does not change sign. Assume $H(x) > 0$. In the case where $\alpha(A)$ is achieved, (3.9) follows from (3.7). The general case follows from the following arguments. Let

$$\alpha(A,\Omega) = \inf_{\psi\in W_0^{1,2}(\Omega)} \frac{\int_\Omega |\nabla_A\psi|^2 dx}{\int_\Omega |\psi|^2 dx}.$$

For $\Omega = B_n = \{x : |x| < n\}$, the minimizer ψ_n exists and

$$\int_{B_n} \left\{4|\partial_{\overline{a}}\psi_n|^2 + (curl\ A)|\psi_n|^2\right\} dx = \alpha(A,B_n)\int_{B_n}|\psi_n|^2 dx.$$

So $\alpha(A,B_n) \geq \inf_{B_n} curl\ A(x)$. Since $\lim_{n\to\infty}\alpha(A,B_n) = \alpha(A)$, we obtain (3.9).

The case when $curl\ A(x) < 0$ is treated in a similar manner. ◻

Proof of Theorem 1. Let $H(x) = curl\ A(x)$. From (3.9) we have

$$\alpha(hA) \geq |h| \inf_{x\in\mathbb{R}^2}|H(x)|.$$

It is sufficient to show that the following inequality holds for all $x_0 \in \mathbb{R}^2$:

$$\limsup_{h\to\infty}\frac{\alpha(hA)}{|h|} \leq |H(x_0)| \tag{3.10}$$

Without loss of generality, we may assume that $h > 0$ and $x_0 = 0$.

Fix $R > 0$. Denote

$$\xi(x) = \frac{1}{2}[\partial_1 A_2(0) + \partial_2 A_1(0)]x_1 x_2,$$
$$\chi(x) = \sqrt{h}A(0)\cdot x + \frac{1}{2}[\partial_1 A_1(0)x_1^2 + \partial_2 A_2(0)x_2^2] + \xi(x),$$

Then

$$\nabla\chi(x) - \sqrt{h}A\left(\frac{x}{\sqrt{h}}\right) = H(0)\omega(x) - \sqrt{h}B\left(\frac{x}{\sqrt{h}}\right),$$

where

$$\left|\sqrt{h}B\left(\frac{x}{\sqrt{h}}\right)\right| \leq \frac{C(R)}{\sqrt{h}}|x|^2 \quad \text{for} \quad x \in B_{R\sqrt{h}}.$$

Let u be a real smooth function supported in B_R, $\psi(x) = u(x)e^{i\chi}$. Denote $\psi_h(x) = \psi(\sqrt{h}x)$. Then,

$$\int_{\mathbb{R}^2} |\nabla_{hA}\psi_h|^2 dx = \int_{B_R}\left\{|\nabla u|^2 + u^2\left|\nabla\chi - \sqrt{h}A\left(\frac{x}{\sqrt{h}}\right)\right|^2\right\}dx$$
$$= \int_{B_R}\left\{|\nabla u|^2 + u^2\left|H(0)\omega(x) - \sqrt{h}B\left(\frac{x}{\sqrt{h}}\right)\right|^2\right\}dx$$
$$\leq \left[1 + \frac{M(R)}{\sqrt{h}}\right]\int_{B_R}\left\{|\nabla u|^2 + \frac{1}{4}|H(0)xu|^2\right\}dx,$$

where $M(R)$ depends only on R. Therefore, for $h > 0$,

$$\frac{\alpha(hA)}{h} \leq \left[1 + \frac{M(R)}{\sqrt{h}}\right]\inf_{u\in W^{1,2}(B_R)} \frac{\int_{B_R}\left\{|\nabla u|^2 + \frac{1}{4}|H(0)x|^2|u|^2\right\}dx}{\int_{B_R}|u|^2 dx}.$$

Letting $h \to \infty$, we obtain

$$\limsup_{h\to+\infty}\frac{\alpha(hA)}{h} \leq \inf_{u\in W^{1,2}(B_R)} \frac{\int_{B_R}\left\{|\nabla u|^2 + \frac{1}{4}|H(0)x|^2|u|^2\right\}dx}{\int_{B_R}|u|^2 dx}. \tag{3.11}$$

To estimate the right hand side of (3.11) we denote

$$\lambda(a) = \inf_{u\in\mathcal{W}(\mathbb{R}^2)} \frac{\int_{\mathbb{R}^2}\{|\nabla u|^2 + a^2|x|^2|u|^2\}dx}{\int_{\mathbb{R}^2}|u|^2 dx},$$

where $a > 0$. It is easy to see that $\lambda(a) = 2a$ and the minimizers are $C\exp(-ar^2/2)$.

Letting R go to $+\infty$ in (3.11), we get

$$\limsup_{h\to+\infty}\frac{\alpha(hA)}{h} \leq \liminf_{R\to+\infty}\inf_{u\in W^{1,2}(B_R)} \frac{\int_{B_R}\left\{|\nabla u|^2 + \frac{1}{4}|H(0)x|^2|u|^2\right\}dx}{\int_{B_R}|u|^2 dx}$$
$$\leq \inf_{u\in\mathcal{W}(\mathbb{R}^2)} \frac{\int_{\mathbb{R}^2}\left\{|\nabla u|^2 + \frac{1}{4}|H(0)x|^2|u|^2\right\}dx}{\int_{\mathbb{R}^2}|u|^2 dx}$$
$$= \lambda\left(\frac{|H(0)|}{2}\right) = |H(0)|.$$

This completes the proof. ∎

From Lemma 3.3 one may see that if *curl* $A \equiv h$, a non-zero constant, then for $h > 0$, $\mathcal{L}(A)$ consists of the L^2-solutions of the first-order equation

$$\partial_{\overline{a}}\psi = 0, \tag{3.12}$$

and for $h < 0$, $\mathcal{L}(A)$ consists of the L^2 solutions of

$$\partial_a \phi = 0. \tag{3.13}$$

We shall give rigorous proofs for the above observation. In the following we call the solutions of equation (3.12) the *self-dual solutions* with respect to the vector field A, and call the solutions of equation (3.13) the *anti-self-dual solutions*.

Lemma 3.5. *Let ψ_0 (respectively ϕ_0) $\in C^1(\Omega)$ be a solution of equation* (3.12) *(respectively* (3.13)*) which does not vanish in Ω. Then every C^1-solution ψ of equation* (3.12) *(respectively ϕ of equation* (3.13)*) can be written as $\psi = f\psi_0$ (respectively $\phi = \overline{f}\phi_0$), where f is any complex analytic function in Ω.*

Proof. Obviously, if f is analytic in Ω then $f\psi_0$ satisfies (3.12). Let ψ be a C^1-solution of (3.12). Since $\psi_0 \neq 0$ in Ω, $\xi = \psi/\psi_0$ is well-defined in Ω. A computation gives $\partial_{\overline{z}}\xi = 0$ in Ω, which is equivalent to the Cauchy-Riemann equation. Hence, ξ is analytic in Ω. Similarly, we have the conclusion for (3.13). □

Let Ω be a simply-connected smooth domain in $\mathbb{R}^2$ and $v = v_1 + iv_2$ be a given complex function, where v_1 and v_2 are real functions. The solutions of

$$\partial_z u = v \quad \text{in } \Omega \tag{3.14}$$

are given by

$$u = \nabla\phi + w + \overline{g}, \tag{3.15}$$

and the solutions of

$$\partial_{\overline{z}} u = \overline{v} \quad \text{in } \Omega \tag{3.16}$$

are given by

$$u = \overline{\nabla\phi} + \overline{w} + g, \tag{3.17}$$

where g is any complex analytic function in Ω, ϕ and w are determined by

$$\begin{cases} \Delta\phi = 2v_1 & \text{in } \Omega, \\ \dfrac{\partial\phi}{\partial\nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} curl\ w = 2v_2 \quad \text{and} \quad div\ w = 0 & \text{in } \Omega, \\ w\cdot\nu = 0 & \text{on } \partial\Omega. \end{cases}$$

(3.15) should be understood as

$$u = (\partial_1\phi + i\partial_2\phi) + (w_1 + iw_2) + (g_1 - ig_2)$$

for $w = (w_1, w_2)$, $g = g_1 + ig_2$.

Given a vector field $A = (A_1,\ A_2)$, let $a = \frac{1}{2}(A_1 - iA_2)$ and let the function ϕ^a and the vector field w^a be the solutions of the following equations respectively:

$$\begin{cases} \Delta\phi^a = A_2 & \text{in } \Omega, \\ \dfrac{\partial\phi^a}{\partial\nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{3.18}$$

and

$$\begin{cases} curl\ w^a = A_1 \quad \text{and} \quad div\ w^a = 0 \quad \text{in } \Omega, \\ w^a \cdot \nu = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.19}$$

When $\Omega = \mathbb{R}^2$, the boundary conditions are ignored. Then, we have

Proposition 3.6. *Let Ω be a simply-connected smooth domain in $\mathbb{R}^2$. The solutions of equation* (3.12) *(self-dual solutions) are given by*

$$\psi = f \exp(-\overline{\nabla\phi^a} - \overline{w}), \tag{3.20}$$

and the solutions of equation (3.13) *(anti-self-dual solutions) are given by*

$$\phi = \overline{f} \exp(\nabla\phi^a + w), \tag{3.21}$$

where f is any complex analytic function in Ω, and ϕ^a and w^a are determined by (3.18) *and* (3.19) *respectively.*

Proof. Obviously, $\psi^a = \exp(-\overline{\nabla\phi^a} - \overline{w})$ (respectively $\phi^a = \exp(\nabla\phi^a + w)$) is a solution of (3.12) (respectively (3.13)) which does not vanish in Ω. By using Lemma 3.5, we complete the proof. □

Proof of Theorem 3.1. For the vector $A = \omega = \frac{1}{2}(-x_2,\ x_1)$ we choose $\phi^a = x_1^3/12$ and $w^a = (x_2^2/4,\ 0)$. Then, from Proposition 3.6, every solution ψ of (3.12) can be represented by $\psi(x) = e^{-r^2/4} f(x)$, where f is any entire function. The L^2 eigenfunctions for $\alpha = \alpha(1)$ are given by such solutions which lie in $L^2(\mathbb{R}^2)$. □

Remark 3.2. Given a vector field A with *curl* $A \not\equiv$ a constant, it will be interesting to classify all the self-dual solutions and the anti-self-dual solutions and to study the existence of such solutions which lie in $L^2(\mathbb{R}^2)$. It will also be interesting to get a more precise estimate for $\alpha(A)$ and study the existence of the minimizers of $\alpha(A)$.

4. Fourier Transform Approach for Eigenvalue Problems in $\mathbb{R}^2$

In this section we use the Fourier transform to study equation (2.1) and prove the following:

Theorem 4.1. *The eigenvalues of equation* (2.1) *are*

$$\alpha_n = 2n + 1, \quad n = 0,\ 1,\ 2,\ \cdots,$$

and all the L^2 eigenfunctions are given by (4.3) *below.*

Proof. Write $\omega = E + \nabla\chi$, where $E(x) = (-x_2, 0)$, $\chi = x_1x_2/2$. Set $\psi = e^{i\chi}\varphi$. Then $\nabla_\omega^2\psi = e^{i\chi}\nabla_E^2\varphi$. Thus, equation (2.1) is transformed to $-\nabla_E^2\varphi = \alpha\varphi$, that is,

$$\Delta\varphi + 2ix_2\partial_1\varphi - |x_2|^2\varphi + \alpha\varphi = 0 \quad \text{in } \mathbb{R}^2. \tag{4.1}$$

Let $\varphi \in L^2(\mathbb{R}^2)$ be a solution of equation (4.1). Let $f(z, x_2) = \mathcal{F}[\varphi]$ be the Fourier transform of $\varphi(\cdot, x_2)$ with respect to the variable x_1. Fix z and set $f(z,t) = y(z, t+z)$. Then, $y(z,t)$ satisfies the Hermite equation in t:

$$y'' - t^2y + \alpha y = 0, \quad y(t) \to 0 \text{ as } |t| \to \infty, \tag{4.2}$$

where $' = \frac{d}{dt}$. It is well-known that the eigenvalues of equation (4.2) are $\alpha = \alpha_n = 2n+1$, $n = 0, 1, 2, \cdots$ and the associated eigenfunctions are $\exp(-t^2/2)H_n(t)$, where $H_n(t)$ is the Hermite polynomial. Therefore

$$f(z,t) = a(z)\exp(-(t+z)^2/2)H_n(t+z), \qquad a(z) \in L^2(\mathbb{R}^2).$$

Hence,

$$\begin{aligned}\varphi(x_1,x_2) &= \mathcal{F}^{-1}\left[a(z)\exp(-(x_2+z)^2/2)H_n(x_2+z)\right]\\ &= b(x_1) \circledast F_n(x_1,x_2),\end{aligned}$$

where $b(x_1) = \mathcal{F}^{-1}[a(z)]$,

$$\begin{aligned}F_n(x_1,x_2) &= \mathcal{F}^{-1}\left[\exp(-(x_2+z)^2/2)H_n(x_2+z)\right]\\ &= \exp(-ix_1x_2)\mathcal{F}^{-1}\left[\exp(-t^2/2)H_n(t)\right]\\ &= i^n n!\exp(-ix_1x_2 - x_1^2/2)\sum_{k=0}^{[n/2]}\frac{2^{(n-2k)/2}}{k!(n-2k)!}H_{n-2k}(\frac{x_1}{\sqrt{2}}),\end{aligned}$$

and $\circledast$ is defined by

$$b(x_1) \circledast g(x_1) = \int_{-\infty}^{+\infty} b(t)g(x_1-t)dt.$$

Thus, the eigenvalues of equation (2.1) are $\alpha_n = 2n+1$, $n = 0, 1, 2, \cdots$, and every L^2 eigenfunction associated with the eigenvalue $\alpha_n = 2n+1$ is of the form

$$\psi = \exp(\frac{ix_1x_2}{2})[b(x_1) \circledast F_n(x_1,x_2)], \tag{4.3}$$

where $b(x_1)$ is any L^2 function. Therefore every eigenspace in $L^2(\mathbb{R}^2)$ is of infinite dimension. □

Remark 4.1. From Theorem 4.1 we can get another representation for the L^2 eigenfunctions of equation (1.4) associated with the minimal eigenvalue $\alpha(h)$. For $n = 0$, $\alpha = \alpha_0 = 1$, and $H_0(t) = 1$, we have $F_0(x_1,x_2) = \exp(-ix_1x_2 - x_1^2/2)$. The L^2 eigenfunctions of (2.1) corresponding to the minimal eigenvalue 1 are given by

$$\begin{aligned}\psi(x_1,x_2) &= \exp(\frac{ix_1x_2}{2})[b \circledast F_0(x_1,x_2)]\\ &= \exp(\frac{ix_1x_2}{2})\int_{-\infty}^{+\infty} b(\tau)\exp(ix_2\tau - (x_1-\tau)^2/2)d\tau\end{aligned} \tag{4.4}$$

After rescaling, we get integral representation for the L^2 eigenfunctions corresponding to the minimal eigenvalue of equation (1.4).

Remark 4.2. In the above discussion we restrict ourselves to the L^2 eigenfunctions only. We mention that equation (4.1) has many bounded solutions which do not belong to $L^2(\mathbb{R}^2)$. In fact, for $\alpha = \alpha_n = 2n+1$, for every constant z_0, the function

$$\varphi_{n,z_0}(x_1,x_2) = \exp(iz_0x_1 - (x_2+z_0)^2/2)H_n(x_2+z_0)$$

is a bounded solution of equation (4.1) which is not an L^2 function. More generally, for a sequence $\{z_j\}$ and an absolutely convergent series $\sum c_j$,

$$\varphi(x_1,x_2) = \sum_j c_j \exp(iz_jx_1 - (x_2+z_j)^2/2)H_n(x_2+z_j)$$

is a bounded solution of equation (4.1) with $\alpha = 2n+1$.

Proof of Theorem 2. Theorem 2 follows from Theorems 3.1, 4.1 and rescaling arguments. □

5. A Neumann Eigenvalue Problem in $\mathbb{R}^2_+$

In this section we consider the eigenvalue problem (1.6) in the half plane. Let β_γ be the number defined in (1.7). We shall show that $0 < \beta_\gamma < 1$ and is not achieved in $\mathcal{W}(\mathbb{R}^2_+)$. Hence, β_γ is *not* a minimal eigenvalue of (1.6). We shall also show that equation (1.6) has no nontrivial bounded solution for all $\beta < \beta_\gamma$ and has bounded eigenfunctions for all $\beta \geq \beta_\gamma$. Thus, β_γ is the first eigenvalue of equation (1.6) in the space of bounded smooth functions.

Lemma 5.1. *For $\beta < \beta_\gamma$, equation* (1.6) *has no nontrivial bounded solution.*

Proof. Suppose that $\psi \in \mathcal{W}(\mathbb{R}^2_+)$ is a solution of equation (1.6) with $\beta < \beta_\gamma$. We shall show that $\psi \equiv 0$.

One can show that $\psi \in C^2$ up to the boundary. Let η be a smooth real cut-off function supported in B_{2R}, $\eta \equiv 1$ in R_R, and $|\nabla\eta| \leq 2/R$. Multiplying (1.6) by $\eta^2\psi$, integrating by parts, and using the boundary condition in (1.6), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^2_+} |\nabla_\omega(\eta\psi)|^2 dx + \gamma\int_{\partial\mathbb{R}^2_+} |\eta\psi|^2 dx_1 - \beta\int_{\mathbb{R}^2_+} |\eta\psi|^2 dx\\ &= \int_{\mathbb{R}^2_+} |\nabla\eta|^2|\psi|^2 dx.\end{aligned}$$

Using (1.7), we have

$$(\beta_\gamma - \beta)\int_{\mathbb{R}^2_+} |\eta\psi|^2 dx \leq \int_{\mathbb{R}^2_+} |\nabla\eta|^2|\psi|^2 dx.$$

Hence,

$$\int_{B_R^+} |\psi|^2 dx \leq \frac{4}{(\beta_\gamma-\beta)R^2}\int_{B_{2R}^+} |\psi|^2 dx \leq \frac{8\pi M^2}{\beta_\gamma - \beta},$$

where $B_R^+ = \mathbb{R}^2_+ \cap B_R$ and $M = \sup|\psi|$. Therefore, $\psi \in L^2(\mathbb{R}^2_+)$, and for every $R > 0$,

$$\int_{B_R^+} |\psi|^2 dx \leq \frac{4}{(\beta_\gamma-\beta)R^2}\int_{\mathbb{R}^2_+} |\psi|^2 dx.$$

Letting R go to $+\infty$, we find that $\psi \equiv 0$. □

Observe that the boundary condition in equation (1.6) is $\partial_2\psi = \left(\frac{i}{2}x_1 + \gamma\right)\psi$ on $\partial\mathbb{R}^2_+$. For convenience, we use gauge transformation to change it to a simpler third boundary condition. As in Section 4, we let $E(x) = (-x_2,\ 0)$, $\chi = \frac{1}{2}x_1x_2$, $\psi = e^{i\chi}\varphi$. Then

$$\beta_\gamma = \inf_{\varphi\in\mathcal{W}(\mathbb{R}^2_+)} \frac{\int_{\mathbb{R}^2_+} |\nabla_E\varphi|^2 dx + \gamma\int_{\partial\mathbb{R}^2_+} |\varphi|^2 dx_1}{\int_{\mathbb{R}^2_+} |\varphi|^2 dx}, \tag{5.1}$$

and equation (1.6) is changed to

$$\begin{cases}\Delta\varphi + 2ix_2\partial_1\varphi - |x_2|^2\varphi + \beta\varphi = 0 & \text{in } \mathbb{R}^2_+,\\ \partial_2\varphi = \gamma\varphi & \text{on } \partial\mathbb{R}^2_+.\end{cases} \tag{5.2}$$

Suppose that $\varphi \in L^2(\mathbb{R}^2_+)$ is a solution of equation (5.2). Denote by $f(z, x_2)$ the L^2 Fourier transform of φ in the variable x_1. Formally, for every fixed z, $f(z, \cdot)$ is a solution of the following problem in t:

$$\begin{cases} u'' = \left[(z+t)^2 - \beta\right] u & \text{for } t > 0, \\ u'(0) = \gamma u(0), \quad u(+\infty) = 0. \end{cases} \tag{5.3}$$

Fix z and denote

$$\beta_\gamma(z) = \inf_{u \in W^{1,2}(\mathbb{R}^1_+)} \frac{\int_0^{+\infty} \{|u'|^2 + (z+t)^2|u|^2\} dt + \gamma |u(0)|^2}{\int_0^{+\infty} |u|^2 dt}. \tag{5.4}$$

The next result gives basic properties of $\beta_\gamma(z)$.

Lemma 5.2. *The following conclusions hold:*

(a) *For every z, $\beta_\gamma(z)$ is achieved and is the first eigenvalue of equation* (5.3).

(b) *$\beta_\gamma(z)$ is continuous in z, $\beta_\gamma(z) > 1$ for $z > -\gamma$, $0 < \beta_\gamma(z) < 1$ for $z < -\gamma$, and*

$$\lim_{z \to -\infty} \beta_\gamma(z) = 1, \qquad \lim_{z \to +\infty} \beta_\gamma(z) = +\infty.$$

(c) *There exists a unique number z_γ, $z_\gamma < -\gamma$, such that*

$$\beta_\gamma(z_\gamma) = \beta^*_\gamma \equiv \inf_{z \in \mathbb{R}^1} \beta_\gamma(z). \tag{5.5}$$

$\beta_\gamma(z)$ is strictly decreasing in $(-\infty, z_\gamma)$ and strictly increasing in $(z_\gamma, +\infty)$.

(d) *The following estimate holds:*

$$0 < \beta^*_\gamma \le 1 - \sup_{t > \gamma} \frac{(t-\gamma)\exp(-t^2)}{\int_{-\infty}^t \exp(-s^2) ds}. \tag{5.6}$$

In particular, for $\gamma = 0$

$$0 < \beta^*_0 < 1 - \frac{1}{\sqrt{2e\pi}};$$

(e) *For all z, the second eigenvalue of equation* (5.3) *is greater than* 1.

Since the proof is quite lengthy, it is postponed to Section 7.

In the following we denote the positive eigenfunction of equation (5.3) with $\beta = \beta_\gamma(z)$ by $u(z, t)$.

Theorem 5.3. *For every $\gamma \ge 0$, let β_γ be the number defined in* (1.7) *and let $\beta^*_\gamma = \beta_\gamma(z_\gamma)$ be given by* (5.5)*, where z_γ is the unique minimum point of $\beta_\gamma(z)$. Then*

$$\beta_\gamma = \beta^*_\gamma \le 1 - \sup_{t > \gamma} \frac{(t-\gamma)\exp(-t^2)}{\int_{-\infty}^t \exp(-s^2) ds}, \tag{5.7}$$

and β_γ is not achieved in $\mathcal{W}(\mathbb{R}^2_+)$. For $\beta = \beta_\gamma$, equation (1.6) *has exactly one linearly independent bounded solution, given by*

$$\psi = \exp(ix_1x_2/2 + z_\gamma x_1) u_\gamma(x_2), \tag{5.8}$$

where $u_\gamma(t)$ is the positive eigenfunction of (5.3) *with $z = z_\gamma$ and $\beta = \beta_\gamma(z_\gamma)$.*

Proof. Step 1. We prove that $\beta_\gamma = \beta^*_\gamma$.

For each $\varphi \in C^2(\mathbb{R}^2_+)$ with a compact support, we denote by $f(z, x_2) = \mathcal{F}[\varphi]$ the Fourier transform of φ in x_1. Then

$$\begin{aligned}&\int_{-\infty}^{+\infty} |\nabla_E \varphi(x_1, x_2)|^2 dx_1 = \int_{-\infty}^{+\infty} \{|\partial_1 \varphi + i x_2 \varphi|^2 + |\partial_2 \varphi|^2\} dx_1 \\ &= \int_{-\infty}^{+\infty} \{|\mathcal{F}[\partial_1 \varphi + i x_2 \varphi]|^2 + |\mathcal{F}[\partial_2 \varphi]|^2\} dz \\ &= \int_{-\infty}^{+\infty} \{|i(x_2 + z) f|^2 + |\partial_2 f|^2\} dz.\end{aligned}$$

Thus,

$$\begin{aligned}&\int_{\mathbb{R}^2_+} |\nabla_E \varphi|^2 dx + \gamma \int_{\partial \mathbb{R}^2_+} |\varphi|^2 dx_1 \\ &= \int_{-\infty}^{+\infty} \left\{ \int_0^{+\infty} \left[|\partial_2 f|^2 + (x_2 + z)^2 |f|^2 \right] dx_2 + \gamma |f(z, 0)|^2 \right\} dz \\ &\geq \int_{-\infty}^{+\infty} \left\{ \beta_\gamma(z) \int_0^{+\infty} |f|^2 dx_2 \right\} dz \\ &\geq \beta_\gamma^* \int_{-\infty}^{+\infty} \int_0^{+\infty} |f|^2 dx_2 dz = \beta_\gamma^* \int_{\mathbb{R}^2_+} |\varphi|^2 dx.\end{aligned}$$

Hence,

$$\int_{\mathbb{R}^2_+} |\nabla_E \varphi|^2 dx + \gamma \int_{\partial \mathbb{R}^2_+} |\varphi|^2 dx_1 \geq \beta_\gamma^* \int_{\mathbb{R}^2_+} |\varphi|^2 dx. \tag{5.9}$$

By approximation, we see that (5.9) holds for all $\varphi \in \mathcal{W}(\mathbb{R}^2_+)$, so $\beta_\gamma \geq \beta_\gamma^*$.

Next, we prove that $\beta_\gamma \leq \beta_\gamma^*$. Set

$$\varphi_\gamma = \exp(i z_\gamma x_1) u_\gamma(x_2). \tag{5.10}$$

Note that φ_γ is a bounded solution of equation (5.2). Let $\eta_n(t)$ be a smooth real cut-off function supported in $|t| < n + 1$ such that $\eta_n(t) = 1$ for $|t| \leq n$, $0 \leq \eta_n(t) \leq 1$, $|\eta_n'(t)| \leq 2$. Let

$$\varphi_n = \eta_n(x_1) \varphi_\gamma(x_1, x_2) = \eta_n(x_1) \exp(i z_\gamma x_1) u_\gamma(x_2).$$

In the following, we use the notation $\|\cdot\|_2$ to denote the L^2 norm — the $L^2(\mathbb{R}^1)$ norm for η_n and η_n', the $L^2(\mathbb{R}^1_+)$ norm for u_γ, and the $L^2(\mathbb{R}^2_+)$ norm for φ_n. Then we compute

$$\begin{aligned}&\int_{\mathbb{R}^2_+} |\nabla_E \varphi_n|^2 dx + \gamma \int_{\partial \mathbb{R}^2_+} |\varphi_n|^2 dx_1 \\ &= \|\eta_n'\|_2^2 \|u_\gamma\|_2^2 + \beta_\gamma^* \|\eta_n\|_2^2 \|u_\gamma\|_2^2 \leq (\beta_\gamma^* + \frac{4}{n}) \|\varphi_n\|_2^2.\end{aligned}$$

Sending n to ∞, we see that $\beta_\gamma \leq \beta_\gamma^*$.

Step 2. The function φ_γ given in (5.10) is a bounded eigenfunction of equation (5.2) with $\beta = \beta_\gamma$; hence $\psi = \exp(i x_1 x_2 / 2) \varphi_\gamma$ is a bounded eigenfunction of (1.6). Next, we show that any bounded eigenfunction of equation (5.2) with $\beta = \beta_\gamma$ is of the form $c\varphi_\gamma$ for some constant c.

Claim 1. There is a constant C such that for any bounded solution φ of equation (5.2) with $\beta = \beta_\gamma$ and for any $a < b$,

$$\int_a^b dx_1 \int_0^{+\infty} |\varphi|^2 dx_2 \leq C\|\varphi\|_*^2 (1 + b - a), \tag{5.11}$$

where

$$\|\varphi\|_* = \sup_{-\infty < x_1 < +\infty, 0 \leq x_2 \leq 2} |\varphi(x)|.$$

Proof of the claim. Let φ be a bounded solution of equation (5.2) with $\beta = \beta_\gamma$. As in the proof of Lemma 5.1, we choose a cut-off function η with compact support. Multiplying (5.2) by $\eta^2\varphi$ and integrating, we get

$$\int_{\mathbb{R}^2_+} \{|\nabla_E(\eta\varphi)|^2 - \beta_\gamma |\eta\varphi|^2\} dx + \gamma \int_{\partial\mathbb{R}^2_+} |\eta\varphi|^2 dx_1 = \int_{\mathbb{R}^2_+} |\nabla\eta|^2 |\varphi|^2 dx. \tag{5.12}$$

Now we choose η such that $spt(\eta) \subset\subset \mathbb{R}^2_+$. Since $curl\ E = 1$, using Theorem 2 we have

$$(1 - \beta_\gamma) \int_{\mathbb{R}^2_+} |\eta\varphi|^2 dx \leq \int_{\mathbb{R}^2_+} |\nabla\eta|^2 |\varphi|^2 dx.$$

Let the cut-off function η approach the function $\eta_1(x_1)\eta_2(x_2)$, where

$$\eta_1(t) = \begin{cases} e^{\varepsilon(m+t)} & \text{if } t \leq -m, \\ 1 & \text{if } -m < t < m, \\ e^{\varepsilon(m-t)} & \text{if } t \geq m, \end{cases}$$

$$\eta_2(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1, \\ 1 & \text{if } 2 < t < n, \\ e^{\varepsilon(n-t)} & \text{if } t > n, \end{cases}$$

where ε is small, and m, n are large. Then we get

$$(1 - \beta_\gamma - 2\varepsilon^2) \int_{-m}^m dx_1 \int_0^{+\infty} |\varphi|^2 dx \leq 10\|\varphi\|_*^2 \int_{-\infty}^{+\infty} \eta_1^2(x_1) dx_1.$$

From this inequality we get Claim 1 by translation. □

It is well known that, a bounded continuous function f on $\mathbb{R}^1$ can be viewed as a distribution (also denoted by f) on $C_0^\infty(\mathbb{R}^1)$ with

$$\langle f, g\rangle = \int_{-\infty}^{+\infty} f(x_1) g(x_1) dx_1$$

for any $g \in C_0^\infty(\mathbb{R}^1)$. The Fourier transform $\mathcal{F}[f]$ of the distribution f is defined by

$$\langle \mathcal{F}[f], g\rangle = \langle f, \mathcal{F}[g]\rangle$$

for any $g \in C_0^\infty(\mathbb{R}^1)$, where $\mathcal{F}[g]$ is the usually Fourier transform of g.

Let φ be a bounded solution of equation (5.2) with $\beta = \beta_\gamma$. For any fixed $x_2 \geq 0$, $\varphi(\cdot, x_2)$ defines a distribution. So φ can be viewed as a distribution (in x_1) with parameter x_2. For any fixed $x_2 \geq 0$, let $\tilde{\varphi}(z, x_2) = \mathcal{F}[\varphi](z, x_2)$ be the Fourier transform of $\varphi(x_1, x_2)$ in x_1 in the sense of distributions, i.e.

$$\langle \tilde{\varphi}(z, x_2), \eta(z)\rangle = \langle \varphi(x_1, x_2), \mathcal{F}[\eta](x_1)\rangle$$

for any $\eta \in C_0^\infty(\mathbb{R}^1)$. Then $\tilde{\varphi}(z, x_2)$ is also a distribution with parameter x_2.

Recall that z_γ is the unique minimum point of $\beta_\gamma(z)$ and $\beta_\gamma = \beta_\gamma(z_\gamma) = \beta_\gamma^*$.

Claim 2. Let φ be a bounded solution of (5.2) with $\beta = \beta_\gamma$, and let $\tilde{\varphi}(z, x_2)$ be the Fourier transform of $\varphi(x_1, x_2)$ in x_1 in the sense of distributions. Then for any $x_2 \geq 0$, the support of $\tilde{\varphi}(\cdot, x_2)$ either is empty or contains only z_γ.

Proof of the claim. Let $\eta(x)$ be a cut-off function. Fix $x_2 \geq 0$, and let $f_\eta(z, x_2) = \mathcal{F}[\eta\varphi](z, x_2)$ be the Fourier transform of the L^2 function $\eta\varphi$ in x_1. As in Step 1 we have

$$\int_{\mathbb{R}^2_+} |\nabla_E(\eta\varphi)|^2 dx + \gamma \int_{\partial\mathbb{R}^2_+} |\eta\varphi|^2 dx_1 \geq \int_{-\infty}^{+\infty} \left\{ \beta_\gamma(z) \int_0^{+\infty} |f_\eta|^2 dx_2 \right\} dz.$$

Since $\|\eta\varphi\|_{L^2(\mathbb{R}^2_+)} = \|f_\eta\|_{L^2(\mathbb{R}^2_+)}$, we have

$$\begin{aligned} &\int_{\mathbb{R}^2_+} \{|\nabla_E(\eta\varphi)|^2 - \beta_\gamma|\eta\varphi|^2\} dx + \gamma \int_{\partial\mathbb{R}^2_+} |\varphi|^2 dx_1 \\ &\geq \int_{-\infty}^{+\infty} [\beta_\gamma(z) - \beta_\gamma] dz \int_0^{+\infty} |f_\eta|^2 dx_2. \end{aligned} \tag{5.13}$$

Using (5.12) and (5.13), we see that

$$\int_{\mathbb{R}^2_+} |\nabla\eta|^2 |\varphi|^2 dx \geq \int_{-\infty}^{+\infty} [\beta_\gamma(z) - \beta_\gamma] dz \int_0^{+\infty} |f_\eta|^2 dx_2. \tag{5.14}$$

From Claim 1, we can choose $\eta(x) = \eta(x_1)$ such that $\eta(x_1) = 1$ for $|x_1| \leq m$ and $\eta(x_1) = 0$ for $|x_1| \geq m + h$, and $|\eta'(x_1)| \leq 2/h$. Then for large h we have

$$\begin{aligned} \int_{\mathbb{R}^2_+} |\nabla\eta|^2 |\varphi|^2 dx &\leq \frac{4}{h^2} \int_{\{m \leq |x_1| \leq m+h, x_2 \geq 0\}} |\varphi|^2 dx \\ &\leq \frac{8C}{h^2}(h+1)\|\varphi\|_*^2 \leq \frac{10C\|\varphi\|_*^2}{h}. \end{aligned}$$

Denote $b(\varepsilon) = \min_{|z - z_\gamma| \geq \varepsilon} [\beta_\gamma(z) - \beta_\gamma]$. From Lemma 5.2, $b(\varepsilon) > 0$ for any $\varepsilon > 0$. Hence, for all large h we have

$$\int_{\{|z - z_\gamma| \geq \varepsilon, x_2 \geq 0\}} |f_\eta|^2 dz dx_2 \leq \frac{10C\|\varphi\|_*^2}{h b(\varepsilon)}.$$

Choose $\varepsilon_n \to 0$, $h_n >> 1/b(\varepsilon_n)$, $m_n \to +\infty$, and choose a sequence of cut-off function $\eta_n(x) = \eta_n(x_1)$ such that $\eta_n(x_1) = 1$ for $|x_1| \leq m_n$, $\eta_n(x_1) = 0$ for $|x_1| \geq m_n + h_n$, and $|\eta_n'(x_1)| \leq 2/h_n$. Denote $f_n(z, x_2) \equiv f_{\eta_n}(z, x_2) = \mathcal{F}[\eta_n\varphi](z, x_2)$. Then we have

$$\lim_{n\to\infty} \int_{\{|z - z_\gamma| \geq \varepsilon_n, x_2 \geq 0\}} |f_n|^2 dz dx_2 = 0. \tag{5.15}$$

Now we show that for any $\xi \in C_0^\infty(\mathbb{R}^2)$ for which $spt(\xi)$ does not intersect with $\{z_\gamma\} \times \mathbb{R}^1$, we have

$$\int_0^{+\infty} \langle \tilde{\varphi}(z, x_2), \xi(z, x_2) \rangle dx_2 = 0. \tag{5.16}$$

To prove (5.16), we choose $N > 0$ such that

$$spt(\xi) \subset \{(z, x_2) : |z - z_\gamma| \geq \varepsilon_n, x_2 \geq 0\}$$

for all $n \geq N$. From (5.15) we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^2_+} f_n \xi \, dz dx_2 = 0. \tag{5.17}$$

On the other hand, since ξ has compact support, we can show that $\mathcal{F}[\xi](x_1, x_2)$ rapidly decays to 0 as x_1 goes to ∞, uniformly in x_2. Therefore,

$$\langle \eta_n(x_1)\varphi(x_1,x_2), \mathcal{F}[\xi](x_1,x_2)\rangle \;\to\; \langle \varphi(x_1,x_2), \mathcal{F}[\xi](x_1,x_2)\rangle$$

as $n \to \infty$, uniformly in x_2. Note that $\xi \equiv 0$ for x_2 large. So $\mathcal{F}[\xi] = 0$ for x_2 large. Thus,

$$\int_0^{+\infty} \langle \eta_n(x_1)\varphi(x_1,x_2), \mathcal{F}[\xi](x_1,x_2)\rangle dx_2 \;\to\; \int_0^{+\infty} \langle \varphi(x_1,x_2), \mathcal{F}[\xi](x_1,x_2)\rangle dx_2$$

as $n \to \infty$. So,

$$\begin{aligned}
\int_{\mathbb{R}^2_+} f_n \xi \, dz dx_2 &= \int_0^{+\infty} \langle f_n(z,x_2), \xi(z,x_2))\rangle dx_2 \\
&= \int_0^{+\infty} \langle \mathcal{F}[\eta_n\varphi](z,x_2), \xi(z,x_2)\rangle dx_2 \\
&= \int_0^{+\infty} \langle \eta_n(x_1)\varphi(x_1,x_2), \mathcal{F}[\xi](x_1,x_2)\rangle dx_2 \\
&\to \int_0^{+\infty} \langle \varphi(x_1,x_2), \mathcal{F}[\xi](x_1,x_2)\rangle dx_2 \\
&= \int_0^{+\infty} \langle \mathcal{F}[\varphi](z,x_2), \xi(z,x_2)\rangle dx_2 \\
&= \int_0^{+\infty} \langle \tilde{\varphi}(z,x_2), \xi(z,x_2)\rangle dx_2.
\end{aligned} \tag{5.18}$$

Combining (5.17) and (5.18) we get (5.16).

Next, we show that for every $x_2 \geq 0$, the support of $\tilde{\varphi}(z,x_2)$ is contained in $\{z_\gamma\}$. Suppose not. Then there is a number $x_2^0 \geq 0$ such that $spt(\tilde{\varphi}(\cdot, x_2^0)) \not\subset \{z_\gamma\}$. So we can choose a function $\chi \in C_0^\infty(\mathbb{R}^1)$ such that

$$spt(\chi) \cap \{z_\gamma\} = \emptyset, \quad \langle \tilde{\varphi}(z,x_2^0), \chi(z)\rangle > 0. \tag{5.19}$$

Note that φ is a continuous function. For any $\psi \in C_0^\infty(\mathbb{R}^1)$ we have

$$\begin{aligned}
&\lim_{x_2\to x_2^0} \langle \tilde{\varphi}(z,x_2), \psi\rangle \;=\; \lim_{x_2\to x_2^0} \langle \mathcal{F}[\varphi](z,x_2), \psi\rangle \\
&= \lim_{x_2\to x_2^0} \langle \varphi(x_1,x_2), \mathcal{F}[\psi](x_1)\rangle \;=\; \langle \varphi(x_1,x_2^0), \mathcal{F}[\psi](x_1)\rangle \\
&= \langle \mathcal{F}[\varphi](z,x_2^0), \psi(z)\rangle \;=\; \langle \tilde{\varphi}(z,x_2^0), \psi(z)\rangle.
\end{aligned}$$

Here we have used the fact that $\mathcal{F}[\psi]$ decays rapidly.

From (5.19) we can choose $\delta > 0$ such that when $|x_2 - x_2^0| \leq \delta$ and $x_2 \geq 0$ we have $\langle \tilde{\varphi}(z,x_2), \chi(z)\rangle > 0$. Choose a cut-off function $\eta(x_2)$ supported in $\{x_2 \geq 0 : |x_2 - x_2^0| < \delta\}$. Then

$$\int_0^{+\infty} \langle \tilde{\varphi}(z,x_2), \eta(x_2)\chi(x_1)\rangle dx_2 > 0,$$

which contradicts (5.16). Now Claim 2 is verified. □

Now we finish the proof of Step 2. Let φ be a non-trivial bounded solution of (5.2) with $\beta = \beta_\gamma$. Let $\tilde{\varphi}(z, x_2) = \mathcal{F}[\varphi](z, x_2)$ be the Fourier transform in x_1. Then $\tilde{\varphi}$ is a distribution with parameter x_2. From Claim 2, for all $x_2 \ge 0$, the support of $\tilde{\varphi}(\cdot, x_2)$ is contained in $\{z_\gamma\}$. Hence $\tilde{\varphi}(z, x_2)$ can be represented by

$$\tilde{\varphi}(z, x_2) = \sum_{k=0}^{N(x_2)} c_k(x_2) \frac{d^k}{dz^k} \delta(z - z_\gamma),$$

where $N(x_2)$ and $c_k(x_2)$ may depend on x_2. For fixed $x_2 \ge 0$, taking the inverse Fourier transform in z we get

$$\varphi(x_1, x_2) = \mathcal{F}^{-1}[\tilde{\varphi}](x_1, x_2) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{N(x_2)} c_k(x_2)(-ix_1)^k \exp(iz_\gamma x_1).$$

Since φ is bounded in $\mathbb{R}^2_+$, we have $c_k(x_2) = 0$ for all $k > 0$ and $x_2 \ge 0$. Denote $v(x_2) = c_0(x_2)/\sqrt{2\pi}$; we have

$$\varphi(x_1, x_2) = v(x_2) \exp(iz_\gamma x_1). \tag{5.20}$$

$v(x_2)$ is a bounded smooth function since φ is. Plugging (5.20) into (5.2), we see that $v(x_2)$ satisfies

$$\begin{cases} v'' - (x_2 + z_\gamma)^2 v + \beta_\gamma v = 0 & \text{for } x_2 > 0, \\ v'(0) = \gamma v(0). \end{cases}$$

Hence $v(x_2) = cu_\gamma(x_2)$ and $\varphi = c\varphi_\gamma$. Now Step 2 is completed and Theorem 5.3 is proved.

Remark 5.1. Let $\varphi(x_1, x_2)$ be a bounded solution of equation (5.2) and $\tilde{\varphi}(z, x_2)$ be the Fourier transform of φ in x_1 in the sense of distributions. We can define the derivative $\partial_2 \tilde{\varphi}(z, x_2)$ as a distribution given by

$$\langle \partial_2 \tilde{\varphi}(z, x_2), \eta \rangle = \lim_{\varepsilon \to 0} \langle \frac{\tilde{\varphi}(z, x_2 + \varepsilon) - \tilde{\varphi}(z, x_2)}{\varepsilon}, \eta \rangle,$$

if the limit exists for all $\eta \in C_0^\infty(\mathbb{R}^1)$. Similarly, we define $\partial_{22}\tilde{\varphi}(z, x_2)$. It is easy to verify that

$$\partial_2 \tilde{\varphi}(z, x_2) = \mathcal{F}[\partial_2 \varphi](z, x_2), \qquad \partial_{22} \tilde{\varphi}(z, x_2) = \mathcal{F}[\partial_{22} \varphi](z, x_2).$$

Then, as a distribution (in z) with parameter x_2, $\tilde{\varphi}$ satisfies the following

$$\begin{cases} \partial_{22}\tilde{\varphi} - (x_2 + z)^2 \tilde{\varphi} + \beta_\gamma \tilde{\varphi} = 0 & \text{for } x_2 > 0, \\ \partial_2 \tilde{\varphi} = \gamma \tilde{\varphi} & \text{for } x_2 = 0. \end{cases}$$

Remark 5.2. Denote $\mathcal{Z}(\beta) = \{z \in \mathbb{R}^1 : \ \beta_\gamma(z) = \beta\}$. Recall that $\beta^*_\gamma = \beta_\gamma$. From Lemma 5.2 it follows that

$$\mathcal{Z}(\beta) = \begin{cases} \emptyset & \text{if } \beta < \beta_\gamma, \\ \{z_\gamma\} & \text{if } \beta = \beta_\gamma, \\ \{z_1(\beta), \ z_2(\beta)\} & \text{if } \beta_\gamma < \beta < 1, \\ \{-\gamma\} & \text{if } \beta = 1, \\ \{z(\beta)\} & \text{if } \beta > 1. \end{cases} \tag{5.21}$$

We know that equation (5.2) has only the trivial solution for $\beta < \beta_\gamma$ (Lemma 5.1), and has only one linearly independent solution φ_γ given in (5.10) (Theorem

5.3). Denote by $u(z, x_2)$ the eigenfunction of equation (5.3) with $\beta = \beta_\gamma(z)$. We see that, when $\beta_\gamma < \beta < 1$, equation (5.2) has 2 linearly independent bounded solutions $\exp(iz_1(\beta)x_1)u(z_1(\beta), x_2)$ and $\exp(iz_2(\beta)x_1)u(z_2(\beta), x_2)$; and when $\beta = 1$, equation (5.2) has a bounded solution $\exp(-i\gamma x_1)u(-\gamma, x_2)$. It would be interesting to know if they are the only bounded solutions for $\beta_\gamma < \beta \le 1$. If the answer is positive, then when $\beta_\gamma < \beta < 1$, the only bounded solution of equation (1.6) is

$$\begin{aligned}\psi(x_1, x_2) = c_1 &\exp(ix_1x_2/2 + iz_1(\beta)x_1)u(z_1(\beta), x_2)\\ &+ c_2 \exp(ix_1x_2/2 + iz_2(\beta)x_1)u(z_2(\beta), x_2),\end{aligned}$$

and when $\beta = 1$, the only bounded solution of (1.6) is

$$c \exp(ix_1x_2/2 - i\gamma x_1)u(-\gamma, x_2),$$

where $u(z, x_2)$ is the eigenfunction of (5.3) with $\beta = \beta_\gamma(z)$.

Remark 5.3. Given a vector field A defined in $\mathbb{R}^2_+$ with *curl* $A \not\equiv$ a constant, and $\gamma \ge 0$, we can define $\beta_\gamma(A)$ in an obvious way. It is interesting to study the existence of the minimizers of $\beta_\gamma(A)$. We have only a partial result for this problem. Furthermore, if $A \in C^2(\overline{\mathbb{R}^2_+})$, then we can use the argument in the proof of Theorem 1 to show that

$$\limsup_{h\to\infty} \frac{\beta_{h\gamma}(hA)}{|h|} \le \min\left\{ \inf_{x\in\mathbb{R}^2_+} |curl\ A(x)|,\ \beta_\gamma \inf_{x\in\partial\mathbb{R}^2_+} |curl\ A(x)| \right\}, \tag{5.22}$$

where $\beta(\gamma)$ is the number given by (1.7). Moreover, if the minimizers of $\beta_{h\gamma}(hA)$ exist for every $h \ne 0$, then by using of the rescaling argument and careful estimation of the asymptotic behavior of the minimizers as $h \to \infty$ we can show that equality holds in (5.22). Since the proof is similar to what we have done in [LP2], we omit the details here.

6. A Dirichlet Eigenvalue Problem in $\mathbb{R}^2_+$

In this section we discuss the Dirichlet problem (1.9) and prove Theorem 4. Let λ_0 be given by (1.10). Similarly to Lemma 5.1, we have

Lemma 6.1. *For all $\lambda < \lambda_0$, equation* (1.9) *has no nontrivial bounded solution.*

As in Sections 4, 5, we set $\psi = \exp(ix_1x_2/2)\varphi$ and change (1.9) to

$$\begin{cases}\Delta\varphi + 2ix_2\partial_1\varphi - |x_2|^2\varphi + \lambda\varphi = 0 & \text{in } \mathbb{R}^2_+,\\ \varphi = 0 & \text{on } \partial\mathbb{R}^2_+.\end{cases} \tag{6.1}$$

Suppose that $\varphi \in L^2(\mathbb{R}^2_+)$ is a solution of (6.1). Denote by $f(z, x_2)$ the L^2 Fourier transform of φ in the variable x_1. For every fixed z, $f(z, t)$ is a solution of the following problem in t:

$$\begin{cases}f'' = \left[(z+t)^2 - \lambda\right] f & \text{for } t > 0,\\ f(0) = 0, \quad f(+\infty) = 0.\end{cases} \tag{6.2}$$

Fix z and denote by $\lambda(z)$ the first eigenvalue of (6.2). Then

$$\lambda(z) = \inf_{u\in W_0^{1,2}(\mathbb{R}^1_+)} \frac{\int_0^{+\infty}\{|u'|^2 + (z+t)^2|u|^2\}dt}{\int_0^{+\infty}|u|^2dt}. \tag{6.3}$$

In the following we denote by $f(z, t)$ the positive eigenfunction of (6.2) with $\lambda = \lambda(z)$.

Lemma 6.2. *$\lambda(z)$ is a continuous and strictly increasing function of z, with*

$$\lambda(z) > 1 \text{ for all } z \in \mathbb{R}^1, \quad \lambda(0) = 3, \quad \lambda(z) > 3 + z^2 \text{ for all } z > 0$$

and

$$\lim_{z\to-\infty} \lambda(z) = \inf_{z\in\mathbb{R}^1} \lambda(z) = 1.$$

Proof. Denote $f_z(t) = f(z,t)$. Then $f_z'(t) \not\equiv -(z+t)f_z(t)$. Hence,

$$\int_0^{+\infty} \{|f_z'|^2 + (z+t)^2|f_z|^2\}dt > -2\int_0^{+\infty} (z+t)f_z f_z' dt$$
$$= -\int_0^{+\infty} (z+t)df_z^2 = \int_0^{+\infty} |f_z|^2 dt.$$

So $\lambda(z) > 1$.

For $z = 0$, f_0 satisfies

$$f'' - t^2 f + \lambda f = 0, \tag{6.4}$$

which is the Hermite equation. Extend f_0 to $\mathbb{R}^1$ by setting $f_0(-t) = -f_0(t)$. Then f_0 is an odd solution of (6.4) in $\mathbb{R}^1$, and $\lambda(0)$ is the least number for which (6.4) has odd eigenfunctions. Hence, $\lambda(0) = 3$.

For $z > 0$ we have $\lambda(z) > \lambda(0) + z^2 \ge 3 + z^2$.

Set $u_n(t) = \eta_n(z+t)\exp(z+t)^2/2)$, where $\eta_n(t)$ is a smooth cut-off function supported in $[-2n, 2n]$ such that $\eta_n(t) = 1$ if $|t| \le n$, $0 \le \eta_n(t) \le 1$, and $|\eta_n'(t)| \le 2/n$. Then

$$\limsup_{z\to-\infty} \lambda(z) \le 1 + \frac{\int_{-\infty}^{+\infty} |\eta_n'(t)|^2 \exp(-t^2)dt}{\int_{-\infty}^{+\infty} |\eta_n(t)|^2 \exp(-t^2)dt}$$
$$\le 1 + \frac{4}{n^2}\frac{\int_{-\infty}^{+\infty} \exp(-t^2)dt}{\int_{-n}^{n} \exp(-t^2)dt}.$$

By letting $n \to \infty$, we get $\limsup_{z\to-\infty} \lambda(z) \le 1$.

The continuity of $\lambda(z)$ can be proved as Lemma 7.7 of Section 7. Obviously, $\lambda(z)$ is strictly increasing for $z \ge 0$. For $z < 0$, the proof is given in Section 7; see Corollary 7.3(2). □

Proof of Theorem 4. First we show that $\lambda_0 = \inf_z \lambda(z) = 1$. This can be done as in Step 1 of the proof of Theorem 5.3. In fact, using the argument there we can show that $\lambda_0 \ge \inf_z \lambda(z) = 1$. For every constant z we replace φ_n used there by

$$\eta_n(x_1)\exp(izx_1)f(z, x_2),$$

where $f(z,t)$ is the eigenfunction of (6.2) with $\lambda = \lambda(z)$. Then we can show that $\lambda_0 \le \lambda(z)$. Hence, $\lambda_0 \le \inf_z \lambda(z) = 1$.

Note that for every z, $\lambda(z) > 1$ and the second eigenvalue of (6.2) is greater than 3. Hence, for every $\lambda \le \lambda_0 = 1$, (6.1) has no nontrivial bounded solution. For $\lambda > 1$, (6.1) has a bounded solution given by

$$\varphi = c\exp(iz(\lambda)x_1)f(z(\lambda), x_2), \tag{6.5}$$

where z is the unique number such that $\lambda(z) = \lambda$. The proof of Theorem 4 is complete. □

Note that when $1 < \lambda \le 3$, (6.5) is the only nontrivial bounded solution of (6.1). So we have the following.

Corollary 6.3. *For $1 < \lambda \le 3$, equation* (1.9) *has exactly one linearly independent bounded solution:*

$$\psi = c\exp(ix_1x_2/2 + iz(\lambda)x_1)f(z(\lambda), x_2), \tag{6.6}$$

where $z(\lambda)$ is the unique number such that $\lambda(z) = \lambda$, and $f(z(\lambda), x_2)$ is the eigenfunction of (6.2) *with $\lambda = \lambda(z)$.*

7. Properties of $\beta_\gamma(z)$ and $\lambda(z)$

In this section we prove Lemma 5.2 concerning the first eigenvalue $\beta_\gamma(z)$ of equation (5.3) with parameter z, and prove other results needed in Sections 5 and 6.

It is easy to see that for every $z \in \mathbb{R}^1$ and $\gamma \ge 0$, $\beta_\gamma(z)$ is achieved. It is a simple eigenvalue, and the associated eigenspace is spanned by a positive eigenfunction $u = u(z,t)$. $\beta_\gamma(z)$ is the only eigenvalue of (5.3) which has a positive bounded eigenfunction. In order to discuss the properties of $\beta_\gamma(z)$ and $u(z,t)$, we make a series of change of variables. Let $x = z + t$, $y(x) = u(t) = u(x - z)$. Then, y satisfies

$$\begin{cases} y'' = (x^2 - \beta)y, & x > z, \\ y'(z) = \gamma y(z), & y(+\infty) = 0. \end{cases} \tag{7.1}$$

Write $y(x) = \exp(-x^2/2)w(x)$. Then

$$\begin{cases} w'' - 2xw' + (\beta - 1)w = 0, & x > z, \\ w'(z) = (z + \gamma)w(z), \\ w(x) = o(\exp(x^2/2 + cx)) & \text{as } x \to +\infty, \end{cases} \tag{7.2}$$

Write $w(x) = |x|^{(\beta-1)/2}v(x)$. Then for $x > z$, $x \ne 0$,

$$v'' - \left(2x + \frac{1-\beta}{x}\right)v' + \frac{(1-\beta)(3-\beta)}{4x^2}v = 0. \tag{7.3}$$

We first discuss the solutions of the equation

$$w'' - 2xw' + (\beta - 1)w = 0, \quad x \in \mathbb{R}^1, \tag{7.4}$$

which are positive near $+\infty$ and grow slower than $\exp(x^2/2 + cx)$.

Lemma 7.1. *For $0 \le \beta \le 3$, let $w(x)$ be a solution of equation* (7.4) *which is positive near $+\infty$. Then:*

(A) *For $0 \le \beta < 1$, $w(x)$ is a monotone function near $+\infty$.*

(A1) *If $w(x)$ is decreasing near $+\infty$, then there exists a positive constant a such that as $x \to +\infty$*

$$w(x) = ax^{(\beta-1)/2}\left[1 - \frac{(1-\beta)(3-\beta)}{16x^2} + O\left(\frac{1}{x^4}\right)\right], \tag{7.5}$$

and as $x \to -\infty$

$$w(x) \ge \frac{|w'(0)|}{2|x|}\exp(x^2) - C. \tag{7.6}$$

For every $0 \le \beta < 1$, equation (7.4) has at most one linearly independent solution with the above properties.

(A2) *If $w(x)$ is increasing, then* (7.6) *holds as $x \to +\infty$.*

(B) *For $\beta = 1$, the solutions of equation* (7.4) *are*

$$w(x) = c_1 + c_2 \int_0^x \exp(t^2) dt. \tag{7.7}$$

(C) *For $1 < \beta < 3$, $w(x)$ is increasing near $+\infty$ and must change its sign. If $w(x)$ grows slower than $\exp(x^2/2 + cx)$, then $w(x)$ satisfies* (7.5) *as $x \to +\infty$, and has a unique zero point, which must be negative. For every $1 < \beta < 3$, equation* (7.4) *has at most one linearly independent solution having these properties.*

(D) *For $\beta = 3$, if $w(x)$ grows slower than $\exp(x^2/2 + cx)$, then $w(x) = cx$.*

(E) *For $\beta > 3$, $w(x)$ has at least two zero points, of which one is positive and the other is negative.*

Proof. We prove this lemma in 4 steps. □

Step 1. Let $0 \le \beta < 1$.

From the assumption that $w(x)$ is positive near $+\infty$,

$$(\exp(-x^2)w')' = (1-\beta)\exp(-x^2)w > 0, \quad \text{for large } x. \tag{7.8}$$

Hence, $\exp(-x^2)w'$ is increasing for large x. Thus, there exists x_0 such that either $w'(x) < 0$ for all $x > x_0$, or $w'(x) > 0$ for all $x > x_0$.

Case (A1). $w'(x) < 0$ for $x > x_0$.

Observe that $w'(x) < 0$ must hold for all x. Moreover, $\lim_{x\to+\infty} \exp(-x^2)w'(x) = 0$. In fact, (7.8) implies that $\lim_{x\to+\infty} \exp(-x^2)w'(x) = -c$ exists and $c \ge 0$. If $c > 0$, then $w'(x) \le -c\exp(x^2)$. So $w(x)$ is negative near $+\infty$, a contradiction.

Integrating (7.8) from x to $+\infty$ we obtain

$$w'(x) = -(1-\beta)\exp(x^2)\int_x^{+\infty} \exp(-t^2)w(t)dt. \tag{7.9}$$

Clearly, $w(+\infty)$ exists, and it must be zero. Otherwise, if $w(+\infty) > 0$, then $w(x) > w(+\infty)$ for all x, and, from (7.9),

$$\begin{aligned} w'(x) &< -(1-\beta)w(+\infty)\exp(x^2)\int_x^{+\infty}\exp(-t^2)dt \\ &= -\frac{(1-\beta)w(+\infty)}{2x} + O\left(\frac{1}{x^2}\right) \end{aligned}$$

as $x \to +\infty$, which implies $\lim_{x\to+\infty} w(x) = -\infty$, a contradiction.

We integrate (7.9) to get

$$\begin{aligned} w(x) &= (1-\beta)\int_x^{+\infty}\exp(s^2)ds\int_s^{+\infty}\exp(-t^2)w(t)dt \\ &= (1-\beta)\int_x^{+\infty}\exp(-t^2)\left[\int_x^t \exp(s^2)ds\right]w(t)dt. \end{aligned} \tag{7.10}$$

Let $v(x) = |x|^{(1-\beta)/2}w(x)$. Then $v(x)$ satisfies (7.3), and hence, for $x > 0$,

$$[|x|^{\beta-1}\exp(-x^2)v(x)']' = -\frac{(1-\beta)(3-\beta)}{4}|x|^{\beta-3}\exp(-x^2)v(x). \tag{7.11}$$

From (7.9) we have $\lim_{x\to+\infty} x^{\beta-1}\exp(-x^2)v'(x) = 0$. For $x > 0$ we integrate (7.11) from x to $+\infty$, and get

$$v'(x) = \frac{(1-\beta)(3-\beta)}{4} x^{1-\beta}\exp(x^2)\int_x^{+\infty} t^{\beta-3}\exp(-t^2)v(t)dt. \tag{7.12}$$

Hence, $v(x)$ is strictly increasing for $x > 0$. Using (7.12) and the fact that $v(x) = |x|^{(1-\beta)/2}w(x) \le C|x|^{(1-\beta)/2}$, we have

$$\begin{aligned} v'(x) &\le \frac{(1-\beta)(3-\beta)C}{4} x^{1-\beta}\exp(x^2)\int_x^{+\infty} t^{-(5-\beta)/2}\exp(-t^2)dt \\ &\le \frac{(1-\beta)(3-\beta)C}{8} x^{-(5+\beta)/2}. \end{aligned}$$

Hence, $v(x)$ is bounded and $v(+\infty)$ exists. Using (7.12) again, we have

$$v'(x) \le Cx^{-3}, \qquad v(x) = v(+\infty)\left[1 - O(\frac{1}{x^2})\right] \quad \text{as } x \to +\infty.$$

Plugging these back to (7.12), we get,

$$v'(x) = \frac{(1-\beta)(3-\beta)v(+\infty)}{8x^3} + O(\frac{1}{x^5}) \quad \text{as } x \to +\infty$$

and

$$v(x) = v(+\infty)\left[1 - \frac{(1-\beta)(3-\beta)}{16x^2}\right] + O(\frac{1}{x^4}) \quad \text{as } x \to +\infty. \tag{7.13}$$

Therefore as $x \to +\infty$, (7.5) holds with $a = v(+\infty) > 0$.

For $x < 0$ we have, from (7.8),

$$\begin{aligned} w'(x) &= w'(0)\exp(x^2) - (1-\beta)\exp(x^2)\int_x^0 \exp(-t^2)w(t)dt, \\ w(x) &= w(0) - w'(0)\int_x^0 \exp(s^2)ds \\ &\qquad + (1-\beta)\int_x^0 \exp(s^2)ds\int_s^0 \exp(-t^2)w(t)dt, \end{aligned}$$

which gives (7.6) for $x \to -\infty$ since $w'(0) < 0$.

Now we show that for $0 \le \beta < 1$, equation (7.4) has at most one linearly independent solution satisfying (7.5). Assume that w_1 and w_2 are two solutions. Then,

$$\lim_{x\to+\infty} x^{(1-\beta)/2}w_j(x) = a_j > 0,$$

$j = 1, 2$. Denote $w(x) = a_2w_1(x) - a_1w_2(x)$. Then, w is a solution of (7.4) and satisfies

$$\lim_{x\to+\infty} x^{(1-\beta)/2}w(x) = 0. \tag{7.14}$$

Suppose $w(x) \not\equiv 0$. By using the above argument, one can show that $w(x)$ is a monotone function near ∞. We may assume $w(x) > 0$ for all large x. Then as $x \to +\infty$, (7.5) must hold, that is, $\lim_{x\to+\infty} x^{(1-\beta)/2}w(x) = a > 0$. But this contradicts (7.14). So $w(x) \equiv 0$; namely, $w_1(x) = (a_1/a_2)w_2(x)$.

Case (A2). $w'(x) > 0$ for all $x > x_0$.

Set $\tilde{w}(x) = w(-x)$. Then, $\tilde{w}'(x) < 0$. So $\tilde{w}(x)$ satisfies (7.6) as $x \to -\infty$; that is, $w(x)$ satisfies (7.6) as $x \to +\infty$.

Now Conclusion (A) is proved. Conclusion (B) is obvious.

Step 2. Let $1 < \beta < 3$, and let $w(x)$ be a solution of (7.4), $w(x) > 0$ for $x > x_0$.

Claim 1. $w'(x) > 0$ for all $x > x_0$, and $\lim_{x\to+\infty} \exp(-x^2)w'(x) = 0$.

In fact, when $x > x_0$, from (7.4) we have

$$(\exp(-x^2)w')' = -(\beta-1)\exp(-x^2)w < 0, \tag{7.15}$$

so $\exp(-x^2)w'(x)$ is decreasing. Since $w(x) > 0$ for all $x > x_0$, we have $w'(x) > 0$ for all $x > x_0$ (this is true for all $\beta > 1$). Now $\lim_{x\to+\infty} \exp(-x^2)w'(x) = c$ exists, $c \geq 0$. If $c > 0$, from $\exp(-x^2)w'(x) > c$ we have

$$w(x) > \frac{c}{2x}\exp(x^2) - C_1. \tag{7.16}$$

Choosing x_1 large enough, integrating (7.15) from x_1 to x, and using (7.16), we get, for $x > x_1$,

$$\exp(-x^2)w'(x) < -\frac{(\beta-1)c}{2}\log(x) + C_2,$$

which contradicts the fact that $w'(x) > 0$ for all $x > x_0$. So Claim 1 is true.

Now (7.9) also holds. Since $w(x)$ is increasing, from (7.9) we have, for x large enough,

$$w'(x) > (\beta-1)w(x)\left[\frac{1}{2x} - \frac{1}{4x^3}\right].$$

Hence,

$$w(x) > Cx^{(\beta-1)/2} \quad \text{for large } x. \tag{7.17}$$

Denote $v(x) = |x|^{-(\beta-1)/2}w(x)$ as before. Then (7.11) holds. For $x > x_0$ we have $v(x) > 0$ and $(|x|^{\beta-1}\exp(-x^2)v')' > 0$. So $|x|^{\beta-1}\exp(-x^2)v'(x)$ is increasing for $x > x_0$.

Claim 2. If $w(x)$ grows slower than $\exp(x^2/2+cx)$ near $+\infty$, we must have $v'(x) < 0$ for all $x > \max\{x_0, 0\}$, which implies

$$v(x) > 0, \quad v'(x) < 0 \quad \text{for all } x > 0. \tag{7.18}$$

Otherwise, there exists a point $x_1 > \max\{x_0, 0\}$ so that $v'(x_1) \geq 0$. But now (7.11) implies

$$x^{\beta-1}\exp(-x^2)v'(x) > c_1 \quad \text{for } x > x_1,$$

where c_1 is a positive constant. Hence, $v'(x) > c_1x^{1-\beta}\exp(x^2)$ and

$$v(x) > v(x_1) + \int_{x_1}^{x} c_1t^{1-\beta}\exp(t^2)dt > \frac{c_1}{2}x^{-\beta}\exp(x^2) - c_2,$$

$$w(x) = |x|^{(\beta-1)/2}v(x) > \frac{c_1}{2}x^{-(1+\beta)/2}\exp(x^2) - c_2x^{(\beta-1)/2}$$

for x large. Therefore, if $w(x)$ grows slower than $\exp(x^2/2 + cx)$, v must satisfy (7.18). So Claim 2 is true.

From (7.17) and (7.18) we have that for $1 < \beta < 3$, if $w(x)$ grows slower than $\exp(x^2/2 + cx)$, then $v(x)$ is decreasing and bounded away from zero for x large enough. Hence, $v(+\infty)$ exists and is positive. Therefore, (7.5) holds for $x \to +\infty$. As in Step 1 we can show that there exists at most one linearly independent solution with these properties.

Step 3. Let $\beta = 3$.

Set $v(x) = w(x)/x$. Again (7.11) holds for x large. So $x^2 \exp(-x^2)v' \equiv c$. As in Step 2, we can show that if $w(x)$ grows slower than $\exp(x^2/2 + cx)$ near $+\infty$, then $c = 0$. Hence, $v' \equiv 0$ and $w(x) = cx$.

Step 4. Proof of Conclusions (C), (E), completed.

We first show the following

Claim 3. When $\beta > 1$, every nontrivial solution of (7.4) must change its sign.

Suppose that there is a solution w such that $w(x) > 0$ for all x. In the proof of Claim 1 in Step 2, we have seen that $w'(x) > 0$ for all x. Set $\tilde{w}(x) = w(-x)$. Then $\tilde{w}$ is also a positive solution of (7.4) but $\tilde{w}'(x) < 0$, which is impossible. So w must change its sign. Claim 3 is proved.

Now we consider the distribution of the zero points of a solution $w(x)$ of (7.4) which is positive near $+\infty$ and grows slower than $\exp(x^2/2 + cx)$. Denote, again, $v(x) = |x|^{(1-\beta)/2}w(x)$.

For $1 < \beta < 3$, $v(x)$ is decreasing and positive for $x > 0$; see (7.18) (here we need the assumption that w grows slower than $\exp(x^2/2 + cx)$ at $+\infty$). Hence, $w(x) > 0$ for all $x > 0$. Next, we show that $w(0) > 0$. Suppose that $w(0) = 0$; then $w'(0) \neq 0$, and as $x \to 0+$,

$$\begin{aligned} v(x) &= x^{(1-\beta)/2}w(x) = x^{(1-\beta)/2}[w'(0)x + O(x^2)] \\ &= w'(0)x^{(3-\beta)/2} + O(x^{(5-\beta)/2}) \to 0, \end{aligned}$$

which implies $v(x) \equiv 0$ since $v(x)$ is decreasing, a contradiction. Therefore, all the zero points of $w(x)$ are negative. Suppose that $w(x)$, and hence $v(x)$, has at least two zero points which, as we proved in the above, are negative. Then, $v(x)$ must have a negative local minimum value. However, (7.3) shows that for $1 < \beta < 3$, $v(x)$ cannot have a negative local minimum value. Hence, $v(x)$ cannot have more than one zero point. Now conclusion (D) is proved.

For $\beta > 3$, using (7.11) we can show that $v(x)$ is increasing. Denote the largest zero point of w by x_0. Then, $x_0 > 0$. In fact, if $x_0 < 0$, then $v(0+) = +\infty$; and if $x_0 = 0$, then $v(x) = w'(0)x^{-(\beta-3)/2}[1 + O(x)] \to +\infty$ again as $x \to 0+$. Now we show that $w(x)$ has at least one negative zero point. Otherwise, $w(x)$ does not change its sign in the region $x < 0$. Set $\tilde{w}(x) = w(-x)$ if $w(x) > 0$ for $x < 0$ and $\tilde{w}(x) = -w(-x)$ if $w(x) < 0$ for $x < 0$. Then, $\tilde{w}(x)$ is positive near $+\infty$, and the above argument shows that $\tilde{w}(x)$ must have at least one positive zero point, a contradiction. Conclusion (E) is proved.

Denote by $\mathcal{B}$ the set of β for which (7.4) has a solution which is positive near $+\infty$ and grows slower than $\exp(x^2/2 + cx)$. From Lemma 7.1 we see that if $\beta \in \mathcal{B}$ and $0 \leq \beta \leq 3$, then (7.4) has exactly one solution having the property

$$\lim_{x \to +\infty} x^{(1-\beta)/2}w(x) = 1.$$

We denote this solution by w_β.

From Lemma 7.1 we have the following consequences.

Corollary 7.2. *Consider the equation*

$$y'' = (x^2 - \beta)y. \tag{7.19}$$

For $0 \le \beta < 1$, *equation* (7.19) *has no bounded positive solution on* $\mathbb{R}^1$. *For* $0 \le \beta \le 3$, *equation* (7.19) *has no solution which changes its sign and remains positive and bounded near* $+\infty$.

Corollary 7.3. *(1) For every* z, *the second eigenvalue of equation* (5.3) *is larger than* 1.

(2) The first eigenvalue $\lambda(z)$ *of equation* (6.2) *is strictly increasing. We have* $\lambda(z) > 1$ *for every* $z \in \mathbb{R}^1$ *and* $\lambda(z) > 3$ *for* $z > 0$. *The second eigenvalue of equation* (6.2) *is larger than* 3.

Proof. We only prove (2). Assume that $f(t)$ is an eigenfunction of (6.2) associated with the eigenvalue β. Let $y(x) = f(x - z)$ and $w(x) = \exp(x^2/2)y(x)$. Then $w(z) = 0$, and w can be extended to a solution of (7.4).

Assume that $\beta = \lambda(z)$ is the first eigenvalue of (6.2) and $0 \le \lambda(z) < 1$. Then $f(t)$ does not change its sign for $t > 0$, and we may assume that $w(x)$ is positive near $+\infty$. From Lemma 7.1(A) we see that $w(x)$ must be decreasing, so $w(x) > 0$ for all $x \ge z$. In particular, $w(z) > 0$, a contradiction. From Lemma 7.1(B), β cannot be 1. Hence, $\lambda(z) > 1$.

If $z > 0$, then w has a positive zero point. Lemma (7.1)(C) implies $\beta = \lambda(z) > 3$.

Moreover, for $1 < \lambda(z) = \beta < 3$, $w(x) = cw_\beta(x)$. From Lemma 7.1(C), w_β has a unique zero point x_β for $1 < \beta < 3$. Therefore, $z = x_\beta$. In other words, $\lambda(z) = \beta$ iff $z = x_\beta$. Hence, the function $\lambda(z)$ is a one-to-one mapping from $(-\infty, 0)$ to $(1, 3)$. From its variational characterization (6.3) we see $\lambda(z)$ is continuous. Since $\lambda(-\infty) = 1$ and $\lambda(0) = 3$, we see that $\lambda(z)$ is strictly increasing in $(-\infty, 0)$. Using (6.3) again, we see that $\lambda(z)$ is strictly increasing in $(0, +\infty)$. So $\lambda(z)$ is strictly increasing on $\mathbb{R}^1$.

Now assume that β is the second eigenvalue of (6.2). Then, $y(t)$ must change its sign on $(z, +\infty)$, and hence $w(x)$ has at least two zero points. From Lemma 7.1(C) we see that $\beta > 3$. The proof is complete. □

Corollary 7.4. *Assume that* $\gamma \ge 0$.

1. *If* $0 < \beta < 1$ *and if* w *is a positive solution of equation* (7.2), *then* $z + \gamma < 0$, *and* $w'(x) < 0$ *for all* $x > z$, *and* (7.5) *holds.*
2. $\beta = 1$ *is an eigenvalue of (7.1) iff* $z = -\gamma$ *and* $u(t) = c\exp(-t^2)$.
3. *If* $\beta > 1$ *and if* w *is a positive solution of equation* (7.2), *then* $z + \gamma > 0$, *and* $w'(x) > 0$ *for all* $x > z$.
4. $\beta_\gamma(z) < 1$ *if and only if* $z < -\gamma$; $\beta_\gamma(z) > 1$ *if and only if* $z > -\gamma$.
5. *For all* $\gamma \ge 0$ *and* $z \le 0$, $\beta_\gamma(z) < 3$.

The proof is straightforward and is omitted.

As in Section 5 we denote $\mathcal{Z}(\beta) = \{z \in \mathbb{R}^1 : \beta_\gamma(z) = \beta\}$, and $\mathcal{Z}^-(\beta) = \{z \le 0 : \beta_\gamma(z) = \beta\}$. From Corollary 7.4(4) we know that $\mathcal{Z}(\beta) = \mathcal{Z}^-(\beta)$ if $0 \le \beta \le 1$. Denote by $\#\mathcal{Z}$ the number of points in the set $\mathcal{Z}$.

Lemma 7.5. $\#\mathcal{Z}(\beta) \le 2$ *if* $0 \le \beta < 1$; $\#\mathcal{Z}^-(\beta) \le 1$ *if* $1 \le \beta \le 3$.

Proof. Obviously, if $0 \le \beta \le 3$ and if $\mathcal{Z}(\beta) \ne \emptyset$, then $\beta \in \mathcal{B}$. If w is a positive solution of (7.2), then $w(x) = cw_\beta(x)$. Therefore, $\beta_\gamma(z) = \beta$ if and only if

$$w'_\beta(z) = (z + \gamma)w_\beta(z). \tag{7.20}$$

Case 1. $1 < \beta \leq 3$. In this case $w_\beta(x)$ has a unique zero point x_β, $x_\beta < 0$, and $w'_\beta(x) > 0$ for all $x > x_\beta$. For $x_\beta < x \leq 0$, from (7.4),

$$\left(\frac{w'_\beta}{w_\beta}\right)' = \frac{w_\beta[2xw'_\beta - (\beta-1)w_\beta] - (w'_\beta)^2}{(w_\beta)^2} < 0.$$

Hence, w'_β/w_β is strictly decreasing over $(x_\beta, 0]$ and the equation $w'_\beta(z)/w_\beta(z) = z+\gamma$ has at most one solution in $(x_\beta, 0]$. That is, there exists at most one $z \leq 0$ such that $\beta_\gamma(z) = \beta$. Hence, $\#\mathcal{Z}^-(\beta) \leq 1$ if $1 < \beta \leq 3$.

Case 2. $\beta = 1$. Then $\#\mathcal{Z}(1) = 1$ because $\mathcal{Z}(1) = \{-\gamma\}$.

Case 3. $0 < \beta < 1$. Using (7.9) and (7.10), we have that, for $0 < \beta < 1$, z is a solution of (7.20) if and only if

$$\int_z^\infty \exp(-t^2)w_\beta(t)\left[\exp(z^2) + (z+\gamma)\int_z^t \exp(s^2)ds\right]dt = 0. \tag{7.21}$$

Denote

$$\begin{aligned} H(z,t) &= \exp(-t^2)\left[\exp(z^2) + (z+\gamma)\int_z^t \exp(s^2)ds\right], \\ G_\beta(z) &= \int_z^{+\infty} H(z,t)w_\beta(t)dt. \end{aligned} \tag{7.22}$$

Then, for every $0 < \beta < 1$, $\beta_\gamma(z) = \beta$ if and only if $G_\beta(z) = 0$.

For $0 < \beta < 1$ and $z + \gamma > 0$ we have $H(z,t) > 0$ for all $t \geq z$, so $G_\beta(z) > 0$.

For $0 < \beta < 1$ and $z + \gamma < 0$, since $w_\beta(t) > 0$ and $w'_\beta(t) < 0$ for all t, we have

$$\begin{aligned} G''_\beta(z) = &-w'_\beta(z) - (z-\gamma)w_\beta(z) \\ &+ 2z(z-\gamma)\exp(z^2)\int_z^{+\infty}\exp(-t^2)w_\beta(t)dt > 0. \end{aligned}$$

Therefore, $G_\beta(z)$ has at most two zero points. This completes the proof.

Next, we estimate $\beta_\gamma(z)$.

Lemma 7.6. *Let $\gamma \geq 0$, and let $u = u(z,t)$ be a positive eigenfunction of equation* (5.3) *with $\beta = \beta_\gamma(z)$. Then*

$$1 + \frac{(z+\gamma)u^2(z,0)}{\int_0^{+\infty}|u(z,t)|^2dt} \leq \beta_\gamma(z) \leq 1 + \frac{(z+\gamma)\exp(-z^2/2)}{\int_z^{+\infty}\exp(-s^2)ds}. \tag{7.23}$$

Therefore,

$$\beta_\gamma(z)\begin{cases} > 1 & \text{if } z+\gamma > 0, \\ = 1 & \text{if } z+\gamma = 0, \\ < 1 & \text{if } z+\gamma < 0. \end{cases} \tag{7.24}$$

Moreover,

$$\lim_{z\to-\infty}\beta_\gamma(z) = 1. \tag{7.25}$$

Proof. Using $u(t) = \exp(-(z+t)^2/2)$ as a test function, we get the right hand of (7.23). Denote $u_z(t) = u(z,t)$. Note that

$$\begin{aligned}\beta_\gamma(z)\int_0^{+\infty}|u_z|^2dt &= \gamma|u_z(0)|^2 + \int_0^{+\infty}\{|u_z'|^2 + (z+t)^2|u_z|^2\}dt\\ &\geq \gamma|u_z(0)|^2 - 2\int_0^{+\infty}(z+t)u_zu_z'dt\\ &= (\gamma+z)|u_z(0)|^2 + \int_0^{+\infty}|u_z|^2dt,\end{aligned}$$

which gives the left hand of (7.23).

Since $u_z(0) \neq 0$, we get (7.24) from (7.23) immediately.

Now we prove (7.25). Assume $\|u_z\|_{L^2(\mathbb{R}^1_+)} = 1$. By (7.23) we only need to show that

$$\lim_{z\to-\infty} z|u_z(0)|^2 = 0. \tag{7.26}$$

Suppose that (7.26) does not hold. Then, there is a sequence $z \to -\infty$ such that $u_z(0) \geq C/\sqrt{|z|}$. Using the boundary condition, we have $u_z'(0) = \gamma u_z(0) \geq C\gamma/\sqrt{|z|}$. For $0 < t < |z|-1$ we have $u_z''(t) = [(z+t)^2 - \beta_\gamma(z)]u_z(t) > 0$; thus, $u_z'(t) > u_z'(0) \geq C\gamma/\sqrt{|z|}$, $u_z(t) > C\gamma t/\sqrt{|z|}$. Therefore,

$$\begin{aligned}1 = \int_0^{+\infty}|u_z|^2dt &> \int_0^{|z|-1}|u_z|^2dt\\ &> \frac{C^2\gamma^2}{3|z|}(|z|-1)^3 \to +\infty \quad \text{as } z\to-\infty,\end{aligned}$$

a contradiction. The proof is complete. □

Remark 7.1. (7.25) can also be proved as follows.

Denote as above by u_z the positive eigenfunction of (5.3) with $\beta = \beta_\gamma(z)$ and $y_z(x) = u_z(x-z)$. Then, y_z is a positive solution of (7.1) with $\beta = \beta_\gamma(z)$. Now we choose u_z so that $\max y_z(x) = y_z(x_z) = 1$. Since $y_z''(x_z) \leq 0$, from (7.1) we find that $|x_z| \leq \sqrt{\beta_\gamma(z)}$. Since $\beta_\gamma(z) < 1$ for $z < -\gamma$ and $0 < y_z(x) \leq 1$, from (7.1) we see that as $z \to -\infty$, $\{y_z'\}$ and $\{y_z''\}$ are uniformly bounded on any fixed bounded interval. Therefore, we can pass to a subsequence and assume $x_z \to x_0$, $\beta_\gamma(z) \to \beta_0$, $y_z \to y$ in $C^2_{loc}(\mathbb{R}^1)$. $0 \leq y(x) \leq 1$, $y(x_0) = 1$, $\beta_0 \leq 1$, $|x_0| \leq \sqrt{\beta_0}$, and y is a solution of (7.19) on $\mathbb{R}^1$. From Corollary 7.2 we see that $\beta_0 \geq 1$. So $\beta_0 = 1$ and $y(x) = \exp(-x^2/2)$.

The above conclusion is true for all sequence $z \to -\infty$, so we get (7.25). Moreover, this argument also shows that $y_z(x)$ must converge to $\exp(-x^2/2)$.

Lemma 7.7. *$\beta_\gamma(z)$ is a continuous function of z.*

Proof. Denote by u_z the positive eigenfunction of (5.3) satisfying $\|u_z\|_{L^2(\mathbb{R}^1_+)} = 1$. Let $z_0 \in \mathbb{R}^1$. By the definition (5.4),

$$\begin{aligned}\beta_\gamma(z_0) &\leq \gamma|u_z(0)|^2 + \int_0^{+\infty}\{|u_z'|^2 + (z_0+t)^2|u_z|^2\}dt\\ &= \beta_\gamma(z) + (z_0-z)^2 + 2(z_0-z)\int_0^{+\infty}(z+t)|u_z|^2dt\\ &\leq \beta_\gamma(z) + (z_0-z)^2 + 4|z_0-z|[1+\beta_\gamma(z)].\end{aligned}$$

Hence, $\beta_\gamma(z_0) \leq \liminf_{z\to z_0}\beta_\gamma(z)$.

On the other hand,

$$\beta_\gamma(z) \le \gamma |u_{z_0}(0)|^2 + \int_0^{+\infty} \{|u'_{z_0}|^2 + (z+t)^2 |u_{z_0}|^2\} dt.$$

Letting $z \to z_0$, we get $\limsup_{z\to z_0} \beta_\gamma(z) \le \beta_\gamma(z_0)$. This completes the proof. □

Proof of Lemma 5.2. Conclusion (a) is obvious. (b) comes from Corollary 7.4 and Lemmas 7.6 and 7.7.

Proof of (c). From (b) we know that $\beta_\gamma(z)$ has a minimum point z_γ, $z_\gamma < -\gamma$, $0 < \beta^*_\gamma = \beta_\gamma(z_\gamma) < 1$. We claim that the minimum point z_γ is unique. Suppose that there exist $z_1 < z_2 < -\gamma$ such that $\beta_\gamma(z_1) = \beta_\gamma(z_2) = \beta^*_\gamma$. Choose a point $z_0 \in (z_1, z_2)$ such that $\beta_\gamma(z_0) = \max_{z_1<z<z_2} \beta_\gamma(z)$. Then, $\beta^*_\gamma \le \beta_\gamma(z_0) < 1$. Furthermore, we have $\beta_\gamma(z_0) > \beta^*_\gamma$. If not, then $\beta_\gamma(z) \equiv \beta^*_\gamma$ for all $z \in (z_1, z_2)$, which contradicts Lemma 7.5. Let β_0 be fixed, $\beta^*_\gamma < \beta_0 < \beta_\gamma(z_0)$. Since $\beta_\gamma(z)$ is continuous and $\beta_\gamma(-\infty) = \beta_\gamma(-\gamma) = 1$, there exist 4 points x_n,

$$x_1 < z_1 < x_2 < z_0 < x_3 < z_2 < x_4 < -\gamma,$$

such that $\beta_\gamma(x_n) = \beta_0$, $n = 1, 2, 3, 4$. Again this contradicts Lemma 7.5.

Thus, for any number $\beta \in (\beta^*_\gamma, 1)$, there exist at least two points $z_1(\beta) < z_\gamma < z_2(\beta) < -\gamma$ such that $\beta_\gamma(z_1(\beta)) = \beta_\gamma(z_2(\beta)) = \beta$. By Lemma 7.5 again, $z_1(\beta)$ and $z_2(\beta)$ are the only two points with this property. Therefore, $\beta_\gamma(z)$ must be decreasing in $(-\infty, z_\gamma)$ and increasing in $(z_\gamma, -\gamma)$.

By Lemma 7.5, $\#\mathcal{Z}^-(\beta) \le 1$ for $1 \le \beta \le 3$. From Corollary 7.4, $1 < \beta_\gamma(z) < 3$ for $-\gamma < z \le 0$. Therefore, $\beta_\gamma(z_1) \ne \beta_\gamma(z_2)$ for any z_1, $z_2 \in [-\gamma, 0]$, $z_1 \ne z_2$. This fact together with the continuity of $\beta_\gamma(z)$ implies that $\beta_\gamma(z)$ is monotone in $[-\gamma, 0]$. Since $\beta_\gamma(z) > 1$ for $z \in (-\gamma, 0)$ and $\beta_\gamma(z) < 1$ for $z < -\gamma$, we see that $\beta_\gamma(z)$ is increasing in $(-\gamma, 0)$.

From the definition (5.4) we easily see that $\beta_\gamma(z)$ is increasing for $z > 0$. Thus, we conclude that $\beta_\gamma(z)$ is increasing in $(z_\gamma, +\infty)$.

(d) follows from Lemma 7.6. For $\gamma = 0$, we choose $t = 1/\sqrt{2}$ in the right hand side of (5.6) to get $0 < \beta^*_0 < 1 - 1/\sqrt{2e\pi}$.

(e) follows from Corollary 7.3(1). The proof Lemma 5.2 is complete. □

Acknowledgements

This work was partially done when the second author visited the Department of Mathematics at Brigham Young University in the 1995 academic year. He would like to thank the Department for its hospitality. This work was partially supported by National Science Foundation grant DMS-9622853 (to Lu), and by the Chinese National Natural Science Foundation, the Science Foundation of the State Education Commission of China, and the Zhejiang Provincial Natural Science Foundation of China (to Pan).

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