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HOW PARABOLIC FREE BOUNDARIES APPROXIMATE HYPERBOLIC FRONTS

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Abstract. A rather complete study of the existence and qualitative behaviour of the boundaries of the support of solutions of the Cauchy problem for nonlinear first-order and second-order scalar conservation laws is presented. Among other properties, it is shown that, under appropriate assumptions, parabolic interfaces converge to hyperbolic ones in the vanishing viscosity limit.

1. Introduction

We investigate phenomena associated with the nonnegative solution of the nonlinear first-order hyperbolic equation

$$u_t + (f(u))_x = 0 \quad \text{in } \mathbb{R}\times\mathbb{R}^+ \tag{1.1}$$

with the initial condition

$$u = u_0 \quad \text{on } \mathbb{R}\times\{0\}, \tag{1.2}$$

and the nonnegative solution of the nonlinear second-order parabolic equation

$$u_t + (f(u))_x = \varepsilon(a(u))_{xx} \quad \text{in } \mathbb{R}\times\mathbb{R}^+ \tag{1.3}$$

in which $\varepsilon > 0$ is a real parameter, with the same initial condition. About the coefficients in these equations and the initial data function we assume the following.

(H$_1$) The function $f \in C([0,\infty)) \cap C^1(0,\infty)$, with f' locally Hölder continuous on $(0,\infty)$, and $f(0) = 0$.

(H$_2$) The function $a \in C([0,\infty)) \cap C^2(0,\infty)$, with a'' locally Hölder continuous on $(0,\infty)$, $a'(s) > 0$ for $s > 0$, and $a(0) = 0$.

(H$_3$) The function $u_0 \in L^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} |u_0(x+h) - u_0(x)|\, dx \to 0 \quad \text{as } h \to 0,$$

u_0 is nonnegative, and is nontrivial in the sense that

$$M := \operatorname{ess\,sup}\{u_0(x) : x \in \mathbb{R}\} > 0. \tag{1.4}$$

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Problem (1.1),(1.2) may be regarded as the limit as $\varepsilon \downarrow 0$ of problem (1.3),(1.2). Indeed, under the assumptions (H_1) and (H_3), it can be shown that problem (1.1),(1.2) admits a unique *entropy* solution [37, 38, 39, 40, 41], while, under the assumptions (H_1)–(H_3), problem (1.3),(1.2) has a unique weak solution for any $\varepsilon > 0$ [21]. Moreover the solution of the parabolic problem converges to the entropy solution of the hyperbolic problem as $\varepsilon \downarrow 0$ in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}))$ for every $0 < T < \infty$ [35, 36, 37, 38, 39, 41].

In this paper we investigate the relationship between fronts associated with the propagation of the support of the entropy solution of the hyperbolic problem (1.1),(1.2) and the corresponding free boundary in the solution of the parabolic problem (1.3),(1.2). More precisely, we shall study how the interface denoting the upper boundary of the support of the solution of (1.3),(1.2), also known as the "right front", approximates the corresponding interface in the solution of (1.1),(1.2) in the vanishing viscosity limit $\varepsilon \downarrow 0$.

To fix ideas, let us denote by $u(x,t;\varepsilon)$ the unique weak solution of problem (1.3),(1.2) if $\varepsilon > 0$, and by $u(x,t;0)$ the unique entropy solution of problem (1.1),(1.2). Define the front

$$\zeta(t;\varepsilon) := \sup\{x \in \mathbb{R} : w(x,t;\varepsilon) > 0\},$$

where

$$w(x,t;\varepsilon) := \int_x^\infty u(y,t;\varepsilon)\,dy \quad \text{ for any } t > 0 \tag{1.5}$$

and $\varepsilon \geq 0$. Our objective is to study the relationship between the fronts $\zeta(\cdot;\varepsilon)$ for $\varepsilon > 0$ and the front $\zeta(\cdot;0)$. A crucial rôle in the analysis is played by the quantities:

$$\sigma_M := \sup\{f(s)/s : 0 < s \leq M\} \tag{1.6}$$

and

$$\sigma_0 := \lim_{\delta \downarrow 0} \sigma_\delta = \limsup_{s \downarrow 0} f(s)/s. \tag{1.7}$$

Obviously, by definition $\sigma_0 \leq \sigma_M$. Setting

$$\zeta_0 := \sup\{x \in \mathbb{R} : w_0(x) > 0\},$$

where

$$w_0(x) := \int_x^\infty u_0(y)\,dy, \tag{1.8}$$

we shall prove the following result concerning the front of problem (1.1),(1.2).

Theorem 1.1. *Let assumptions* (H_1) *and* (H_3) *hold.*

(a) *If* $\sigma_0 = \infty$, *then* $\zeta(t;0) = \infty$ *for all* $t > 0$.
(b) *If* $\sigma_0 < \infty$, *then as an extended function* $\zeta(\cdot;0)$ *is lower semi-continuous and continuous from the right on* $[0,\infty)$ *with* $\zeta(0;0) = \zeta_0$, *and*

$$\zeta(t;0) \leq \zeta(t_0;0) + \sigma_M(t - t_0) \quad \textit{for all } t > t_0 \geq 0. \tag{1.9}$$

Moreover if $\sigma_0 > -\infty$, *then* $\zeta(\cdot;0)$ *is continuous on* $[0,\infty)$ *with*

$$\zeta(t;0) \geq \zeta(t_0;0) + \sigma_0(t - t_0) \quad \textit{for all } t > t_0 \geq 0. \tag{1.10}$$

We shall also characterize the speed of the front $\zeta'(t;0)$ for $t \geq 0$ in terms of the behaviour of $u(\cdot,t;0)$ near $\zeta(t;0)$.

Such a qualitative description of the behaviour of the fronts of problem (1.1),(1.2) appears to be new, at least in such generality. When the function $f \in C^2([0,\infty))$ and is strictly convex or concave, and solutions of problem (1.1),(1.2) are assumed to be of locally bounded variation, a quite complete theory was developed in [10] by using generalized characteristics. This theory was later extended to the case in which the flux function f has a single inflection point in [11].

In the regular case that $f \in C^1([0,\infty))$ the quantity $\sigma_0 = f'(0)$. Hence if the flux function f is concave, then $\sigma_M = \sigma_0 = f'(0)$, and the front $\zeta(\cdot;0)$ corresponds to a classical characteristic [10]. For the Riemann problem, with $u_0 = M\chi_{(-\infty,0]}$, the ensuing regular wave is called a rarefaction fan. On the other hand, if f is convex and the front moves at a speed $\zeta'(t;0) > \sigma_0 = f'(0)$, we recognize that it corresponds to a shock wave [8, 26, 43, 50] and its speed is given by the Rankine-Hugoniot formula. In general, the situation is more involved. When no convexity or concavity is assumed, even for the Riemann problem the solution exhibits a complicated structure and is a composition of rarefaction fans, shocks and contact discontinuities, i.e. shock waves moving at characteristic speed.

Similar results for problem (1.3),(1.2) have been proven in [20] and easily extend to the present situation. These results involve the condition

(H$_4$)

$$\int_0^\delta \frac{a'(s)}{\max\{s,-f(s)\}}\,ds < \infty \qquad \text{for some } \delta > 0.$$

Theorem 1.2. *Let $\varepsilon > 0$ and assumptions* (H$_1$)–(H$_3$) *hold.*

(a) *If $\sigma_0 = \infty$ or assumption* (H$_4$) *is negated, then $\zeta(t;\varepsilon) = \infty$ for all $t > 0$.*

(b) *If $\sigma_0 < \infty$ and* (H$_4$) *is satisfied, then as an extended function $\zeta(\cdot;\varepsilon)$ is lower semi-continuous and continuous from the right on $[0,\infty)$ with $\zeta(0;\varepsilon) = \zeta_0$, and*

$$\zeta(t;\varepsilon) \leq \zeta(t_0;\varepsilon) + \sigma_M(t-t_0) + Q_M(t-t_0,\varepsilon) \qquad \text{for all } t > t_0 \geq 0, \tag{1.11}$$

where

$$Q_M(t,\varepsilon) := \inf_{\sigma > \sigma_M} \left\{ (\sigma - \sigma_M)t + \varepsilon \int_0^M \frac{a'(s)}{\sigma s - f(s)}\,ds \right\}.$$

Moreover if $\sigma_0 > -\infty$, then $\zeta(\cdot;\varepsilon)$ is continuous on $[0,\infty)$ with

$$\zeta(t;\varepsilon) \geq \zeta(t_0;\varepsilon) + \sigma_0(t-t_0) \qquad \text{for all } t > t_0 \geq 0. \tag{1.12}$$

According to the above results, if $\sigma_0 = \infty$, then, irrespective of the initial data u_0, there holds $\zeta(t;\varepsilon) = \infty$ for all $t > 0$ and $\varepsilon \geq 0$. In this case both the hyperbolic problem and the parabolic problem display *infinite speed of propagation.* On the other hand, if $\sigma_0 < \infty$, then the hyperbolic problem (1.1),(1.2) displays *finite speed of propagation*; i.e. if $\zeta_0 < \infty$ there holds $\zeta(t;0) < \infty$ for all $t > 0$. For the parabolic equation, finite speed of propagation depends on the additional condition (H$_4$). Therefore, the condition $\sigma_0 < \infty$ may be viewed as of a hyperbolic nature, while (H$_4$) ensures that the parabolic character of (1.3) is not too dominant, or, in other words, that the diffusion is ‘slow’ enough.

To give a better quantitative description of the balance between the effects of the convective and the diffusion terms in equation (1.3), it is informative to recapitulate the above results for the model equations

$$u_t + \lambda(u^n)_x = 0 \tag{1.13}$$

and

$$u_t + \lambda(u^n)_x = \varepsilon(u^m)_{xx} \tag{1.14}$$

where $\lambda \in \{-1, 0, 1\}$, $n > 0$ and $m > 0$ are real parameters. Equation (1.14) has a number of well-known special cases including the *Burgers equation* when $n = 2$ and $m = 1$ and the *porous media equation* when $\lambda = 0$. Properties of interfaces in solutions of the porous media equation have been intensively studied [5, 32, 45, 54]. The more general case of (1.14) has also been widely investigated [2, 3, 12, 14, 15, 18, 22, 23, 27, 28, 31, 34, 47, 48, 49]. For these particular equations Theorems 1.1 and 1.2 were already known [12, 14, 20, 23, 24, 25, 35]. If $\lambda = 1$ and $n < 1$, we have the situation that $\sigma_0 = \infty$ and thus $\zeta(t;\varepsilon) = \infty$ for all $t > 0$ and $\varepsilon \geq 0$. If $\lambda = 0$ or $n \geq 1$, then σ_0 is finite and thus $\zeta(\cdot;0)$ is a continuous function on $[0,\infty)$. Moreover if $m \leq 1$, then (H_4) is negated and thus $\zeta(t;\varepsilon) = \infty$ for every $t > 0$ and $\varepsilon > 0$, whereas if $m > 1$, then (H_4) is satisfied and $\zeta(\cdot;\varepsilon)$ is continuous on $[0,\infty)$ for every $\varepsilon \geq 0$. In the remaining case $\lambda = -1$ and $n < 1$ we have $\sigma_0 = -\infty$ and (H_4) holds if and only if $m > n$. This situation displays exceptional behaviour, since in this instance both equations (1.13) and (1.14) admit *instantaneous shrinking*, this is to say that one can have $\zeta_0 = \infty$ and $\zeta(t;\varepsilon) < \infty$ for all $t > 0$, and *deferred instantaneous shrinking*, where $\zeta(t;\varepsilon) = \infty$ for all $0 \leq t < \tau$ and $\zeta(t;\varepsilon) < \infty$ for all $t > \tau$ for some $0 < \tau < \infty$ [23, 35]. In particular in [35] a family of solutions of (1.13) for which $\zeta(t;\varepsilon) = \infty$ for all $0 \leq t < \tau$ and $\zeta(t;\varepsilon) < \infty$ for all $t \geq \tau$ for some $0 < \tau < \infty$ was constructed. This family shows that the continuity stated in Theorem 1.1 is the best possible.

In the general situation with $\sigma_0 > -\infty$ we shall completely clarify the relation between the parabolic free boundaries and the hyperbolic front. From Theorems 1.1 and 1.2 it turns out that there are three mutually exclusive cases when $\sigma_0 > -\infty$. The first is $\sigma_0 = \infty$ or $\zeta_0 = \infty$. In this case $\zeta(t;\varepsilon) = \infty$ for all $t > 0$ and $\varepsilon \geq 0$. So the interfaces $\zeta(.;\varepsilon)$ do not effectively exist for any $\varepsilon \geq 0$. The second is $\sigma_0 < \infty$, (H_4) does not hold, and $\zeta_0 < \infty$. In this case $\zeta(t;0) < \infty = \zeta(t;\varepsilon)$ for all $t > 0$ and $\varepsilon > 0$. So $\zeta(\cdot;0)$ denotes a well-defined front, whereas the parabolic free boundary $\zeta(\cdot;\varepsilon)$ for $\varepsilon > 0$ does not exist. In both of these cases, the relation between the parabolic free boundaries and the hyperbolic front is clear. The final case is $\sigma_0 < \infty$, (H_4) holds, and, $\zeta_0 < \infty$, which implies that $\zeta(t;\varepsilon) < \infty$ for all $t > 0$ and all $\varepsilon \geq 0$. Concerning this case, we shall establish the following result on the convergence of the interfaces.

Theorem 1.3. *Suppose that assumptions* (H_1)–(H_4) *hold and* $-\infty < \sigma_0 < \infty$. *Suppose furthermore that* $\zeta_0 < \infty$. *Then*

$$\zeta(\cdot;\varepsilon) \to \zeta(\cdot;0) \quad \text{as } \varepsilon \downarrow 0$$

in $C([0,T]) \cap C^{0+\alpha}([\tau,T])$ *for all* $0 < \tau < T < \infty$ *and* $0 < \alpha < 1$.

We conclude that in the case that σ_0 is finite, the hyperbolic front is the vanishing viscosity limit of the free boundary of the parabolic problem when the latter displays finite speed of propagation.

With regard to the singular situation $\sigma_0 = -\infty$ we shall obtain a weaker result. Recall that this is the situation in which instantaneous shrinking and deferred instantaneous shrinking can occur, and that the interface $\zeta(\cdot;0)$ is not necessarily continuous. The result we shall obtain in this instance is the following.

Theorem 1.4. *Suppose that assumptions* (H_1)–(H_4) *hold and* $\sigma_0 = -\infty$. *Then*

$$\liminf_{\varepsilon\downarrow 0} \zeta(t;\varepsilon) \geq \zeta(t;0) \quad \text{for all } t > 0,$$

and, if there exists a $t_0 \geq 0$ *such that* $\limsup_{\varepsilon\downarrow 0}\zeta(t_0;\varepsilon) < \infty$, *there holds*

$$\limsup_{\varepsilon\downarrow 0} \zeta(t;\varepsilon) \leq \limsup_{s\uparrow t} \zeta(s;0) \quad \text{for all } t > t_0.$$

It follows that if $\zeta_0 < \infty$, then $\zeta(\cdot;\varepsilon)$ converges pointwise to $\zeta(\cdot;0)$ as $\varepsilon \downarrow 0$ at all times t at which $\zeta(\cdot;0)$ is continuous while the cluster points of $\zeta(t;\varepsilon)$ as $\varepsilon \downarrow 0$ lie within the range of the jump of $\zeta(\cdot;0)$ at any time t at which $\zeta(\cdot;0)$ is discontinuous. If $\zeta_0 = \infty$, then either $\zeta(t;0) = \infty$ for all $t > 0$, or there is an "instantaneous-shrinking" time $0 \leq \tau < \infty$ such that $\zeta(t;0) = \infty$ for all $0 \leq t < \tau$ and $\zeta(t;0) < \infty$ for any $t > \tau$. In the first event, $\zeta(t;\varepsilon) \to \infty = \zeta(t;0)$ as $\varepsilon \downarrow 0$ for all $t > 0$. In the second event, $\zeta(t;\varepsilon) \to \infty = \zeta(t;0)$ as $\varepsilon \downarrow 0$ for all $0 < t < \tau$, and, there exists a τ^*, with $\tau \leq \tau^* \leq \infty$, such that the cluster points of $\zeta(t;\varepsilon)$ as $\varepsilon \downarrow 0$ lie between $\zeta(t;0)$ and ∞ for any $\tau \leq t \leq \tau^*$, while $\zeta(t;\varepsilon)$ converges to $\zeta(t;0)$ as $\varepsilon \downarrow 0$ for $t > \tau^*$ in the same manner as when $\zeta_0 < \infty$.

The major tool in our investigation will be a comparison principle for the primitive (1.5) of the solution of problems (1.1),(1.2) and (1.3),(1.2). We recall that solutions of problem (1.3),(1.2) are known to converge to the entropy solution of (1.1),(1.2) as $\varepsilon \downarrow 0$ only in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}))$ for any $0 < T < \infty$ — a convergence which is too weak to investigate the behaviour of the support of solutions directly. So it can be viewed as both necessary and natural to investigate properties of the fronts using this variable. Applying this principle we shall obtain our results by comparing the solution of problems (1.1),(1.2) and (1.3),(1.2) with suitably constructed self-similar solutions. For the hyperbolic equation we employ solutions of the Riemann problem. For the parabolic equation we rely on travelling waves.

In Section 2 we present some useful results on the second-order parabolic problem (1.3),(1.2); and, in Section 3 we do the same for the first-order hyperbolic problem (1.1),(1.2). We prove Theorems 1.1 and 1.2 and further results concerning the speed of propagation of the fronts and their qualitative behaviour. Section 4 is devoted to the question of the convergence of the parabolic free boundaries to the hyperbolic front. In particular Theorems 1.3 and 1.4 will be proven there, and a number of situations in which the convergence can be improved upon will be described. Finally, in Section 5, we present a review of the consequences for the power-law equations (1.13) and (1.14).

Note added in proof. Since the present paper was submitted for publication, results comparable to Theorem 1.1 and Theorem 3.10 part (ii) have appeared in: J. I. Diaz and S. N. Kruzhkov, *Propagation properties for scalar conservation laws*, C. R. Acad. Sci. Paris Sér. I Math. **323** (1996), 463–468. MR **97e:**35105

2. The second-order equation

Since equation (1.3) is not necessarily of uniformly parabolic type, problem (1.3),(1.2) need not admit a classical solution, even for smooth initial data [13, 18, 19, 20, 21]. Hence the following notion of a weak solution is introduced.

Definition 2.1. A solution of problem (1.3),(1.2) is a nonnegative function $u \in L^\infty(\mathbb{R} \times \mathbb{R}^+)$ such that

$$\iint_{\mathbb{R}\times\mathbb{R}^+} \{u\varphi_t + f(u)\varphi_x + \varepsilon\, a(u)\varphi_{xx}\}\, dx\; dt = 0 \tag{2.1}$$

for every nonnegative function $\varphi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^+)$, and

$$\operatorname*{ess\,lim}_{t\downarrow 0} \int_{x_1}^{x_2} |u(x,t) - u_0(x)|\, dx = 0 \quad \text{for all } -\infty < x_1 < x_2 < \infty. \tag{2.2}$$

Concerning the existence and uniqueness of such a solution, the following can be established.

Lemma 2.2. *Let $\varepsilon > 0$ and hypotheses* (H_1)–(H_3) *hold. Then problem* (1.3),(1.2) *admits a unique solution $u(x,t;\varepsilon)$. Moreover, $u(\cdot,\cdot;\varepsilon) \in C([0,\infty); L^1_{\text{loc}}(\mathbb{R})) \cap C(\mathbb{R} \times \mathbb{R}^+)$, $u(x,t;\varepsilon) \le M$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}^+$,*

$$\int_{\mathbb{R}} |u(x+h,t;\varepsilon) - u(x,t;\varepsilon)|\, dx \le \int_{\mathbb{R}} |u_0(x+h) - u_0(x)|\, dx \tag{2.3}$$

for all $t > 0$ and $h > 0$, and

$$\int_{\mathbb{R}} u(x,t;\varepsilon)\, dx = \int_{\mathbb{R}} u_0(x)\, dx \quad \text{for all } t > 0. \tag{2.4}$$

The major assertions of this lemma, albeit with an alternative definition of a solution of problem (1.3),(1.2), have been established in [21] under the additional assumption that $u_0 \in C(\mathbb{R})$. The present result can be obtained by combining the ideas in [21] with those of Kruzhkov and co-workers in [36, 38, 39, 40, 41].

Let us now introduce some notation which will be adhered to throughout the remainder of this paper. We define

$$F_M := \max\{|f(s)| : 0 \le s \le M\} \quad \text{and} \quad A_M := \left(\int_0^M a(s)\, ds\right)^{1/2}.$$

We let σ_M be given by (1.6) for $M > 0$, and σ_0 by (1.7). Observing that $\sigma_M < \infty$ for $M > 0$ if and only if $\sigma_0 < \infty$; we set

$$I_M := \int_0^M \frac{a'(s)}{\sigma_M s - f(s)}\, ds, \tag{2.5}$$

and note that $0 < I_M \le \infty$, if $\sigma_0 < \infty$. Finally, if $\sigma_0 < \infty$ we define

$$q_M(t,\varepsilon,\sigma) := (\sigma - \sigma_M)t + \varepsilon \int_0^M \frac{a'(s)}{\sigma s - f(s)}\, ds \tag{2.6}$$

for any $\sigma > \sigma_M$, and

$$Q_M(t,\varepsilon) := \inf\{q_M(t,\varepsilon,\sigma) : \sigma > \sigma_M\}. \tag{2.7}$$

The last quantity has the properties stated below.

Lemma 2.3. *Let $\sigma_0 < \infty$ and assumptions* (H_1), (H_2) *and* (H_4) *hold. Then, for any $M > 0$:* (i) *$Q_M \in C([0,\infty) \times [0,\infty))$;* (ii) *$Q_M$ is increasing in both arguments t and ε;* (iii) *$Q_M(t,\varepsilon) \le \varepsilon I_M$ for all $t > 0$ and $\varepsilon > 0$;* (iv) *$Q_M(0,\varepsilon) = 0$ for all $\varepsilon \ge 0$; and* (v) *$Q_M(t,0) = 0$ for all $t \ge 0$.*

Proof. Following analysis in [20, 24, 25], under the hypotheses of the lemma, the quantity $q_M(t,\varepsilon,\sigma)$ is finite for each $\sigma > \sigma_M$, $t \geq 0$ and $\varepsilon \geq 0$. Therefore Q_M is well defined, and the continuity, the monotonicity and upper bound of Q_M are straightforward. In fact it can be shown that $Q_M \in C([0,\infty) \times [0,\infty)) \cap C^1(\mathbb{R}^+ \times \mathbb{R}^+)$ with $(\partial Q_M/\partial t)(t,\varepsilon) \geq 0$ and $(\partial Q_M/\partial \varepsilon)(t,\varepsilon) > 0$ for any $t > 0$ and $\varepsilon > 0$. With respect to the remaining assertions, we note that by definition $Q_M(0,\varepsilon) \leq q_M(0,\varepsilon,\sigma)$ for all $\sigma > \sigma_M$, whence, letting $\sigma \to \infty$, (iv) holds while, $Q_M(t,0) \leq q_M(t,0,\sigma) = (\sigma - \sigma_M)t$ for all $\sigma > \sigma_M$, whence, letting $\sigma \to \sigma_M$, (v) is obtained. □

We shall now outline the proof of Theorem 1.2.

Proof of Theorem 1.2. Using travelling-wave solutions for comparison, the first two claims of the theorem were proven in [20] under the additional assumption that $u_0 \in C(\mathbb{R})$. Furthermore the estimate $\sigma_0(t - t_0) \leq \zeta(t;\varepsilon) - \zeta(t_0;\varepsilon) \leq \sigma_M(t - t_0) + q(t - t_0, \varepsilon, \sigma)$ for any $\sigma > \sigma_M$ and $t > t_0 \geq 0$ such that $\zeta(t_0;\varepsilon) < \infty$ was also obtained in [20] under this assumption. These results readily extend to the present situation. Application of Lemma 2.3 to optimize the upper bound completes the proof. □

To show further properties of the right interface of the solution of problem (1.3),(1.2), we need the following estimate on the regularity of the solution.

Proposition 2.4. *Suppose that* $\varepsilon > 0$ *and assumptions* (H_1)–(H_3) *hold. Let* u *denote the unique solution of problem* (1.3),(1.2). *Then the derivative* $(a(u))_x$ *exists in the classical sense and satisfies*

$$|f(u) - \varepsilon(a(u))_x|(x,t) \leq F_M + \varepsilon^{1/2} A_M t^{-1/2} \tag{2.8}$$

at every point $(x,t) \in \mathbb{R} \times \mathbb{R}^+$. *Moreover, if* $\sigma_0 < \infty$ *and assumption* (H_4) *holds, then*

$$(f(u) - \varepsilon(a(u))_x)(x,t) \leq \{\sigma_M + Q_M(1, \varepsilon/4t)\}\, u(x,t) \tag{2.9}$$

for all $(x,t) \in \mathbb{R} \times \mathbb{R}^+$.

Proof. We refine the Bernstein technique as applied in [7, 20, 21]. Suppose that u is a classical solution of problem (1.3),(1.2) which is bounded below by a positive constant, and which has a bounded derivative $(a(u))_x$. Consider the function

$$z(x,t) := \left(\frac{f(u) - \varepsilon(a(u))_x - \sigma u - \beta}{\theta(u)}\right)(x,t)$$

where σ and β are constants which will be chosen later and $\theta \in C^2((0,M])$ is a positive function which will also be defined later. Then by calculation it can be verified that $N(z) = 0$ in $\mathbb{R} \times \mathbb{R}^+$, where N is the nonlinear parabolic differential operator

$$
\begin{aligned}
N(\eta) \quad = \quad & \varepsilon a'(u)\eta_{xx} - \left(2\theta'(u) + \frac{a''(u)\theta(u)}{a'(u)}\right)\eta\eta_x \\
& - \left(f'(u) + \frac{\{\sigma u + \beta - f(u)\}a''(u)}{a'(u)} + 2\frac{\{\sigma u + \beta - f(u)\}\theta'(u)}{\theta(u)}\right)\eta_x \\
& + \frac{\theta(u)\theta''(u)}{\varepsilon a'(u)}\eta^3 + 2\frac{\{\sigma u + \beta - f(u)\}\theta''(u)}{\varepsilon a'(u)}\eta^2 \\
& + \frac{\{\sigma u + \beta - f(u)\}^2\theta''(u)}{\varepsilon a'(u)\theta(u)}\eta - \eta_t.
\end{aligned}
$$

Our objective is to apply a comparison principle argument using the operator N to estimate z in terms of a suitable test-function η.

Our first choice is the combination $\sigma = 0$, $\beta = F_M$,

$$
\theta(s) = \left(\gamma + \varepsilon \int_s^M a(r)\, dr\right)^{1/2} \tag{2.10}
$$

for some constant $\gamma > 0$, and $\eta = (t + \tau)^{-1/2}$ for some constant $\tau > 0$. With this combination we find

$$
N(\eta) = \left(\frac{\theta(u)\theta''(u)}{\varepsilon a'(u)} + \frac{1}{2}\right)\eta^3 + 2\frac{\{F_M - f(u)\}\theta''(u)}{\varepsilon a'(u)}\eta^2 + \frac{\{F_M - f(u)\}^2\theta''(u)}{\varepsilon a'(u)\theta(u)}\eta.
$$

By differentiation of (2.10), it can be checked that this function has the property $\theta''(s) = -[\varepsilon a'(s) + 2\{\theta'(s)\}^2]/2\theta(s) \leq -\varepsilon a'(s)/2\theta(s) < 0$ for all $0 < s \leq M$. Additionally $f(s) \leq F_M$ for all such s. Hence we deduce

$$
N(\eta) \leq 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^+. \tag{2.11}
$$

Simultaneously we can choose τ so small that

$$
z \leq \eta \quad \text{on } \mathbb{R} \times \{0\}. \tag{2.12}
$$

By the comparison principle in [33] we then obtain

$$
z \leq \eta \quad \text{in } \mathbb{R} \times \mathbb{R}^+. \tag{2.13}
$$

This yields $f(u) - \varepsilon(a(u))_x - F_M \leq \theta(u)\eta \leq (\gamma + \varepsilon A_M^2)^{1/2} t^{-1/2}$, whence letting $\gamma \downarrow 0$ we obtain

$$
f(u) - \varepsilon(a(u))_x \leq F_M + \varepsilon^{1/2} A_M t^{-1/2}. \tag{2.14}
$$

As a second option, we take $\sigma = 0$, $\beta = -F_M$, θ as in (2.10) and $\eta = -(t+\tau)^{-1/2}$ for some $\tau > 0$. Arguing as above, we obtain (2.11)–(2.13) with the inequalities reversed. By the comparison principle [33] we now obtain

$$
f(u) - \varepsilon(a(u))_x \geq -F_M - \varepsilon^{1/2} A_M t^{-1/2} \tag{2.15}
$$

in analogous fashion.

Our third and final choice, when $\sigma_0 < \infty$ and (H_4) is satisfied, is $\sigma > \sigma_M$, $\beta = 0$,

$$
\theta(s) = \gamma + \varepsilon \int_0^s \int_r^M \frac{a'(\xi)}{\sigma\xi - f(\xi)}\, d\xi\, dr \tag{2.16}
$$

for some constant $\gamma > 0$, and $\eta = \frac{1}{4}(t+\tau)^{-1}$ for some constant $\tau > 0$. The outcome is

$$\begin{aligned} N(\eta) &= \frac{\theta(u)\theta''(u)}{\varepsilon a'(u)}\eta^3 + 2\left(\frac{\{\sigma u - f(u)\}\theta''(u)}{\varepsilon a'(u)} + 2\right)\eta^2 + \frac{\{\sigma u - f(u)\}^2\theta''(u)}{\varepsilon a'(u)\theta(u)}\eta \\ &= \frac{\{\sigma u - f(u) - \theta(u)\eta\}^2\theta''(u)}{\varepsilon a'(u)\theta(u)}\eta + 4\left(\frac{\{\sigma u - f(u)\}\theta''(u)}{\varepsilon a'(u)} + 1\right)\eta^2. \end{aligned}$$

At the same time, differentiation of (2.16) yields $\theta''(s) = -\varepsilon a'(s)/\{\sigma s - f(s)\} < 0$ for $0 < s \le M$. Thus once again (2.11) holds, while a suitable choice of τ gives (2.12). The conclusion is once more (2.13). In this particular case this can be reformulated as

$$f(u) - \varepsilon(a(u))_x - \sigma u \le \theta(u)\eta \le \left(\gamma + \varepsilon u \int_0^M \frac{a'(s)}{\sigma s - f(s)}\, ds\right)\frac{1}{4}t^{-1},$$

whence, letting $\gamma \downarrow 0$, we obtain

$$f(u) - \varepsilon(a(u))_x \le \{\sigma_M + q_M(1, \varepsilon/4t, \sigma)\}\, u. \tag{2.17}$$

Combining (2.14) and (2.15) yields (2.8), while (2.17) provides (2.9) in view of the arbitrariness of $\sigma > \sigma_M$. By applying a standard parabolic regularization argument, these estimates for a positive classical solution of equation (1.3) can be extended to the solution of problem (1.3),(1.2) in the sense of Definition 2.1. Indeed, the estimate (2.8) can be used to construct the weak solution as the limit of a sequence of classical solutions of the equation, as was done in [13, 19, 21]. This also gives the existence of $(a(u))_x$ as a generalized derivative in $\mathbb{R} \times \mathbb{R}^+$ and as a continuous classical derivative in $\{(x,t) \in \mathbb{R} \times \mathbb{R}^+ : u(x,t) > 0\}$. To prove that $(a(u))_x$ actually exists in the classical sense everywhere in $\mathbb{R} \times \mathbb{R}^+$, four different situations need to be considered. The first is that $\sigma_0 = \infty$ or (H_4) does not hold, and that the analogous condition for the left front denoting the infimum of the support of u to be infinite irrespective of the initial data holds. In this case, u is necessarily positive everywhere in $\mathbb{R} \times \mathbb{R}^+$, and thus $(a(u))_x$ exists classically everywhere in $\mathbb{R} \times \mathbb{R}^+$. The second case is that $\sigma_0 < \infty$ and (H_4) holds, but the condition for the left front to be infinite irrespective of the initial data still holds. In this case u is necessarily positive for all $x \in \mathbb{R}$ with $x < \zeta(t)$ and $t > 0$. Thus $(a(u))_x$ exists everywhere except possibly at the interfacial points $(\zeta(t), t)$ with $t > 0$. However, for such a point there holds

$$a(u(x,t)) - a(u(\zeta(t),t)) = 0 \quad \text{for all } x > \zeta(t), \tag{2.18}$$

$$a(u(\zeta(t),t)) - a(u(x,t)) \le 0 \quad \text{for all } x < \zeta(t), \tag{2.19}$$

while (2.9) implies

$$\begin{aligned} &a(u(\zeta(t),t)) - a(u(x,t)) \\ &\qquad \ge \varepsilon^{-1} \int_x^{\zeta(t)} [f(u(y,t)) - \{\sigma_M + Q_M(1, \varepsilon/4t)\}u(y,t)]\, dy \end{aligned} \tag{2.20}$$

for all $x < \zeta(t)$. Dividing (2.18)–(2.20) by $|x - \zeta(t)|$, letting $x \to \zeta(t)$ and using the continuity of u demonstrate that $(a(u))_x(\zeta(t), t)$ exists and equals 0. Thus $(a(u))_x$ exists everywhere in $\mathbb{R} \times \mathbb{R}^+$ in this case also. The third case in which conditions are such that ζ is infinite irrespective of the initial data, but the corresponding left interface may exist, can be treated by analogy. In the final case, where the conditions on f and a are such that both right and left interfaces may occur, a lower bound on $f(u)-\varepsilon(a(u))_x$ analogous to (2.9) may be obtained alongside (2.9). These two bounds can be employed, as in the argument immediately above, to show that necessarily $(a(u))_x$ exists and equals 0 at every point $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ such that $u(x,t) = 0$. Thus in this case too, $(a(u))_x$ is defined in the classical sense everywhere in $\mathbb{R} \times \mathbb{R}^+$. In fact, in this case, the bounds can be used to show that $(a(u))_x$ is continuous in $\mathbb{R} \times \mathbb{R}^+$. □

To show further properties of the right interface of problem (1.3),(1.2), let us also recall the notation (1.5) previously introduced, and for completeness define

$$w(x,0;\varepsilon) = w_0(x) \tag{2.21}$$

for any $x \in \mathbb{R}$, where w_0 is given by (1.8). About the function w we can then state the following.

Lemma 2.5. *Suppose that $\varepsilon > 0$ and assumptions* (H$_1$)–(H$_3$) *hold.*

(a) *If $w_0(x) = \infty$ for some $x \in \mathbb{R}$, then $w(x,t;\varepsilon) = \infty$ for all $(x,t) \in \mathbb{R} \times [0,\infty)$.*
(b) *If $w_0(x) < \infty$ for some $x \in \mathbb{R}$, then $w(x,t;\varepsilon) < \infty$ for all $(x,t) \in \mathbb{R} \times [0,\infty)$, and*

$$\begin{aligned} &|w(x_1,t_1;\varepsilon) - w(x_2,t_2;\varepsilon)| \\ &\qquad \le M|x_1 - x_2| + F_M|t_2 - t_1| + 2\varepsilon^{1/2}A_M|t_2 - t_1|^{1/2} \end{aligned} \tag{2.22}$$

for all $(x_1,t_1),(x_2,t_2) \in \mathbb{R} \times [0,\infty)$.

Proof. Let $\psi \in C^\infty(\mathbb{R})$ be such that $\psi(x) = 0$ for $x \le x_1$, $\psi'(x) \ge 0$ for $x_1 < x < x_2$, and $\psi(x) = 1$ for $x \ge x_2$, for some $-\infty < x_1 < x_2 < \infty$. Let $T > 0$. Denote by $\{\psi_i\}_{i=1}^\infty$ a sequence of $C^\infty(\mathbb{R})$ functions such that $\psi_i(x) = \psi(x)$ for $x \le i$, $\psi_i(x) = 0$ for $x \ge i+1$ and $|\psi_i''(x)| \le K$ for all $x \in \mathbb{R}$ for some constant K which does not depend on i. Pick $0 < \tau < T$ and denote by $\{\phi_j\}_{j=1}^\infty$ a sequence of $C_0^\infty(\mathbb{R}^+)$ functions such that $\phi_j(t) = 0$ for $t \le \tau - 1/j$, $\phi_j'(t) \ge 0$ for $\tau - 1/j < t < \tau$, $\phi_j(t) = 1$ for $\tau \le t \le T$, $\phi_j'(t) \le 0$ for $T < t < T + 1/j$, and $\phi_j(t) = 0$ for $t \ge T + 1/j$. Substitute $\varphi = \psi_i(x)\phi_j(t)$ in (2.1). By following standard procedures and adapting arguments in [18, 20, 21], one may take the successive limits $j \to \infty$, $\tau \downarrow 0$ and $i \to \infty$ to deduce

$$\begin{aligned} &\int_{\mathbb{R}} u(x,T;\varepsilon)\psi(x)\,dx \\ &\qquad = \int_{\mathbb{R}} u_0(x)\psi(x)\,dx + \int_0^T \int_{x_1}^{x_2} \{f(u)\psi' + \varepsilon a(u)\psi''\}\,dx\,dt. \end{aligned} \tag{2.23}$$

Part (a) of the lemma and the first conclusion of part (b) follow directly from this identity in view of the arbitrariness of ψ and T. To confirm (2.22), we see that (2.23) implies

$$\begin{aligned} \left| \int_{\mathbb{R}} \{u(x,t_2;\varepsilon) - u(x,t_1;\varepsilon)\}\psi(x)\,dx \right| &= \left| \int_{t_1}^{t_2} \int_{x_1}^{x_2} \{f(u) - \varepsilon(a(u))_x\}\psi'\,dx\,dt \right| \\ &\le \int_{t_1}^{t_2} \int_{x_1}^{x_2} \{F_M + \varepsilon^{1/2}A_M t^{-1/2}\}\psi'\,dx\,dt \\ &= F_M(t_2 - t_1) + 2\varepsilon^{1/2}A_M(t_2^{1/2} - t_1^{1/2}) \end{aligned}$$

for any $-\infty < x_1 < x_2 < \infty$ and $0 \le t_1 < t_2 < \infty$, by (2.8). This gives (2.22) easily. □

With the above notation, we can state the following preliminary result.

Lemma 2.6. *Let $\varepsilon > 0$, assumptions* (H$_1$)–(H$_4$) *hold, and $\sigma_0 < \infty$. Fix $\delta > 0$ and define $x_\delta(t)$ by*

$$\int_{x_\delta(t)}^{\infty} w(x,t;\varepsilon)\,dx = \delta.$$

Then for any $t_2 > t_1 \geq 0$ such that $\zeta(t_1;\varepsilon) < \infty$ there holds

$$x_\delta(t_2) - x_\delta(t_1) = \int_{t_1}^{t_2} \frac{\int_{x_\delta(t)}^{\infty} f(u(x,t;\varepsilon))\,dx + \varepsilon a(u(x_\delta(t),t;\varepsilon))}{w(x_\delta(t),t;\varepsilon)}\,dt. \tag{2.24}$$

Proof. Assume u is a positive classical solution of (1.3) in $\mathbb{R}\times\mathbb{R}^+$. Fix $x^* \in \mathbb{R}$ and define $x_\delta(t) < x^*$ by

$$X(x_\delta(t),t) = \delta, \tag{2.25}$$

where

$$X(x,t) := \int_x^{x^*}\int_y^{x^*} u(z,t)\,dz\,dy.$$

Since u is continuously differentiable and positive, (2.25) defines a function $x_\delta \in C^1(\mathbb{R}^+)$ according to the Implicit Function Theorem. Furthermore,

$$0 = \frac{d}{dt}X(x_\delta(t),t) = -x_\delta'(t)\int_{x_\delta(t)}^{x^*} u(x,t)\,dx + \int_{x_\delta(t)}^{x^*}\int_x^{x^*} u_t(y,t)\,dy\,dx.$$

Using (1.3) to eliminate u_t and expanding the right-hand side, this gives

$$x_\delta'(t)\int_{x_\delta(t)}^{x^*} u(x,t)\,dx = \int_{x_\delta(t)}^{x^*} f(u(x,t))\,dx + \varepsilon a(u(x_\delta(t),t)) - r(x^*,t), \tag{2.26}$$

where

$$r(x,t) := \varepsilon a(u(x,t)) + \{x - x_\delta(t)\}\left(f(u) - \varepsilon(a(u))_x\right)(x,t),$$

whence, dividing (2.26) by the integral on the left-hand side of this identity and integrating with respect to t yield

$$x_\delta(t_2) - x_\delta(t_1) = \int_{t_1}^{t_2} \frac{\int_{x_\delta(t)}^{x^*} f(u(x,t))\,dx + \varepsilon a(u(x_\delta(t),t)) - r(x^*,t)}{\int_{x_\delta(t)}^{x^*} u(x,t)\,dx}\,dt.$$

The result now follows if we choose $x^* > \zeta(t;\varepsilon)$ for all $t_1 \leq t \leq t_2$ and approximate $u(\cdot,\cdot;\varepsilon)$ by a sequence of positive solutions. □

The promised "finer" properties of the right interface are given in the next statement.

Theorem 2.7. *Suppose that $\varepsilon > 0$, assumptions* (H$_1$)–(H$_4$) *hold, and $\sigma_0 < \infty$. Let $u(\cdot,\cdot;\varepsilon)$ denote the unique solution of problem* (1.3),(1.2) *and define*

$$\underline{V}(t) := \liminf_{\substack{x\uparrow\zeta(t;\varepsilon)\\ u(x,t;\varepsilon)>0}} \left(\frac{f(u)-\varepsilon(a(u))_x}{u}\right)(x,t;\varepsilon),$$

$$\overline{V}(t) := \limsup_{\substack{x\uparrow\zeta(t;\varepsilon)\\ u(x,t;\varepsilon)>0}} \left(\frac{f(u)-\varepsilon(a(u))_x}{u}\right)(x,t;\varepsilon)$$

for all $t > 0$. Then:

(i) *For any* $t_0 \geq 0$ *such that* $\zeta(t_0;\varepsilon) < \infty$ *there holds*

$$\int_{t_0}^{t} \underline{V}(s)ds \leq \zeta(t;\varepsilon) - \zeta(t_0;\varepsilon) \leq \int_{t_0}^{t} \overline{V}(s)\, ds \quad \text{for all } t > t_0. \tag{2.27}$$

(ii) *For any* $t_0 > 0$ *such that* $\zeta(t_0;\varepsilon) < \infty$ *and any* $\gamma > 0$, *there exists a* $t_1 > t_0$ *such that*

$$\{\underline{V}(t_0) - \gamma\}(t - t_0) \leq \zeta(t;\varepsilon) - \zeta(t_0;\varepsilon) \leq \{\overline{V}(t_0) + \gamma\}(t - t_0)$$

for all $t_0 < t \leq t_1$. *Moreover, if* $a(u_0)$ *is absolutely continuous on* $\mathbb{R}$, *this conclusion extends to* $t_0 = 0$ *when the limits in the definition of* $\overline{V}$ *and* $\underline{V}$ *are interpreted as essential limits.*

Proof. The first claim is a consequence of Lemma 2.6. Using the notation of that lemma, for any $t > 0$ such that $\zeta(t;\varepsilon) < \infty$ and any $\delta > 0$ there holds

$$\begin{aligned} &\int_{x_\delta(t)}^{\infty} f(u(x,t;\varepsilon))\, dx + \varepsilon a(u(x_\delta(t),t;\varepsilon)) \\ &\qquad = \int_{x_\delta(t)}^{\infty} (f(u) - \varepsilon(a(u))_x)\,(x,t;\varepsilon)\, dx, \end{aligned} \tag{2.28}$$

while, noting that $f(u) - \varepsilon(a(u))_x = 0$ if $u = 0$ by Proposition 2.4, we can estimate

$$\begin{aligned} &\int_{x_\delta(t)}^{\infty} (f(u) - \varepsilon(a(u))_x)\,(x,t;\varepsilon)\, dx \\ &\quad \leq \operatorname*{ess\,sup}_{\substack{x_\delta(t)<x<\zeta(t;\varepsilon)\\ u(x,t;\varepsilon)>0}} \left\{ \left(\frac{f(u) - \varepsilon(a(u))_x}{u} \right)(x,t;\varepsilon) \right\} \int_{x_\delta(t)}^{\infty} u(x,t;\varepsilon)\, dx. \end{aligned} \tag{2.29}$$

Combining (2.24), (2.28) and (2.29) yields

$$x_\delta(t_2) - x_\delta(t_1) \leq \int_{t_1}^{t_2} \operatorname*{ess\,sup}_{\substack{x_\delta(t)<x<\zeta(t;\varepsilon)\\ u(x,t;\varepsilon)>0}} \left\{ \left(\frac{f(u) - \varepsilon(a(u))_x}{u} \right)(x,t;\varepsilon) \right\} dt$$

for any $t_2 > t_1$ and $t_1 \geq 0$ such that $\zeta(t_1;\varepsilon) < \infty$. Letting $\delta \downarrow 0$ subsequently gives the right-hand inequality in (2.27). The proof of the left-hand inequality in (2.27) is similar. The second claim of the theorem was proved in [20] using a comparison argument with travelling-wave solutions under the additional assumption that $u_0 \in C(\mathbb{R})$. This proof readily extends to the present situation. □

Corollary 2.8. *For any* $t_0 \geq 0$ *such that* $\zeta(t_0;\varepsilon) < \infty$ *there holds*

$$\zeta(t;\varepsilon) - \zeta(t_0;\varepsilon) \leq \sigma_M(t - t_0) + \int_{t_0}^{t} Q_M(1, \varepsilon/4s)\, ds \quad \text{for all } t > t_0. \tag{2.30}$$

Proof. From (2.9) it follows that $\overline{V}(t) \leq \sigma_M + Q_M(1, \varepsilon/4t)$ for all $t > 0$. Substitution in (2.27) gives the result. □

As mentioned in the introduction, our major tool in the ensuing investigation will be a comparison principle for the primitive with respect to x of the solution of problems (1.3),(1.2) and (1.1),(1.2). Such a comparison principle was first introduced in the study of the porous media equation by Vázquez [52] and was designated a *shifting comparison principle*. Extensions were made in [2, 9]. The principle is the following.

Lemma 2.9. *Consider two problems* (1.3),(1.2) *with respective initial data functions* $u_0^{(1)}$ *and* $u_0^{(2)}$, *functions* f_1 *and* f_2, *and parameters* ε_1 *and* ε_2, *but with the same coefficient* a *for which hypotheses* (H_1)–(H_3) *hold. For* $i = 1, 2$, *let* M_i, $w^{(i)}$ *and* $w_0^{(i)}$ *denote the corresponding variables defined by* (1.4), (1.5) *and* (1.8) *respectively. Finally, let* D *denote a domain of the form* $D := (x_0, \infty) \times (0, T]$ *with* $-\infty \leq x_0 < \infty$ *and* $0 < T < \infty$. *If*

$$f_1(s) \geq f_2(s) \quad \text{for all } 0 \leq s \leq M_2, \tag{2.31}$$

$$\varepsilon_1 \geq \varepsilon_2, \tag{2.32}$$

$$w_0^{(1)}(x) \geq w_0^{(2)}(x) \quad \text{for all } x \in (x_0, \infty), \tag{2.33}$$

$$(\varepsilon_1 - \varepsilon_2) \operatorname{ess\,sup}\{u_0^{(1)}(x+h) - u_0^{(1)}(x) : (x, h) \in \mathbb{R} \times \mathbb{R}^+\} = 0, \tag{2.34}$$

and, in the event that $x_0 > -\infty$,

$$w^{(1)}(x_0, t; \varepsilon_1) > w^{(2)}(x_0, t; \varepsilon_2) \quad \text{for all } t \in [0, T], \tag{2.35}$$

then

$$w^{(1)}(x, t; \varepsilon_1) \geq w^{(2)}(x, t; \varepsilon_2) \quad \text{for all } (x, t) \in D. \tag{2.36}$$

Proof. In the case $\varepsilon_1 = \varepsilon_2$ this lemma may be found in [2]. We adapt that proof for the case $\varepsilon_1 > \varepsilon_2$. In light of Lemma 2.5, without loss of generality, we may suppose that $w^{(i)}(x, t) < \infty$ for all $(x, t) \in \mathbb{R} \times \mathbb{R}^+$ and $i = 1, 2$. Furthermore it suffices to consider positive classical solutions $u^{(i)}$ of problem (1.3),(1.2), since any nonnegative weak solution may be constructed as the limit of a sequence of such solutions [2, 13, 19, 21]. In this case (2.32) and (2.34) may be interpreted as reading $\varepsilon_1 = \varepsilon_2$, or $\varepsilon_1 > \varepsilon_2$ and $u_0^{(1)}$ is nonincreasing on $\mathbb{R}$. In this latter event, though, it follows that $u^{(1)}(\cdot, t)$ is also nonincreasing on $\mathbb{R}$ for any $t > 0$ [44, 46, 51]. Thus, (2.32) and (2.34) imply

$$(\varepsilon_1 - \varepsilon_2) u_x^{(1)} \leq 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^+. \tag{2.37}$$

Now we observe that for $i = 1, 2$ the function $w^{(i)}$ satisfies $w_x^{(i)} = -u^{(i)}$ and $w_t^{(i)} = -\varepsilon_i a'(u^{(i)}) u_x^{(i)} + f_i(u^{(i)}) = \varepsilon_i a'(u^{(i)}) w_{xx}^{(i)} + f_i(u^{(i)})$ in $\mathbb{R} \times \mathbb{R}^+$. Therefore defining

$$z(x, t) := w^{(2)}(x, t) - w^{(1)}(x, t),$$

we can compute

$$\begin{aligned} z_t \;=\; & \varepsilon_2 a'(u^{(2)}) z_{xx} - \varepsilon_2 u_x^{(1)} \{a'(u^{(2)}) - a'(u^{(1)})\} + \{f_1(u^{(2)}) - f_1(u^{(1)})\} \\ & + (\varepsilon_1 - \varepsilon_2) a'(u^{(1)}) u_x^{(1)} + f_2(u^{(2)}) - f_1(u^{(2)}), \end{aligned}$$

whence, setting

$$\alpha(x, t) := \int_0^1 \left(\varepsilon_2 u_x^{(1)} a''(\omega u^{(2)} + (1 - \omega) u^{(1)}) - f_1'(\omega u^{(2)} + (1 - \omega) u^{(1)}) \right)(x, t)\, d\omega,$$

there holds

$$\begin{aligned} \varepsilon_2 a'(u^{(2)}) z_{xx} + \alpha z_x - z_t \;&=\; f_1(u^{(2)}) - f_2(u^{(2)}) - (\varepsilon_1 - \varepsilon_2) a'(u^{(1)}) u_x^{(1)} \\ &\geq\; 0 \end{aligned}$$

by (2.31) and (2.37), Simultaneously, by (2.33) and (2.35) we have $z \leq 0$ on $\overline{D} \setminus D$. This implies $z \leq 0$ in D [33, 42], which proves (2.36). □

3. The first-order equation

In this section we present some results on the nonlinear first-order conservation law (1.1). Following Kruzhkov [37] we employ the following definition.

Definition 3.1. An entropy solution of problem (1.1),(1.2) is a nonnegative function $u \in L^\infty(\mathbb{R} \times \mathbb{R}^+)$ such that

$$\iint_{\mathbb{R}\times\mathbb{R}^+} \{|u-k|\varphi_t + \operatorname{sgn}(u-k)(f(u)-f(k))\varphi_x\}\,dx\,dt \geq 0 \tag{3.1}$$

for every nonnegative function $\varphi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^+)$ and constant $k \geq 0$, and (2.2) holds.

Concerning existence and uniqueness for problem (1.1),(1.2), the following has already been established [37, 38, 39, 40, 41].

Lemma 3.2. *Under hypotheses* (H_1) *and* (H_3) *problem* (1.1),(1.2) *admits a unique entropy solution* $u(x,t;0)$. *Moreover,* $u(\cdot,\cdot;0) \in C([0,\infty); L^1_{\mathrm{loc}}(\mathbb{R}))$, $u(x,t;0) \leq M$ *for almost all* $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, (2.3) *holds with* $\varepsilon = 0$ *for all* $t > 0$ *and* $h > 0$, *and* (2.4) *holds with* $\varepsilon = 0$.

Using the vanishing viscosity method it is also possible to show that entropy solutions are the limit, as $\varepsilon \downarrow 0$, of the corresponding solutions to problem (1.3),(1.2) [35, 36, 37, 38, 39, 41].

Lemma 3.3. *Under the assumptions* (H_1)–(H_3) *one has* $u(\cdot,\cdot;\varepsilon) \to u(\cdot,\cdot;0)$ *in* $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}))$ *as* $\varepsilon \downarrow 0$ *for every* $0 < T < \infty$.

The above makes it possible to extend the properties of the primitive of the solution of problem (1.3),(1.2) with respect to x to the solution of problem (1.1),(1.2). Let $w(x,t;0)$ be defined by (1.5) and (2.21) with $\varepsilon = 0$.

Lemma 3.4. *Suppose that assumptions* (H_1) *and* (H_3) *hold.*

(a) *If* $w_0(x) = \infty$ *for some* $x \in \mathbb{R}$, *then* $w(x,t;0) = \infty$ *for all* $(x,t) \in \mathbb{R} \times [0,\infty)$.
(b) *If* $w_0(x) < \infty$ *for some* $x \in \mathbb{R}$, *then* $w(x,t;0) < \infty$ *for all* $(x,t) \in \mathbb{R} \times [0,\infty)$, *and*

$$|w(x_1,t_1;0) - w(x_2,t_2;0)| \leq M|x_1 - x_2| + F_M|t_1 - t_2| \tag{3.2}$$

for all $(x_1,t_1), (x_2,t_2) \in \mathbb{R} \times [0,\infty)$. *Moreover, if hypothesis* (H_2) *holds, then* $w(\cdot,\cdot;\varepsilon) \to w(\cdot,\cdot;0)$ *as* $\varepsilon \downarrow 0$ *uniformly on compact subsets of* $\mathbb{R} \times [0,\infty)$.

Proof. Taking $k = 0$ and $k = M$ in (3.1) we obtain an identity for $u(\cdot,\cdot;0)$ similar to (2.1). The proof of all but the last assertion may subsequently be completed along the lines of the proof of Lemma 2.5. To verify the last assertion, we use the identity (2.23) where $\psi \in C^\infty(\mathbb{R})$ is such that $\psi(x) = 0$ for $x \leq x_1$, $\psi(x) = 1$ for $x \geq x_2$, and $\psi'(x) \geq 0$ for $x_1 < x < x_2$, for some $-\infty < x_1 < x_2 < \infty$. Since, by Lemma 3.3, $u(\cdot,\cdot;\varepsilon) \to u(\cdot,\cdot;0)$ in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}))$, we deduce from (2.23) that

$$\int_{\mathbb{R}} u(x,T;\varepsilon)\psi(x)\,dx \to \int_{\mathbb{R}} u(x,T;0)\psi(x)\,dx \quad \text{as } \varepsilon \downarrow 0$$

for any function ψ of the chosen type and any $T > 0$. Thus $\limsup_{\varepsilon\downarrow 0} w(x_2,T;\varepsilon) \leq w(x_1,T;0)$ and $\liminf_{\varepsilon\downarrow 0} w(x_1,T;\varepsilon) \geq w(x_2,T;0)$ for any $x_1 < x_2$ and $T > 0$.

This yields the pointwise convergence of $w(\cdot,\cdot;\varepsilon)$ to $w(\cdot,\cdot;0)$ in $\mathbb{R}\times\mathbb{R}^+$. The uniform convergence on compact subsets of $\mathbb{R}\times[0,\infty)$ subsequently follows from the continuity estimate (2.22). □

With the aid of the above lemma, we have the following simple result.

Lemma 3.5. *For any* $t_0\geq 0$ *there holds* $\liminf_{t\to t_0}\zeta(t;0)\geq\zeta(t_0;0)$.

Proof. Pick $x_0<\zeta(t_0;0)$. Then $w(x_0,t_0;0)>0$. By Lemma 3.4 this implies $w(x_0,t;0)>0$ for all $t\geq 0$ sufficiently close to t_0, whence $\zeta(t;0)>x_0$ for all such t. Subsequently, $\liminf_{t\to t_0}\zeta(t;0)\geq x_0$, and, since $x_0<\zeta(t_0;0)$ was arbitrary, the result is proved. □

Lemma 3.4 also provides the extension of the comparison principle, given for the parabolic problem by Lemma 2.9, to the hyperbolic problem.

Lemma 3.6. *Consider two problems* (1.1),(1.2) *with respective initial data functions* $u_0^{(1)}$ *and* $u_0^{(2)}$, *and functions* f_1 *and* f_2, *for which hypotheses* (H_1) *and* (H_3) *hold. For* $i=1,2$, *let* M_i, $w^{(i)}$ *and* $w_0^{(i)}$ *denote the corresponding variables defined by* (1.4), (1.5) *with* $\varepsilon=0$, *and* (1.8) *respectively. Finally, let* D *denote a domain of the form* $D:=(x_0,\infty)\times(0,T]$ *with* $-\infty\leq x_0<\infty$ *and* $0<T<\infty$. *If* (2.31) *and* (2.33) *are satisfied, and, in the event that* $x_0>-\infty$, $w^{(1)}(x_0,t;0)>w^{(2)}(x_0,t;0)$ *for all* $t\in[0,T]$, *then* $w^{(1)}(x,t;0)\geq w^{(2)}(x,t;0)$ *for all* $(x,t)\in D$.

In deriving the results on the interface in solutions of the parabolic equation (1.2) a crucial rôle was played by travelling-wave solutions [20]. For the hyperbolic equation (1.1) we would like a similar class of special solutions for comparison. Our choice falls upon solutions of the Riemann problem for equation (1.1), i.e. problem (1.1),(1.2) with initial data $u_0=M\chi_{(-\infty,0]}$. From classical theory, see for instance [8, 17, 26], it can be deduced that the solution of this problem is given by

$$u(x,t)=U(x/t), \tag{3.3}$$

where

$$U(\eta)\in\begin{cases} M & \text{for } \eta\leq\widetilde{f}'(M),\\ \left(\widetilde{f}'\right)^{-1}(\eta) & \text{for } \widetilde{f}'(M)<\eta<\sigma_M,\\ 0 & \text{for } \eta\geq\sigma_M,\end{cases} \tag{3.4}$$

and $\widetilde{f}$ denotes the concave hull of f on the interval $[0,M]$, i.e.

$$\widetilde{f}(s):=\sup_{\substack{0\leq s_0\leq s\leq s_1\leq M\\ s_0<s_1}}\left\{\frac{(s-s_0)f(s_1)+(s_1-s)f(s_0)}{s_1-s_0}\right\}. \tag{3.5}$$

An early proof that (3.3)–(3.5) is the vanishing viscosity solution when $f\in C^2([0,M])$ and f'' has at most a finite number of zeros on $[0,M]$ can be found in [30]. Since, however, a rigorous proof under the general conditions which we are considering in the present paper appears to be absent, we shall provide one below. We observe beforehand that, by elementary but somewhat lengthy arguments, $\widetilde{f}\in C([0,M])\cap C^1((0,M])$, with $\widetilde{f}'$ Hölder continuous on every set $[\delta,M]$ with $0<\delta<M$, $\widetilde{f}(0)=0$, and $\widetilde{f}(M)=f(M)$. Moreover, $\widetilde{f}'$ is decreasing on $(0,M]$, and $\widetilde{f}'(s)\to\sigma_M$ as $s\downarrow 0$.

Proposition 3.7. *Suppose that hypothesis* (H$_1$) *holds and* $u_0 = M\chi_{(-\infty,0]}$ *for some* $M > 0$. *Then the unique entropy solution of problem* (1.1),(1.2) *is given by* (3.3)–(3.5).

Proof. It is easily verified that u satisfies (2.2). The proof of the proposition boils down to showing that (3.1) holds for all $\varphi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^+)$ and real numbers $k \geq 0$. We prove this in three steps under increasingly weaker conditions on the function f.

Step 1. *Suppose that* $f \in C^2([0,M])$ *and there exists a partition* $0 = s_0^- \leq s_0^+ < s_1^- \leq s_1^+ < \cdots < s_{n-1}^- \leq s_{n-1}^+ < s_n^- \leq s_n^+ = M$ *such that* $\widetilde{f}$ *is affine on* $[s_i^-, s_i^+]$ *for* $i = 0, \ldots, n$ *and* $\widetilde{f} = f$ *with* $f'' < 0$ *on* (s_{i-1}^+, s_i^-) *for* $i = 1, \ldots, n$. Fix $k \geq 0$, and if necessary add an extra element to the partition of $[0,M]$ so that without loss of generality one can assume that $k \notin (s_{i-1}^+, s_i^-)$ for $i = 1, \ldots, n$. Set $\eta_i := \widetilde{f}'(s_i^-) = \widetilde{f}'(s_i^+)$ for $i = 0, \ldots, n$. Then making the change of variables $x = \eta\tau$ and $t = \tau$, we deduce that

$$\begin{aligned}
&\iint_{\mathbb{R}\times\mathbb{R}^+} \{|u-k|\varphi_t + \operatorname{sgn}(u-k)(f(u)-f(k))\varphi_x\}\,dx\,dt \\
&\quad= \iint_{\mathbb{R}\times\mathbb{R}^+} \{|U-k|(\tau\varphi_\tau - \eta\varphi_\eta) + \operatorname{sgn}(U-k)(f(U)-f(k))\varphi_\eta\}\,d\eta\,d\tau \\
&\quad= \iint_{\mathbb{R}\times\mathbb{R}^+} \{|U-k|(-\varphi - \eta\varphi_\eta) + \operatorname{sgn}(U-k)(f(U)-f(k))\varphi_\eta\}\,d\eta\,d\tau
\end{aligned}$$

for all $\varphi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^+)$. Subsequently, to prove (3.1) it suffices to confirm that

$$J := \int_{\mathbb{R}} \{|U-k|(-\psi - \eta\psi') + \operatorname{sgn}(U-k)(f(U)-f(k))\psi'\}\,d\eta \geq 0$$

for any nonnegative function $\psi \in C_0^\infty(\mathbb{R})$. However, noting that $U = s_n^+$ in $(-\infty, \eta_n)$,

$$\begin{aligned}
&|U-k|(-\psi - \eta\psi') + \operatorname{sgn}(U-k)(f(U)-f(k))\psi' \\
&\qquad= \{\operatorname{sgn}(U-k)[f(U)-f(k)-(U-k)f'(U)]\psi\}'
\end{aligned}$$

in (η_i, η_{i-1}) for $i = n, \ldots, 1$, and $u = s_0^-$ in (η_0, ∞), one computes

$$\begin{aligned}
J \;=\; \sum_{i=0}^{n} \big[&\operatorname{sgn}(s_i^+ - k)\{f(s_i^+) - f(k) - (s_i^+ - k)f'(s_i^+)\} \\
&- \operatorname{sgn}(s_i^- - k)\{f(s_i^-) - f(k) - (s_i^- - k)f'(s_i^-)\}\big]\,\psi(\eta_i),
\end{aligned}$$

whence, since $f'(s_i^-) = f'(s_i^+)$ for $i = 0, \ldots, n$, and in particular $f'(s_i^-) = f'(s_i^+) = \{f(s_i^+) - f(s_i^-)\}/(s_i^+ - s_i^-)$ if $s_i^- < s_i^+$, we deduce that

$$J = 2\left\{\frac{f(s_i^+)(k - s_i^-) + f(s_i^-)(s_i^+ - k)}{s_i^+ - s_i^-} - f(k)\right\}\psi(\eta_i) \geq 0$$

if $s_i^- < k < s_i^+$ for some i, and $J = 0$ otherwise.

Step 2. *Suppose that* $f \in C^{1+\alpha}([0,M])$ *for some* $0 < \alpha \leq 1$. Fix $\gamma > 0$ and choose n so large that if $\delta := M/n$, there holds $|f'(s) - f'(r)| < \gamma$ for all $0 \leq r, s \leq M$ with $|s-r| < 4\delta$. Subsequently, define the function ϕ on $(-2\delta, M+2\delta)$ by $\phi(s) := f'(4i\delta)$ for $(4i-2)\delta < s \leq (4i+2)\delta$ and $i = 0, \ldots, n$, and thereafter the function ψ on

$[0,M]$ by

$$\psi(s) := \frac{\int_{s-\delta}^{s+\delta} \exp\left(-1/\{1-(r-s)^2\delta^{-2}\}\right)\phi(r)\,dr}{\int_{-\delta}^{\delta} \exp\left(-1/\{1-r^2\delta^{-2}\}\right)\,dr}.$$

Finally, define the functions g and h on $[0,M]$ by

$$g(s) := \int_0^s \psi(r)\,dr + \gamma s \quad \text{and} \quad h(s) := \int_0^s \psi(r)\,dr - \gamma s.$$

Using the standard theory of mollifiers [1], it can be verified that g and h both satisfy the requirements for Step 1 above, while $f'(s)+2\gamma \geq g'(s) \geq f'(s) \geq h'(s) \geq f'(s)-2\gamma$ for all $0 \leq s \leq M$, whence $g(s) \geq f(s) \geq h(s)$, and if $\widetilde{g}$ and $\widetilde{h}$ denote the concave hull of g and h on $[0,M]$ respectively, $\widetilde{f}'(s)+2\gamma \geq \widetilde{g}'(s) \geq \widetilde{f}'(s) \geq \widetilde{h}'(s) \geq \widetilde{f}'(s)-2\gamma$ for all $0 \leq s \leq M$. Subsequently, by Lemma 3.6, the primitive (1.5) of the entropy solution of problem (1.1),(1.2) is boxed in by the corresponding primitives of the solution of this problem with f replaced by g and h respectively, and in the limit $\gamma \downarrow 0$ we obtain the asserted result.

Step 3. *Suppose that* (H$_1$) *holds.* Let $\delta \in (0,M)$ be fixed. By Step 2, the explicit solution $u^{(\delta)}$ of problem (1.1),(1.2) with the initial condition

$$u_0^{(\delta)} := \delta + (M-\delta)\chi_{(-\infty,0]}$$

is given by (3.3) and (3.4) where $\widetilde{f}$ is replaced by the concave hull of f on $[\delta,M]$, σ_M by $\sup\{(f(s)-f(\delta))/(s-\delta) : \delta < s \leq M\}$, and "0" by "$\delta$". Moreover, by monotonicity [39, 40, 41], the sequence $u^{(\delta)}$ converges almost everywhere to the entropy solution of problem (1.1),(1.2) with the initial condition $u_0 = M\chi_{(-\infty,0]}$. ☐

Corollary 3.8. *Suppose that hypothesis* (H$_1$) *holds and* $u_0 = M\chi_{(-\infty,\zeta_0]}$ *for some* $M > 0$ *and* ζ_0. *Then* $\zeta(t;0) = \zeta_0 + \sigma_M t$ *for all* $t > 0$.

Utilizing the comparison principle and the known behaviour of the solution of a Riemann problem for equation (1.1), we are able to obtain the next key estimate.

Lemma 3.9. *Suppose that* $\zeta_0 < \infty$. *Then*

$$\zeta_0 + \sigma_0 t \leq \zeta(t;0) \leq \zeta_0 + \sigma_M t \quad \text{for all } t > 0. \tag{3.6}$$

Proof. The right-hand inequality in (3.6) is immediate from Corollary 3.8 and Lemma 3.6. To confirm the left-hand inequality, fix $T > 0$ and $x_1 < \zeta_0$. By Lemma 3.5 there exists an $x_0 < x_1$ such that $\zeta(t;0) > x_0$ for all $t \in [0,T]$, whence we can find a $\mu > 0$ such that $w(x_0,t;0) > \mu$ for all $t \in [0,T]$ and $w_0(x) > \mu$ for all $x \in [x_0,x_1]$. Next, let $\delta > 0$ be so small that $\delta(x_1-x_0)+F_\delta T < \mu$ and consider the entropy solution $\widetilde{u}$ of the Riemann problem (1.1),(1.2) with initial data $\widetilde{u}_0 := \delta\chi_{(-\infty,x_1]}$. If $\widetilde{w}$ is the corresponding variable given by (1.5) and (2.21) with $\varepsilon = 0$, by (3.2) there holds $\widetilde{w}(x,t) \leq \delta(x_1-x)+F_\delta t$ for all $(x,t) \in [x_0,x_1]\times[0,T]$. Summarizing, this gives $w(x_0,t;0) > \mu > \widetilde{w}(x_0,t)$ for all $t \in [0,T]$, $w(x,0;0) > \mu > \widetilde{w}(x,0)$ for all $x \in [x_0,x_1]$, and $w(x,0;0) \geq 0 = \widetilde{w}(x,0)$ for all $x \in [x_1,\infty)$. So all the assumptions of Lemma 3.6 are satisfied. Consequently, if $\widetilde{\zeta}$ denotes the supremum of the support of $\widetilde{w}$, we have $\zeta(t;0) \geq \widetilde{\zeta}(t)$ for all $t \in [0,T]$. However, by Corollary 3.8, $\widetilde{\zeta}(t) = x_1 + \sigma_\delta t$. This yields $\zeta(t;0) \geq x_1 + \sigma_0 t$ for all $t \in [0,T]$. Letting x_1 approach ζ_0 and T approach infinity, the proof is complete. ☐

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1. Identifying $\zeta(0;0)$ with ζ_0, Lemma 3.5 infers that $\zeta(\cdot;0)$ is lower semi-continuous on $[0,\infty)$. Furthermore, noting that to prove (1.9) and (1.10) without loss of generality we may take $t_0 = 0$, by Lemmata 3.6 and 3.9 we have (1.9), and (1.10) if $\sigma_0 > -\infty$. Now, the inequality (1.9) implies $\limsup_{t\downarrow t_0}\zeta(t;0) \le \zeta(t_0;0)$ for all $t_0 \ge 0$, which, together with the lower semi-continuity, gives the continuity of $\zeta(\cdot;0)$ from the right, while, if $\sigma_0 > -\infty$, the inequality (1.10) implies $\limsup_{t_0\uparrow t}\zeta(t_0;0) \le \zeta(t;0)$ for all $t > 0$, which, together with the previous conclusions, gives the full continuity. Thus the theorem is proved. ◻

It is also possible to give some further refinements for the qualitative description of the hyperbolic fronts, in the spirit of Theorem 2.7.

Theorem 3.10. *Let assumptions* (H$_1$) *and* (H$_3$) *hold and* $\sigma_0 < \infty$. *Let* $u(\cdot,\cdot;0)$ *denote the unique entropy solution of problem* (1.1),(1.2) *and define*

$$\underline{V}(t) := \operatorname*{lim\,ess\,inf}_{\substack{x\uparrow\zeta(t;0)\\ u(x,t;0)>0}} \left(\frac{f(u)}{u}\right)(x,t;0) \quad \text{and} \quad \overline{V}(t) := \operatorname*{lim\,ess\,sup}_{\substack{x\uparrow\zeta(t;0)\\ u(x,t;0)>0}} \left(\frac{f(u)}{u}\right)(x,t;0)$$

for all $t > 0$. *Then*:

(i) *For any* $t_0 \ge 0$ *such that* $\zeta(t_0;0) < \infty$ *there holds*

$$\zeta(t;0) - \zeta(t_0;0) \le \int_{t_0}^{t} \overline{V}(s)\,ds \quad \text{for all } t > t_0.$$

Furthermore, if $\liminf_{s\downarrow 0} f(s)/s > -\infty$ *there holds*

$$\int_{t_0}^{t} \underline{V}(s)ds \le \zeta(t;0) - \zeta(t_0;0) \quad \text{for all } t > t_0.$$

(ii) *For any* $t_0 \ge 0$ *such that* $\zeta(t_0;0) < \infty$ *and any* $\gamma > 0$, *there exists a* $t_1 > t_0$ *such that*

$$\{\sigma_{\underline{u}} - \gamma\}(t - t_0) \le \zeta(t;0) - \zeta(t_0;0) \le \{\sigma_{\overline{u}} + \gamma\}(t - t_0) \tag{3.7}$$

for all $t_0 < t \le t_1$, *where*

$$\underline{u} := \operatorname*{lim\,ess\,inf}_{x\uparrow\zeta(t_0;0)} u(x,t_0;0) \quad \text{and} \quad \overline{u} := \operatorname*{lim\,ess\,sup}_{x\uparrow\zeta(t_0;0)} u(x,t_0;0).$$

Proof. Letting $\varepsilon \downarrow 0$ in Lemma 2.6, the conclusions of Lemma 2.6 hold with $\varepsilon = 0$. The proof of part (i) of the present theorem may subsequently be completed along the lines of the proof of part (i) of Theorem 2.7. As regards the proof of part (ii), without loss of generality we may take $t_0 = 0$. Furthermore, we may suppose that $\sigma_{\underline{u}} - \gamma > \sigma_0$, for otherwise the left-hand inequality in (3.7) holds by Theorem 1.1. In this case, let $0 < \delta < \underline{u}$ be such that $\sigma_\delta > \sigma_{\underline{u}} - \gamma$. Consider next the entropy solution $\widetilde{u}$ of problem (1.1),(1.2) with initial data $\widetilde{u}_0(x) := \delta\chi_{(-\infty,\zeta_0]}$. If $\widetilde{w}$ denotes the corresponding variable defined by (1.5) and (2.21), then $w_0(x) \ge \widetilde{w}(x,0)$ for all $x \ge x_0$ and in particular $w_0(x_0) > \widetilde{w}(x_0,0)$ for some $x_0 < \zeta_0$, whence by the continuity of w and $\widetilde{w}$ there exists a $T > 0$ such that $w(x_0,t;0) > \widetilde{w}(x_0,t)$ for all $t \in [0,T]$. Lemma 3.6 and Corollary 3.8 then imply $\zeta(t;0) \ge \zeta_0 + \sigma_\delta t$ for all $t \in [0,T]$. Identifying t_1 with T yields the left-hand inequality in (3.7). The proof of the right-hand inequality is similar. ◻

4. Convergence of the fronts

In this section, we address the question of the convergence of $\zeta(\cdot;\varepsilon)$ to $\zeta(\cdot;0)$ as $\varepsilon \downarrow 0$. Since it is not clear in advance that $\zeta(\cdot;\varepsilon)$ has a limit as $\varepsilon \downarrow 0$, we need to introduce the quantities:

$$\underline{\zeta}(t) := \liminf_{\varepsilon\downarrow 0} \zeta(t;\varepsilon) \quad \text{and} \quad \overline{\zeta}(t) := \limsup_{\varepsilon\downarrow 0} \zeta(t;\varepsilon)$$

for $t \geq 0$. Furthermore, since $\zeta(\cdot;0)$ is not necessarily continuous when $\sigma_0 = -\infty$, for convenience we define

$$\widetilde{\zeta}(t) := \limsup_{s\uparrow t} \zeta(s;0)$$

for all $t > 0$. As an extended function $\widetilde{\zeta}$ is upper semi-continuous and continuous from the left on $\mathbb{R}^+$ with $\widetilde{\zeta}(t) \geq \zeta(t;0)$ for all $t > 0$.

To prove our main results, namely Theorems 1.3 and 1.4, we present three preliminary lemmata.

Lemma 4.1. *Suppose that assumptions* (H_1)–(H_4) *hold and* $\sigma_0 < \infty$. *Then* $\underline{\zeta}(t) \geq \zeta(t;0)$ *for every* $t > 0$.

Proof. The proof of this result bears much resemblance to the proof of the lower semi-continuity of $\zeta(\cdot;0)$ in Lemma 3.5. Fix $t > 0$ and $x_0 < \zeta(t;0)$. Then $w(x_0,t;0) > 0$. Subsequently, since $w(\cdot,\cdot;\varepsilon)$ converges to $w(\cdot,\cdot;0)$ as $\varepsilon \downarrow 0$ uniformly on compact subsets of $\mathbb{R}\times[0,\infty)$ by Lemma 3.4, $w(x_0,t;\varepsilon) > 0$ for sufficiently small $\varepsilon > 0$, whence $\zeta(t;\varepsilon) > x_0$ for such ε, and thus $\underline{\zeta}(t) \geq x_0$. Letting x_0 approach $\zeta(t;0)$ yields the desired result. □

Lemma 4.2. *Suppose that assumptions* (H_1)–(H_4) *hold and* $\sigma_0 < \infty$. *Then*

$$\overline{\zeta}(t) \leq \overline{\zeta}(t_0) + \sigma_M(t-t_0) \quad \text{for all } t > t_0 \geq 0. \tag{4.1}$$

Moreover if $\sigma_0 > -\infty$, *then*

$$\overline{\zeta}(t) \geq \overline{\zeta}(t_0) + \sigma_0(t-t_0) \quad \text{for all } t > t_0 \geq 0. \tag{4.2}$$

Proof. The inequalities (4.1) and (4.2) follow from (1.11) and (1.12) respectively, in view of Lemma 2.3. □

The next lemma is the more difficult part of our proof.

Lemma 4.3. *Suppose that* $\sigma_0 < \infty$ *and that* $\widetilde{\zeta}(T) < \overline{\zeta}(T)$ *for some* $T > 0$. *Then there exists a* $\tau \in [0,T)$ *such that*

$$\overline{\zeta}(T) \leq \overline{\zeta}(t_0) + \sigma(T-t_0) \tag{4.3}$$

for all $t_0 \in [\tau,T)$ *and* $\sigma > \sigma_0$.

Proof. Pick an x_0 such that $\widetilde{\zeta}(T) < x_0 < \overline{\zeta}(T)$ and let $0 \leq \tau < T$ be such that $\zeta(t;0) < x_0$ and $x_0 + \sigma_M(T-t) < \overline{\zeta}(T)$ for all $t \in [\tau,T]$. Next fix $t_0 \in [\tau,T)$ and $\sigma > \sigma_0$. Now, (4.3) is trivially true if $\overline{\zeta}(t_0) = \infty$, while (4.3) is a corollary of Lemma 4.2 when $\sigma \geq \sigma_M$. It remains therefore to verify (4.3) when $\overline{\zeta}(t_0) < \infty$ and $\sigma_0 < \sigma < \sigma_M$. For this case, fix x_1 so large that

$$x_1 > \max\{x_0, x_0 - \sigma(T-t_0), \overline{\zeta}(t_0)\}, \tag{4.4}$$

and define

$$\mu := \sup\{\delta > 0 : \sigma_\delta < \sigma\}.$$

Subsequently, as in [20], for any $\varepsilon > 0$ we can construct a travelling-wave solution $U(\cdot,\cdot;\varepsilon)$ of equation (1.3) in $\mathbb{R}\times[0,\infty)$, via the identity

$$\int_0^{U(x,t;\varepsilon)} \frac{\varepsilon a'(s)}{\sigma s - f(s)}\,ds = \max\{x_1 - x + \sigma(t-t_0), 0\}.$$

It is easy to check that $U(x,t;\varepsilon) = 0$ for all $x \geq x_1 + \sigma(t-t_0)$, $U(x,t;\varepsilon) > 0$ for all $x < x_1 + \sigma(t-t_0)$, and $U(x,t;\varepsilon) \to \mu$ as $x \to -\infty$, for every $t \geq 0$ and $\varepsilon > 0$. Furthermore, $U(x,t;\varepsilon) \to \mu$ as $\varepsilon \downarrow 0$, uniformly on compact subsets of $\{(x,t) \in \mathbb{R}\times[0,\infty) : x < x_1 + \sigma(t-t_0)\}$. Let $W(\cdot,\cdot;\varepsilon)$ denote the variable (1.5) associated with $U(\cdot,\cdot;\varepsilon)$. Observe that $W(x_0,t;\varepsilon) \to \mu\{x_1 - x_0 + \sigma(t-t_0)\}$ and $w(x_0,t;\varepsilon) \to w(x_0,t;0) = 0$ as $\varepsilon \downarrow 0$ for any $t \in [t_0,T]$. Also, $W(x,t_0;\varepsilon) \to \mu(x_1 - x_0)$ and $w(x,t_0;\varepsilon) \to w(x,t_0;0) = 0$ as $\varepsilon \downarrow 0$ for all $x \in [x_0,x_1)$, while, by definition, $W(x,t_0;\varepsilon) = 0$ for all $x \in [x_1,\infty)$ and $w(x,t_0;\varepsilon) = 0$ for all $x \in [\zeta(t_0;\varepsilon),\infty)$ for every $\varepsilon > 0$. Since $x_1 > \overline{\zeta}(t_0)$, it follows that we can choose an $\varepsilon^* > 0$ so small that $W(x_0,t;\varepsilon) > w(x_0,t;\varepsilon)$ for all $t \in [t_0,T]$ and $W(x,t_0;\varepsilon) \geq w(x,t_0;\varepsilon)$ for all $x \in [x_0,\infty)$, for every $0 < \varepsilon < \varepsilon^*$, whence, by the comparison principle given by Lemma 2.9, there holds $W(x,t;\varepsilon) \geq w(x,t;\varepsilon)$ for all $(x,t) \in (x_0,\infty)\times(t_0,T]$ and $0 < \varepsilon < \varepsilon^*$. This implies $\zeta(T;\varepsilon) \leq x_1 + \sigma(T-t_0)$ for all $0 < \varepsilon < \varepsilon^*$. Hence, letting $\varepsilon \downarrow 0$ and thereafter letting x_1 approach its lower bound in (4.4), we obtain

$$\overline{\zeta}(T) \leq \max\{x_0 + \sigma(T-t_0), x_0, \overline{\zeta}(t_0) + \sigma(T-t_0)\}.$$

However, since $x_0 + \sigma(T-t_0) < x_0 + \sigma_M(T-t_0) < \overline{\zeta}(T)$ and $x_0 < \overline{\zeta}(T)$, this yields (4.3). □

We are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3. In light of Lemma 4.1, to obtain the pointwise convergence of $\zeta(\cdot;\varepsilon)$ to $\zeta(\cdot;0)$ it suffices to show that

$$\overline{\zeta}(t) \leq \zeta(t;0) \quad \text{for all } t > 0. \tag{4.5}$$

By Theorem 1.1 and Lemma 4.2, $\zeta(\cdot;0)$ and $\overline{\zeta}$ are both continuous on $[0,\infty)$ with $\zeta(0;0) = \overline{\zeta}(0) = \zeta_0$. It follows that if (4.5) is not true, then there exist a $\tau \geq 0$ and a $T > \tau$ such that

$$\overline{\zeta}(\tau) = \zeta(\tau;0) \tag{4.6}$$

and

$$\overline{\zeta}(t) > \zeta(t;0) \quad \text{for all } t \in (\tau,T]. \tag{4.7}$$

In this event, for every $t_1 \in (\tau,T)$ there is a $t_0 \in [\tau,t_1)$ such that $\overline{\zeta}(t_1) \leq \overline{\zeta}(t_0) + \sigma_0(t_1 - t)$ for all $t \in [t_0,t_1)$ by Lemma 4.3. From the arbitrariness of t_1 and a continuation argument we thus obtain

$$\overline{\zeta}(t) \leq \overline{\zeta}(\tau) + \sigma_0(t-\tau) \quad \text{for all } t \in (\tau,T]. \tag{4.8}$$

On the other hand, by (1.10)

$$\zeta(\tau;0) + \sigma_0(t-\tau) \leq \zeta(t;0) \quad \text{for all } t > \tau. \tag{4.9}$$

Combining (4.6), (4.8) and (4.9) contradicts (4.7). This proves the pointwise convergence in $\mathbb{R}^+$ of $\zeta(\cdot;\varepsilon)$ to $\zeta(\cdot;0)$ as $\varepsilon \downarrow 0$. The remainder of the proof is now straightforward. From Theorem 1.2, Corollary 2.8 and the properties of the function Q_M proved in Lemma 2.3, we have a uniform estimate of the continuity of $\zeta(\cdot;\varepsilon)$ on $[0,T]$ and a uniform estimate of the Lipschitz continuity of $\zeta(\cdot;\varepsilon)$ on $[\tau,T]$

for any $0 < \tau < T < \infty$ and $0 < \varepsilon < \varepsilon^*$ with $\varepsilon^* > 0$ fixed. The asserted convergence follows via the Ascoli-Arzelà Theorem. $\square$

A direct consequence of Theorem 1.3 is given by the following refinement. Recall the definition of σ_M and σ_0 by (1.6) and (1.7), and the definition of I_M by (2.5).

Proposition 4.4. *Under the conditions of Theorem* 1.3 *the following additional convergence occurs*: (i) $\zeta(\cdot;\varepsilon) \to \zeta(\cdot;0)$ *in* $C^{0+\alpha}([0,T])$ *for all* $0 < T < \infty$ *and* $0 < \alpha < 1/2$ *if* $a'(s)/s \in L^1(0,M)$; (ii) $\zeta'(\cdot;\varepsilon) \to \zeta'(\cdot;0)$ *in* $L^\infty(\tau,\infty)$ *for all* $0 < \tau < \infty$ *if* $\sigma_M = \sigma_0$; (iii) $\zeta(\cdot;\varepsilon) - \zeta(\cdot;0) \to 0$ *in* $C^{0+1}([\tau,\infty))$ *for all* $0 < \tau < \infty$ *if* $\sigma_M = \sigma_0$ *and* $I_M < \infty$; *and* (iv) $\zeta(\cdot;\varepsilon) \to \zeta(\cdot;0)$ *in* $C^{0+1}([\tau,\infty))$ *for all* $0 < \tau < \infty$ *if* $\sigma_M = \sigma_0 = 0$ *and* $I_M < \infty$.

Proof. (i) We polish an estimation in [20]. By definition

$$\begin{aligned} q_M(t,\varepsilon,\sigma) &= (\sigma - \sigma_M)t + \varepsilon \int_0^M \frac{a'(s)}{\sigma s - f(s)}\,ds \\ &\le (\sigma - \sigma_M)t + \varepsilon \int_0^M \frac{a'(s)}{\sigma s - \sigma_M s}\,ds. \end{aligned}$$

Hence, setting

$$c_M := 2\left(\int_0^M \frac{a'(s)}{s}\,ds\right)^{1/2},$$

we can estimate $Q_M(t,\varepsilon) \le q_M(t,\varepsilon,\sigma_M + c_M\varepsilon^{1/2}t^{-1/2}/2) \le c_M\varepsilon^{1/2}t^{1/2}$ for any $\varepsilon > 0$ and $t > 0$. By Theorem 1.2, for every $\varepsilon^* > 0$ this gives a uniform bound for $\zeta(\cdot;\varepsilon)$ in $C^{0+1/2}([0,T])$ for any $T > 0$ and $0 < \varepsilon < \varepsilon^*$, from which the assertion follows. (ii) If $\sigma_M = \sigma_0$, then by Theorem 1.1 we have explicitly that $\zeta'(t;0) = \sigma_0$ for all $t \ge 0$, while by Theorem 1.2 and Corollary 2.8 the function $\zeta(\cdot;\varepsilon)$ is absolutely continuous on $\mathbb{R}^+$ with $\sigma_0 \le \zeta'(t;\varepsilon) \le \sigma_0 + Q_M(1,\varepsilon/4\tau)$ for almost all $t > \tau$ and all $\varepsilon > 0$. This provides the required result. (iii) This case builds on the previous one, since now by Theorem 1.2 and Lemma 2.3, $\zeta(t;0) = \zeta_0 + \sigma_0 t$ and $\zeta_0 + \sigma_0 t \le \zeta(t;\varepsilon) \le \zeta_0 + \sigma_0 t + \varepsilon I_M$ for all $t \ge 0$. (iv) This case is covered by the proof of the previous one. $\square$

We remark that the equation $u_t = (a(u))_{xx}$ admits a unique similarity solution of the form $u = U(xt^{-1/2})$ in the sense of distributions in $\mathbb{R} \times \mathbb{R}^+$ where $a(U)$ is continuously-differentiable and nonincreasing on $\mathbb{R}$, $U(\eta) \to M$ as $\eta \to -\infty$, and $U(\eta) \to 0$ as $\eta \to \infty$. Furthermore, if $a'(s)/s \in L^1(0,M)$, then there is a $c_M > 0$ such that $U(\eta) > 0$ for all $\eta < c_M$ and $U(\eta) = 0$ for all $\eta \ge c_M$ [16]. By a change of variables, it can subsequently be verified that when $f(s) = \sigma_M s$ for all $0 \le s \le M$ the solution of problem (1.3),(1.2) with $u_0 = M\chi_{(-\infty,0]}$ is given by $u(x,t;\varepsilon) = f((x - \sigma_M t)\varepsilon^{-1/2}t^{-1/2})$ so that in this particular instance one finds $\zeta(t;\varepsilon) = \sigma_M t + c_M\varepsilon^{1/2}t^{1/2}$ for a constant c_M which depends only on a and M. This shows that the convergence established in parts (i) and (ii) above is the best possible in general.

Let us turn now to the singular situation, $\sigma_0 = -\infty$.

Proof of Theorem 1.4. Recalling Lemma 4.1 to prove the theorem it suffices to show that if $\overline{\zeta}(t_0) < \infty$ for some $t_0 \ge 0$, then

$$\overline{\zeta}(t) \le \widetilde{\zeta}(t) \quad \text{for all } t > t_0. \tag{4.10}$$

Let us therefore hypothesize, contrary to (4.10), that

$$\widetilde{\zeta}(T) < \overline{\zeta}(T) \quad \text{for some } T > t_0. \tag{4.11}$$

Then, by Lemma 4.3 there exists a $t_1 \in (t_0, T)$ such that

$$\overline{\zeta}(T) \leq \overline{\zeta}(t_1) + \sigma(T - t_1) \tag{4.12}$$

for all $\sigma > -\infty$. However, by Lemma 4.2 we have

$$\overline{\zeta}(t_1) \leq \overline{\zeta}(t_0) + \sigma_M(t_1 - t_0). \tag{4.13}$$

Combining (4.11)–(4.13) we obtain $\widetilde{\zeta}(T) < \overline{\zeta}(t_0) + \sigma_M(t_1 - t_0) + \sigma(T - t_1)$ for all $\sigma > -\infty$. In the limit $\sigma \to -\infty$ this is inadmissible. We therefore conclude that (4.10) must hold. $\square$

There are particular situations in which we can strengthen the results of Theorem 1.4.

Proposition 4.5. *Suppose that assumptions* (H$_1$)–(H$_4$) *hold and* $\sigma_0 < \infty$. (i) *If* $w_0(x) = \infty$ *for some* $x \in \mathbb{R}$, *then* $\zeta(t; \varepsilon) = \infty$ *for all* $t \geq 0$ *and* $\varepsilon \geq 0$. (ii) *If* $\zeta_0 < \infty$ *and*

$$u_0(x) \geq \mu \quad \textit{for almost all } x \in (-\infty, \zeta_0) \tag{4.14}$$

for some $\mu > 0$ *such that* $\sigma_\mu = \sigma_M$, *then* $\zeta(\cdot; \varepsilon) \to \zeta(\cdot; 0)$ *in* $L^\infty(0, T)$ *for all* $0 < T < \infty$. (iii) *If* $\zeta_0 < \infty$ *and* (4.14) *holds for some* $\mu > 0$ *such that* $\sigma_\mu = \sigma_M$ *and* $I_M < \infty$, *then* $\zeta(\cdot; \varepsilon) - \zeta(\cdot; 0) \to 0$ *in* $L^\infty(\mathbb{R}^+)$. (iv) *If* $\zeta_0 < \infty$ *and* (4.14) *holds for some* $\mu > 0$ *such that* $\sigma_\mu = \sigma_M = 0$ *and* $I_M < \infty$, *then* $\zeta(\cdot; \varepsilon) \to \zeta(\cdot; 0)$ *in* $L^\infty(\mathbb{R}^+)$.

Proof. Part (i) is an immediate consequence of the first part of Lemma 2.5 and the first part of Lemma 3.4. The key to the remaining parts is the estimate

$$\zeta(t; \varepsilon) \geq \zeta_0 + \sigma_\mu t \quad \text{for all } t > 0 \text{ and } \varepsilon \geq 0. \tag{4.15}$$

With this in hand, parts (ii)–(iv) of the proposition can be deduced similarly to the corresponding parts of Proposition 4.4. To obtain (4.15) in the first-order case $\varepsilon = 0$, we may compare the given entropy solution of (1.1),(1.2) under the restriction (4.14), with the solution of the same problem with initial data function $\widetilde{u}_0 := \mu\chi_{(-\infty,\zeta_0]}$. Then Lemma 3.6 and Corollary 3.8 give (4.15) with $\varepsilon = 0$. For the second-order case the inequality (4.15) is similarly proved by a comparison argument with the travelling-wave solution U of problem (1.3),(1.2) given by

$$\int_0^{U(x,t)} \frac{\varepsilon a'(s)}{\sigma_\mu s - f(s)}\, ds = \max\{\zeta_0 - x + \sigma_\mu t, 0\}$$

with appropriate initial data and $\varepsilon > 0$. $\square$

Note that Proposition 4.5 is also valid in the situation that $\sigma_0 > -\infty$, and in this light parts (iii) and (iv) complement the corresponding parts of Proposition 4.4.

Our final result in this section concerns the monotonic convergence of the fronts $\zeta(\cdot; \varepsilon)$ in both the cases $\sigma_0 > -\infty$ and $\sigma_0 = -\infty$.

Theorem 4.6. *Let hypotheses* (H$_1$)–(H$_3$) *hold. Suppose that* u_0 *is nonincreasing in the sense that* $\operatorname{ess\,inf}\{u_0(y) : y \in (-\infty, x)\} \geq \operatorname{ess\,sup}\{u_0(y) : y \in (x, \infty)\}$ *for all* $x \in \mathbb{R}$. *Then*

$$\zeta(t; \varepsilon_1) \geq \zeta(t; \varepsilon_2) \geq \zeta(t; 0) \quad \textit{for any } \varepsilon_1 > \varepsilon_2 > 0 \textit{ and } t > 0.$$

Proof. By Lemma 2.9, there holds $w(x,t;\varepsilon_1) \geq w(x,t;\varepsilon_2)$ for all $(x,t) \in \mathbb{R}\times\mathbb{R}^+$ and $\varepsilon_1 > \varepsilon_2 > 0$, whence, by Lemma 3.4, $w(x,t;\varepsilon_2) \geq w(x,t;0)$ for all $(x,t) \in \mathbb{R}\times\mathbb{R}^+$ also. ☐

5. The special case of power laws

In this final section we provide a synopsis of the previous results for the prototype equations (1.13) and (1.14) with $\lambda \in \{-1,0,1\}$, $n > 0$ and $m > 0$ real parameters. Although these results do not present any new qualitative information, there is a novelty in that the power-law structure of equations (1.13) and (1.14) enables us to obtain explicit estimates of the continuity of the interfaces.

Theorem 5.1. *Let $\lambda \in \{-1,0,1\}$, $n > 0$ and $m > 0$ be real constants, and suppose that u_0 satisfies assumption* (H$_3$)*. As described in the introduction, let ζ_0 denote the supremum of the support of u_0, $\zeta(\cdot;0)$ denote the supremum of the support of the entropy solution of problem* (1.13),(1.2)*, and, $\zeta(\cdot;\varepsilon)$ the supremum of the support of the solution of problem* (1.14),(1.2) *for $\varepsilon > 0$.*

(i) *If $n < 1$ and $\lambda = 1$, there holds $\zeta(t;\varepsilon) = \infty$ for all $t > 0$ and $\varepsilon \geq 0$.*

(ii) *If $n = 1$, there holds $\zeta(t;0) = \zeta_0 + \lambda t$ for all $t > 0$. Moreover:*
  (a) *If $m \leq 1$, then $\zeta(t;\varepsilon) = \infty$ for all $t > 0$ and $\varepsilon > 0$.*
  (b) *If $m > 1$, then $\zeta_0 + \lambda t \leq \zeta(t;\varepsilon) \leq \zeta_0 + \lambda t + R(t,\varepsilon,M)$ for all $t > 0$ and $\varepsilon > 0$, where*

$$R(t,\varepsilon,M) = \mathcal{C}\varepsilon^{1/2}t^{1/2}M^{(m-1)/2} \tag{5.1}$$

  for some constant $\mathcal{C}$ which depends only on m. Furthermore if $\zeta_0 < \infty$ then the generalized derivative $\zeta'(t;\varepsilon)$ exists and satisfies $\lambda \leq \zeta'(t;\varepsilon) \leq \lambda + t^{-1}R(t,\varepsilon/4,M)$ for almost all $t > 0$ and all $\varepsilon \geq 0$.

(iii) *If $n > 1$ and $\lambda = -1$, there holds $\zeta(t;0) = \zeta_0$ for all $t > 0$. Moreover:*
  (a) *If $m \leq 1$, then $\zeta(t;\varepsilon) = \infty$ for all $t > 0$ and $\varepsilon > 0$.*
  (b) *If $m > 1$, then $\zeta_0 \leq \zeta(t;\varepsilon) \leq \zeta_0 + R(t,\varepsilon,M)$ for all $t > 0$ and $\varepsilon > 0$, where $R(t,\varepsilon,M)$ is given by* (5.1) *or by*

$$R(t,\varepsilon,M) = \begin{cases} \mathcal{C}\varepsilon^{(n-1)/(2n-m-1)}t^{(n-m)/(2n-m-1)} & \text{for } m < n, \\ \mathcal{C}\varepsilon\{\ln(\varepsilon^{-1}tM^{n-1}+1)+1\} & \text{for } m = n, \\ \mathcal{C}\varepsilon M^{m-n} & \text{for } m > n, \end{cases} \tag{5.2}$$

  for some constant $\mathcal{C}$ which depends only on m and n. Furthermore if $\zeta_0 < \infty$, then the generalized derivative $\zeta'(t;\varepsilon)$ exists and satisfies $0 \leq \zeta'(t;\varepsilon) \leq t^{-1}R(t,\varepsilon/4,M)$ for almost all $t > 0$ and all $\varepsilon \geq 0$.

(iv) *If $n > 1$ and $\lambda = 1$, there holds $\zeta_0 \leq \zeta(t;0) \leq \zeta_0 + M^{n-1}t$ for all $t > 0$. Moreover:*
  (a) *If $m \leq 1$, then $\zeta(t;\varepsilon) = \infty$ for all $t > 0$ and $\varepsilon > 0$.*
  (b) *If $m > 1$, then $\zeta_0 \leq \zeta(t;\varepsilon) \leq \zeta_0 + M^{n-1}t + R(t,\varepsilon,M)$ for all $t > 0$ and $\varepsilon > 0$, where $R(t,\varepsilon,M)$ is given by* (5.1) *or by*

$$R(t,\varepsilon,M) = \mathcal{C}\varepsilon M^{m-n}\{\ln(\varepsilon^{-1}tM^{2n-m-1}+1)+1\} \tag{5.3}$$

  for some constant $\mathcal{C}$ which depends only on m and n. Furthermore if $\zeta_0 < \infty$, then the generalized derivative $\zeta'(t;\varepsilon)$ exists and satisfies $0 \leq \zeta'(t;\varepsilon) \leq M^{n-1} + t^{-1}R(t,\varepsilon/4,M)$ for almost all $t > 0$ and all $\varepsilon \geq 0$.

(v) *If $n < 1$ and $\lambda = -1$, there holds $\zeta(t;0) \leq \zeta_0 - M^{n-1}t$ for all $t > 0$. Moreover:*
  (a) *If $m \leq n$, then $\zeta(t;\varepsilon) = \infty$ for all $t > 0$ and $\varepsilon > 0$.*

(b) *If $m > n$, then $\zeta(t;\varepsilon) \le \zeta_0 - M^{n-1}t + R(t,\varepsilon,M)$ for all $t > 0$ and $\varepsilon > 0$, where $R(t,\varepsilon,M)$ is given by*

$$R(t,\varepsilon,M) = \begin{cases} \mathcal{C}\varepsilon^{(1-n)/(m+1-2n)}t^{(m-n)/(m+1-2n)} & \text{for } m < 1, \\ \mathcal{C}\varepsilon^{1/2}t^{1/2}\ln^{1/2}(\varepsilon t^{-1}M^{2(1-n)}+1) & \text{for } m = 1, \\ \mathcal{C}\varepsilon^{1/2}t^{1/2}M^{(m-1)/2} & \text{for } m > 1, \end{cases} \tag{5.4}$$

or by

$$R(t,\varepsilon,M) = \mathcal{C}\varepsilon M^{m-n}\{\ln(\varepsilon^{-1}tM^{2n-m-1}+1)+1\} \tag{5.5}$$

for some constant $\mathcal{C}$ which depends only on m and n. Furthermore there holds $\zeta'(t;\varepsilon) \le -M^{n-1} + t^{-1}R(t,\varepsilon/4,M)$ for almost all $t > 0$ and all $\varepsilon \ge 0$ such that $\zeta(t;\varepsilon) < \infty$ and the generalized derivative $\zeta'(t;\varepsilon)$ exists.

In cases (iii), (iv) and (v) above, two alternatives for the function R have been presented. In these cases the first estimate is sharper for small values of t, whereas the second one is sharper for small values of ε and large values of t.

Proof of Theorem 5.1. The major conclusions are covered by Theorems 1.1 and 1.2 and Corollary 2.8. It remains to obtain the explicit estimates on $\zeta(\cdot;\varepsilon)$ for $\varepsilon > 0$ in terms of the function R. In light of the inequalities (1.11) and (2.30) and the observation that $Q_M(1,\varepsilon/4t) = t^{-1}Q_M(t,\varepsilon/4)$, these estimates may be obtained by showing that, in each case, $Q_M(t,\varepsilon) \le R(t,\varepsilon,M)$. To achieve this, recalling that Q_M is defined by (2.7), it suffices to show that for each $t > 0$ and $\varepsilon > 0$ we can find a $\sigma \ge \sigma_M$ such that

$$q_M(t,\varepsilon,\sigma) \le R(t,\varepsilon,M), \tag{5.6}$$

where σ_M is given by (1.6) and q_M by (2.6), for a suitable constant $\mathcal{C}$ in the function R. We apply this strategy case by case below.

(ii) Retracing the proof of part (i) of Proposition 4.4 there holds

$$q_M(t,\varepsilon,\sigma) \le \mathcal{C}\varepsilon^{1/2}t^{1/2}M^{(m-1)/2}$$

for $\sigma = \sigma_M + \mathcal{C}\varepsilon^{1/2}t^{-1/2}M^{(m-1)/2}/2$, where $\mathcal{C} = 2m^{1/2}(m-1)^{-1/2}$. This yields (5.6) with R given by (5.1). Note that this proof also partially covers cases (iii), (iv) and (v).

(iii) In view of the last remark, we only have to consider (5.2). We begin with the case $m < n$. In this case

$$\begin{aligned} q_M(t,\varepsilon,\sigma) &= (\sigma-\sigma_M)t + \varepsilon\int_0^M \frac{ms^{m-1}}{\sigma s + s^n}\,ds \\ &\le (\sigma-\sigma_M)t + \varepsilon\int_0^\infty \frac{ms^{m-1}}{\sigma s + s^n}\,ds \\ &= (\sigma-\sigma_M)t + \varepsilon\sigma^{-(n-m)/(n-1)}\int_0^\infty \frac{mr^{m-1}}{r + r^n}\,dr. \end{aligned}$$

Choosing $\sigma = \sigma_M + \varepsilon^{(n-1)/(2n-m-1)}t^{-(n-1)/(2n-m-1)}$ we arrive at (5.6) with a suitable constant $\mathcal{C}$ in (5.2). Note that this argument applies just as well when $n < m < 1$. For $m = n$ the first integral above can be evaluated explicitly to yield $q_M(t,\varepsilon,\sigma) = \sigma t + \varepsilon n(n-1)^{-1}\ln(\sigma^{-1}M^{n-1}+1)$. Taking $\sigma = \varepsilon t^{-1}$ we obtain the required result. Finally, for $m > n$ we have $q_M(t,\varepsilon,0) = \varepsilon m(m-n)^{-1}M^{m-n}$, which yields the last part of (5.2).

(iv) Only (5.3) still has to be verified. Let $p = \min\{m, n\}$. Then

$$\begin{aligned} q_M(t,\varepsilon,\sigma) &= (\sigma - \sigma_M)t + \varepsilon \int_0^M \frac{m s^{m-2}}{\sigma - s^{n-1}}\, ds \\ &\leq (\sigma - \sigma_M)t + \varepsilon \int_0^M \frac{m M^{m-p} s^{p-2}}{\sigma - M^{n-p} s^{p-1}}\, ds \\ &= (\sigma - M^{n-1})t + \varepsilon m (p-1)^{-1} M^{m-n} \{\ln \sigma - \ln(\sigma - M^{n-1})\}. \end{aligned}$$

Taking $\sigma = M^{n-1} + \varepsilon t^{-1} M^{m-n}$, we obtain (5.6) with $\mathcal{C} = \max\{1, m(p-1)^{-1}\}$ in (5.3).

(v) The proof of (5.6) with R given by (5.4) in the case $m > 1$ has been covered with part (ii), while the proof in the case $m < 1$ has been covered with part (iii). This leaves (5.4) for $m = 1$. In this instance, one has $q_M(t,\varepsilon,\sigma) = (\sigma + M^{n-1})t + \varepsilon(1-n)^{-1}\sigma^{-1}\ln(\sigma M^{1-n} + 1)$ if $\sigma \neq 0$ and $q_M(t,\varepsilon,0) = M^{n-1}t + \varepsilon(1-n)^{-1}M^{1-n}$. Taking $\sigma = -M^{n-1} + \varepsilon^{1/2} t^{-1/2} \ln^{1/2}(\varepsilon t^{-1} M^{2(1-n)} + 1)$, we derive our bound after some tedious but elementary calculations. Lastly, we have to verify the bound in (5.5). This we can do similarly to (5.3). Setting $p = \min\{m, 1\}$, we observe that

$$\begin{aligned} q_M(t,\varepsilon,\sigma) &= (\sigma - \sigma_M)t + \varepsilon \int_0^M \frac{m s^{m-n-1}}{1 + \sigma s^{1-n}}\, ds \\ &\leq (\sigma - \sigma_M)t + \varepsilon \int_0^M \frac{m M^{m-p} s^{p-n-1}}{1 + \sigma M^{1-p} s^{p-n}}\, ds \\ &= (\sigma + M^{n-1})t + \varepsilon m (p-n)^{-1} M^{m-1} \sigma^{-1} \ln(\sigma M^{1-n} + 1), \end{aligned}$$

whenever $\sigma < 0$, while $q_M(t,\varepsilon,0) = M^{n-1}t + \varepsilon m(m-n)^{-1}M^{m-n}$. The conclusion follows, again after some calculations, taking $\sigma = \min\{-M^{n-1} + \varepsilon t^{-1} M^{m-n}, 0\}$. $\square$

Under the assumption that u_0 is continuous, further estimates of the kind in Theorem 5.1, both from above and below, can be found in the literature. Estimates specifically concerned with the dependence of the initial growth rate on the smoothness of u_0 near ζ_0 have been obtained in [2, 3]. In [22, 34] results on the large time behaviour in relation to the decay of the solution in some reference point have been obtained, while in [12, 14, 15, 18, 23, 27, 31, 47] the large time behaviour when u_0 has compact support has been investigated. In a number of instances it is known that the interface $\zeta(\cdot;\varepsilon)$ in the solution of problem (1.14),(1.2) is smoother than suggested by Theorem 5.1 [15, 28, 47, 48, 49]. In particular, for the case $n = 1$ the theory of the porous media equation [4, 6, 29, 53] may be applied to deduce especially refined regularity.

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