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A CONDITION FOR THE STABILITY OF $\mathbb{R}$-COVERED ON FOLIATIONS OF 3-MANIFOLDS

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Abstract. We give a sufficient condition for a codimension one, transversely orientable foliation of a closed 3-manifold to have the property that any foliation sufficiently close to it be $\mathbb{R}$-covered. This condition can be readily verified for many examples. Further, if an $\mathbb{R}$-covered foliation has a compact leaf L, then any transverse loop meeting L lifts to a copy of the leaf space, and the ambient manifold fibers over S^1 with L as fiber.

The focus in this paper is codimension one, transversely oriented, C^1 foliations of closed 3-manifolds. The property of being $\mathbb{R}$-covered (that is, covered by a trivial product of planes) has been important in the study of foliations, particularly those arising from Anosov flows. Solodov [So] and Barbot [Ba1], [Ba2] have shown that an $\mathbb{R}$-covered Anosov foliation implies that the associated Anosov flow is transitive. Ghys [Gh], in the case of Seifert manifolds, proved that all Anosov foliations are $\mathbb{R}$-covered; Plante [Pl1], in the case that the fundamental group of the ambient 3-manifold is solvable, proved the same. These results are essential in showing the associated Anosov flows are conjugate to standard models—geodesic flows or suspensions of Anosov diffeomorphisms [Pl1, Pl2], [Gh]. Fenley [Fe1, Fe2] has used the hypothesis of $\mathbb{R}$-covered to uncover the rich structure of metric and homotopy properties of the flow lines in many Anosov flows. In general, $\mathbb{R}$-covered foliations are particularly nice since the action of the fundamental group $\pi_1(M)$ of the manifold on the universal cover induces a homomorphism from $\pi_1(M)$ to the group of homeomorphisms of $\mathbb{R}$ (where $\mathbb{R}$ is the leaf space of the lifted foliation).

Taut foliations have been well-studied, especially by Thurston and Gabai. Tautness is the key to Roussarie's [R] and Thurston's [T] results on isotoping incompressible tori, and in Thurston's study of norm-minimizing leaves. Gabai [Ga1, Ga2, Ga3], in turn, used these results, by tautly foliating knot complements, to find the minimal genus spanning surface for a large class of knots and links.

In 3-dimensions, an $\mathbb{R}$-covered foliation is easily shown to be taut as long as $M \neq S^2 \times S^1$ (Lemma B in section 3). However, while tautness indicates the absence of dead-end components, it does not imply $\mathbb{R}$-covered as the many non-$\mathbb{R}$-covered Anosov foliations show.

In this paper, we give a sufficient condition for an $\mathbb{R}$-covered foliation to have the property that all foliations sufficiently close to it in the C^1 metric are also $\mathbb{R}$-covered. This dates back to a question posed by W. Thurston in 1976. A key element of the proof lies in finding a property of a branched surface which carries only foliations

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which are $\mathbb{R}$-covered. First, we show that a taut foliation is carried by a branched surface with transitive dual graph, and that all foliations carried by such a branched surface must indeed be taut (Theorems 1 and 2 in section 2). Second, we define an '$\mathbb{R}$-covered' branched surface and show that any foliation carried by an $\mathbb{R}$-covered branched surface is in fact $\mathbb{R}$-covered. An $\mathbb{R}$-covered branched surface is formally defined in section 3, but intuitively it is a branched surface with a transitive dual graph which does not contain the local behavior one would expect if there were a pair of planar leaves in the universal cover corresponding to a pair of nonseparable points in a non-Hausdorff leaf space. Using [Sh1], it follows readily (Corollary 5) that any foliation F which is carried by an $\mathbb{R}$-covered branched surface is 'stably $\mathbb{R}$-covered' in the sense described above. Further we give a procedure to construct such a branched surface for many $\mathbb{R}$-covered foliations, hence providing numerous examples of stably $\mathbb{R}$-covered foliations.

In section 4, we give some related results, including a simple topological condition equivalent to a foliation being $\mathbb{R}$-covered. We also show that if we assume the existence of a compact leaf L in an $\mathbb{R}$-covered foliation, then any closed transverse curve meeting L lifts to a copy of the leaf space, and in fact, the manifold must fiber over S^1 with L as fiber.

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1. Preliminaries

Throughout this paper, M will be a closed Riemannian 3-manifold, and all foliations of M will be C^1, codimension one, and transversely oriented. The universal cover of M will be denoted $\hat{M}$, and $p : \hat{M} \to M$ will be the covering map. Given a foliation F of M, $\hat{F}$ will denote the foliation obtained by lifting F to $\hat{M}$.

Branched surface construction. One of the main tools used in this paper is that of a branched surface constructed from a foliation. Since the procedure for that construction, first suggested to Goodman by C. Danthony, is in an unpublished paper of Christy and Goodman [C-G]; we include an outline. We note that our definition gives Williams' regular branched surface [W].

We begin with a foliation F, a nonsingular flow ϕ transverse to F, and a finite ***generating set*** $\Delta = \{D_i\}$ of disjoint imbedded compact surfaces with boundary, satisfying the following:

(i) each D_i lies in a leaf of F (hence is transverse to ϕ);
(ii) every orbit of ϕ meets the interior of some element of Δ in forward and backward time;
(iii) $\{x \in \bigcup(Bdy D_i) \mid$ the forward orbit of x meets $Bdy(D_j)$ some j before meeting $\bigcup \operatorname{int}(D_i)\}$ is finite;
(iv) any orbit of ϕ meetings $\bigcup(Bdy\ D_i)$ at most twice.

We cut M open along the interior of each element of Δ to produce a compact manifold M^* with boundary which can be imbedded in M. Each orbit of ϕ gives rise to a disjoint union of interval orbits, called ***fibers*** of M^*, in an induced flow ϕ^* on M^*. We then construct the branched surface W as a quotient space, identifying fiber to a point in W, i.e., W is homeomorphic to $M^*/\sim$, where $x \sim y$ if x and y lie on the same fiber of M^*. Then W is called the ***branched surface corresponding to*** ($\boldsymbol{F}$,ϕ,Δ), and has a transverse orientation induced by ϕ^*.

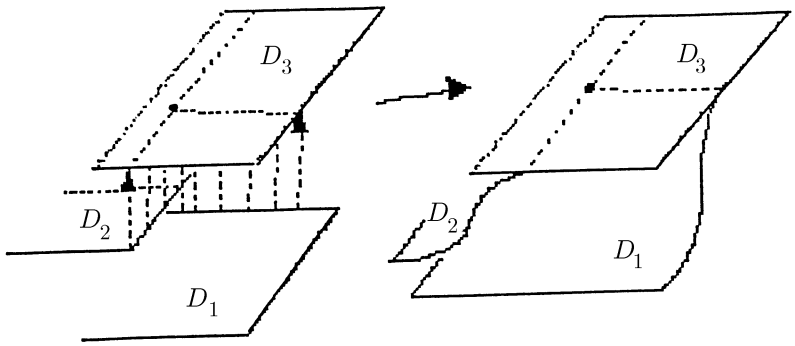

FIGURE 1.1.

It is much more natural to view the branched surface W as imbedded in M. The set $\mathrm{Int}(\Delta) = \{x \mid x \in \mathrm{int}(\bigcup D_i)\}$ in M contains the upper most representative (in terms of the orientation of ϕ^*) of each equivalence class. Each x in the boundary of an element of Δ is identified with an element of $\mathrm{Int}(\Delta)$ according to what point of $\mathrm{Int}(\Delta)$ is first met by the forward orbit of x. To identify points so $\mathrm{Int}(\Delta)$ is homeomorphic to W, we modify the elements of Δ near their boundaries to branch into $\mathrm{Int}(\Delta)$, staying transverse to the flow. See Figure 1.1. Throughout the paper, we will assume W is imbedded in M.

The requirements for Δ imply that W is a connected 2-dimensional complex with a set of charts defining orientation preserving local diffeomorphisms onto one of the models in Figure 1.2, such that the transition maps are smooth and preserve transverse orientation. Each local model projects horizontally onto a vertical model of $\mathbb{R}^2$; therefore $T\mathbb{R}^2$ induces a smooth structure on W when we pull back each local projection.

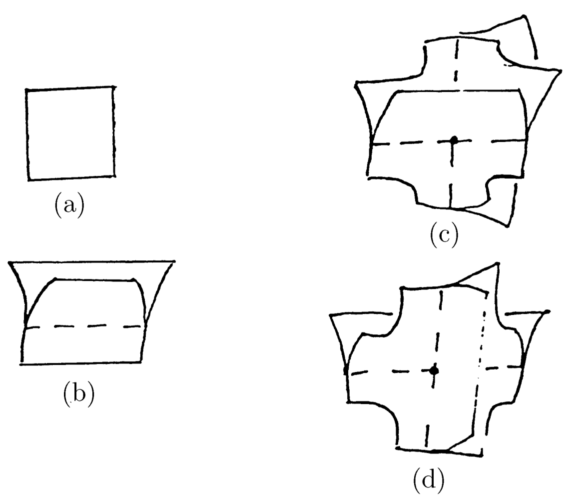

FIGURE 1.2.

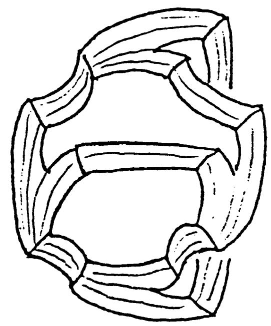

FIGURE 1.3. Example of $N(W)$ locally with foliation F^*

Each branched surface obtained in the above fashion is a connected 2-manifold except on a dimension one subset B called the ***branch set of W***. The elements of Δ can be assumed to be chosen large enough to ensure that B is connected. The set B is a 1-manifold except at a finite number of points, called ***double points of B***, where B crosses itself transversely. We see the double points in B precisely when an orbit of ϕ meets $\bigcup(Bdy\ D_i)$ twice, which by choice of Δ, occurs finitely often. Each component of $W - B$ is called a ***sector of W***.

By "smooth arc or surface in W" we will mean an arc or surface respectively in W which is smooth under the structure inherited from W. We will not consider arcs which cross the branch set, turn and "backtrack" to the same sector.

Note that if we thicken W along the interval orbits of ϕ^*, we retrieve M^*, which, for that reason, we shall henceforth call ***N(W)***, ***the neighborhood of W***. We refer to $N(W)$ as a manifold with ***furrowed boundary***. That is, $N(W)$ has a C^1 atlas with three types of local models: neighborhoods in 3-dimensional Euclidean space, neighborhoods in 3-dimensional Euclidean half-space, and neighborhoods of the origin in the product of $\{(x, y) \text{ in } \mathbb{R}^2 \mid x < 0, \text{ or } y \geq x^2, \text{ or } y \leq -x^2\}$ with the real line. Boundary points of $N(W)$ with a local neighborhood of the third type are the ***furrow points***.

Foliations carried by branched surfaces. A foliation F clearly gives rise to a foliation F^* of $N(W)$ where the leaves are transverse to the fibers and the set of branch points of the leaves is the same as the set of furrow points of $N(W)$. Each boundary component of $N(W)$ is contained in a branched leaf of F^* and this leaf corresponds to a leaf of F which contains an element of Δ. Figure 1.3 shows how F^* appears locally.

There are, of course, many possible foliations of $N(W)$ with the properties of F^* above. When we 'collapse' the components of $M - N(W)$ by identifying points bounding fibers to recover ϕ, each of these foliations yields a foliation of M, also transverse to ϕ, which we say is ***carried by W***. In particular, if F gives rise to F^* on $N(W)$, the foliation F^* yields F under this collapsing process.

Throughout the paper, $\pi_W : N(W) \to W$ will denote the quotient map which identifies points in the same fiber. We say the image of a point x under this map is the ***projection*** of that point. Accordingly, we say points in the fiber lie over x.

2. Tautness and branched surfaces

Recall that a foliation F is ***taut*** if there is a single transverse loop meeting every leaf of F, i.e., M consists of a single Novikov component for F. In Theorems 1 and 2 in this section, we show that a foliation is taut if and only if it is carried by a branched surface with a dual graph that is transitive (defined later in this section). This characterization of tautness will be useful in the next section.

We now proceed to a branched surface characterization of tautness, beginning with some definitions and a simple lemma.

For a branched surface W, we define the ***dual graph*** to W as follows: each component of $M - W$ will contain one vertex. There will be an edge through each sector of W, oriented according to the transverse orientation of W, joining two vertices. So the dual graph is imbedded in M and transverse to W. If for any ordered pair of vertices (v, w) of the dual graph there is a positively oriented path from v to w, then the dual graph is said to be ***transitive***.

For a foliation F with a transverse flow ϕ and a generating set Δ, we say a curve is a positive ***staircase curve*** for (F, ϕ, Δ) if it is a finite composition $\gamma_1 * \alpha_1 * \cdots * \gamma_{n-1} * \alpha_{n-1} * \gamma_n$, where each α_k is contained in the interior of some $D_k \in \Delta$, each γ_k is contained in an orbit of ϕ with the same orientation as ϕ, and $\mathrm{int}(\gamma_k) \cap \mathrm{int}(\Delta) = \emptyset$ for all k. Let W be the branched surface corresponding to (F, ϕ, Δ).

Lemma A. *The dual graph to W is transitive if and only if, for any ordered pair (D, D') of elements of Δ, there is a positive staircase curve for (F, ϕ, Δ) from D to D'.*

Proof. ($\leftarrow$) Given (v_A, v_B), an ordered pair of vertices of the dual contained in components A and B of $M - W$ respectively, let D_A and D_B be elements of Δ corresponding to A and B. By hypothesis, there is a positive staircase curve $\gamma_1 * \alpha_1 * \cdots * \gamma_{n-1} * \alpha_{n-1} * \gamma_n$ from D_A to D_B. For $1 \leq k \leq n$, γ_k corresponds to a fiber I_k in $N(W)$, hence, in W, projects to a point y_k on a sector. Let $D_0 = D_A$, $D_n = D_B$, and D_k be the element of Δ containing α_k for $0 < k < n$. Since the fibers of $N(W)$ induce the transverse orientation on W, there exists a positively oriented arc through y_k, $k \geq 1$, from the component of $M - W$ corresponding to D_{k-1} to the component corresponding to D_k. Clearly this indicates a positively oriented path in the dual from v_A to v_B.

($\rightarrow$) Now suppose the dual is transitive. Let (D_A, D_B) be an ordered pair of elements of Δ, (v_A, v_B) the corresponding pair of vertices in the dual contained in components (A, B) of $M - W$. There is a positively oriented path of edges of the

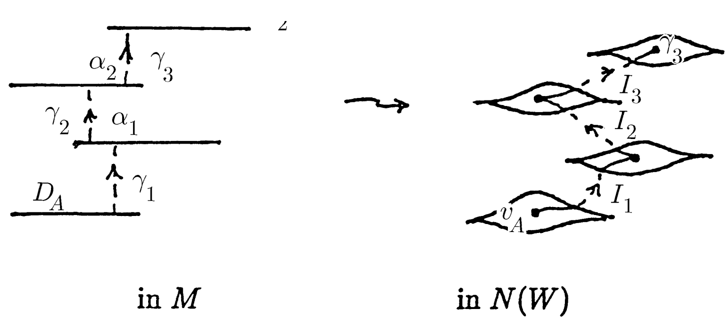

Figure 2.1.

dual $t_1 * t_2 * \cdots * t_n$, from v_A to v_B. For each k, there is a point y_k in t_k lying on a sector of W. Let I_k be the fiber of $N(W)$ over y_k. So $t = t_1 * t_2 * \cdots * t_n$ corresponds in $N(W)$ to $t_1' * I_1 * t_1'' * t_2' * I_2 * t_2'' * \cdots * t_n' * I_n * t_n''$, where the two components of $t_k - \{y_k\}$, which lie in adjacent components of $M - W$, correspond to arcs t_k' and t_k'' respectively in $M - N(W)$. Note that each $t_{k-1}'' \cup t_k'$ collapses to a curve in an element of Δ (when we collapse the components of $M - N(W)$), yielding a curve containing the desired positive staircase curve. □

Theorem 1. *F taut implies that F is carried by a branched surface W such that the dual of W is transitive.*

Proof. The key fact is that F taut implies that there exists a volume-preserving ϕ transverse to F [Pl3]. We take a branched surface W, generated by a finite set of disks $\Delta = \{D_1, \cdots, D_n\}$ chosen for F and ϕ as described in section 1 (e.g., cover the manifold with foliation boxes and choose Δ to contain a slice from each box). Consider the dual graph to W with vertices $V = \{v_1, \cdots, v_n\}$ corresponding to the disks of Δ.

Suppose the dual graph to W is not transitive. Then for some vertex v_i, there is a vertex v_j with no positively-oriented path in the dual from v_i to v_j. Since the manifold is connected, it follows that we may assume that v_j is the 'first' such vertex; that is, there is a vertex v in V which is met by a positively-oriented path from v_i, and v is adjoined to v_j by an edge in the dual graph (apparently oriented from v_j to v).

By construction of the dual graph, there is an orbit of ϕ going from some point x in the interior of D_j to some y in the interior of D (where D_j is the disk in Δ corresponding to v_j, and similarly for D and v). Continuity of the flow ensures that there is a $d > 0$ such that any point in the interior of D_j within d of x also flows into the interior of D.

Since ϕ is volume-preserving, the non-wandering set of ϕ is all of M; in particular, the point x is non-wandering. So there is a point x' in D_j within d of x whose positive orbit returns to D_j, passing through D along the way. Therefore, there exists a positive staircase path from D to D_j (see Figure 2.2). Adjoining this path with one from D_i to D, we obtain a positive staircase path from D_i to D_j (corresponding to a positively-oriented path in the dual from v_i to v_j) contradicting the assumption on v_j. Hence the dual graph to W is transitive. □

Theorem 2. *If W has a transitive dual, then every foliation carried by W is taut.*

Proof. Let (F, ϕ) be a foliation and a transverse flow carried by W with a transitive dual and let Δ be the generating set for W. Let L be any leaf of F and $\mathcal{O}$ be

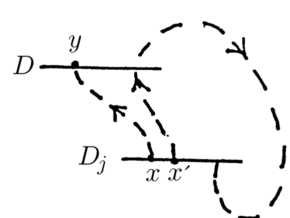

FIGURE 2.2.

any orbit of the transverse flow ϕ meeting L. The positive semi-orbit of $\mathcal{O}, \mathcal{O}^+$, eventually meets some $D \in \Delta$, the generating set for W. Further, the negative semi-orbit of $\mathcal{O}$, $\mathcal{O}^-$, meet some D' in Δ. By the hypothesis of a transitive dual and Lemma A, there is a positive staircase curve t from D to D'. Modifying t produces a positively oriented transverse arc from D to D', and hence (joining t with that part of $\mathcal{O}$ from D' to D) a transverse loop through L. Since L was arbitrary, F is taut. □

Although it is known that the property of being taut is stable for codimension one foliations of any dimension (see [Su], [Sch], and implicitly in [Pl3]), we note that the result in dimension 3 easily follows from the above theorems.

Corollary 3. *The property of being taut is stable; that is, if F is a taut foliation, then every foliation sufficiently near F (in the C^1-metric of* [Hi]*) is also taut.*

Proof. Let F be a taut foliation. There exists a branched surface W with a transitive dual carrying F (Theorem 1). Then by Theorem 2, any foliation carried by W is also taut. By [Sh1], each foliation sufficiently close to F is topologically equivalent to a foliation that is carried by W, so is taut. □

3. Stability of ℝ-covered

A stronger condition than being taut for the foliation F is that it is ℝ***-covered***; that is, when F is lifted to the universal cover of M, the leaf space of $\hat{F}$ is homeomorphic to $\mathbb{R}$. The following lemma makes clear that, for manifolds $M \neq S^2 \times S^1$, ℝ-covered is indeed a stronger condition than taut, and in fact, that in this case ℝ-covered is equivalent to $\hat{F}$ being a trivial product of planes. There are well-known examples of taut foliation which are not ℝ-covered, like the many non-ℝ-covered Anosov foliations.

Lemma B. *For $M \neq S^2 \times S^1$, F being ℝ-covered implies that $\hat{F}$ is a trivial product of planes and F is taut.*

Proof. First note that F being ℝ-covered implies there are no null-homotopic closed transverse curves, since such a curve would lift to the universal cover and the leaf space of $\hat{F}$ could not be $\mathbb{R}$. Therefore any Reeb component lifts to $D^2 \times \mathbb{R}$ in the universal cover with the generator of the bounding torus which yields a vanishing cycle having trivial holonomy on the outside of the Reeb component. So a neighborhood of the Reeb component can be excised and replaced by a solid torus foliated by a product of disks. This excision process leaves M intact and keeps the property that the leaf space is still ℝ. Once all the Reeb components have been erased in this manner, the leaves in the universal cover are simply connected, hence planes (since $M \neq S^2 \times S^1$). Therefore $\hat{M}$ is homeomorphic to $\mathbb{R}^3$ and M is irreducible [Ro].

We claim that any toral leaf L in F has multiple lifts. Otherwise, taking a basepoint x_0 in L, any loop in M based at x_0 lifts to an arc in $\hat{M}$ (homeomorphic to $\mathbb{R}^3$) with both ends on the single lift $\hat{L}$; hence the loop in M is homotopic to a loop in L. However, by [He] and our hypotheses, $\pi_1 M$ cannot be isomorphic to a subgroup of the fundamental group of the torus.

Now, if L is a toral leaf of F with at least two lifts L_1 and L_2, then since the leaf space of $\hat{F}$ is ℝ, there is a transverse path from L_1 to L_2. Hence, there exists a closed

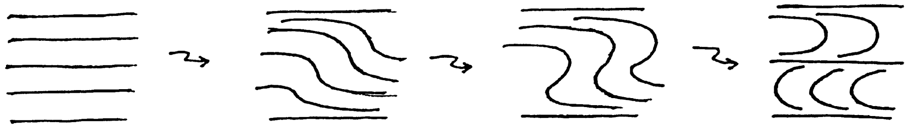

FIGURE 3.1.

transverse curve through L. In particular, L cannot bound a Reeb component, so F is Reebless and $\hat{F}$ a trivial product of planes.

Further, any nontoral leaf has a closed transversal through it [Go] and we've just shown any toral leaf must also. Therefore F is taut. □

To illustrate some of the subtlety of the question of stability of $\mathbb{R}$-covered, consider the following example of a foliation of a ***noncompact*** manifold, $\mathbb{R}^2$, which although it is an $\mathbb{R}$-foliation, has non-$\mathbb{R}$-covered foliations arbitrarily near it.

On the lower half-plane, let the foliation F be a product of horizontal lines. In the upper half-plane, make each $\mathbb{R} \times \{n\}$, n a nonnegative integer, a leaf. In the strips in between, we insert discrete samples from a 1-parameter family of foliations as follows. The product foliation on $\mathbb{R} \times I$, I a closed interval, is homotopic to the foliation on $\mathbb{R} \times I$ with two Reeb-like components as shown above in Figure 3.1 on the right. Let F_t indicate the family of foliations in this homotopy parameterized by t in $[0,1]$, where F_0 is the product foliation on the left, and F_1 is the far right foliation. Let $\{t_j\}$ be a sequence in $[0,1]$ monotonically approaching 1. Then in each strip of the upper half-plane, $\mathbb{R} \times [n, n+1]$, n a nonnegative integer, insert the foliation F_{t_n}.

Clearly the foliation of $\mathbb{R}^2$ thus produced is an $\mathbb{R}$-foliation, but arbitrarily small perturbations can move it to one that is not.

In this section we provide a characterization of a nice branched surface that carries only $\mathbb{R}$-covered foliations, as long as M is compact and $M \neq S^2 \times S^1$. Using [Sh1], we obtain a verifiable condition for all foliations sufficiently near a given one to be $\mathbb{R}$-covered (Corollary 5). The motivation for the characterization of this branched surface is that if a foliation F is taut (hence all the leaves of $\hat{F}$ are planes) then the only obstruction to $\hat{F}$ being an $\mathbb{R}$-foliation is that the leaf space is non-Hausdorff, i.e., there is a pair of leaves A and B in $\hat{F}$ which are approached on the same side by a sequence of leaves $\{K_n\}$ as shown in Figure 3.2.

When one considers what such a picture might yield in a branched surface, one is eventually led to the following definitions.

Definition. A smooth arc γ in W is said to be ***negatively*** (***positively***) ***branching*** if the ends of γ branch into the negative (positive) side of two smooth local subsets

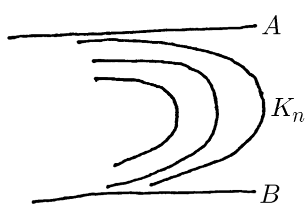

FIGURE 3.2.

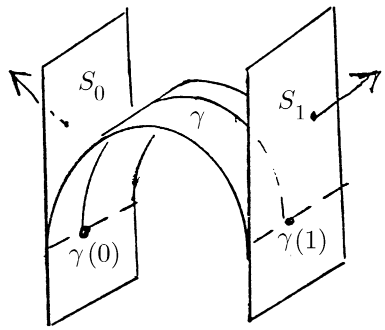

FIGURE 3.3.

of W. We assume the endpoints of γ, $\gamma(0)$ and $\gamma(1)$ in the branch set B are not double points of B. Then there is a smoothly imbedded strip $\gamma \times I$ in W with $\gamma = \gamma \times \{\frac{1}{2}\}$, $\gamma(0) \times I$ and $\gamma(1) \times I$ contained in B-{double points in B} . Let S_0 and S_1 be smooth disks in W, unbranched on the positive (negative) side, which contain $\gamma(0) \times I$ and $\gamma(1) \times I$ respectively.

Figure 3.3 shows a negatively branching arc γ.

Remark. It is easy to see that any element of Δ (generating W) with endpoints in the boundary of that generating surface (whose ends are not double points) gives rise to a branching arc.

Definition. A branched surface W imbedded in a closed manifold $M \neq S^2 \times S^1$ is said to be $\mathbb{R}$-covered if it has a transitive dual and if each negative (respectively, positively) branching arc γ is a fixed point homotopic in M to a smooth arc δ in W, called a ***bypass for*** γ, where δ has the following properties:

i. δ contains no negatively (respectively, positively) branching arc,
ii. for some $\varepsilon > 0$, $\delta((0, \varepsilon))$ lies in S_0 and $\delta((1-\varepsilon, 1))$ lies in S_1 (where S_0 and S_1 are as in the preceding definition),

The main result is as follows:

Theorem 4. *If a branched surface W is an $\mathbb{R}$-covered branched surface, then every foliation carried by W is $\mathbb{R}$-covered.*

The following corollary follows from the above theorem, using [Sh1].

Corollary 5. *If F is carried by an $\mathbb{R}$-covered branched surface, then every foliation sufficiently C^1-close to F is also $\mathbb{R}$-covered.*

Remark. Because of the above corollary, we call an $\mathbb{R}$-covered foliation carried by an $\mathbb{R}$-covered branched surface W ***stably $\mathbb{R}$-covered***.

Proof of Theorem 4. Let W be an $\mathbb{R}$-covered branched surface and let G be a foliation carried by W. By Theorem 2, we know G has no Reeb components; hence when we lift G to the universal cover, we obtain a foliation $\hat{G}$, where all leaves are topologically closed planes and the leaf space is a 1-manifold, possibly non-Hausdorff [Ha].

Now suppose the leaf space of $\hat{G}$ is not $\mathbb{R}$, i.e., there exist distinct leaves A and B of $\hat{G}$ which correspond to a pair of nonseparable points in the leaf space. We want to arrive at a contradiction. Without loss of generality, assume A and B are nonseparable on the negative side. Then there is a sequence of leaves, $\{K_n\}$, and points x_n, y_n in K_n for each n, such that $\{x_n\}$ converges to a point x in A along an orbit and $\{y_n\}$ converges to a point y in B along an orbit. Choose N large enough so that the arcs in the orbits, o_A and o_B, from x_N to x and y_N to y respectively do not meet the lift of any generating surface for W, hence $p(o_A)$ and $p(o_B)$ are contained in fibers of $N(W)$. Let $\hat{\gamma}$ be an arc in K_N joining x_N and y_N, and assume $\hat{\gamma}(0) = x_N$, $\hat{\gamma}(1) = y_N$.

Passing to bypasses if necessary (which necessarily exist if γ contains a branching arc), we see that there is a path δ in W, homotopic to $\gamma = p(\hat{\gamma})$, from $\gamma(0)$ to $\gamma(1)$ not containing a branching arc (see Figure 3.4).

In each case there is a transverse arc joining A and B, contradicting the fact that they correspond to nonseparable points in the leaf space.

Therefore G must be $\mathbb{R}$-covered. □

It seems likely that any $\mathbb{R}$-covered foliation is carried by some $\mathbb{R}$-covered branched surface. For many such foliations, the following modification of a branched surface W with a transitive dual and carrying the $\mathbb{R}$-covered foliation F will produce an $\mathbb{R}$-covered branched surface. The idea is that bypasses for branching arcs are quite easily constructed in W carrying an $\mathbb{R}$-covered foliation.

Suppose there is a branching arc γ in W which is not homotopic in M to an arc δ as in the definition. Without loss of generality, assume γ is negatively branching.

Let t_0 and t_1 be the fibers through $\gamma(0)$ and $\gamma(1)$ respectively. The positive end of these fibers lies in elements of Δ, say D_0 and D_1, and lie in the boundary of $N(W)$. Let L_0 and L_1 be the leaves of F containing D_0 and D_1 respectively.

In the natural ordering of leaves of $\hat{F}$ (induced by an oriented copy of the leaf space $\mathbb{R}$), we have either i) $\hat{L}_0 < \hat{L}_1$, ii) $\hat{L}_0 > \hat{L}_1$, or iii) $\hat{L}_0 = \hat{L}_1$. Consider case i.

Then $\hat{L}_0$ meets $\hat{t}_1$, hence L_0 meets t_1, so we can extend D_0 along L_0 so that the fiber t_1 is cut. This amounts to splitting $N(W)$ along L_0 so that the ends of t_0 and the new fiber t_1^* through $\gamma(1)$ lie in the same boundary component of $N(W)$ (see Figure 3.5). Now a portion of the positive (i.e., outwardly oriented) boundary of $N(W)$ corresponds to the extended D_0 hence to a smooth surface in W that is unbranched on the positive side. Further, there is an arc δ in this portion of $N(W)$ which is homotopic to $s_0^{-1} \circ \gamma \circ s_1$, where s_0 and s_1 are contained in t_0 and t_1^* respectively. Hence, as arcs in W, γ and δ are homotopic.

Cases ii and iii are similar, except that in case iii, D_0 is extended to merge with D_1. Hence in any case, a bypass for the branching arc γ is created.

This procedure words quite well for many explicit examples of $\mathbb{R}$-covered foliations to create an $\mathbb{R}$-covered branched surface carrying that foliation [see Figure 3.6

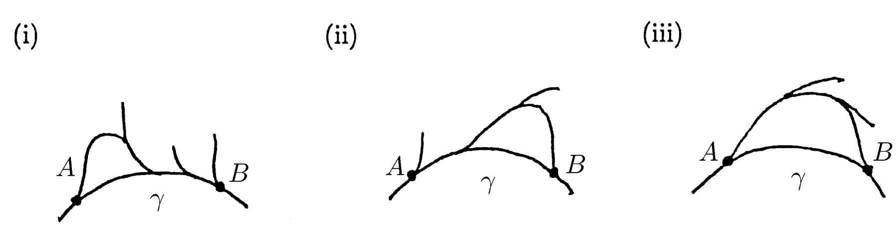

Figure 3.4.

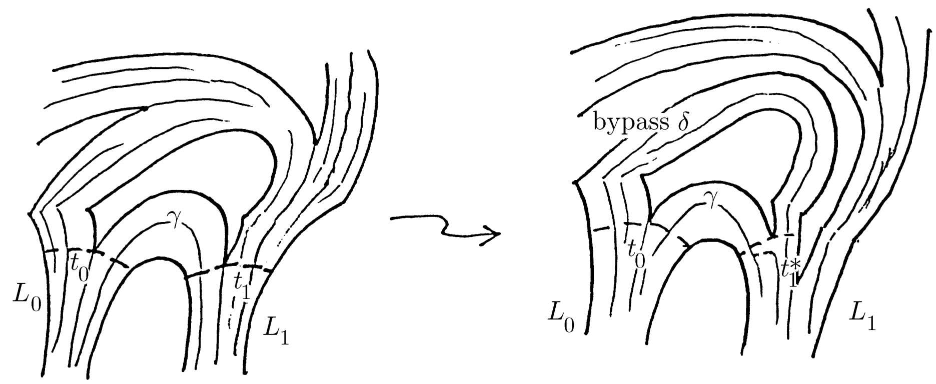

FIGURE 3.5.

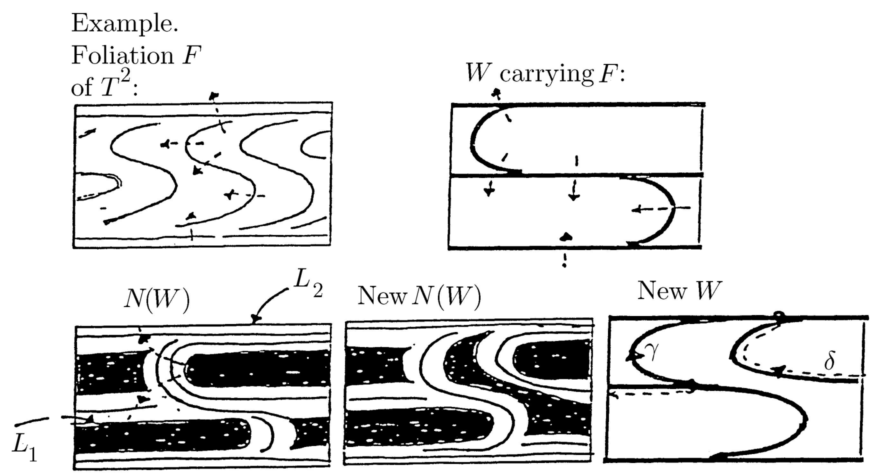

FIGURE 3.6.

for an illustration in 2 dimensions], hence by Corollary 5, these examples are stably ℝ-covered. However it is, unfortunately, not clear that in general in the process of creating a bypass for a branching arc, we do not create a new branching arc (or destroy an existing bypass for another branching arc). Hence we cannot yet claim that <u>any</u> ℝ-covered foliation is stably ℝ-covered, although we conjecture that it is so.

4. FURTHER RESULTS

In this section, we give several further results about ℝ-covered foliations which can be obtained, particularly when one assumes there is a compact leaf in the ℝ-covered foliation F. In this case, we will see that much more can be said about the ambient manifold (Proposition 9).

As before, $\hat{F}$ indicates the foliation obtained when F is lifted to the universal cover $\hat{M}$ of M.

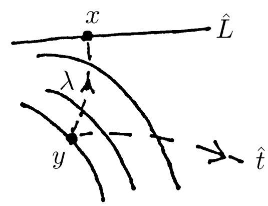

FIGURE 4.1.

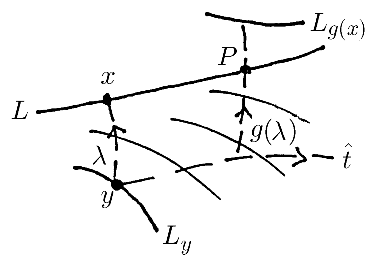

FIGURE 4.2.

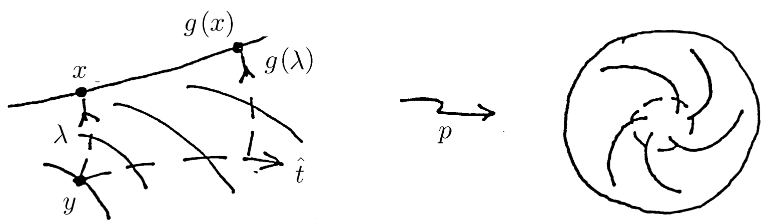

FIGURE 4.3.

Proposition 6. *Suppose F is $\mathbb{R}$-covered and that there is a transverse loop t with a lift $\hat{t}$ which does not meet every leaf of $\hat{F}$. Then t is freely homotopic to a loop on any leaf L such that $\hat{L}$ is in the boundary of the saturation of $\hat{t}$.*

Proof. Suppose F is $\mathbb{R}$-covered and t is a transverse loop with a lift $\hat{t}$ which does not meet every leaf of $\hat{F}$. Without loss of generality, assume t is positively oriented. Let $\hat{L}$ be a leaf in the boundary of the saturation of $\hat{t}$ with x a point on $\hat{L}$. (There are at most 2 such leaves.) Let λ be a transverse arc from a point y on $\hat{t}$ (y on a leaf L_y) to x. Assume, without loss of generality, that $Ly < \hat{L}$ in the induced ordering on the leaves of $\hat{F}$. See Figure 4.1.

We note that $\hat{L}$ being on the boundary of the saturation of $\hat{t}$ implies that every leaf meeting λ also meets $\hat{t}$.

Now let g be the deck transformation corresponding to the loop t. There are three cases to consider:

Case 1. $L_{g(x)} < \hat{L}$, where $L_{g(x)}$ denotes the leaf containing $g(x)$.

Since $L_y < L_{g(y)} < L_{g(x)} < \hat{L}$, $L_{g(x)}$ meets λ and hence meets $\hat{t}$. But then $g^{-1}(L_{g(x)}) = \hat{L}$ meets $\hat{t}$, a contradiction.

Case 2. $\hat{L} < L_{g(x)}$.

The point $g(y)$ is on $\hat{t}$ so $g(\lambda)$ is a transverse arc from $g(y) \in \hat{t}$ to $g(x)$, hence $g(\lambda)$ crosses $\hat{L}$ at a point p. Then $g^{-1}(p) \in$ interior λ, i.e., $g^{-1}(\hat{L})$ meets the interior of λ hence $\hat{t}$. So $\hat{L}$ meets $\hat{t}$, giving a contradiction as above. See Figure 4.2.

Hence we have

Case 3. $\hat{L} = L_{g(x)}$, i.e., g takes $\hat{L}$ to itself, and the free homotopy is show in Figure 4.3. □

Note. Alternate versions of the above proposition appear in [Sh2], [So].

Proposition 7. *If F is $\mathbb{R}$-covered and has a compact leaf L, then any transverse loop meeting L lifts to a copy of the leaf space of $\hat{F}$ (i.e., a curve that meets every leaf of $\hat{F}$ exactly once).*

Proof. Let t be a closed transverse curve through L. Lift t to $\hat{t}$. If there is a leaf in $\hat{F}$ not met by $\hat{t}$, then the above proposition shows that t is freely homotopic to a loop on a leaf L' such that $\hat{L}'$ is in the boundary of the saturation of $\hat{t}$. The immersed annulus of free homotopy, as we see in Figure 4.3, may be taken transverse to F. So there is a curve in L that enters but cannot leave this annulus. Hence, L is not closed, a contradiction. □

Proposition 8. *On a manifold not homeomorphic to $S^2 \times S^1$, F is $\mathbb{R}$-covered if and only if F has no Reeb components and every arc can be homotoped to be transverse to F or tangent to F.*

Proof. ($\Rightarrow$) Take an arc α and lift it to $\hat{\alpha}$ in $\hat{F}$. Then the endpoints of $\hat{\alpha}$ are either on the same leaf of $\hat{F}$ (hence homotopic to an arc in that leaf) or different leaves of $\hat{F}$ (hence homotopic to a transverse arc in $\hat{F}$, and therefore in F). That there are no Reeb components in an $\mathbb{R}$-covered foliation follows from Lemma B.

($\Leftarrow$) As in the proof of Theorem 4, the hypothesis of no Reeb components, together with $M \neq S^2 \times S^1$, implies all the leaves of $\hat{F}$ are topologically closed planes and the leaf space of $\hat{F}$ is a 1-manifold, possibly non-Hausdorff.

Now suppose the leaf space of $\hat{F}$ is non-Hausdorff, so there exists a pair of non-Hausdorff points in the leaf space, corresponding to $L_1 \neq L_2$ in $\hat{F}$. Without loss of generality, suppose L_1 and L_2 are nonseparated on their negative side. Then L_1 is on the back side of L_2 and L_2 is on the back side of L_1. Consider a curve from L_1 to L_2. By hypothesis, we can homotope this curve to a transversal since L_1 is not equal to L_2. This implies that either L_1 is on the front side of L_2 or L_2 is on the front side of L_1, both contradicting L_1 not separated from L_2. □

Proposition 9. *If F is $\mathbb{R}$-covered and has a compact leaf L, M fibers over S^1 with L as fiber.*

Proof. The idea is to first show that $\pi_1(L)$ is a normal subgroup of $\pi_1(M)$. Then, since $\pi_1(L)$ is not isomorphic to $\mathbb{Z}$, M is irreducible [Ro], and L cannot separate M, we have by Theorem 11.1 of Hempel [He] that M fibers over S^1 with L as fiber.

To show that $\pi_1(L)$ is normal in $\pi_1(M)$, we note that since L is compact, the lifts of L to $\hat{M}$ correspond to a discrete set in the leaf space $\mathbb{R}$. Since this set contains more than one element (see proof of Lemma B), it is order isomorphic to the set of integers $\mathbb{Z}$. We index the lifts by $\mathbb{Z}$, i.e., denote each lift L_i, i in $\mathbb{Z}$ the integer to which the leaf corresponds under this isomorphism. Let x_0 in L_0 be a basepoint and let g be in $\pi_1 L$. The covering translation induced by g takes L_0 to L_0, and since the corresponding action of g on the leaf space preserves the ordering of $\mathbb{Z}$, it fixes all elements of $\mathbb{Z}$. Now let t be any element of $\pi_1 M$ with basepoint in L. Then the action of t takes L_0 to L_i, some i in $\mathbb{Z}$, and $g(L_i) = L_i$, so $t^{-1} * g * t(L_0) = L_0$. Hence $t^{-1} * g * t$ is homotopic to a loop in L, giving $\pi_1 L$ normal in $\pi_1 M$. □

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