q-KRAWTCHOUK POLYNOMIALS AS SPHERICAL FUNCTIONS ON THE HECKE ALGEBRA OF TYPE B

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ABSTRACT. The Hecke algebra for the hyperoctahedral group contains the Hecke algebra for the symmetric group as a subalgebra. Inducing the index representation of the subalgebra gives a Hecke algebra module, which splits multiplicity free. The corresponding zonal spherical functions are calculated in terms of q-Krawtchouk polynomials using the quantised enveloping algebra for $\mathfrak{sl}(2,\mathbb{C})$. The result covers a number of previously established interpretations of (q-)Krawtchouk polynomials on the hyperoctahedral group, finite groups of Lie type, hypergroups and the quantum $SU(2)$ group.

1. INTRODUCTION

The hyperoctahedral group is the finite group of signed permutations, and it contains the permutation group as a subgroup. The functions on the hyperoctahedral group, which are left and right invariant with respect to the permutation group, are spherical functions. The zonal spherical functions are the spherical functions which are contained in an irreducible subrepresentation of the group algebra under the left regular representation. The zonal spherical functions are known in terms of finite discrete orthogonal polynomials, the so-called symmetric Krawtchouk polynomials. This result goes back to Vere-Jones in the beginning of the seventies, and is also contained in the work of Delsarte on coding theory; see [9] for information and references.

There are also group theoretic interpretations of q-Krawtchouk polynomials. Here q-Krawtchouk polynomials are q-analogues of the Krawtchouk polynomials in the sense that for q tending to one we recover the Krawtchouk polynomials. There is more than just one q-analogue for the Krawtchouk polynomial, but we consider the q-analogue which is commonly called q-Krawtchouk polynomial; see §7 for its definition in terms of basic hypergeometric series. In particular there is the interpretation by Stanton [22], [24] of q-Krawtchouk polynomials (for specific values of its parameter and of q) as spherical functions on finite groups of Lie type which have the hyperoctahedral group as Weyl group; see [4] for a list of these finite groups of Lie type.

Outside the group theoretic setting there are some relevant interpretations of the (q-)Krawtchouk polynomials. The non-symmetric Krawtchouk polynomials have

an interpretation as symmetrised characters on certain hypergroups (cf. Dunkl and Ramirez [10]) and the hyperoctahedral group case is recovered by a suitable specialisation. For the quantum $SU(2)$ group case the q-Krawtchouk polynomials enter the picture as matrix elements of a basis transition in the representation space of the irreducible unitary representations; see Koornwinder [17].

The hyperoctahedral group is the Weyl group for the root system of type B, and we can associate a Hecke algebra to it. There is a subalgebra corresponding to the Hecke algebra for the symmetric group, and we can consider elements which are left and right invariant with respect to the index representation of the Hecke algebra for the symmetric group. The index representation is the analogue of the trivial representation. We show that the zonal spherical functions on this Hecke algebra can be expressed in terms of q-Krawtchouk polynomials by deriving and solving a second-order difference equation for the zonal spherical functions. This interpretation of q-Krawtchouk polynomials gives a unified approach to the interpretations of $(q$-$)$Krawtchouk polynomials as zonal spherical functions on the hyperoctahedral group and the appropriate finite groups of Lie type, since these can be obtained by suitable specialisation. Moreover, it contains the Dunkl and Ramirez result on the interpretation of Krawtchouk polynomials as symmetrised characters on certain hypergroups, and it gives a conceptual explanation of the occurrence of the q-Krawtchouk polynomials in the quantum $SU(2)$ group setting.

We consider the Hecke algebra module obtained by inducing the index representation from the Hecke subalgebra for the symmetric group. This module splits multiplicity free as does the induced representation from the symmetric group to the hyperoctahedral group. This can also be done for other finite Coxeter groups with a maximal parabolic subgroup such that the induced representation splits multiplicity free. A list of these situations can be found in [3, Thm. 10.4.11]. In order to obtain interesting spherical functions we have to restrict to the cases A_n, B_n, D_n, for which we have a parameter n, with I_2^m as a possible addition to this list; cf. [24, Table 1]. For the Weyl group cases, i.e. A_n, B_n, D_n, only the non-simply-laced B_n is interesting, since the corresponding Hecke algebra has two parameters. The Hecke algebras for the simply-laced A_n and D_n only have one parameter, so we cannot expect an extension of the results tabulated in Stanton [24, Table 2]. The case studied here corresponds to $B_{n,n}$ (notation of [3, p. 295, Thm. 10.4.11]).

The contents of the paper are as follows. In §2 we investigate the hyperoctahedral group in more detail. In particular we study the coset representatives of minimal length with respect to the permutation subgroup. The Hecke algebra $\mathcal{H}_n$ for the hyperoctahedral group and the subalgebra $\mathcal{F}_n$ corresponding to the Hecke algebra for the symmetric group are introduced in §3. The representation of $\mathcal{H}_n$ obtained by inducing the index representation of $\mathcal{F}_n$ is also studied in §3. The induced module V_n is an analogue of $\mathbb{C}[\mathbb{Z}_2^n]$ and is a commutative algebra carrying a non-degenerate bilinear form. In §4 we let the quantised universal enveloping algebra $U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C}))$ for $\mathfrak{sl}(2,\mathbb{C})$ act on V_n, which by Jimbo's analogue of the Frobenius-Schur-Weyl duality gives the commutant of the action of $\mathcal{F}_n$ on V_n the $\mathcal{F}_n$-invariant elements, i.e. the elements transforming according to the index representation of $\mathcal{F}_n$, in V_n are identified with an irreducible module of $U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C}))$.

Next, in §5 we calculate the characters of the algebra V_n, and we give the corresponding orthogonality relations. Using the non-degenerate bilinear form we identify V_n with its dual. The contragredient representation is investigated in §6, and we decompose $V_n^* \cong V_n$ multiplicity free into irreducible $\mathcal{H}_n$-modules. In §6 we also

investigate the $U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C}))$-module of $\mathcal{F}_n$-invariant elements in greater detail by giving the action of the generators of $U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C}))$ in an explicit basis. A second-order difference equation as well as orthogonality relations for the zonal spherical elements are derived in §7. The second-order difference equation is used to identify the zonal spherical element with q-Krawtchouk polynomials. The relation with the interpretations of $(q\text{-})$Krawtchouk polynomials described in the first few paragraphs is worked out in some more detail in §7.

Notation. The notation for q-shifted factorials $(a;q)_k = \prod_{i=0}^{k-1}(1 - aq^i)$, $k \in \mathbb{Z}_+$, and basic hypergeometric series

$$_{r+1}\varphi_r \left(\begin{array}{c} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{array}; q, z \right) = \sum_{k=0}^{\infty} \frac{(a_1;q)_k \ldots (a_{r+1};q)_k}{(b_1;q)_k \ldots (b_r;q)_k} \frac{z^k}{(q;q)_k}$$

follows Gasper and Rahman [11]. If one of the upper parameters a_i equals q^{-d}, $d \in \mathbb{Z}_+$, then the series terminates. If one of the lower parameters b_j equals q^{-n}, $n \in \mathbb{Z}_+$, then the series is not well-defined unless one of the upper parameters equals q^{-d}, $d \in \{0, 1, \ldots, n\}$ with the convention that if $d = n$ we consider it as a terminating series of $n + 1$ terms.

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2. The hyperoctahedral group

The hyperoctahedral group is the semi-direct product $H_n = \mathbb{Z}_2^n \rtimes S_n$, where $\mathbb{Z}_2 = \{-1, 1\}$ is considered as the multiplicative group of two elements and S_n is the symmetric group, the group of permutations on n letters. So H_n is the wreath product $\mathbb{Z}_2 \wr S_n$. The order of H_n is $2^n\, n!$. In this section we study the coset representatives for H_n/S_n for which we give explicit formulas. This section is included for completeness and notational purposes. See also [1], [2], [4], [14].

The hyperoctahedral group is the Weyl group for the root system of type B_n (or C_n); cf. [2, Planche II]. The hyperoctahedral group operates on the standard n-dimensional Euclidean space $\mathbb{R}^n$ with coordinates ϵ_i, $i = 1, \ldots, n$. The group S_n acts by permutation of the coordinates and $x \in \mathbb{Z}_2^n$ acts by sign changes, i.e. x maps ϵ_i to $x_i\epsilon_i$. Observe that $x \in \mathbb{Z}_2^n \subset H_n$ is involutive. The roots, denoted by R, are $\pm\epsilon_i$, $1 \le i \le n$, and $\pm\epsilon_i \pm \epsilon_j$, $1 \le i < j \le n$. The simple roots are $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $i = 1, \ldots, n-1$, and $\alpha_n = \epsilon_n$ and the corresponding positive roots, denoted by R^+, are ϵ_i, $1 \le i \le n$, and $\epsilon_i - \epsilon_j$, $1 \le i < j \le n$, and $\epsilon_i + \epsilon_j$, $1 \le i < j \le n$. The other roots form the negative roots R^-.

Using this realisation of H_n we see that $\sigma x \sigma^{-1} = x^\sigma \in \mathbb{Z}_2^n$ for $x \in \mathbb{Z}_2^n$, $\sigma \in S_n$, where $x^\sigma = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$. Note that $(x^\sigma)^\tau = x^{(\tau\sigma)}$.

The hyperoctahedral group is a Coxeter group [2], [14] with the generating set of simple reflections S given by s_i, $i = 1, \ldots, n$, with s_i the reflection in the simple root α_i. Thus H_n is generated by s_i, $i = 1, \ldots, n$, subject to the quadratic relations $s_i^2 = 1$, and the braid relations

$$s_i s_j = s_j s_i, \quad |i - j| > 1, \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad i = 1, \ldots, n-2,$$

$$s_n s_{n-1} s_n s_{n-1} = s_{n-1} s_n s_{n-1} s_n.$$

In particular, the permutation group $S_n \subset H_n$ is the maximal parabolic subgroup generated by $I = \{s_1, \ldots, s_{n-1}\} \subset S$. Hence there exist distinguished coset representatives for H_n/S_n; cf. [2, p. 37, ex. 3], [14, §1.10].

Proposition 2.1. *For $x \in \mathbb{Z}_2^n$ let $w(x) = \#\{i \mid x_i = -1\}$ the (Hamming) weight of $x \in \mathbb{Z}_2^n$ and $x_{i_1} = \ldots = x_{i_{w(x)}} = -1$. Then $\ell(x) = (1 + 2n)w(x) - 2\sum_{j=1}^{w(x)} i_j$. Let $u_x \in H_n$ be the unique element of minimal length in the coset $x\,S_n$. Then $u_x = x\,\sigma_x$ with*

$$\ell(u_x) = (1+n)w(x) - \sum_{j=1}^{w(x)} i_j, \qquad \ell(\sigma_x) = n\,w(x) - \sum_{j=1}^{w(x)} i_j.$$

If $w \in xS_n$, then $w = u_x\sigma$ with $\ell(w) = \ell(u_x) + \ell(\sigma)$.

Proof. The last statement holds in greater generality (cf. the general theorem on the distinguished coset representatives [2, p. 37, ex. 3], [14, §1.10]) and it implies the uniqueness of the coset representatives of minimal length.

Let $\sigma \in S_n$ be arbitrary, then we can consider the action of σx on the negative roots and count how many negative roots are made positive. Since $x \in \mathbb{Z}_2^n$ is involutive $\ell(x\sigma^{-1}) = \ell((\sigma x)^{-1}) = \#(R^+ \cap (\sigma x)R^-)$. There are $\#\{i \mid x_i = -1\}$ positive roots in $\sigma x(-\epsilon_i)$, $1 \le i \le n$, there are $\#\{i < j \mid (x_i = x_j = 1 \wedge \sigma(i) > \sigma(j)) \vee (x_i = -1 \wedge x_j = 1) \vee (x_i = x_j = -1 \wedge \sigma(i) < \sigma(j))\}$ positive roots in $\sigma x(-\epsilon_i + \epsilon_j)$, $i < j$, and $\#\{i < j \mid (x_i = -1 \wedge x_j = 1 \wedge \sigma(i) < \sigma(j)) \vee (x_i = x_j = -1) \vee (x_i = 1 \wedge x_j = -1 \wedge \sigma(i) > \sigma(j))\}$ positive roots in $\sigma x(-\epsilon_i - \epsilon_j)$, $i < j$. We rewrite this in a σ-dependent part and a σ-independent part as

$$\ell(x\sigma^{-1}) = w(x) + \#\{i < j \mid x_i = -1\} + \#\{i < j \mid x_i = x_j = 1 \,\wedge\, \sigma(i) > \sigma(j)\}$$
$$+ \#\{i < j \mid x_i = x_j = -1 \,\wedge\, \sigma(i) < \sigma(j)\}$$
$$+ \#\{i < j \mid x_i = -1 \,\wedge\, x_j = 1 \,\wedge\, \sigma(i) < \sigma(j)\}$$
$$+ \#\{i < j \mid x_i = 1 \,\wedge\, x_j = -1 \,\wedge\, \sigma(i) > \sigma(j)\}.$$

In particular, if $\sigma = 1$, we obtain the result for $\ell(x)$. The σ-dependent part is zero, and thus minimal, for σ defined inductively by $\sigma(1) = n$ if $x_1 = -1$ and $\sigma(1) = 1$ if $x_1 = 1$, and $\sigma(i)$ is as large, respectively small, as possible if $x_i = -1$, respectively $x_i = 1$. Now define σ_x to be the inverse of the σ defined in this way, and let $u_x = x\,\sigma_x$. Then $u_x = x\,\sigma_x$ has length $\ell(u_x) = w(x) + \#\{i < j \mid x_i = -1\}$, which is minimal in the coset $x\,S_n$. The statement on $\ell(\sigma_x)$ follows directly, or from the general last statement of the proposition. $\square$

Remark 2.2. (i) If we write u_x as the $n \times n$ signed permutation matrix, then it is characterised by the conditions (i) all -1's occur columnwise to the right of all $+1$'s, (ii) the -1's decrease in columns as they increase by rows, (iii) the $+1$'s increase in columns as they increase by rows, (iv) the non-zero entry in the i-th row is x_i. The permutation matrix for σ_x is obtained from the one for u_x by replacing all -1's by $+1$'s.

(ii) The cardinality of H_n/S_n is 2^n and Proposition 2.1 shows how this can be parametrised by $\mathbb{Z}_2^n$.

(iii) Proposition 2.1 gives the length of $x \in \mathbb{Z}_2^n$. To give an explicit reduced expression we first observe that $x = x^{i_1} \ldots x^{i_{w(x)}}$ is a decomposition in commuting elements, where x^j is defined by $w(x^j) = 1$ and $i_1(x^j) = j$, or $x^j = (1, \ldots, 1, -1, 1, \ldots, 1)$ with -1 at the j-th spot. Proposition 2.1 implies that

$\ell(x) = \sum_{j=1}^{w(x)} \ell(x^{i_j})$, so that a reduced expression for x is obtained by inserting a reduced expression for each x^j. Finally, note that

$$(2.1) \qquad x^j = s_j s_{j+1} \dots s_{n-1} s_n s_{n-1} s_{n-2} \dots s_j$$

is a reduced expression.

Proposition 2.3. *Let u_x, $x \in \mathbb{Z}_2^n$, be the coset representatives of minimal length as in Proposition 2.1, then $\ell(s_i u_x) = \ell(u_x) + 1 \iff x_i \geq x_{i+1}$, $1 \leq i < n$, and $\ell(s_n u_x) = \ell(u_x) + 1 \iff x_n = 1$. Moreover, for $1 \leq i < n$ we have $s_i u_x = u_{x^{s_i}}$ if $x_i \neq x_{i+1}$ and $s_i u_x = u_x s_j$ if $x_i = x_{i+1}$ with $j = \min\left(\sigma_x^{-1}(i), \sigma_x^{-1}(i+1)\right)$ and $s_n u_x = u_{xx^n}$.*

Proof. Recall (see [14, §§1.6-7]) that $\ell(s_i w) = \ell(w) + 1$ if and only if $w^{-1} \alpha_i \in R^+$. So, taking $w = u_x$, we have to calculate $u_x^{-1} \alpha_i = \sigma_x^{-1} x \, \alpha_i = x_i \epsilon_{\sigma_x^{-1}(i)} - x_{i+1} \epsilon_{\sigma_x^{-1}(i+1)}$ for $1 \leq i < n$. For $x_i = -1$, $x_{i+1} = 1$ this root is negative, and for $x_i = 1$, $x_{i+1} = -1$ this root is positive. By construction of the permutation σ_x (cf. the proof of Proposition 2.1) we have $\sigma_x^{-1}(i) < \sigma_x^{-1}(i+1)$ if $x_i = x_{i+1} = 1$ and $\sigma_x^{-1}(i) > \sigma_x^{-1}(i+1)$ if $x_i = x_{i+1} = -1$, so that in these cases $u_x^{-1} \alpha_i$ is a positive root as well. Similarly, $u_x^{-1} \alpha_n = \sigma_x^{-1} x \, \epsilon_n = x_n \, \epsilon_{\sigma_x^{-1}(n)}$ and this is a positive root if and only if $x_n = 1$.

To prove the second statement recall that $\sigma x = x^\sigma \sigma$ for $x \in \mathbb{Z}_2^n$, $\sigma \in S_n$. Hence, $s_i u_x = s_i x \sigma_x = x^{s_i} s_i \sigma_x = u_{x^{s_i}} \sigma_{x^{s_i}}^{-1} s_i \sigma_x \in u_{x^{s_i}} S_n$ for $1 \leq i < n$. Now we have to consider some cases separately. If $x_i = x_{i+1}$, then $x^{s_i} = x$, and by the part of the proposition already proved, $\ell(s_i u_x) = \ell(u_x) + 1 = \ell(u_{x^{s_i}}) + 1$. Hence by Proposition 2.1, $\ell(\sigma_x^{-1} s_i \sigma_x) = 1$, so that $\sigma_x^{-1} s_i \sigma_x = s_j$ for some $1 \leq j \leq n$. Since $\sigma_x^{-1} s_i \sigma_x$ interchanges $\epsilon_{\sigma_x^{-1}(i)}$ and $\epsilon_{\sigma_x^{-1}(i+1)}$ and fixes the other coordinates ϵ_k and since $\sigma_x^{-1}(i)$ and $\sigma_x^{-1}(i+1)$ differ by 1 by construction of σ_x, it follows that $j = \sigma_x^{-1}(i)$ if $x_i = x_{i+1} = 1$ and $j = \sigma_x^{-1}(i+1)$ if $x_i = x_{i+1} = -1$ by construction of σ_x, or $j = \min\left(\sigma_x^{-1}(i), \sigma_x^{-1}(i+1)\right)$.

In case $x_i = 1$, $x_{i+1} = -1$ we have by Proposition 2.1 and by the part of the proposition already proved $\ell(u_{x^{s_i}}) = \ell(u_x) + 1 = \ell(s_i u_x)$. Since $u_{x^{s_i}}$ and $s_i u_x$ are in the same coset $x^{s_i} S_n$ and have the same length, it follows that $s_i u_x = u_{x^{s_i}}$ by the uniqueness of the minimal coset representative. Replace x by x^{s_i} to find the same conclusion in case $x_i = -1$, $x_{i+1} = 1$.

It remains to consider $s_n u_x = s_n x \sigma_x = x x^n \sigma_x = u_{xx^n} \sigma_{xx^n}^{-1} \sigma_x \in u_{xx^n} S_n$. By Proposition 2.1 and by the part of the proposition already proved we see that for both $x_n = 1$ and $x_n = -1$ we have $\ell(u_{xx^n}) = \ell(s_n u_x)$ and thus $s_n u_x = u_{xx^n}$ by uniqueness of the minimal coset representative. $\qquad \square$

3. The Hecke algebra for H_n and the induced module

The Hecke algebra can be associated to any Coxeter group; cf. [2, p. 54, ex. 22], [8, §68A], [14, §7.1]. In particular, we define the Hecke algebra $\mathcal{H}_n = \mathcal{H}(H_n)$ associated with the hyperoctahedral group as the algebra over the field $\mathbb{C}(p^{\frac{1}{2}}, q^{\frac{1}{2}})$ with elements T_w, $w \in H_n$, subject to the relations

$$(3.1) \qquad \begin{aligned} &T_i T_w = T_{s_i w}, &&\text{if} \quad \ell(s_i w) > \ell(w), \\ &T_i^2 = (q-1)T_i + q, &&\text{for } 1 \leq i < n, \qquad T_n^2 = (p-1)T_n + p, \end{aligned}$$

where we use the notation $T_i = T_{s_i}$, $1 \leq i \leq n$. The identity of $\mathcal{H}_n$ is $1 = T_e$ with $e \in H_n$ the identity of the hyperoctahedral group. Note that the elements T_i, $1 \leq i \leq n$, generate $\mathcal{H}_n$.

The Hecke algebra $\mathcal{H}_n$ can be defined over the ring $\mathbb{Z}[p, q]$, but then we would have to extend it later.

Similarly, we define the Hecke algebra $\tilde{\mathcal{F}}_n = \mathcal{H}(S_n)$ as the algebra over $\mathbb{C}(q^{\frac{1}{2}})$ with generators T_σ, $\sigma \in S_n$, subject to the relations

$$T_i T_\sigma = T_{s_i\sigma}, \qquad \text{if} \quad \ell(s_i\sigma) > \ell(\sigma), \quad 1 \leq i < n,$$
$$T_i^2 = (q-1)T_i + q, \qquad 1 \leq i < n,$$

and we view $\mathcal{F}_n = \mathbb{C}(p^{\frac{1}{2}}, q^{\frac{1}{2}}) \otimes_{\mathbb{C}(q^{\frac{1}{2}})} \tilde{\mathcal{F}}_n$ as a (maximal parabolic) subalgebra of $\mathcal{H}_n$.

The Hecke algebra $\mathcal{F}_n$ has two one-dimensional representations (cf. [6, §10]), namely the index representation $\iota \colon \mathcal{F}_n \to \mathbb{C}(p^{\frac{1}{2}}, q^{\frac{1}{2}})$, $\iota(T_i) = q$, $1 \leq i < n$ and the sign representation $T_w \mapsto (-1)^{\ell(w)}$. Observe that $\iota(T_\sigma) = q^{\ell(\sigma)}$.

The Hecke algebra $\mathcal{H}_n$ has four one-dimensional representations of which two representations restricted to $\mathcal{F}_n$ give the index representation ι of $\mathcal{F}_n$. They are defined by

$$\iota, \iota' \colon \mathcal{H}_n \to \mathbb{C}(p^{\frac{1}{2}}, q^{\frac{1}{2}}), \quad \iota|_{\mathcal{F}_n} = \iota, \ \iota'|_{\mathcal{F}_n} = \iota, \quad \iota(T_n) = p, \ \iota'(T_n) = -1.$$

Again, ι is called the index representation of $\mathcal{H}_n$ and denoted by the same symbol. The other two one-dimensional representations of $\mathcal{H}_n$ when restricted to $\mathcal{F}_n$ give the sign representation of $\mathcal{F}_n$; see [6, §10]. The complete representation theory of $\mathcal{H}_n$ and $\mathcal{F}_n$ can be found in Hoefsmit's thesis [13].

By V_n we denote the induced module obtained from inducing the one-dimensional representation ι of $\mathcal{F}_n$ to $\mathcal{H}_n$. So $V_n = \mathcal{H}_n \otimes_{\mathcal{F}_n} \mathbb{C}(p^{\frac{1}{2}}, q^{\frac{1}{2}}) = \mathrm{Ind}_{\mathcal{F}_n}^{\mathcal{H}_n} \iota$, and $\mathcal{H}_n$ acts from the left by multiplication. We denote the corresponding representation of $\mathcal{H}_n$ in V_n by ρ. Define the mapping $\pi \colon \mathcal{H}_n \to V_n$, $\pi(T) = \rho(T)(1 \otimes 1) = T \otimes 1$.

Theorem 3.1. (i) $\{u(x) = \pi(T_{u_x}) \mid x \in \mathbb{Z}_2^n\}$ *forms a basis for* V_n. *With respect to this basis the action of the generators of* $\mathcal{H}_n$ *is given by* $1 \leq i < n$,

$$\rho(T_i)\, u(x) = \begin{cases} u(x^{s_i}), & \text{if } x_i = 1, \, x_{i+1} = -1, \\ q\, u(x), & \text{if } x_i = x_{i+1}, \\ (q-1)\, u(x) + q\, u(x^{s_i}), & \text{if } x_i = -1, \, x_{i+1} = 1, \end{cases}$$

$$\rho(T_n)\, u(x) = \begin{cases} u(xx^n), & \text{if } x_n = 1, \\ (p-1)\, u(x) + p\, u(xx^n), & \text{if } x_n = -1. \end{cases}$$

(ii) *Define the product* $u(x)u(y) = q^{-\ell(\sigma_x)-\ell(\sigma_y)}\pi(T_x T_y)$, *then* V_n *is a commutative algebra over* $\mathbb{C}(p^{\frac{1}{2}}, q^{\frac{1}{2}})$. *As an algebra* V_n *is generated by* $u(x^j)$, $1 \leq j \leq n$, *with* x^j *defined in Remark 2.2(iii), subject to the relations*

$$u(x^j)u(x^k) = u(x^k)u(x^j) = u(x^k x^j), \qquad k \neq j,$$

(3.2)
$$\left(u(x^j)\right)^2 = (q-1)u(x^j)\left(u(x^{j+1}) + \ldots + u(x^n)\right)$$
$$+ (p-1)u(x^j) + pq^{n-j}u(1).$$

(iii) *The* $\mathbb{C}(p^{\frac{1}{2}}, q^{\frac{1}{2}})$*-linear form* τ *on* V_n *defined by* $\tau\left(u(x)\right) = \delta_{x,1} = \delta_{x_1,1}\ldots\delta_{x_n,1}$ *induces a non-degenerate symmetric associative bilinear form* $B(\xi, \eta) = \tau(\xi\eta)$ *on* V_n

and $B\big(u(x), u(y)\big) = \delta_{x,y}\, \iota(T_{u_x})$, so $\{u(x)\}_{\{x \in \mathbb{Z}_2^n\}}$ and $\{v(x) = \iota(T_{u_x})^{-1}u(x)\}_{\{x \in \mathbb{Z}_2^n\}}$ are dual bases for the bilinear form B.

Remark 3.2. (i) The product in $V_n \cong \mathcal{H}_n P$, with P the central idempotent of (4.6), is defined by $(TP)(SP) = (TS)P$ instead of $(TP)(SP) = (TPS)P$.

(ii) Instead of the basis $\pi(T_{u_x})$ we can also use the basis $\pi(T_x) = q^{\ell(\sigma_x)}\pi(T_{u_x})$, by Proposition 2.1, for the induced module V_n. We obtain the same action of the generators for $\mathcal{H}_n$ as in Theorem 3.1(i) by using Lemma 3.3 of Ariki and Koike [1].

Proof. (i) $\mathcal{H}_n$ and $\mathcal{F}_n$ are split semisimple algebras with the same numerical invariants as the group algebras $\mathbb{C}[H_n]$ and $\mathbb{C}[S_n]$ (see [13], [21]) and since the index representation corresponds to the trivial representation, it follows that V_n has dimension 2^n over $\mathbb{C}(p^{\frac{1}{2}}, q^{\frac{1}{2}})$. Proposition 2.1 implies that $\{u(x) \mid x \in \mathbb{Z}_2^n\}$ forms a basis for V_n. The action of the generators T_i, $1 \le i \le n$, follows directly from the definition of V_n, Proposition 2.3 and the relations in $\mathcal{H}_n$;

$$T_i T_w = (q-1)T_w + qT_{s_i w}, \text{ if } \ell(s_i w) < \ell(w), \ 1 \le i < n,$$
$$T_n T_w = (p-1)T_w + pT_{s_n w}, \text{ if } \ell(s_n w) < \ell(w).$$

(ii) The product is defined by $\pi(T_x)\pi(T_y) = \pi(T_x T_y)$ and it defines an algebra structure on V_n, since $\mathcal{H}_n$ is an algebra and π is an algebra morphism. To show that it is a commutative algebra we remark that $T_x T_y = T_y T_x$ for $x, y \in \mathbb{Z}_2^n$. Since $T_x = T_{x^{i_1}} \ldots T_{x^{i_{w(x)}}}$ (see Remark 2.2(iii)), this follows from $T_{x^j}T_{x^k} = T_{x^k}T_{x^j}$. This is trivial for $j = k$. For $j \ne k$ we have from Proposition 2.1 $\ell(x^j) + \ell(x^k) = \ell(x^j x^k)$ implying $T_{x^j}T_{x^k} = T_{x^j x^k} = T_{x^k x^j} = T_{x^k}T_{x^j}$, since $\mathbb{Z}_2^n$ is commutative.

V_n is generated by $u(x^j)$, $1 \le j \le n$, since $u(x^{i_1})\ldots u(x^{i_{w(x)}}) = q^{-\ell(\sigma_x)}\pi(T_x)$, which is proved by induction on the weight of x and Proposition 2.1. By Proposition 2.1 we have $T_x = T_{u_x}T_{\sigma_x^{-1}}$, and hence $\pi(T_x) = q^{\ell(\sigma_x)}u(x)$. So we conclude that

$$(3.3) \qquad u(x^{i_1})\ldots u(x^{i_{w(x)}}) = u(x).$$

It remains to calculate $\big(u(x^j)\big)^2 = q^{-2(n-j)}\pi(T_{x^j}T_{x^j})$. For $j = n$ we obtain the result from $T_n^2 = (p-1)T_n + p$. In general we use the reduced expression (2.1) and $T_j^2 = (q-1)T_j + q$ to find $T_{x^j}T_{x^j} = (q-1)T_{x^j}T_{x^{j+1}}T_j + qT_j T_{x^{j+1}}T_{x^{j+1}}T_j$. Hence, taking into account Proposition 2.1, we find $\big(u(x^j)\big)^2 = (q-1)u(x^j)u(x^{j+1}) + \rho(T_j)\big(u(x^{j+1})\big)^2$. We can now use downward induction on j to find

$$\big(u(x^j)\big)^2 = (q-1)u(x^j)u(x^{j+1}) + (q-1)\sum_{p=j+2}^{n}\rho(T_j)u(x^{j+1}x^p)$$
$$+ (p-1)\rho(T_j)u(x^{j+1}) + pq^{n-j-1}\rho(T_j)u(1)$$

by (3.3). Now use part (i) of the theorem to find (3.2).

(iii) The bilinear form is associative by construction and symmetric since V_n is a commutative algebra. Since the last equality implies that B is non-degenerate, it suffices to prove $\tau\big(u(x)u(y)\big) = \delta_{x,y}\iota(T_{u_x})$. This can be proved by an induction argument. First we use induction on $w(y)$, the case $w(y) = 0$ being trivial. Let $k = i_{w(y)}$ so that $y_k = -1$ and $y_{k+1} = \ldots = y_n = 1$. There are two cases to be considered, namely $x_k = 1$ or $x_k = -1$. If $x_k = 1$ we have $\tau\big(u(x)u(y)\big) = \tau\big(u(xx^k)u(yx^k))\big) = 0 = \delta_{x,y}\iota(T_{u_x})$ by (3.3) and $w(yx^k) = w(y) - 1$. If $x_k = -1$

we use downward induction on k. So let us first consider $k = n$. Then

$$\tau\big(u(x)u(y)\big) = \tau\big(u(xx^n)u(yx^n)u(x^n)^2\big)$$
$$= (p-1)\,\tau\big(u(x)u(yx^n)\big) + p\,\tau\big(u(xx^n)u(yx^n)\big)$$

by (3.3) and (3.2). By the induction hypothesis on $w(y)$, the first term equals zero and the second term equals $p\delta_{xx^n,yx^n}\iota(T_{u_{xx^n}}) = \delta_{x,y}\iota(T_{u_x})$ by Proposition 2.1. To proceed with the downward induction consider

$$\tau\big(u(x)u(y)\big) = \tau\big(u(xx^k)u(yx^k)u(x^k)^2\big) = (p-1)\,\tau\big(u(x)u(yx^k)\big)$$
$$+ \sum_{l=k+1}^{n} \tau\big(u(x)u(yx^kx^l)\big) + pq^{n-k}\,\tau\big(u(xx^k)u(yx^k)\big).$$

The first term is zero by the induction hypothesis for $w(y)$ and all terms in the sum are zero by the induction hypothesis on k and the case $x_k = 1$ already considered. By the induction hypothesis on $w(y)$ we obtain $pq^{n-k}\delta_{xx^k,yx^k}\iota(T_{u_{xx^k}}) = \delta_{x,y}\iota(T_{u_x})$ by Proposition 2.1. $\qquad\blacksquare$

The last statement of Theorem 3.1 is the analogue of the more general statement that T_w and $\iota(T_w)^{-1}T_{w^{-1}}$ are dual bases for the bilinear form associated to the linear form $T_w \mapsto \delta_{w,1}$. This holds for the Hecke algebra associated to any finite Coxeter group; cf. [4, §10.9].

4. $\mathcal{F}_n$-INVARIANT ELEMENTS IN THE INDUCED REPRESENTATION

Let us introduce the orthonormal basis $\hat{u}(x) = u(x)\big(\iota(T_{u_x})\big)^{-\frac{1}{2}}$, $x \in \mathbb{Z}_2^n$, of V_n, so that $B\big(\hat{u}(x), \hat{u}(y)\big) = \delta_{x,y}$. Then we have (cf. Theorem 3.1(i))

$$(4.1) \qquad \rho(T_i)\,\hat{u}(x) = \begin{cases} q^{\frac{1}{2}}\,\hat{u}(x^{s_i}), & \text{if } x_i = 1,\, x_{i+1} = -1, \\ q\,\hat{u}(x), & \text{if } x_i = x_{i+1}, \\ (q-1)\,\hat{u}(x) + q^{\frac{1}{2}}\,\hat{u}(x^{s_i}), & \text{if } x_i = -1,\, x_{i+1} = 1. \end{cases}$$

Let $y \in \mathbb{Z}_2^{i-1}$, $z \in \mathbb{Z}_2^{n-i-1}$ fixed and define the linearly independent elements $f_{-1,-1} = \hat{u}(y,-1,-1,z)$, $f_{-1,1} = \hat{u}(y,-1,1,z)$, $f_{1,-1} = \hat{u}(y,1,-1,z)$, $f_{1,1} = \hat{u}(y,1,1,z)$. Then T_i leaves the space spanned by these four elements invariant and with respect to this basis $\rho(T_i)$ is represented by the 4×4-matrix

$$(4.2) \qquad R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-1 & q^{\frac{1}{2}} & 0 \\ 0 & q^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix},$$

which is closely related to the R-matrix in the fundamental representation for the quantised universal enveloping algebra $U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C}))$. So let us recall the definition of $U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C}))$; see e.g. [5, Def. 9.1.1].

Definition 4.1. $U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C}))$ is the associative algebra with unit over $\mathbb{C}(q^{\frac{1}{2}})$ with generators K, K^{-1}, E and F subject to the relations

$$KK^{-1} = 1 = K^{-1}K, \quad KE = qEK, \quad KF = q^{-1}FK,$$
$$EF - FE = \frac{K - K^{-1}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}.$$

There exists a Hopf-algebra structure on $U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C}))$ with the comultiplication

$$\Delta\colon U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C})) \to U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C})) \otimes U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C})),$$

which is an algebra homomorphism, given by

$$\Delta(K) = K \otimes K, \qquad \Delta(E) = K \otimes E + E \otimes 1,$$
$$\Delta(F) = 1 \otimes F + F \otimes K^{-1}, \qquad \Delta(K^{-1}) = K^{-1} \otimes K^{-1}.$$

Remark 4.2. The correspondence with the quantised universal enveloping algebra as in [17, §3] is obtained by identifying K, K^{-1}, E and F with A^2, D^2, $q^{-1/4}AB$, $q^{1/4}CD$ of [17, §3], where we have replaced q by $q^{\frac{1}{2}}$.

The representation theory of $U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C}))$ is well-known (cf. e.g. [5, §10.1], [17, §3]) and is similar to the representation theory of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. We recall some of these results in the next theorem.

Theorem 4.3. (i) *There is precisely one irreducible $U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C}))$-module W_N of each dimension $N+1$ over $\mathbb{C}(q^{\frac{1}{2}})$ with highest weight vector v_+, i.e. $K \cdot v_+ = q^{N/2}v_+$, $E \cdot v_+ = 0$.*
(ii) *The Clebsch-Gordan decomposition holds; $W_N \otimes W_M \cong \bigoplus_{k=0}^{\min(N,M)} W_{M+N-2k}$ as $U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C}))$-modules.*

The tensor product representation of $U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C}))$ is defined using the comultiplication Δ. If ν^1, ν^2 are two representations of $U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C}))$ acting in W^1, W^2, then the representation $\nu^1 \otimes \nu^2$ acts in $W^1 \otimes W^2$ and is defined by

$$\left(\nu^1 \otimes \nu^2\right)(X)\, w^1 \otimes w^2 = \sum_{(X)} \nu^1(X_{(1)})\, w^1 \otimes \nu^2(X_{(2)})\, w^2,$$

where $\Delta(X) = \sum_{(X)} X_{(1)} \otimes X_{(2)}$.

We can define an antilinear $*$-operator on $U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C}))$ by $(q^{\frac{1}{2}})^* = q^{\frac{1}{2}}$ and

$$(4.3) \qquad K^* = K, \quad E^* = q^{-\frac{1}{2}}FK, \quad F^* = q^{\frac{1}{2}}K^{-1}E, \quad (K^{-1})^* = K^{-1}.$$

There are other $*$-structures possible on $U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C}))$ corresponding to different real forms of $\mathfrak{sl}(2,\mathbb{C})$. This $*$-structure corresponds to the compact real form $\mathfrak{su}(2)$ and for this $*$-structure the irreducible $U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C}))$-modules described in Theorem 4.3, and all the tensor product representations built from the irreducible representations, are $*$-representations of $U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C}))$. For more information on this subject we refer to [5, Ch. 9].

Let us now consider the fundamental 2-dimensional representation in $W = W_1$ of $U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C}))$. Let $\{e_{-1}, e_1\}$ be the standard orthonormal basis of W, then we have

$$(4.4) \qquad K \mapsto \begin{pmatrix} q^{\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} \end{pmatrix}, \quad E \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Consider the n-fold tensor product representation t in $W^{\otimes n}$ and extend t to a representation of $U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C}))$ in $\mathbb{C}(p^{\frac{1}{2}}, q^{\frac{1}{2}}) \otimes_{\mathbb{C}(q^{\frac{1}{2}})} W^{\otimes n}$ by $t(X)(f \otimes w) = f \otimes t(X)w$. Similarly we extend the modules of $U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C}))$ to modules over $\mathbb{C}(p^{\frac{1}{2}}, q^{\frac{1}{2}})$. By identifying $\mathbb{C}(p^{\frac{1}{2}}, q^{\frac{1}{2}}) \otimes_{\mathbb{C}(q^{\frac{1}{2}})} W^{\otimes n}$ with V_n by the unitary mapping

$$(4.5) \qquad f \otimes e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_n} \longmapsto f\, \hat{u}(i_1, i_2, \ldots, i_n), \qquad f \in \mathbb{C}(p^{\frac{1}{2}}, q^{\frac{1}{2}}),$$

we have the representation t of $U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C}))$ acting in V_n. The inner product on V_n arises from the antilinear map defined by $(q^{\frac{1}{2}})^* = q^{\frac{1}{2}}$, $(p^{\frac{1}{2}})^* = p^{\frac{1}{2}}$, $(\hat{u}(x))^* = \hat{u}(x)$ and $\langle v_1, v_2 \rangle = B(v_1, v_2^*)$ for $v_1, v_2 \in V_n$. This representation is a $*$-representation of $U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C}))$ for the $*$-operator defined by (4.3). It is straightforward to check that for $n = 2$ the representation t of $U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C}))$ in V_2 commutes with the matrix R of (4.2) and this is part of Jimbo's theorem on the analogue of the Frobenius-Schur-Weyl duality; cf. [15, Prop. 3], [5, §10.2.B].

Theorem 4.4 (Jimbo). *The algebras $t\big(U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C}))\big)$ and $\rho(\mathcal{F}_n)$ are each others commutant in* $\mathrm{End}_{\mathbb{C}(p^{\frac{1}{2}},q^{\frac{1}{2}})}(V_n)$.

Jimbo's theorem can be used to determine the $\mathcal{F}_n$-invariant elements in V_n. We call an element $v \in V_n$ a $\mathcal{F}_n$-invariant element if $\rho(T_\sigma) v = \iota(T_\sigma) v = q^{\ell(\sigma)} v$ for all $\sigma \in S_n$. So v is $\mathcal{F}_n$-invariant if it realises the one-dimensional index representation of $\mathcal{F}_n$. Let us define the corresponding idempotent, central in $\mathcal{F}_n$,

$$(4.6) \qquad\qquad P = \frac{1}{P_A(q)} \sum_{\sigma \in S_n} T_\sigma \in \mathcal{F}_n \subset \mathcal{H}_n.$$

Then in any representation of $\mathcal{H}_n$ the operator corresponding to P acts as a projection operator on the $\mathcal{F}_n$-invariant elements since $P(T_i - q) = (T_i - q)P = 0$, $1 \le i < n$. Here $P_A(q) = \sum_{\sigma \in S_n} q^{\ell(\sigma)}$ is the Poincaré polynomial for the Coxeter group S_n. So for arbitrary $v \in V_n$ the Hecke symmetrised vector $\rho(P) v = \frac{1}{P_A(q)} \sum_{\sigma \in S_n} \rho(T_\sigma) v$ is $\mathcal{F}_n$-symmetric.

Proposition 4.5. *The space of $\mathcal{F}_n$-invariant elements in V_n is the unique irreducible $n+1$-dimensional highest weight $U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C}))$-module with highest weight vector $u(-1,-1,\dots,-1)$.*

Proof. By Jimbo's Theorem 4.4 we see that the space of $\mathcal{F}_n$-invariant elements is an irreducible invariant $U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C}))$-module. Since $u(-1,-1,\dots,-1)$ is a $\mathcal{F}_n$-invariant vector by (4.1), it suffices to check

$$(4.7) \qquad \begin{aligned} t(K)\, u(-1,-1,\dots,-1) &= q^{n/2}\, u(-1,-1,\dots,-1), \\ t(E)\, u(-1,-1,\dots,-1) &= 0, \end{aligned}$$

by Theorem 4.3(i). Let $\Delta^{(2)} = \Delta$ and inductively

$$\Delta^{(n)} = (1 \otimes \dots \otimes 1 \otimes \Delta) \circ \Delta^{(n-1)} \colon U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C})) \to \Big(U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C}))\Big)^{\otimes n},$$

then it follows by induction on n from Definition 4.1 that

$$(4.8) \qquad \begin{aligned} \Delta^{(n)}(K) &= K \otimes K \otimes K \otimes \dots \otimes K, \\ \Delta^{(n)}(E) &= \sum_{i=1}^{n} K \otimes \dots \otimes K \otimes E \otimes 1 \otimes \dots \otimes 1, \end{aligned}$$

where the E occurs at the i-th component of the tensor product. From (4.4) and the identification (4.5) we directly obtain (4.7), since $K \cdot e_{-1} = q^{\frac{1}{2}} e_{-1}$ and $E \cdot e_{-1} = 0$.

From iteration of the Clebsch-Gordan decomposition in Theorem 4.3 it follows that the $n + 1$-dimensional irreducible module W_n occurs with multiplicity one in $W^{\otimes n}$, which implies uniqueness. $\qquad\qquad\square$

In §6 we give an explicit description of this $U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C}))$-module of $\mathcal{F}_n$-invariant elements.

5. Characters of V_n and orthogonality relations

Since V_n is a commutative algebra by Theorem 3.1(ii), all irreducible representations are one-dimensional, so we now investigate the characters of V_n. There are 2^n characters and they span $V_n^* = \operatorname{Hom}_{\mathbb{C}(p^{\frac{1}{2}}, q^{\frac{1}{2}})}(V_n, \mathbb{C}(p^{\frac{1}{2}}, q^{\frac{1}{2}}))$.

Theorem 5.1. V_n^* is spanned by the characters χ_y, $y \in \mathbb{Z}_2^n$, defined by

$$\chi_y\big(u(x^j)\big) = \begin{cases} pq^{m_j(y)}, & \text{if } y_j = 1, \\ -q^{n-j-m_j(y)}, & \text{if } y_j = -1, \end{cases}$$

where $m_j(y) = \#\{p > j \mid y_p = 1\}$.

Proof. The χ_y are obviously different for $y \in \mathbb{Z}_2^n$. It suffices to check that χ_y preserves the quadratic relations of (3.2). So we calculate

$$\sum_{k=j+1}^{n} \chi_y\big(u(x^k)\big) = \sum_{\substack{k=j+1 \\ y_k=1}}^{n} \chi_y\big(u(x^k)\big) + \sum_{\substack{k=j+1 \\ y_k=-1}}^{n} \chi_y\big(u(x^k)\big),$$

which is a sum of two geometric series, where the first contains $m_j(y)$ elements and the second contains $n - j - m_j(y)$ elements. Hence, this equals

$$\sum_{l=0}^{m_j(y)-1} pq^l + \sum_{l=0}^{n-j-m_j(y)-1} (-q^l) = p\frac{1-q^{m_j(y)}}{1-q} - \frac{1-q^{n-j-m_j(y)}}{1-q}$$

and so $p-1+(q-1)\sum_{k=j+1}^{n} \chi_y\big(u(x^k)\big) = pq^{m_j(y)} - q^{n-j-m_j(y)}$. Thus the quadratic relation is preserved if $\chi = \chi_y\big(u(x^j)\big)$ satisfies $\chi^2 = \big(pq^{m_j(y)} - q^{n-j-m_j(y)}\big)\chi + pq^{n-j}$ so $\chi = pq^{m_j(y)}$ or $\chi = -q^{n-j-m_j(y)}$. □

Remark 5.2. Observe that two of these characters can be easily calculated for arbitrary $u(x) \in V_n$;

$$\chi_{(1,\dots,1)}\big(u(x)\big) = \iota(T_{u_x}) = p^{w(x)} q^{nw(x)-\sum_{j=1}^{w(x)} i_j},$$

$$\chi_{(-1,\dots,-1)}\big(u(x)\big) = \iota'(T_{u_x}) = (-1)^{w(x)} q^{nw(x)-\sum_{j=1}^{w(x)} i_j}.$$

The characters $\chi_{(1,\dots,1)}$, respectively $\chi_{(-1,\dots,-1)}$, are the images of the one-dimensional representations ι, respectively ι', of $\mathcal{H}_n$ under the projection $\pi\colon \mathcal{H}_n \to V_n$.

The algebra V_n is a split semisimple algebra over $\mathbb{C}(p^{\frac{1}{2}}, q^{\frac{1}{2}})$, since it is a commutative 2^n-dimensional algebra with 2^n different one-dimensional representations. So we can apply Kilmoyer's results [7, §9B] to obtain part of the following orthogonality relations.

Proposition 5.3. The characters χ_y, $y \in \mathbb{Z}_2^n$, of V_n satisfy the following orthogonality relations; for $y, z \in \mathbb{Z}_2^n$

$$\sum_{x \in \mathbb{Z}_2^n} \chi_y\big(\hat{u}(x)\big) \chi_z\big(\hat{u}(x^j)\big) = \delta_{y,z} h_y, \quad h_y = \prod_{j=1}^{n} \big(1 + \big(pq^{2m_j(y)+j-n}\big)^{y_j}\big).$$

Corollary 5.4. The dual orthogonality relations hold; for $x, z \in \mathbb{Z}_2^n$

$$\sum_{y \in \mathbb{Z}_2^n} \frac{1}{h_y} \chi_y\big(\hat{u}(x)\big) \chi_y\big(\hat{u}(z)\big) = \delta_{x,z}.$$

Proof of Proposition 5.3. Apart from the squared norm the proposition follows immediately from [7, Prop. (9.17), (9.19)]. To calculate the squared norm we observe that

$$h_y = \sum_{x \in \mathbb{Z}_2^n} \frac{\left(\chi_y\big(u(x)\big)\right)^2}{\iota(T_{u_x})} = \left(1 + \left(\chi_y\big(u(x^1)\big)\right)^2 p^{-1} q^{1-n}\right) \sum_{z \in \mathbb{Z}_2^{n-1}} \frac{\left(\chi_y\big(u(1,z)\big)\right)^2}{\iota(T_{u(1,z)})}$$

by writing $x \in \mathbb{Z}_2^n$ as $(1, z)$ and $(-1, z)$ for $z \in \mathbb{Z}_2^{n-1}$ and using (3.3), χ_y being a character and Proposition 2.1. Note that, by (3.3) and Theorem 5.1, the sum is independent of y_1. Theorem 5.1 implies that $\left(\chi_y\big(u(x^1)\big)\right)^2 p^{-1} q^{1-n} = \left(pq^{2m_1(y)+1-n}\right)^{y_1}$. Now iterate to find the value for h_y. $\qquad\square$

Remark 5.5. By Remark 5.2 we have $h_{(1,\dots,1)} = \sum_{x \in \mathbb{Z}_2^n} \iota(T_{u_x})$. In this case we also have, by Proposition 2.1,

$$h_{(1,\dots,1)} P_A(q) = \sum_{x \in \mathbb{Z}_2^n} \iota(T_{u_x}) \sum_{\sigma \in S_n} \iota(T_\sigma) = \sum_{w \in H_n} \iota(T_w) = P_B(p, q),$$

where $P_B(p, q)$ is the Poincaré polynomial for the hyperoctahedral group, which is defined by the last equality. Hence, $h_{(1,\dots,1)} = P_B(p, q)/P_A(q) = (-p; q)_n$ which can be checked directly from the explicit expressions for the Poincaré polynomials [18];

$$P_A(q) = \frac{(q; q)_n}{(1 - q)^n}, \qquad P_B(p, q) = \frac{(-p; q)_n (q; q)_n}{(1 - q)^n}.$$

Similarly, since $\iota(T_\sigma) = \iota'(T_\sigma)$, $\sigma \in S_n$, and $\iota(T_n) = p$, $\iota'(T_n) = -1$, we get

$$h_{(-1,\dots,-1)} P_A(q) = \sum_{x \in \mathbb{Z}_2^n} \frac{\left(\iota'(T_{u_x})\right)^2}{\iota(T_{u_x})} \sum_{\sigma \in S_n} \frac{\left(\iota'(T_\sigma)\right)^2}{\iota(T_\sigma)} = \sum_{w \in H_n} \frac{\left(\iota'(T_w)\right)^2}{\iota(T_w)} = P_B(p^{-1}, q).$$

Hence, $h_{(-1,\dots,-1)} = P_B(p^{-1}, q)/P_A(q) = (-p^{-1}; q)_n$.

Remark 5.6. Since we have a non-degenerate bilinear form B on V_n (see Theorem 3.1(iii)) we can identify $\chi_y \in V_n^*$ with $\xi_y \in V_n$ by $\chi_y(v) = B(v, \xi_y)$ for all $v \in V_n$. Then Proposition 5.3 states that $\{\xi_y \mid y \in \mathbb{Z}_2^n\}$ is an orthogonal basis of V_n, or $B(\xi_y, \xi_z) = \delta_{y,z} h_y$. From [7, Prop. (9.17)] it follows that $h_y^{-1} \xi_y$ is the (central) primitive idempotent corresponding to the one-dimensional representation χ_y of V_n. Hence, we have $\xi_y \xi_z = \delta_{y,z} h_y \xi_y$ and $\sum_{y \in \mathbb{Z}_2^n} h_y^{-1} \xi_y = u(1, \dots, 1)$ as identities in V_n, since $u(1, \dots, 1)$ is the identity of V_n.

More generally, let the $\mathbb{C}(p^{\frac{1}{2}}, q^{\frac{1}{2}})$-linear isomorphism $b \colon V_n^* \to V_n$, $b(f) = \xi_f$, be defined by $f(v) = B(v, \xi_f)$ for all $v \in V_n$. Then b is the analogue of the Fourier transform. To see this define $\hat{\ } \colon V_n^* \to V_n^{**}$ by

$$\hat{f}(\chi) = \sum_{x \in \mathbb{Z}_2^n} f\big(\hat{u}(x)\big) \chi\big(\hat{u}(x)\big), \qquad \chi \in V_n^*,$$

and let $a \colon V_n^{**} \to V_n$ be the standard isomorphism, i.e. for $\eta \in V_n^{**}$ we have $\eta(\chi) = \chi\big(a(\eta)\big)$ for all $\chi \in V_n^*$. Then $a(\hat{f}) = \xi_f = b(f)$.

Using the Fourier transform b we put a commutative $\mathbb{C}(p^{\frac{1}{2}}, q^{\frac{1}{2}})$-algebra structure on V_n^* by $\chi \star \eta = b^{-1}\big(b(\chi) b(\eta)\big)$, which is an analogue of the convolution product. Since $b(\chi_y)$, $y \in \mathbb{Z}_2^n$ are primitive idempotents, we get the following analogue of the convolution product of characters; $\chi_y \star \chi_z = \delta_{y,z} h_y \chi_y$, $y, z \in \mathbb{Z}_2^n$. Also $\tau =$

$b^{-1}\big(u(1,\dots,1)\big) = \sum_{y\in\mathbb{Z}_2^n} h_y^{-1}\chi_y$ is the identity of the algebra V_n^* given in Theorem 3.1(iii).

6. Contragredient representations

We can define the contragredient representation ρ^* of $\mathcal{H}_n$ in V_n^* as follows; for $\chi \in V_n^*$ put

$$\big(\rho^*(\sum_{w\in H_n} c_w\,T_w)\chi_y\big)(v) = \chi_y\big(\rho((\sum_{w\in H_n} c_w\,T_w)^*)v\big), \qquad v \in V_n,$$

for some antimultiplicative map $*\colon \mathcal{H}_n \to \mathcal{H}_n$ which also preserves the quadratic relations of (3.1). There are two candidates for the $*$-operator; $*_1$ is the antilinear map defined by $T_w^{*_1} = T_{w^{-1}}$, $(f(p^{\frac{1}{2}},q^{\frac{1}{2}}))^{*_1} = \overline{f(p^{\frac{1}{2}},q^{\frac{1}{2}})}$, and $*_2$ is the antilinear map defined by $T_w^{*_2} = T_w^{-1}$ and $(f(p^{\frac{1}{2}},q^{\frac{1}{2}}))^{*_2} = \overline{f(p^{-\frac{1}{2}},q^{-\frac{1}{2}})}$ for $f \in \mathbb{C}(p^{\frac{1}{2}},q^{\frac{1}{2}})$. Observe that these maps are involutions and that $*_1$ and $*_2$ commute. The composition $*_1 \circ *_2 \colon \mathcal{H}_n \to \mathcal{H}_n$ is the multiplicative involution which gives rise to the Kazhdan-Lusztig basis of $\mathcal{H}_n$; cf. [14, Ch. 7]. Let ρ^* be the contragredient representation defined using $*_1 = *$; the contragredient representation ρ_0^* using $*_2$ is related to it; cf. Remark 6.2.

Since the basis $\hat{u}(x)$, $x \in \mathbb{Z}_2^n$ is orthonormal with respect to the bilinear form B of Theorem 3.1(iii) and $\rho(T_i)$, $1 \le i < n$ is given by the orthogonal matrix R in (4.2), we see that $B(\rho(T_w)v,w) = B(v,\rho(T_w^*)w)$ for $w \in S_n$. Since $\rho(T_n)$ is also given by an orthogonal matrix with respect to the orthonormal basis $\{\hat{u}(x)\}$, we see that this holds for arbitrary $w \in H_n$. This implies the following commutation relations for the action of $\mathcal{H}_n$ with the Fourier transform b introduced in Remark 5.6;

$$(6.1) \qquad b\big(\rho^*(T)\chi\big) = \rho(T)\,b(\chi), \qquad b\big(\rho_0^*(T)\chi\big) = \rho((T^{*_1})^{*_2})\,b(\chi),$$

for $\chi \in V_n^*$, $T \in \mathcal{H}_n$.

Theorem 6.1. *The action of the generators of $\mathcal{H}_n$ on the characters $\chi_y \in V_n^*$ in ρ^* is given by*

$$\rho^*(T_n)\,\chi_y = \chi_y \begin{cases} p, & \text{if } y_n = 1, \\ -1, & \text{if } y_n = -1, \end{cases}$$

and for $1 \le i < n$ by

$$\rho^*(T_i)\,\chi_y = q\chi_y, \qquad \text{if } y_i = y_{i+1},$$

$$\rho^*(T_i)\,\chi_y = \frac{pq^{m_i(y)}(q-1)\chi_y + (pq^{m_i(y)} + q^{n-i-m_i(y)})\chi_{y^{s_i}}}{pq^{m_i(y)} + q^{n-i-1-m_i(y)}}, \quad \text{if } y_i = 1 = -y_{i+1},$$

$$\rho^*(T_i)\,\chi_y = \frac{q^{n-i-m_i(y)}(q-1)\chi_y + (pq^{m_i(y)} + q^{n-i-m_i(y)})\chi_{y^{s_i}}}{pq^{m_i(y)-1} + q^{n-i-m_i(y)}}, \quad \text{if } -y_i = 1 = y_{i+1},$$

where $y^{s_i} = (y_1,\dots,y_{i-1},y_{i+1},y_i,y_{i+1},\dots,y_n)$ and $m_i(y) = \#\{p > i \mid y_p = 1\}$.

Remark 6.2. For $1 \le i < n$ the result of Theorem 6.1 can be written uniformly as

$$\rho^*(T_i)\chi_y = \frac{y_i\chi_y\big((\pi(T_{u(x^i)}))\big)(q-1)\chi_y + (pq^{m_i(y)} + q^{n-i-m_i(y)})\chi_{y^{s_i}}}{pq^{m_{i+1}(y)} + q^{n-i-1-m_{i+1}(y)}}.$$

If we write this as $\rho^*(T_i)\chi_y = A_i(y)\chi_y + B_i(y)\chi_{y^{s_i}}$, then $\rho^*(T_i - q)\chi_y = B_i(y)(\chi_{y^{s_i}} - \chi_y)$. The action of $\rho_0^*(T_i)$ on χ_y also follows, since by (6.1)

$$\rho_0^*(T_i)\chi_y = b^{-1}\big(\rho((T_i^{*_1})^{*_2})b(\chi_y)\big) = b^{-1}\big(\rho(T_i^{-1})b(\chi_y)\big) = \rho^*(T_i^{-1})\chi_y.$$

Now use $T_i^{-1} = q^{-1}T_i + q^{-1} - 1$ $(1 \le i < n)$ and a similar expression for T_n^{-1} with q replaced by p.

Proof. Let us first consider the action of $\rho^*(T_n)$, then using Theorem 3.1, (3.3) and χ_y being a character we obtain

$$\big(\rho^*(T_n)\chi_y\big)\big(u(x)\big) = \chi_y\big(u(x)\big) \begin{cases} \chi_y\big(u(x^n)\big), & \text{if } x_n = 1, \\ (p-1) + p\big(\chi_y\big(u(x^n)\big)\big)^{-1}, & \text{if } x_n = -1. \end{cases}$$

Now by Theorem 5.1 $\chi_y\big(u(x^n)\big)$ equals -1 if $y_n = -1$ and p if $y_n = 1$, which implies the result for the action of T_n.

Theorem 3.1 gives for $0 \le i < n$

(6.2)
$$\big(\rho^*(T_i)\chi_y\big)\big(u(x)\big) = \begin{cases} \chi_y\big(u(x^{s_i})\big), & \text{if } x_i = 1, \, x_{i+1} = -1, \\ q\chi_y\big(u(x)\big), & \text{if } x_i = x_{i+1}, \\ (q-1)\chi_y\big(u(x)\big) + q\chi_y\big(u(x^{s_i})\big), & \text{if } x_i = -1, \, x_{i+1} = 1. \end{cases}$$

In the case $y_i = y_{i+1}$ we have $\chi_y\big(u(x^{s_i})\big) = q\chi_y\big(u(x)\big)$ if $x_i = 1$, $x_{i+1} = -1$. This follows from (3.3) and χ_y being a character and $\chi_y\big(u(x^i)\big) = q\chi_y\big(u(x^{i+1})\big)$, which follows from Theorem 5.1. Using this in (6.2) shows $\rho^*(T_i)\chi_y = q\chi_y$ for $y_i = y_{i+1}$.

We now consider the case $y_i = 1$, $y_{i+1} = -1$, so we have to show that (6.2) equals

(6.3)
$$\frac{pq^{m_i(y)}(q-1)\chi_y\big(u(x)\big) + (pq^{m_i(y)} + q^{n-i-m_i(y)})\chi_{y^{s_i}}\big(u(x)\big)}{pq^{m_i(y)} + q^{n-i-1-m_i(y)}}$$

for all choices of x. To treat the case $x_i = x_{i+1}$ we note that $\chi_y\big(u(x)\big) = \chi_{y^{s_i}}\big(u(x)\big)$, which follows from Theorem 5.1 and $m_p(y^{s_i}) = m_p(y)$ for $p \ne i$ and $m_i(y^{s_i}) = m_i(y) + 1$. This shows that for $x_i = x_{i+1}$ (6.3) reduces to $q\chi_y\big(u(x)\big)$.

To treat the case $x_i \ne x_{i+1}$ we introduce $z \in \mathbb{Z}_2^n$ defined by $z_p = x_p$ for $p \ne i, i+1$ and $z_i = z_{i+1} = 1$. From the previous paragraph we get $\chi_y\big(u(z)\big) = \chi_{y^{s_i}}\big(u(z)\big)$. Using Theorem 5.1 and χ_y being a character shows that (6.2) equals

(6.4)
$$\chi_y\big(u(z)\big) \begin{cases} pq^{m_i(y)}, & \text{if } x_i = 1, \, x_{i+1} = -1, \\ (q-1)pq^{m_i(y)} - q^{n-i-m_i(y)}, & \text{if } x_i = -1, \, x_{i+1} = 1, \end{cases}$$

since $m_i(y) = m_{i+1}(y)$ in this case. Because, by Theorem 5.1,

$$\chi_y\big(u(x)\big) = \chi_y\big(u(z)\big) \begin{cases} -q^{n-i-1-m_i(y)}, & \text{if } x_i = 1, \, x_{i+1} = -1, \\ pq^{m_i(y)} & \text{if } x_i = -1, \, x_{i+1} = 1, \end{cases}$$

and

$$\chi_{y^{s_i}}\big(u(x)\big) = \chi_y\big(u(z)\big) \begin{cases} pq^{m_i(y)}, & \text{if } x_i = 1, \, x_{i+1} = -1, \\ -q^{n-i-m_i(y)-1} & \text{if } x_i = -1, \, x_{i+1} = 1, \end{cases}$$

it is straightforward to check that (6.3) equals (6.4). This gives the action of T_i on χ_y with $y_i = 1$, $y_{i+1} = -1$.

To prove the last identity of the theorem, we apply $\rho^*(T_i)$ once more to $\rho^*(T_i)\chi_y$ with $y_i = 1$, $y_{i+1} = -1$. Using the quadratic relation for T_i we get an expression for $\rho^*(T_i)\chi_{y^{s_i}}$ in terms of χ_y and $\rho^*(T_i)\chi_y$. Using the result already proved for the last term gives an expression for $\rho^*(T_i)\chi_{y^{s_i}}$ in terms of χ_y and $\chi_{y^{s_i}}$. Now replace y by y^{s_i} to find the result after some calculations. □

Corollary 6.3. *Define for $f = 0, \ldots, n$ the space U_f of characters χ_y with $w(y) = f$;*

$$U_f = \bigoplus_{w(y)=f} \mathbb{C}(p^{\frac{1}{2}}, q^{\frac{1}{2}}) \chi_y \subset V_n^*,$$

so that $\dim_{\mathbb{C}(p^{\frac{1}{2}}, q^{\frac{1}{2}})} U_f = \binom{n}{f}$. Then $V_n^* = \bigoplus_{f=0}^n U_f$ is the decomposition of the representation ρ^* in V_n^* into irreducible $\mathcal{H}_n$-modules.

Remark 6.4. The irreducible representations of $\mathcal{H}_n$ have been classified by Hoefsmit [13, Def. (2.2.6), Thms. (2.2.7), (2.2.14)] and are parametrised by double partitions of n. The irreducible $\mathcal{H}_n$-module U_f corresponds to $\big((n-f), (f)\big)$ and χ_y corresponds to the standard tableau of shape $\big((n-f), (f)\big)$ given by

$n+1-j_{n-f}$	$n+1-j_{n-f-1}$	$\ldots$	$n+1-j_1$	$(n-f)$,

$n+1-i_f$	$n+1-j_{f-1}$	$\ldots$	$n+1-i_1$	(f),

where $y_{i_1} = \ldots = y_{i_f} = -1$, $i_1 < \ldots < i_f$, and $y_{j_1} = \ldots = y_{j_{n-f}} = 1$, $j_1 < \ldots < j_{n-f}$. Applying [13, Prop. (3.3.3)] shows that for $x \in \mathbb{Z}_2^n$ the operator $\rho^*(T_x)$ is diagonal with respect to the basis $\{\chi_y\}_{y \in \mathbb{Z}_2^n}$ of V_n^*.

As before we call $\chi \in V_n^*$ a $\mathcal{F}_n$-invariant element of V_n^* if $\rho^*(T_\sigma)\chi = q^{\ell(\sigma)}\chi$ for all $\sigma \in S_n$. So χ is $\mathcal{F}_n$-invariant if it realises the index representation ι of $\mathcal{F}_n$ in V_n^*.

Let $V_n^d = \bigoplus_{\{x \in \mathbb{Z}_2^n | w(x)=d\}} \mathbb{C}(p^{\frac{1}{2}}, q^{\frac{1}{2}}) u(x) \subset V_n$, then $V_n = \bigoplus_{d=0}^n V_n^d$ and by (4.1) it follows that each V_n^d is invariant under the action of $\mathcal{F}_n$ via ρ. In general V_n^d is not an irreducible $\mathcal{F}_n$-module, and it can be obtained from inducing the index representation of $\mathcal{F}_d \otimes \mathcal{F}_{n-d}$. For the decomposition as $\mathcal{F}_n$-module we can proceed as in Dunkl [9, §2].

We can now describe the $U_{q^{1/2}}(\mathfrak{sl}(2, \mathbb{C}))$-module of $\mathcal{F}_n$-invariant elements in V_n explicitly.

Theorem 6.5. *Define the $\mathcal{F}_n$-invariant elements w_d, $d = 0, 1, \ldots, n$, in V_n by*

$$w_d = \rho(P)\, v(\underbrace{1, \ldots, 1}_{n-d}, \underbrace{1, \ldots, -1}_{d},) \in (V_n)^{\mathcal{F}_n},$$

with $P = \frac{1}{P_A(q)} \sum_{\sigma \in S_n} T_\sigma$ and $v(x) = \iota(T_{u_x})^{-1} u(x)$ defined in (4.6) and Theorem 3.1(iii), then w_d is non-zero and forms an orthogonal basis for the space of $\mathcal{F}_n$-invariant elements in V_n;

$$B(w_d, w_e) = \delta_{d,e}\, p^{-d} q^{-d(d-1)/2} \begin{bmatrix} n \\ d \end{bmatrix}_q^{-1}.$$

Moreover, for $x \in \mathbb{Z}_2^n$ with $w(x) = d$ we have $\rho(P)\, v(x) = w_d$. The action of $U_{q^{1/2}}(\mathfrak{sl}(2, \mathbb{C}))$ in this basis is given by $t(K)\, w_d = q^{d-n/2}\, w_d$ and

$$t(E)\, w_d = p^{\frac{1}{2}} q^{(1-n)/2} q^d \frac{1 - q^{n-d}}{1 - q}\, w_{d+1}, \quad t(F)\, w_d = p^{-\frac{1}{2}} q^{1-d} \frac{1 - q^d}{1 - q}\, w_{d-1}.$$

Here $\begin{bmatrix} n \\ d \end{bmatrix}_q = \dfrac{(q^n; q^{-1})_d}{(q; q)_d}$ denotes the q-binomial coefficient

Proof. From (4.8), (4.4) and the identification (4.5) we see that $t(K)\colon V_n^d \to V_n^d$, $t(E)\colon V_n^d \to V_n^{d+1}$ and $t(F)\colon V_n^d \to V_n^{d-1}$ using a similar expression for $\Delta^{(n)}(F)$. From Theorem 4.3 and Proposition 4.5 it follows that the space of $\mathcal{F}_n$-invariant elements in V_n^d is one-dimensional.

Theorem 6.1 implies that $\chi_1 = \chi_{(1,\dots,1)}$ is $\mathcal{F}_n$-invariant in V_n^* under ρ^*. Using Remark 5.2 we get $1 = \chi_1(v(x)) = (\rho^*(P)\chi_1)(v(x)) = \chi_1(\rho(P)\,v(x))$. So Hecke symmetrising any basis element yields a non-zero $\mathcal{F}_n$-invariant element in $V_n^{w(x)}$. Thus $\rho(P)\,v(x) = C(x)\,w_{w(x)}$ for some non-zero $C(x) \in \mathbb{C}(p^{\frac{1}{2}}, q^{\frac{1}{2}})$. So we obtain $1 = C(x)\,\chi_1(w_{w(x)})$. Since $C(1,\dots,1,-1,\dots,-1) = 1$ we have $\chi_1(w_d) = 1$ and $C(x) = 1/\chi_1(w_{w(x)}) = 1$.

To calculate the action of K we note that by (4.4) and (4.8)

$$t(K)\,\hat{u}(1,\dots,1,-1,\dots,-1) = q^{-(n-d)/2} q^{d/2}\,\hat{u}(1,\dots,1,-1,\dots,-1),$$

in terms of the orthonormal basis $\hat{u}(x)$ of V_n. Since $t(K)$ commutes with the Hecke symmetrisator by Jimbo's Theorem 4.4, we find $t(K)w_d = q^{d-n/2}w_d$.

Similarly,

$$t(E)\,\hat{u}(1,\dots,1,-1,\dots,-1) = \sum_{i=1}^{n-d} q^{-(i-1)/2}\,\hat{u}(1,\dots,1,-1,1,\dots,1,-1,\dots,-1)$$

where the first -1 occurs at the i-th place. Now apply the Hecke symmetrisator, use Jimbo's Theorem 4.4, and $v(x) = \iota(T_{u_x})^{-\frac{1}{2}}\hat{u}(x)$ to find

$$t(E)\,w_d = \left(\sum_{i=1}^{n-d} q^{-(i-1)/2} p^{\frac{1}{2}} q^{(n-i)/2}\right) w_{d+1},$$

which gives the result.

The action of $t(F)$ can be calculated similarly, but can also be derived from the commutation relation for E and F in $U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C}))$; see Definition 4.1. So if $t(F)w_d = c_d w_{d-1}$, we obtain

$$p^{\frac{1}{2}} q^{d-1} \frac{q^{(1-n)/2}}{1-q}\left(c_d(1 - q^{n-d+1}) - c_{d+1}q(1 - q^{n-d})\right) = \frac{q^{d-n/2} - q^{-d+n/2}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}.$$

Together with the initial condition $c_0 = 0$ this two-term recurrence relation determines c_d.

Using the fact that V_n is a $*$-module of $U_{q^{1/2}}(\mathfrak{sl}(2,\mathbb{C}))$ we get

$$p^{\frac{1}{2}} q^{(1-n)/2} q^d \frac{1 - q^{n-d}}{1-q}\,B(w_{d+1}, w_{d+1}) = B(t(E)w_d, w_{d+1})$$

$$= B(w_d, t(q^{-\frac{1}{2}}FK)w_{d+1}) = p^{-\frac{1}{2}} q^{(1-n)/2} \frac{1 - q^{d+1}}{1-q}\,B(w_d, w_d).$$

Solve this recurrence relation with the condition $B(w_0, w_0) = 1$ to find the result. $\square$

Proposition 6.6. *The space of $\mathcal{F}_n$-invariant elements in V_n^* is $n+1$-dimensional, and each U_f contains a one-dimensional space of $\mathcal{F}_n$-invariant elements. Moreover, $\phi_f = \rho^*(P)\,\chi_y = \rho^*(P)\,\chi_z$ is a non-zero $\mathcal{F}_n$-invariant element in U_f for $y, z \in \mathbb{Z}_2^n$ with $w(y) = w(z) = f$.*

Remark 6.7. Extend $\phi_f\colon V_n \to \mathbb{C}(p^{\frac{1}{2}}, q^{\frac{1}{2}})$ to $\mathcal{H}_n$ by defining $\phi_f(T) = \phi\big(\pi(T)\big)$ for $T \in \mathcal{H}_n$ and $\pi\colon \mathcal{H}_n \to V_n$ as in §3. Then ϕ_f is left and right $\iota|_{\mathcal{F}_n}$-invariant; $\phi_f(T_\sigma T T_\tau) = \iota(T_\sigma)\iota(T_\tau)\phi_f(T)$ for all $\sigma, \tau \in S_n$. Moreover, ϕ_f is contained in the irreducible $\mathcal{H}_n$-module U_f, so that we may regard ϕ_f as a 'zonal spherical function' on the Hecke algebra $\mathcal{H}_n$.

Proof. First observe that $\chi(v(x)) = (\rho^*(P)\chi)(v(x)) = \chi(\rho(P)v(x)) = \chi(w_{w(x)})$ for χ $\mathcal{F}_n$-invariant by Theorem 6.5. So χ is completely determined by $\chi(w_d)$, $d = 0, 1, \ldots, n$. The dimension of the space of $\mathcal{F}_n$-invariant elements in V_n^* is at most $n + 1$. If we can show that U_d has at least a one-dimensional subspace of $\mathcal{F}_n$-invariant elements, it follows that U_d has a subspace of $\mathcal{F}_n$-invariant elements of dimension one and the space of $\mathcal{F}_n$-invariant elements in V_n^* is $n+1$-dimensional.

Obviously $\rho^*(P)\chi_y$ is a $\mathcal{F}_n$-invariant element in $U_{d(y)}$ and

$$\rho^*(P)\chi_y\big(u(1, \ldots, 1)\big) = \chi_y\big(u(1, \ldots, 1)\big) = 1,$$

since the unit element $u(1, \ldots, 1)$ of V_n is $\mathcal{F}_n$-invariant. So this is a non-zero $\mathcal{F}_n$-invariant element in U_f, $f = w(y)$, and by the previous paragraph the same element with y replaced by z with $w(z) = f$ is a multiple of this $\mathcal{F}_n$-invariant element in U_f. The constant is 1, since evaluation at $u(1, \ldots, 1)$ gives 1 in both cases. $\square$

Remark 6.8. Since the representation t of $U_{q^{1/2}}(\mathfrak{sl}(2, \mathbb{C}))$ acts in V_n, we can also define contragredient representations of $U_{q^{1/2}}(\mathfrak{sl}(2, \mathbb{C}))$ in V_n^* using antimultiplicative mappings of $U_{q^{1/2}}(\mathfrak{sl}(2, \mathbb{C}))$ to itself. An obvious candidate is the antipode of the Hopf-algebra $U_{q^{1/2}}(\mathfrak{sl}(2, \mathbb{C}))$. Another candidate is the $*$-operator defined by (4.3). We define a representation t^* by $\big(t^*(X)\chi\big)(v) = \chi\big(t(X^*)v\big)$ for $X \in U_{q^{1/2}}(\mathfrak{sl}(2, \mathbb{C}))$, $\chi \in V_n^*$ and $v \in V_n$, and we do not use the contragredient representation defined using the antipode. Observe that $B(t(X)v, w) = B(v, t(X^*)w)$ implies $b\big(t^*(X)\chi\big) = t(X)b(\chi)$, with b the Fourier transform defined in Remark 5.6.

7. $\mathcal{F}_n$-INVARIANT ELEMENTS AND q-KRAWTCHOUK POLYNOMIALS

It follows from Proposition 6.6 that $\phi_f = \rho^*(P)\chi_y \in (V_n^*)^{\mathcal{F}_n}$ only depends on $f = w(y)$. Then $\phi_f\big(u(x)\big)$ is determined by $d = w(x)$ by Theorem 6.5 and $\phi_f(w_d) = \chi_y(w_d)$, $w(y) = f$. Proposition 5.3 implies that the $\phi_f(w_d)$ satisfy certain orthogonality relations for d running through the finite set $\{0, 1, \ldots, n\}$. This is not sufficient to determine $\phi_f(w_d)$ since we do not a priori know if $\phi_f(w_d)$ is a polynomial, but we come back to the orthogonality properties later. Using the fact that ϕ_f is a sum of characters, we give in Theorem 7.1 an explicit expression for a weighted average of ϕ_f evaluated in w_{d-1}, w_d and w_{d+1}, which corresponds to the finite Laplacian in Stanton [22, Def. 5.2].

Theorem 7.1. *The values $\phi_f(w_d)$ satisfy the recurrence relation for $0 \le f \le n$*

$$\big(p(1 - q^{n-f}) - (1 - q^f)\big)\phi_f(w_d) = pq^d(1 - q^{n-d})\phi_f(w_{d+1})$$
$$+ (1 - q^d)(p - 1)\phi_f(w_d) + (1 - q^d)\phi_f(w_{d-1})$$

with initial conditions $\phi_f(w_0) = 1$ and $\phi_f(w_n) = (-p)^{f-n}q^{f(f-n)}$.

Corollary 7.2. *The basis ϕ_f of the $n + 1$-dimensional irreducible $U_{q^{1/2}}(\mathfrak{sl}(2, \mathbb{C}))$-module $(V_n^*)^{\mathcal{F}_n}$ are eigenvectors of the following self-adjoint operator;*

$$t^* \left(E + E^* + \frac{p^{\frac{1}{2}} - p^{-\frac{1}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} (K - 1) \right) \phi_f = \frac{p^{\frac{1}{2}} q^{n/2 - f} - p^{-\frac{1}{2}} q^{f - n/2} + p^{-\frac{1}{2}} - p^{\frac{1}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \phi_f.$$

Proof. Combine the results of Theorems 7.1 and 6.5 with the definition of t^*. $\quad\square$

Proof of Theorem 7.1. The values $\phi_f(w_0)$ and $\phi_f(w_n)$ follow from Theorem 5.1 since V_n^0 and V_n^n are one-dimensional.

Let $\phi_f = \sum_{w(y)=f} c_f(y)\chi_y$, then for $v \in V_n$ and $w \in V_n^{\mathcal{F}_n}$

$$(7.1) \qquad \phi_f(wv) = \sum_{w(y)=f} c_f(y)\chi_y(w)\chi_y(v) = \chi_y(w)\phi_f(v) = \phi_f(w)\phi_f(v),$$

since χ_y is a character of V_n and $\chi_y(w)$ only depends on $w(y) = f$. By taking w a simple $\mathcal{F}_n$-invariant element, a multiple of w_1, and by working out $\rho(P)(wv)$ as a suitable sum of neighbouring elements w_{d+1}, w_d, w_{d-1} for suitably chosen $v \in V_n^d$ we see that (7.1) may be regarded as averaging over neighbours, or as a finite Laplacian.

We now use this with $w = \sum_{i=1}^n u(x^i)$, which is $\mathcal{F}_n$-invariant as easily follows from Theorem 3.1(i), and by Theorem 6.5 this is a constant multiple of w_1. In order to evaluate $\phi_f(w)$ we may take $y = (-1, \dots, -1, 1, \dots, 1)$ with $w(y) = f$ to get

$$\phi_f(w) = \sum_{i=1}^n \chi_y\big(u(x^i)\big) = \sum_{i=1}^f -q^{f-i} + \sum_{i=f+1}^n pq^{n-i} = -\frac{1 - q^f}{1 - q} + p\frac{1 - q^{n-f}}{1 - q}.$$

We take $v = v(1, \dots, 1, -1, \dots, -1)$ with d minus signs, so that $\phi_f(v) = \phi_f(w_d)$.

So it remains to calculate $\phi_f(wv)$, and for this we need an explicit expression for $\rho(P)(vw)$ in terms of the basis w_e, $0 \le e \le n$, for $\mathcal{F}_n$-invariant elements of Theorem 6.5. Now observe that

$$p^d q^{d(d-1)/2} vw = u(1, \dots, 1, -1, \dots, -1)\, w$$

$$= \sum_{i=1}^{n-d} u(1, \dots, 1, -\overset{i}{1}, 1, \dots, 1, \underbrace{1, \dots, -1}_{d})$$

$$+ \sum_{i=n-d+1}^n u(1, \dots, 1, \underbrace{-1, \dots, -1}_{i-n-d+1}, \overset{i}{1}, \underbrace{-1, \dots, -1}_{n-i})\,\big(u(x^i)\big)^2.$$

Using Theorem 6.5 we see that Hecke symmetrising the first sum yields a multiple of w_{d+1}; explicitly

$$\sum_{i=1}^{n-d} p^{d+1} q^{d(d-1)/2 + n - i}\, w_{d+1} = p^{d+1} q^{d(d-1)/2} q^d \frac{1 - q^{n-d}}{1 - q}\, w_{d+1}.$$

In order to treat the second sum we need the following lemma.

Lemma 7.3. *For $z \in \mathbb{Z}_2^{i-1}$ we have in V_n*

$$u(z, \overset{i}{1}, \underbrace{-1, \ldots, -1}_{n-i}) \left(u(x^i) \right)^2 = q^{n-i}(p-1)\, u(z, -1, \ldots, -1)$$

$$+ pq^{n-i-1}(q-1) \sum_{j=i+1}^{n} u(z, -1, \ldots, -1, \overset{j}{1}, -1, \ldots, -1)$$

$$+ pq^{n-i}\, u(z, 1, -1, \ldots, -1).$$

Assuming Lemma 7.3 we see that the second sum can be written as

$$(p-1) \sum_{i=n-d+1}^{n} q^{n-i}\, u(1, \ldots, 1, \underbrace{-1, \ldots, -1}_{d})$$

$$+ (q-1) \sum_{i=n-d+1}^{n} pq^{n-i-1} \sum_{j=i+1}^{n} u(1, \ldots, 1, \underbrace{-1, \ldots, -1}_{j-n+d-1}, \overset{j}{1}, \underbrace{-1, \ldots, -1}_{n-j})$$

$$+ \sum_{i=n-d+1}^{n} pq^{n-i} u(1, \ldots, 1, \underbrace{-1, \ldots, -1}_{i-n+d-1}, \overset{i}{1}, \underbrace{-1, \ldots, -1}_{n-i}).$$

Now we can use Theorem 6.5 to Hecke symmetrise each of these sums. Hecke symmetrising the first sum gives

$$p^d q^{d(d-1)/2}(p-1) \sum_{i=n-d+1}^{n} q^{n-i}\, w_d = p^d q^{d(d-1)/2}(p-1) \frac{1-q^d}{1-q}\, w_d,$$

and Hecke symmetrising the last sum gives

$$\sum_{i=n-d+1}^{n} pq^{n-i} p^{d-1} q^{d(d-1)/2 - (n-i)}\, w_{d-1} = p^d q^{d(d-1)/2}(n-d)\, w_{d-1}$$

and Hecke symmetrising the double sum gives

$$(q-1) \sum_{i=n-d+1}^{n} pq^{n-i-1} \sum_{j=i+1}^{n} p^{d-1} q^{d(d-1)/2 - (n-j)}\, w_{d-1}$$

$$= (q-1) p^d q^{d(d-1)/2} \sum_{i=n-d+1}^{n} \frac{1-q^{n-i}}{1-q}\, w_{d-1}$$

$$= p^d q^{d(d-1)/2} \left(\frac{1-q^d}{1-q} - (n-d) \right) w_{d-1}.$$

Collecting all terms shows

$$(1-q)\rho(P)(wv) = pq^d(1-q^{n-d})\, w_{d+1} + (1-q^d)(p-1)\, w_d + (1-q^d)\, w_{d-1}.$$

Now apply ϕ_f to get

$$(1-q)\phi_f(wv) = pq^d(1-q^{n-d})\, \phi_f(w_{d+1})$$

$$+ (1-q^d)(p-1)\, \phi_f(w_d) + (1-q^d)\, \phi_f(w_{d-1}).$$

Combine this with the first paragraphs of the proof to find the result. □

Proof of Lemma 7.3. First recall from Theorem 3.1(ii) that

$$u(x^i)^2 = (q-1) \sum_{l=i+1}^{n} u(x^i x^l) + (p-1)v(x^i) + pq^{n-i}.$$

This in particular proves the case $i = n$, and for the general case we obtain

$$u(z, \overset{i}{1}, \underbrace{-1, \dots, -1})\left(u(x^i)\right)^2 = (p-1)u(z, -1, \dots, -1) + pq^{n-i}\, u(z, \overset{i}{1}, -1, \dots, -1)$$
$$ + (q-1) \sum_{l=i+1}^{n} u(z, -1, \dots, -1, \overset{l}{1}, -1 \dots, -1)\left(u(x^l)\right)^2.$$

Proceeding by downward induction on i we can rewrite each term in the last sum. Collecting the coefficients of each basis element $u(x)$ then proves the lemma. $\square$

Remark 7.4. The second-order difference equation for $\phi_f(w_d)$ with respect to d of Theorem 7.1 is the consequence of the homomorphism property (7.1) for $w = w_1$ (up to a constant) and (sufficiently) arbitrary v. So we have calculated the coefficients in the expansion, or product formula, $\phi_f(w_k)\,\phi_f(w_d) = \sum_{l=0}^{n} c_l(k,d)\,\phi_f(w_l)$ for $k = 1$. The coefficients $c_l(k,d)$ are determined by $\rho(P)\left(w_k w_d\right) = \sum_{l=0}^{n} c_l(k,d)\, w_l$, or with w_d replaced by v as in the proof of Theorem 7.1. It follows from Theorem 6.5 that

$$c_l(k,d)\, B(w_l, w_l) = B(\rho(P)\left(w_k w_d\right), w_l) = B(w_k w_d, w_l) = \tau(w_k w_d w_l),$$

which is symmetric in k, l and d, since V_n is a commutative algebra. In general these constants seem hard to calculate, but for $q = 1$ they have been calculated explicitly; cf. Remark 7.10.

Remark 7.5. From (7.1) we can derive a convolution property for ϕ_f; cf. Remark 5.6. Note that $b(\phi_f) \in (V_n)^{\mathcal{F}_n}$ by (6.1), so that for all $v \in V_n$, $(\phi_f \star \phi_g)(v) = \phi_f\left(b(\phi_g)v\right) = \phi_f\left(b(\phi_g)\right)\phi_f(v)$. Since $\chi_y \star \chi_z = \delta_{y,z} h_y \chi_y$ (cf. Remark 5.6), we see that $\phi_f \star \phi_g = 0$ for $f \neq g$. So $\phi_f\left(b(\phi_g)\right) = 0$ for $f \neq g$ and $\phi_f \star \phi_g = \delta_{f,g}\, \phi_f\left(b(\phi_f)\right)\phi_f$. The convolution property of $\phi_f = \sum_{y \in \mathbb{Z}_2^n : \, w(y)=f} c_f(y)\,\chi_y$ can be used to calculate $c_f(y)$, since it implies $h_y\left(c_f(y)\right)^2 = \phi_f\left(b(\phi_f)\right)c_f(y)$, so that $c_f(y) = 0$ or $c_f(y) = \phi_f\left(b(\phi_f)\right)h_y^{-1}$. The first case is excluded by the proof of the next proposition, where $\phi_f\left(b(\phi_f)\right)$ is also explictly given.

Proposition 7.6. *The following orthogonality relations hold for $0 \leq d, e, f \leq n$;*

$$\sum_{d=0}^{n} p^d q^{d(d-1)/2} \begin{bmatrix} n \\ d \end{bmatrix}_q \phi_g(w_d)\,\phi_f(w_d) = \delta_{g,f}\, H_f,$$

$$\sum_{f=0}^{n} \frac{1}{H_f}\, \phi_f(w_g)\,\phi_f(w_d) = \delta_{d,g}\, p^d q^{d(d-1)/2} \begin{bmatrix} n \\ d \end{bmatrix}_q^{-1}$$

with $H_f = \begin{bmatrix} n \\ f \end{bmatrix}_q^{-1} \dfrac{(-pq^{-f}; q)_{n+1}}{pq^{n-f} + q^f} q^{f(f+1)/2} p^{-f}.$

Proof. The explicit value for the weights follows from the observation $\phi_f\left(b(\phi_g)\right) = 0$ for $f \neq g$ of Remark 7.5. Since $b(\phi_g) \in (V_n)^{\mathcal{F}_n}$ we develop it in the orthogonal

basis w_d, $0 \le d \le n$;

$$b(\phi_g) = \sum_{d=0}^{n} \frac{B\big(b(\phi_g), w_d\big)}{B(w_d, w_d)} w_d = \sum_{d=0}^{n} \frac{\phi_g(w_d)}{B(w_d, w_d)} w_d.$$

Hence, for $f \ne g$ we have $0 = \phi_f\big(b(\phi_g)\big) = \sum_{d=0}^{n} \frac{\phi_g(w_d)\phi_f(w_d)}{B(w_d, w_d)}$, so that the value for the weights follows from Theorem 6.5 and $H_f = \phi_f\big(b(\phi_f)\big) = B\big(b(\phi_f), b(\phi_f)\big)$.

In order to calculate squared norm H_f we first observe that H_f is non-zero for every f and that $H_0 = h_{(1,\dots,1)} = (-p; q)_n$, $H_n = h_{(-1,\dots,-1)} = (-p^{-1}; q)_n$; cf. Proposition 5.3 and Remark 5.5. By Corollary 5.4

$$\frac{1}{H_f} = \sum_{y \in \mathbb{Z}_2^n \,:\, w(y)=f} \frac{1}{h_y}$$

with h_y as in Proposition 5.3. At first this holds up to a scalar independent of f and n, and by the initial conditions this scalar equals 1.

We now use the notation $h_y(n) = h_y$ to stress the n-dependence. From Proposition 5.3 we see that for $y = (1, z)$, $z \in \mathbb{Z}_2^{n-1}$, with $f = w(y) = d(z)$

$$h_y(n) = (1 + pq^{2m_1(y)+1-n})\, h_z(n-1) = (1 + pq^{n-1-2f})\, h_z(n-1)$$

and for $y = (-1, z)$, $z \in \mathbb{Z}_2^{n-1}$, with $f = w(y) = d(z) + 1$

$$h_y(n) = (1 + p^{-1}q^{n-1-2m_1(y)})\, h_z(n-1) = (1 + p^{-1}q^{2f-n-1})\, h_z(n-1).$$

Writing the sum over $y \in \mathbb{Z}_2^n$ as two sums of the form $y = (1, z)$ for $z \in \mathbb{Z}_2^{n-1}$ and $y = (-1, z)$ for $z \in \mathbb{Z}_2^{n-1}$ gives the recurrence relation, with $H_f = H_f(n)$,

$$\frac{1}{H_f(n)} = \frac{1}{(1 + pq^{n-1-2f})H_f(n-1)} + \frac{1}{(1 + p^{-1}q^{2f-n-1})H_{f-1}(n-1)}.$$

This recurrence relation together with the initial conditions completely determines $H_f(n)$.

Put $H_f(n)^{-1} = (pq^{n-f} + q^f)\, D_f^n$ to find the recurrence relation

$$(1 + pq^{n-2f})\, D_f^n = D_f^{n-1} + pq^{n-2f}\, D_{f-1}^{n-1},$$

which, by substitution of $D_f^n = p^f q^{-f(f+1)/2}(-pq^{-f}; q)_{n+1}^{-1} E_n^f$, can be rephrased as

$$(1 + pq^{n-2f})\, E_f^n = (1 + pq^{n-f})\, E_f^{n-1} + (q^{n-f} + pq^{n-2f})\, E_{f-1}^{n-1}.$$

The initial condition is now $E_0^n = 1 = E_n^n$. Comparing this recurrence relation with

$$\begin{bmatrix} n \\ f \end{bmatrix}_q = q^f \begin{bmatrix} n-1 \\ f \end{bmatrix}_q + \begin{bmatrix} n-1 \\ f-1 \end{bmatrix}_q = \begin{bmatrix} n-1 \\ f \end{bmatrix}_q + q^{n-f} \begin{bmatrix} n-1 \\ f-1 \end{bmatrix}_q$$

gives $E_f^n = \begin{bmatrix} n \\ f \end{bmatrix}_q$. $\qquad\square$

Theorem 7.1 completely determines the values of $\phi_f(w_d)$ and this can be expressed using q-Krawtchouk polynomials. Define the q-Krawtchouk polynomials of degree n, $0 \le n \le N$, and of argument q^{-x} by

(7.2) $$K_n(q^{-x}; a, N; q) = {}_3\varphi_2\left(\begin{matrix} q^{-n}, q^{-x}, -q^{n-N}/a \\ q^{-N}, \ 0 \end{matrix} ; q, q \right);$$

see e.g. [22, §3], [11, ex. 7.8(i)]. The following proposition can be found in Stanton [22, Prop. 3.7].

Proposition 7.7. *The q-Krawtchouk polynomials satisfy the second order q-diffe-rence equation*

$$(q^n - aq^{N-n}) K_n(q^{-x}; a, N; q) = aq^x(1 - q^{N-x}) K_n(q^{-(x+1)}; a, N; q)$$
$$+ q^x(a-1) K_n(q^{-x}; a, N; q) + (1 - q^x) K_n(q^{-(x-1)}; a, N; q)$$

with initial conditions $K_n(1; a, N; q) = 1$, $K_n(q^{-N}; a, N; q) = (-a)^{-n} q^{n(n-N)}$.

Theorem 7.8. *We have $\phi_f(w_d) = K_f(q^{-d}; p, n; q)$ with the q-Krawtchouk polyno-mial K_f defined by (7.2).*

Proof. Compare Theorem 7.1 with Proposition 7.7. $\square$

Now that we have the zonal spherical function ϕ_f in terms of explicit polynomials, we can interpret some of the identities derived for ϕ_f as identities for q-Krawtchouk polynomials. First of all, the orthogonality relations of Proposition 7.6 correspond to the orthogonality relations for the q-Krawtchouk polynomials and for the dual q-Krawtchouk polynomials, where the dual q-Krawtchouk polynomials are defined by

$$R_n(q^{-x} - q^{x-N}/a; a, N; q) = K_x(q^{-n}; a, N; q).$$

The second order q-difference equation for the q-Krawtchouk polynomials of Theorem 7.1 is the three-term recurrence relation for the dual q-Krawtchouk polynomi-als.

Secondly, Corollary 7.2 corresponds to the fact that the matrix elements of the transition of the basis of eigenvectors for the action of K to the basis of eigenvectors for the action of $E + E^* + \frac{p^{\frac{1}{2}} - p^{-\frac{1}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}(K - 1)$ is given by q-Krawtchouk polynomials; see Koornwinder [17, Thm. 4.3]. Using this interpretation Koornwinder [17] is able to give an interpretation of Askey-Wilson polynomials on the quantum $SU(2)$ group as zonal spherical functions.

Remark 7.9. V_{n-1} can be viewed as a subalgebra of V_n by identifying $u(x) \in V_{n-1}$ for $x \in \mathbb{Z}_2^{n-1}$ with $u(1, x) \in V_n$. Since $\chi_y\big((u(1, x))\big)$ does not depend on y_1 we can view $\chi_y|V_{n-1}$ as an element of V_{n-1}^*. If we let $w_d^{n-1} \in V_{n-1} \subset V_n$ be the $\mathcal{F}_{n-1}$-invariant elements as in Theorem 6.5, then for $0 \le d \le n-1$

$$w_d^n = \frac{1 - q}{1 - q^n} \sum_{l=0}^{n-1} \rho(T_l T_{l-1} \dots T_2 T_1) w_d^{n-1}$$

by choosing minimal coset representatives in S_n/S_{n-1}. For arbitrary $y \in \mathbb{Z}_2^n$ we get

$$\phi_{w(y)}(w_d^n) = \frac{1 - q}{1 - q^n} \sum_{l=0}^{n-1} \big(\rho^*(T_1 T_2 \dots T_{l-1} T_l) \chi_y\big)(w_d^{n-1}).$$

Using Theorem 6.1 we can calculate $\rho^*(T_1 T_2 \dots T_{l-1} T_l) \chi_y$ explicitly for suitably choosen $y \in \mathbb{Z}_2^n$, and we obtain an explicit recurrence relation expressing $\phi_f(w_d^n)$ in terms of $\phi_f(w_d^{n-1})$ and $\phi_{f-1}(w_d^{n-1})$. Together with the initial conditions for $\phi_0(w_d^n)$ and $\phi_n(w_d^n)$ this recurrence relation determines $\phi_f(w_d^n)$. This recurrence relation is

equivalent to the following contiguous relation for the q-Krawtchouk polynomials:

$$(1-q^N)(1+aq^{N-2n})K_n(q^{-x};a,N;q) = (1-q^{N-n})(1+aq^{N-n})K_n(q^{-x};a,N-1;q)$$
$$+ q^{N-n}(1-q^n)(1+aq^{-n})K_{n-1}(q^{-x};a,N-1;q).$$

In the next two remarks we discuss how Theorem 7.8 is related to known interpretations of (q-)Krawtchouk polynomials on finite (hyper)groups.

Remark 7.10. In the specialisation $q=p=1$ we have $V_n = \mathbb{C}[\mathbb{Z}_2^n]$ and $K_f(d;\frac{1}{2},n)$ are the spherical functions on H_n with respect to subgroup S_n. This result goes back to Vere-Jones in 1971 (also in the context of statistics) and Delsarte in 1973 (related to the Hamming scheme in coding theory); see [9] and references given there.

In case we specialise only $q=1$, we see that V_n is the n-fold tensor product of V_1. Here V_1 is generated by two elements; a unit element and ω, say, satisfying $\omega^2 = (p-1)\omega + p$. Then this can be considered as a commutative hypergroup; see Dunkl and Ramirez [10, §5] with their a corresponding to $-p$. The interpretation of Krawtchouk polynomials as symmetrised characters [10, Thm. 5.1] corresponds to the result in Theorem 7.8 specialised to $q=1$; see also Koornwinder [16, §6]. In this case the coefficients $c_l(k,d)$ occurring in the product formula of Remark 7.4 can be determined by a counting argument and leads to the product formula for Krawtchouk polynomials; see [10, §5]. In the general case V_n gives rise to a 2^n-point hypergroup, which is not an n-fold tensor product. Its characters are described in §5.

Remark 7.11. Stanton [22], [24] has shown that the q-Krawtchouk polynomials appear as spherical functions on certain finite groups of Lie type for specific values of p and q; see Carter [4] for information on finite groups of Lie type. For a finite group G with subgroup B the Hecke algebra $H(G,B)$ is defined as the algebra $e_B\mathbb{C}[G]e_B$, where $e_B = |B|^{-1}\sum_{b\in B} b$ is the idempotent corresponding to the subgroup B. This can also be viewed as the convolution algebra of the left and right B-invariant functions on G, or as the intertwiner algebra of the induced representation 1_B^G. In particular, (G,B) is a Gelfand pair if and only if $H(G,B)$ is commutative and in this case we want to determine the spherical functions. See [4], [7], [8] for more information and references.

Let G be a finite group of Lie type, then it has a BN-pair implying the existence of subgroups B and N such that there is a Weyl group $W \cong N/(B\cap N)$. Moreover, the Bruhat decomposition holds, $G = \bigcup_{w\in W} BwB$, and from this we can associate to each parabolic subgroup of W a parabolic subgroup of G. We now consider the cases in which the corresponding Weyl group is the hyperoctahedral group H_n. Using the classification of simple finite groups of Lie type [4, §1.19, p. 464] there are 5 types; the Chevalley groups of type B_n and C_n and the twisted groups of type $^2D_{n+1}$, $^2A_{2n-1}$ and $^2A_{2n}$. These groups have realisations as classical groups over finite fields; cf. [4, p. 40]. Moreover, $H(G,B)$ is obtained from the generic Hecke algebra $\mathcal{H}_n$ by suitable specialisation of p and q. Denote by P the corresponding maximal parabolic subgroup corresponding to the maximal parabolic subgroup $S_n \subset H_n$, so $P = \bigcup_{\sigma\in S_n} B\sigma B$. Then $H(G,P) \cong H(H_n,S_n)$, which is commutative; see Curtis, Iwahori and Kilmoyer [6, §§2], [3] and also [3, Thm. 10.4.11].

The spherical functions corresponding to the Gelfand pair (G,P) have been determined by Stanton [22] in terms of q-Krawtchouk polynomials (7.2); let $p_0(\neq 2)$

be a prime and q_0 an integral power of p_0, then the zonal spherical functions can be expressed in terms of q-Krawtchouk polynomials $K_f(\cdot; p, n; q)$ with p and q as in the following table:

	B_n	C_n	$^2D_{n+1}$	$^2A_{2n-1}$	$^2A_{2n}$
p	q_0	q_0	q_0^2	q_0	q_0^3
q	q_0	q_0	q_0	q_0^2	q_0^2

(For these specialisations the values of the coefficients $c_l(k, d)$ occurring in the product formula of Remark 7.4 can be determined, since spherical functions satisfy a product formula; see [22, §7].) These specialisations are also the specialisations of p and q needed to obtain $H(G, B)$ from $\mathcal{H}_n$ in these five cases according to [4, p. 464], so Theorem 7.8 unifies and generalises Stanton's results.

The reason for this is the following. $V_n \cong V_n^* \cong e_B \mathbb{C}[G] e_P = \bigoplus_{f=0}^n U_f$ as $\mathcal{H}_n = H(G, B)$-module and the action of the characteristic function of BwB can be considered as $\lambda(Bw)$ on $L(B \backslash G / P) = e_B \mathbb{C}[G] e_P$, where λ denotes the left regular representation. By Stanton [22, Thm. 5.4], which corresponds nicely to Theorem 7.1 and Corollary 7.2, $L(G/P) = \mathbb{C}[G] e_P$ decomposes as $\bigoplus_{f=0}^n X_f$ under the left regular representation λ of G. Then $U_f = e_B X_f$ and $U_f^{\mathcal{F}_n} = e_P U_f = e_P X_f$ is the one-dimensional space spanned by the zonal spherical function.

Remark 7.12. Macdonald [19] studies the representation of the (extended) affine Hecke algebra obtained from inducing the index representation of the corresponding finite Hecke algebra for the Weyl group W_0. The representation space can be identified with the group algebra of the weight lattice P, and there exist orthogonal polynomials E_λ ($\lambda \in P$) acting on P and Weyl group invariant orthogonal polynomials P_λ ($\lambda \in P^+$, the dominant weights), where the P_λ can be obtained from Hecke symmetrising over W_0 from E_μ for μ in the Weyl group orbit of λ. The situation in this paper is analogous to the situation in Macdonald [19]; P, E_λ, P_λ, W correspond to $\mathbb{Z}_2^n$, χ_y, ϕ_f, S_n. In general, it seems not possible to derive the results of this paper from Macdonald [19]; but see Matsumoto [20] for the case $n = 2$.

Remark 7.13. As mentioned in Remarks 7.10 and 7.11, Theorem 7.8 covers a number of cases in which ϕ_f is a spherical function on a finite group. In most of these cases this can be used to derive an addition formula for $(q$-$)$Krawtchouk polynomials; see Dunkl [9] and Stanton [23]. The interpretation of general q-Krawtchouk polynomials as 'zonal spherical functions' on the Hecke algebra $\mathcal{H}_n$ might lead to an addition formula as well, but this is not clear. It should be noted that there is also an addition formula for so-called quantum q-Krawtchouk polynomials, which is derived by Groza and Kachurik [12] from their relation to matrix elements of irreducible representations of the quantum $SU(2)$ group and explicit knowledge of the Clebsch-Gordan coefficients for this quantum group. This interpretation of quantum q-Krawtchouk polynomials on the quantum $SU(2)$ group is different from the interpretation of q-Krawtchouk polynomials on the quantum $SU(2)$ group as described in Koornwinder [17]; cf. Corollary 7.2.

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