TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 352, Number 12, Pages 5511–5524
S 0002-9947(00)02677-5
Article electronically published on August 8, 2000

GERMS OF HOLOMORPHIC VECTOR FIELDS IN $\mathbb{C}^m$ WITHOUT A SEPARATRIX

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Abstract. We prove the existence of families of germs of holomorphic vector fields in $\mathbb{C}^m$ without a separatrix, in every complex dimension m bigger than or equal to 4.

A separatrix of a germ of a holomorphic vector field X at 0 in $\mathbb{C}^m$ is a germ of an irreducible analytic curve C at 0 such that X restricted to $C \setminus \{0\}$ is tangent to C. Every germ X in $\mathbb{C}^2$ has at least one separatrix (see [1]), but there exist families of germs X in $\mathbb{C}^3$ without separatrices (see [5] and [7]).

In this paper we prove the existence of families of germs X in $\mathbb{C}^m$ without separatrices, in every complex dimension m bigger than or equal to 4. More concretely, we begin by recalling a result from [8] which states that the existence of germs X without separatrices in $\mathbb{C}^n$, for $n = 3, 4$ and 5, implies the existence of such germs in any other higher dimension (see Remark 3.2 below). Then, in Sections 4 and 5 we give families of such germs in $\mathbb{C}^4$ and $\mathbb{C}^5$ respectively.

The way to accomplish these results may be divided into two parts: The first part comes from [8], and consists on the obtention of a set of sufficient conditions, actually depending only on the first two non-trivial jets of a germ X, which guarantee that X is separatrices-free ($m \geq 3$). The second part consists of the computations, for vector fields in $\mathbb{C}^4$ and $\mathbb{C}^5$, of a set of algebraic equations which correspond to the conditions above. To prove that the quasi-projective varieties thus defined are not empty, we compute explicit elements for them.

The main technique to obtain the separatrices-free conditions on germs X at 0 in $\mathbb{C}^m$, relies on the resolution of singularities for foliations: Consider the foliation $\mathcal{F}$ defined by X and the blow-up $\sigma : \tilde{\mathbb{C}}^m \to \mathbb{C}^m$ with center at 0. The strict transform $\mathcal{F}_1$ of $\mathcal{F}$ under σ is a foliation by curves on the manifold $\tilde{\mathbb{C}}^m$ leaving the exceptional divisor $E \simeq \mathbb{CP}^{m-1}$ invariant. The conditions for X to be separatrices-free are expressed in terms of the linear part of $\mathcal{F}_1$ at each of its singular points, on the exceptional divisor E. To wit, if at every such singular point, all the generalized eigenspaces of the linear part of $\mathcal{F}_1$ are tangent to E, and the remaining eigenvalues of the linear part satisfy an additional condition, then the germ X has no separatrices through 0 (see Remark 2.2, Definition 3.1 and Theorem 3.3 for precise statements).

Received by the editors December 5, 1997.
2000 *Mathematics Subject Classification.* Primary 32S65.
Supported by DGICYT (Spain) PB97-0284-C02-01.
Partially supported by CONACYT (Mexico) Projects 3398-E9307, 0324P-E9506 and Postdoctoral Grant 963052, at Dto. Álgebra, Geometría y Topología, U. Valladolid.

The exposition is divided into five sections. In Section 1 we recall some well known facts concerning foliations by curves on compact holomorphic manifolds, as well as the construction of the strict transform of a foliation by curves under the blow-up centered at a point. In Section 2 we give a quick review on foliations by curves in projective spaces. In Section 3 we recall the results from [8] dealing with the aforesaid set of separatrices-free conditions. These results justify that the computations on four- and five-dimensional vector fields that take effect respectively in Sections 4 and 5 do produce examples of separatrices-free germs of holomorphic vector fields.

1. Preliminaries on foliations

Let M be an m-dimensional complex manifold.

A *holomorphic foliation by curves (with singularities)* $\mathcal{F}$ on M may be defined by a familiy of holomorphic vector fields $\{X_\alpha\}$ on an open cover $\{U_\alpha\}$ of M, which satisfy $X_\alpha = f_{\alpha\beta}X_\beta$ in $U_\alpha \cap U_\beta$, where $f_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ is a holomorphic non-vanishing function in $U_\alpha \cap U_\beta$. The *singular set* of $\mathcal{F}$ is the analytic subvariety defined by

$$\mathrm{Sing}(\mathcal{F}) = \{p \in M \mid X_\alpha(p) = 0\}.$$

We have the following objects associated to a singular point p of a (for short) holomorphic foliation by curves $\mathcal{F}$ on M.

Let z be a coordinate on M near p such that $z(p) = 0$ and $\mathcal{F}$ is generated by a vector field $Y(z) = \sum_{j=1}^{m} Y_j(z)\partial/\partial z_j$.

1. *The linear part of* $\mathcal{F}$ *at* p, which is the linear endomorphism on the tangent space T_pM of M at p, induced by Y. It corresponds to the terms of degree 1 in the power series expansion of Y and will be denoted by $DY(p)$ or by $L(\mathcal{F}, p)$. It is well defined up to multiplication by non-zero constants.
2. The *multiplicity* (or *index*) $\mu(\mathcal{F}, p)$ *of the foliation* $\mathcal{F}$ *at* p, which is the codimension in the local ring $\mathcal{O}_{M,p}$ of the ideal generated by $\{Y_j\}_{j=1}^{m}$:
$$\mu(\mathcal{F}, p) = \dim_{\mathbb{C}}(\mathcal{O}_{M,p}/(Y_1, ..., Y_m)).$$
$\mu(\mathcal{F}, p)$ is finite if and only if p is an isolated singularity.
3. The *algebraic multiplicity* of $\mathcal{F}$ at p, which is the degree of the smallest non-zero coefficient in the power series expansion of Y. We will say that $\mathcal{F}$ is *dicritical* (resp. *non-dicritical*) at p if the terms of the smallest degree of Y are (resp. are not) a multiple of the radial vector field.

Let $\sigma : \tilde{M} \to M$ be the blow-up (or quadratic transformation) centered at $p \in M$. The *strict transform of* $\mathcal{F}$ *under* σ (or the *(adapted) foliation* obtained by pulling back under σ the foliation $\mathcal{F}$) is the foliation $\mathcal{F}_1$ on $\tilde{M}$ defined as follows: If $\mathcal{F}$ is generated at p by a vector field Y, of algebraic multiplicity $d+1$, and σ is given by

$$\sigma(\zeta_1, ..., \zeta_m) = (\zeta_1, \zeta_1\zeta_2, ..., \zeta_1\zeta_m) = (z_1, z_2, ..., z_m), \tag{1.1}$$

then $\mathcal{F}_1$ is generated by

$$Y_1 = \zeta_1^{-d}(D\sigma)^{-1}(Y \circ \sigma) \tag{1.2}$$

where $(D\sigma)^{-1} : \sigma^*TM \to T\widetilde{M}$ is the rational bundle map over $\tilde{M}$ having poles of order 1 along the exceptional divisor $E = \sigma^{-1}(0)$. Y_1 is both holomorphic and tangent to E.

By the comments that follow the twisted Euler sequence (2.1) below, it is easy to see that Y_1 does not vanish identically along E if and only if $\mathcal{F}$ is non-dicritical at p. In this case we will also say that the exceptional divisor is non-dicritical. On the opposite, if Y is dicritical at 0, then Y has infinitely many separatrices through 0, therefore dicritical vector fields are uninteresting to our study.

2. Foliations in projective spaces

We now recall some well-known facts on holomorphic foliations by curves on projective spaces.

A polynomial vector field $X(z) = \sum_{j=1}^{m} X_j(z)\partial/\partial z_j$ in $\mathbb{C}^m$ will be called a $(d+1)-$*homogeneous vector field* if each component $X_j(z)$ is a homogeneous polynomial of degree $d+1$. Each component may be thought of as a global section of the line bundle on $\mathbb{CP}^{m-1}$ with Chern class d, which we denote by L_d (see [3], p. 165). Let $\Pi : \mathbb{C}^m \setminus \{0\} \to \mathbb{CP}^{m-1}$ be the natural projection and denote by $T\mathbb{CP}^{m-1}$ the tangent bundle to $\mathbb{CP}^{m-1}$. The twisted Euler sequence

$$0 \longrightarrow L_d \longrightarrow \bigoplus_m L_{d+1} \xrightarrow{\Pi_*} L_d \otimes T\mathbb{CP}^{m-1} \longrightarrow 0 \tag{2.1}$$

states then that a $(d+1)$-homogeneous vector field X induces a section of the bundle $L_d \otimes T\mathbb{CP}^{m-1}$ (see [6], p. 409). Two such vector fields define the same section if and only if its difference is of the form $g \cdot R$, where g is a homogeneous polynomial of degree d, and R is the radial vector field. Such a section gives, by definition, a holomorphic foliation by curves $\mathcal{F}$ on $\mathbb{CP}^{m-1}$. The projective space of lines through 0 in $H^0(\mathbb{CP}^{m-1}, L_d \otimes T\mathbb{CP}^{m-1})$ might be called then the space of foliations by curves of degree d in $\mathbb{CP}^{m-1}$. We shall denote it by

$$\mathcal{F}\mathrm{ol}(d, \mathbb{CP}^{m-1}) = \mathrm{Proj}H^0(\mathbb{CP}^{m-1}, L_d \otimes T\mathbb{CP}^{m-1}). \tag{2.2}$$

The singular set $\mathrm{Sing}(\mathcal{F})$ of a foliation $\mathcal{F}$ in $\mathcal{F}\mathrm{ol}(d, \mathbb{CP}^{m-1})$ can be now easily recognized as the set of points $[\zeta] \in \mathbb{CP}^{m-1}$ where $X(\zeta)$ is proportional to the vector ζ. If $\mathrm{Sing}(\mathcal{F})$ consists only of isolated singularities, then, by a theorem of Chern [2], one knows that the sum of the multiplicities $\mu(\mathcal{F}, p)$ of the foliation at its singular points equals the top Chern class $c_{m-1}(L_d \otimes T\mathbb{CP}^{m-1})$ of the associated bundle, which may be seen to be

$$c_{m-1}(L_d \otimes T\mathbb{CP}^{m-1}) = \begin{cases} ((d+1)^m - 1)/d, & \text{if } d \geq 1\ , \\ m, & \text{if } d = 0\ , \end{cases} \tag{2.3}$$

by applying the Whitney product formula to the twisted Euler sequence (2.1). For further details, see [4].

A direct consequence from Lemma 1.1 in [4], and from the Theorem on the Dimension of the Fibers ([9], p. 77), is the following:

Remark 2.1. Let $m \geq 3$ and $d \geq 1$ be integers, then the subset

$$\{\mathcal{F} \in \mathcal{F}\mathrm{ol}(d, \mathbb{CP}^{m-1})\ :\ \dim(\mathrm{Sing}(\mathcal{F})) \geq 1\}$$

is closed in $\mathcal{F}\mathrm{ol}(d, \mathbb{CP}^{m-1})$.

Consider $\sigma : \tilde{\mathbb{C}}^m \to \mathbb{C}^m$ the blow-up at $0 \in \mathbb{C}^m$. Given a germ at 0 of a holomorphic vector field X, of algebraic multiplicity $d+1$ having a non-dicritical singularity at 0, consider the strict transform $\mathcal{F}_1$ under σ of the foliation $\mathcal{F}$ induced by X. Recall from the first section that the exceptional divisor $E \simeq \mathbb{CP}^{m-1}$ is invariant by $\mathcal{F}_1$. One may easily verify that the restriction $\mathcal{F}_1\mid_E$ of the foliation

$\mathcal{F}_1$ to E is isomorphic to the one obtained by pushing forward under the map Π_* in (2.1) the $(d+1)$-homogeneous part X_{d+1} of X.

We shall now illustrate some of the results described above for the case of 2-homogeneous (or quadratic) vector fields on $\mathbb{C}^4$. On the one hand, this will help us give a clearer exposition of the results of the following section (see Remark 2.2 and Definition 3.1), and on the other hand, the computations involved here will be applied in Section 4.

A quadratic vector field Y in $\mathbb{C}^4$ is a vector field of the form

$$(2.4)\quad Y = \sum a_{j,k} z_j z_k \frac{\partial}{\partial z_1} + \sum b_{j,k} z_j z_k \frac{\partial}{\partial z_2} + \sum c_{j,k} z_j z_k \frac{\partial}{\partial z_3} + \sum d_{j,k} z_j z_k \frac{\partial}{\partial z_4}$$

where $a_{j,k}, b_{j,k}, c_{j,k}, d_{j,k} \in \mathbb{C}$, and the subindices in the sums run over $1 \leq j \leq k \leq 4$. The set $\{Y\}$ of quadratic vector fields has a natural $\mathbb{C}$-vector space structure, whose projective space of lines through 0 has dimension 39. We will denote it by Ξ_4. The group $GL(4,\mathbb{C})$ of linear automorphisms of $\mathbb{C}^4$ has dimension 16 and acts naturally on Ξ_4.

By the twisted Euler sequence (2.1), the space (2.2) of foliations

$$\mathcal{F}\,\mathrm{ol}(1, \mathbb{CP}^3) = \mathrm{Proj} H^0(\mathbb{CP}^3, L_1 \otimes T\mathbb{CP}^3)$$

induced by vector fields of the form (2.4) is isomorphic to $\mathbb{P}^{35}$. If the singular set of such a foliation consists only of isolated points, then the sum (2.3) of the multiplicities equals 15.

Take $m = 4$ in (1.1): In these coordinates, the exceptional divisor $E \simeq \mathbb{CP}^3$ is given by $\zeta_1 = 0$ and $\bar{\zeta} = (\zeta_2, \zeta_3, \zeta_4)$ is an affine coordinate system whose origin corresponds to the point $q_1 = [1, 0, 0, 0]$.

Consider the vector field (2.4) plus a linear multiple of the radial vector field

$$(2.5)\qquad Y(z) + L(z) \cdot R(z) = Y(z) + (\sum_{j=1}^{4} r_j\, z_j) \cdot (\sum_{j=1}^{4} z_j \cdot \frac{\partial}{\partial z_j}),$$

and let $Y_1(\zeta)$ be the corresponding generator (1.2) (with $d = 1$) of the adapted foliation $\mathcal{F}_1$ induced by the vector field (2.5). Then, $\mathcal{F}_1$ is singular at $q_1 = 0$ if and only if

$$(2.6)\qquad b_{1,1} = c_{1,1} = d_{1,1} = 0,$$

conditions that we will assume in what follows. $Y_1(\zeta)$ is a polynomial vector field of degree 3 having the following form, when written as a sum of polynomial homogeneous vector fields:

$$(2.7)\qquad Y_1(\zeta) = A(\zeta) + B(\zeta) + g(\zeta)\, \tau(\zeta),$$

where the linear part $L(\mathcal{F}_1, q_1)$ is given by

$$(2.8)\qquad A(\zeta) = \begin{pmatrix} a_{1,1} + r_1 & 0 & 0 & 0 \\ 0 & b_{1,2} - a_{1,1} & b_{1,3} & b_{1,4} \\ 0 & c_{1,2} & c_{1,3} - a_{1,1} & c_{1,4} \\ 0 & d_{1,2} & d_{1,3} & d_{1,4} - a_{1,1} \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \end{pmatrix},$$

the quadratic part $B(\zeta)$, by

$$B(\zeta) = \begin{pmatrix} \zeta_1\left(\left(a_{1,2}+r_2\right)\zeta_2+\left(a_{1,3}+r_3\right)\zeta_3+\left(a_{1,4}+r_4\right)\zeta_4\right), \\ \left(-a_{1,2}+b_{2,2}\right)\zeta_2{}^2+\left(-a_{1,3}+b_{2,3}\right)\zeta_2\zeta_3 \\ \quad +\left(-a_{1,4}+b_{2,4}\right)\zeta_2\zeta_4+b_{3,3}\zeta_3{}^2+b_{3,4}\zeta_3\zeta_4+b_{4,4}\zeta_4{}^2, \\ c_{2,2}\zeta_2{}^2+\left(-a_{1,2}+c_{2,3}\right)\zeta_2\zeta_3+c_{2,4}\zeta_2\zeta_4 \\ \quad +\left(c_{3,3}-a_{1,3}\right)\zeta_3{}^2+\left(c_{3,4}-a_{1,4}\right)\zeta_3\zeta_4+c_{4,4}\zeta_4{}^2, \\ d_{2,2}\zeta_2{}^2+d_{2,3}\zeta_2\zeta_3+\left(-a_{1,2}+d_{2,4}\right)\zeta_2\zeta_4 \\ \quad +d_{3,3}\zeta_3{}^2+\left(d_{3,4}-a_{1,3}\right)\zeta_3\zeta_4+\left(d_{4,4}-a_{1,4}\right)\zeta_4{}^2 \end{pmatrix},$$

and the cubic part, by

$$\text{(2.9)} \quad g(\bar{\zeta})\tau(\zeta) = -(a_{2,2}\zeta_2{}^2+a_{2,3}\zeta_2\zeta_3 + a_{2,4}\zeta_2\zeta_4+a_{3,3}\zeta_3{}^2+a_{3,4}\zeta_3\zeta_4+a_{4,4}\zeta_4{}^2)\cdot\begin{pmatrix} -\zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \end{pmatrix}.$$

As was pointed out before, the exceptional divisor ($\zeta_1 = 0$) is invariant by $Y_1(\zeta)$. The restriction $\mathcal{F}_1 \mid_E$ of $\mathcal{F}_1$ to the exceptional divisor is generated by the vector field

$$\text{(2.10)} \qquad \bar{Y}(\bar{\zeta}) = \sum_{j=2}^{4} Y_{1,j}(0,\zeta,\zeta_3,\zeta_4)\frac{\partial}{\partial\zeta_j} = \sum_{j=2}^{4}\bar{Y}_j(\bar{\zeta})\frac{\partial}{\partial\zeta_j},$$

whose cubic part is the quadratic polynomial $g(\bar{\zeta})$ times the radial vector field $R(\bar{\zeta})$ (see (2.9)).

Adding a general polynomial 3-homogeneous (cubic) vector field to the quadratic one given by (2.5), subject to the conditions (2.6), one obtains that the linear part (2.8) becomes

$$\text{(2.11)} \qquad \begin{pmatrix} a_{1,1}+r_1 & 0 & 0 & 0 \\ b_{3,0,0} & -a_{1,1}+b_{1,2} & b_{1,3} & b_{1,4} \\ c_{3,0,0} & c_{1,2} & c_{1,3}-a_{1,1} & c_{1,4} \\ d_{3,0,0} & d_{1,2} & d_{1,3} & -a_{1,1}+d_{1,4} \end{pmatrix},$$

where $b_{3,0,0}$, $c_{3,0,0}$ and $d_{3,0,0}$ are the coefficients of $z_1{}^3\frac{\partial}{\partial z_2}$, $z_1{}^3\frac{\partial}{\partial z_3}$ and $z_1{}^3\frac{\partial}{\partial z_4}$, respectively.

Remark 2.2. Observe that $a_{1,1}+r_1$ is an eigenvalue of (2.8) whose generalized eigenspace is not tangent to the exceptional divisor. We shall call it *the normal eigenvalue of* $L(\mathcal{F}_1, q_1)$.

If the normal eigenvalue is simple, then its generalized eigenspace is one-dimensional and transversal to the exceptional divisor ($\zeta_1 = 0$). From the theory of normal forms, one may conclude the existence of a separatrix $\tilde{\mathcal{C}}$ of the vector field (2.7) transversal to ($\zeta_1 = 0$). This separatrix would then be the strict transform of a separatrix $\mathcal{C}$ of the quadratic vector field (2.5), subject to the conditions (2.6).

One way to avoid this possibility is by forcing the normal eigenvalue to be double and non-semisimple: Under these conditions, the generalized eigenspace of the normal eigenvalue is tangent to the exceptional divisor. Observe now from (2.11) that, for the case of quadratic vector fields, one may impose the conditions above by adding a suitable generic cubic homogeneous polynomial vector field to the quadratic one given by (2.5), subject to the conditions (2.6).

All these considerations form the content of Definition 3.1, (extended there to germs of arbitrary algebraic multiplicity), and they turn out to be sufficient conditions on a germ of a vector field to be separatrices-free, as shown in Remark 3.2 and Theorem 3.3 below.

3. Separatrices-free conditions

Definition 3.1. Let $\mathbf{m}$ denote the maximal ideal in the local ring of holomorphic functions $\mathcal{O}_{\mathbb{C}^m,0}$ at $0 \in \mathbb{C}^m$. Let $d \geq 1$ and $m \geq 3$ be integers and let

$$\mathcal{V}_{m,\,d} \subset \frac{\mathbf{m}^{d+1}_{\mathbb{C}^m,0} \cdot \Theta_{\mathbb{C}^m,0}}{\mathbf{m}^{d+3}_{\mathbb{C}^m,0} \cdot \Theta_{\mathbb{C}^m,0}}$$

be the set of polynomial vector fields $\{X\}$ in $\mathbb{C}^m$, with non-vanishing terms only in degrees $d+1$ and $d+2$, that satisfy the following conditions:

(a) X is non-dicritical at 0 and the adapted foliation $\mathcal{F}_1 \mid_E$ induced by X and the blowing up at 0, restricted to the exceptional divisor E, has only isolated singularities q_j.
(b) For each singular point q_j, the normal eigenvalue of the linear part $L(\mathcal{F}_1\,, q_j)$ of $\mathcal{F}_1$ is non-zero and double.
(c) For each singular point q_j, the Jordan block associated to the normal eigenvalue of $L(\mathcal{F}_1\,, q_j)$ is non-semisimple.
(d) For each singular point q_j, the quotients of the normal eigenvalue with the remaining distinct eigenvalues of $L(\mathcal{F}_1\,, q_j)$ are not strictly positive rational numbers.

Remark 3.2 ([8]). Let X be a holomorphic vector field at $0 \in \mathbb{C}^m$, of algebraic multiplicity $d+1$ whose $(d+2)$-jet lies in $\mathcal{V}_{m,\,d}$, and let p in E be an isolated singularity of $\mathcal{F}_1$. Consider a coordinate system ζ in $\tilde{\mathbb{C}}^m$ around p such that $\zeta(p) = 0$ and the exceptional divisor E is given by $(\zeta_1 = 0)$. The $(d+2)$-jet of X determines the linear part of $\mathcal{F}_1$ at p, and hence, a generator Y_1 of $\mathcal{F}_1$ around p satisfies the conditions (a), (b) and (c) from Definition 3.1. The conditions (a) and (b) above imply that X has an isolated singularity at 0.

Theorem 3.3 ([8])**.** *Let X be a germ of a holomorphic vector field at $0 \in \mathbb{C}^m$, of algebraic multiplicity $d+1$ such that its $(d+2)$-jet belongs to the set $\mathcal{V}_{m,\,d}$ from* Definition 3.1*. Then X has an isolated singularity at* 0 *and does not have a separatrix through* 0*.*

Concerning the existence of germs of holomorphic vector fields without separatrices in higher dimensions, the following results were also given in [8]:

Theorem 3.4 ([8])**.** *Let $n \geq 2$ be an integer. For each $j = 1, ..., n$, let $z^j = (z^j_1, z^j_2, z^j_3)$ be coordinates on $\mathbb{C}^3_j$ and let $X_j(z^j)$ be a germ at* 0 *of a holomorphic vector field in $\mathbb{C}^3_j$, with an isolated singularity at* 0*, and without a separatrix. Then the germ at* 0 *in $\mathbb{C}^{3n}$ given by $X(z^1, ..., z^n) = (X_1(z^1), ..., X_n(z^n))$ has an isolated singularity at* 0 *and does not have a separatrix through* 0*.*

Proof. It is clear that X has an isolated singularity at $0 \in \mathbb{C}^{3n}$, so that it only remains to prove that X does not have separatrices outside the 3-planes $\mathbb{C}^3_j$. We will prove it by contradiction: Let C be a germ at $0 \in \mathbb{C}^{3n}$ of an irreducible curve which is invariant by X in $C \setminus \{0\}$, and which is not contained in any $\mathbb{C}^3_j$, for $j = 1, ..., n$. Let $z_0 = (z_0^1, ..., z_0^n)$ be a regular point of X in $C \setminus \{0\}$, and denote by α the solution of $\dot{z} = X(z)$ with initial condition, say, $z(0) = z_0$. By assumption, the closure $\bar{\alpha}$ of α in a small polydisk $\triangle$ at 0 is $C \cap \triangle$, and $z_0^k \neq 0$ for every $k \in \{1, ..., n\}$. Now, since the (local) flow of X splits as the product of the (local) flows of its components X_j, then the image $\alpha_k = \pi_k(\alpha)$ of α under the projection $\pi_k : \mathbb{C}^{3n} \to \mathbb{C}^3_k$ is a separatrix through $0 \in \mathbb{C}^3_k$ of the vector field $X_k(z^k)$. This contradiction finishes the proof. ∎

It is easy to see that every integer $m \geq 3$ may be written as a linear combination $m = 3m_1 + 4m_2 + 5m_3$ with non-negative integer coefficients m_j. Taking this for granted, and following the argument from the proof of Theorem 3.4, one can prove the following result:

Remark 3.5 ([8]). Let $m \geq 3$ be an integer and write it as an ordered sum $m = 3m_1 + 4m_2 + 5m_3$. For $i = 1, ..., m_1$, $j = 1, ..., m_2$ and $k = 1, ..., m_3$, let X^i, Y^j and Z^k be germs at 0 for holomorphic vector fields respectively in $\mathbb{C}^3$, $\mathbb{C}^4$ and $\mathbb{C}^5$, having an isolated singularity at 0 and without a separatrix through 0. Then the germ of vector field X at $0 \in \mathbb{C}^m$ given by

$$X = (X^1, ..., X^{m_1}, Y^1, ..., Y^{m_2}, Z^1, ..., Z^{m_3}),$$

has an isolated singularity at $0 \in \mathbb{C}^m$ and does not have a separatrix through 0.

4. Four-dimensional examples

Definition 4.1. Let

$$\mathcal{X}_{4,\,1} \subset \frac{\mathbf{m}^2_{\mathbb{C}^4,0} \cdot \Theta_{\mathbb{C}^4,0}}{\mathbf{m}^4_{\mathbb{C}^4,0} \cdot \Theta_{\mathbb{C}^4,0}} \tag{4.1}$$

be the set of polynomial vector fields $\{X\}$ in $\mathbb{C}^4$, with non-vanishing terms only in degrees 2 and 3, that satisfy the following conditions:

(a) X is non-dicritical at 0 and the adapted foliation $\mathcal{F}_1 \mid_E$ induced by X and the blowing up at 0, restricted to the exceptional divisor E, has only one singular point q of multiplicity 15.
(b) The linear part $L(\mathcal{F}_1\,, q)$ of $\mathcal{F}_1$ at the singular point q satisfies that the normal eigenvalue to E is non-zero and double.
(c) The Jordan block in $L(\mathcal{F}_1\,, q)$ associated to the normal eigenvalue is non-semisimple.
(d) The quotients of the normal eigenvalue with the remaining distinct eigenvalues of $L(\mathcal{F}_1\,, q)$ are not strictly positive rational numbers.

Remark 4.2. By Theorem 3.3, any vector field of algebraic multiplicity 2 whose 3-jet lies on $\mathcal{X}_{4,\,1}$ has an isolated singularity at 0 and does not have a separatrix through 0. We will prove that the set $\mathcal{X}_{4,\,1}$ is not empty by showing an element in there.

Recall from Section 2 that Ξ_4 denotes the projective space of lines through 0 in the vector space of quadratic vector fields in $\mathbb{C}^4$. Now we prove:

Lemma 4.3.

(1) *The set of quadratic homogeneous vector fields* $\{Y\}$ *in* $\mathbb{C}^4$ *that satisfy conditions* (a) *and* (b) *from Definition 4.1 contains a quasiprojective subvariety* $\mathcal{Y}_2$ *of* Ξ_4 *of codimension at most 15.*

(2) *The set of polynomial vector fields* $\{Y\}$ *in* $\mathbb{C}^4$ *with non-vanishing terms only in degrees 2 and 3 that satisfy conditions* (a), (b) *and* (c) *from* Definition 4.1 *contains a quasiprojective subvariety* $\mathcal{Y}$ *of codimension at most 15.*

Proof. **1)** We will first use the group $GL(4,\mathbb{C})$ to normalize the position of the singular point q, to be at $q_1 = [1,0,0,0]$. Let Y_1 be the generator (2.7) of the adapted foliation $\mathcal{F}_1$ obtained by pulling back the vector field (2.5), subject to the conditions (2.6), under the quadratic transformation at 0.

To simplify the exposition, denote by $P_c(\lambda)$ the characteristic polynomial of the linear part $L(\mathcal{F}_1 \mid_E, q_1)$ (the lower-right 3×3 submatrix of (2.8)). It follows that the conditions for the normal eigenvalue to be non-zero and double are given respectively by

$$a_{1,1} + r_1 \neq 0 \tag{4.2}$$

and

$$P_c(a_{1,1} + r_1) = 0. \tag{4.3}$$

Equations for $\mu(\mathcal{F}_1 \mid_E, q_1) = 15$ may be given in the following way: Let $\bar{Y}$ be the generator (2.10) of the restriction $\mathcal{F}_1 \mid_E$ of $\mathcal{F}_1$ to the exceptional divisor E, and let $\bar{\zeta} = (\zeta_2, \zeta_3, \zeta_4)$ be the coordinates with $\zeta(q_1) = 0$.

The idea is the following: From (2.8), if we assume that

$$D_{23} = \begin{vmatrix} b_{1,2} - a_{1,1} & b_{1,3} \\ c_{1,2} & c_{1,3} - a_{1,1} \end{vmatrix} \neq 0, \tag{4.4}$$

then the hypersurfaces $(\bar{Y}_2 = 0)$ and $(\bar{Y}_3 = 0)$ intersect transversally at $q_1 = 0$, and this intersection defines an analytic curve $\mathcal{C}$. By the Implicit Function Theorem, $\mathcal{C}$ may be locally parametrized by a map α of the form

$$\alpha : \zeta_4 \mapsto (\alpha_2(\zeta_4), \alpha_3(\zeta_4), \zeta_4), \tag{4.5}$$

that is, we may use ζ_4 as a local parameter and, moreover, one can approximate this map up to any desired jet-order.

Denote by $J_{15}\alpha(\zeta_4) = (J_{15}\alpha_2(\zeta_4), J_{15}\alpha_3(\zeta_4), \zeta_4)$ the 15th-order jet of α. Then the intersection of $\mathcal{C}$ with the remaining hypersurface $(\bar{Y}_4 = 0)$ is expressed as a polynomial in ζ_4 of the form

$$\bar{Y}_4(J_{15}\alpha(\zeta_4)) = a_1\,\zeta_4 + a_2\,\zeta_4^2 + ... + a_{15}\,\zeta_4^{15} + ... \tag{4.6}$$

whose coefficients depend algebraically on those from (2.10).

It is easy to see that for $k = 1, 2, ...15$,

$$\mu(\mathcal{F}_1, q_1) = k \iff a_j = 0, \text{ for } j = 1, ..., k-1 \text{ and } a_k \neq 0. \tag{4.7}$$

For example, a manipulation in the explicit expression for the coefficient a_1 from (4.7), shows that the polynomial a_1 is equal to the product of the determinants (4.4) and that of $L(\mathcal{F}_1 \mid_E, q_1)$. For obvious reasons we have omitted the remaining explicit expressions of the coefficients a_j in (4.7).

Let $\mathcal{Y}_1 \subset \Xi_4 \simeq \mathbb{P}^{39}$ be the quasi-projective subvariety defined by the equations (2.6), $\{a_j = 0,\ j = 1, ..., 14\}$, (4.3), and the open conditions (4.2), (4.4) and $a_{15} \neq 0$, in the *open* subset of Ξ_4 that corresponds to foliations in $\mathcal{F}\,\mathrm{ol}(1,\mathbb{CP}^3)$ with only isolated singularities (Remark 2.1). $\mathcal{Y}_1$ consists of quadratic homogeneous vector fields with a single isolated singularity at q_1 that satisfy conditions (a) and (b) from Definition 4.1. If non-empty, each irreducible component has dimension at least $39-(3+14+1)=21$. Let $GL(4,\mathbb{C})\cdot\mathcal{Y}_1$ be the constructible set obtained by the action of the linear group on $\mathcal{Y}_1$ and let $\mathcal{Y}_2$ be the interior of this set in its Zariski closure $(\overline{GL(4,\mathbb{C})\cdot\mathcal{Y}_1})^O$. If non-empty, $\mathcal{Y}_2$ is a quasi-projective subvariety of Ξ_4 of dimension at least $21+3=24$, hence of codimension at most 15.

To prove that $\mathcal{Y}_2$ is not empty, we give an element in $\mathcal{Y}_1$:

Let t be a complex number satisfying

$$576\,t^3-880\,t^2-400\,t-25=0. \tag{4.8}$$

Then, the vector field X given by

$$\begin{aligned} X&(z_1,z_2,z_3,z_4)\\ &=\Big({z_1}^2+\big(-\tfrac{7}{162}-\tfrac{121}{162}\,t-\tfrac{179}{81}\,t^2\big)\,{z_2}^2+\big(\tfrac{5}{36}-\tfrac{5}{3}\,t^2+\tfrac{16}{9}\,t\big)\,z_2z_3\\ &\quad+\big(-\tfrac{5}{36}-\tfrac{19}{9}\,t\big)\,z_2z_4+\big(\tfrac{1}{18}+\tfrac{13}{9}\,t\big)\,{z_3}^2-tz_3z_4\Big)\tfrac{\partial}{\partial z_1}\\ &\quad+\Big(2\,z_1z_2-{z_3}^2\Big)\tfrac{\partial}{\partial z_2}+\Big(2\,z_1z_3-{z_4}^2\Big)\tfrac{\partial}{\partial z_3}\\ &\quad+\Big(\big(\tfrac{1}{3}+\tfrac{8}{3}\,t\big)\,z_1z_2-z_1z_3+z_1z_4+\big(\tfrac{34}{27}\,t-\tfrac{98}{27}\,t^2+\tfrac{13}{108}\big)\,{z_2}^2\\ &\quad+\big(-\tfrac{1}{12}-\tfrac{2}{3}\,t\big)\,z_2z_3+\big(\tfrac{4}{9}\,t+\tfrac{1}{18}\big)\,z_2z_4+\big(-\tfrac{1}{3}-\tfrac{11}{3}\,t\big)\,{z_3}^2+{z_4}^2\Big)\tfrac{\partial}{\partial z_4} \end{aligned} \tag{4.9}$$

belongs to $\mathcal{Y}_1$.

The construction of this example is divided into two steps: The first step consists on the construction of a polynomial vector field V having the same shape as (2.10), with a singularity at $q_1=0$ of multiplicty 15, whose linear part has at least one non-zero eigenvalue. With this in hand, we easily find a quadratic homogeneous vector field Z in $\mathbb{C}^4$ (4.13) whose strict transform, when restricted to the exceptional divisor, coincides with V. The second step makes use of the twisted Euler sequence (2.1): We shall find a linear multiple $L(z)\cdot R(z)$ of the radial vector field in $\mathbb{C}^4$ such that the strict transform of $X(z)=Z(z)+L(z)\cdot R(z)$ induces the same foliation as V in the exceptional divisor and has the desired repeated-eigenvalue condition (b).

We carry out the construction, and apologize for using z instead of ζ for the coordinates (1.1) in $\tilde{\mathbb{C}}^4$. The starting point then of the first step of the construction is the following vector field:

$$\begin{aligned} V&(z_2,z_3,z_4)\\ &=\Big(z_2-{z_3}^2+z_2\,Q(z_2,z_3,z_4)\Big)\frac{\partial}{\partial z_2}\\ &\quad+\Big(z_3-{z_4}^2+z_3\,Q(z_2,z_3,z_4)\Big)\frac{\partial}{\partial z_3}\\ &\quad+\Big(a_{1,0,0}z_2+a_{0,1,0}z_3+a_{0,0,1}z_4+a_{2,0,0}{z_2}^2+a_{1,1,0}z_2z_3+a_{1,0,1}z_2z_4\\ &\quad+a_{0,2,0}{z_3}^2+a_{0,1,1}z_3z_4+a_{0,0,2}{z_4}^2+z_4\,Q(z_2,z_3,z_4)\Big)\frac{\partial}{\partial z_4} \end{aligned} \tag{4.10}$$

where the quadratic homogeneous polynomial $Q(z_2, z_3, z_4)$ is given by

$$Q(z_2, z_3, z_4) = e_1 {z_2}^2 + e_2 z_2\, z_3 + e_3 {z_3}^2 + e_4 z_2\, z_4 + e_5 z_3\, z_4 + e_6 {z_4}^2. \tag{4.11}$$

The analytic curve $\mathcal{C}$ given by $(V_2 = 0).(V_3 = 0)$ may be locally parametrized by a map of the form (4.5). We used the symbolic manipulator Maple V to compute a true parametrization (4.5) of the curve $\mathcal{C}$ up to order 15.

To wit, we took

$$\alpha_2(z_4) = \sum_{j=1}^{15} s_j\, z_4^j$$
$$\alpha_3(z_4) = \sum_{j=1}^{15} t_j\, z_4^j\,,$$

as (4.5), evaluated the explicit expressions $\{A_{2,\,j}, A_{3,\,j},\ j = 1, ..., 15\}$ given by

$$V_2(\sum_{j=1}^{15} s_j\, z_4^j, \sum_{j=1}^{15} t_j\, z_4^j, z_4) = \sum_{j=1}^{15} A_{2,\,j}\, z_4^j\,,$$
$$V_3(\sum_{j=1}^{15} s_j\, z_4^j, \sum_{j=1}^{15} t_j\, z_4^j, z_4) = \sum_{j=1}^{15} A_{3,\,j}\, z_4^j\,,$$

and solved the set of equations $\{A_{2,\,j} = 0, A_{3,\,j} = 0,\ j = 1, ..., 15\}$, for $\{t_j, s_j\ ,\ j = 1, ..., 15\}$, in terms of the coefficients of (4.10).

Later, we computed the corresponding set of equations (4.7) to find a solution. We found the following one:

$$\begin{aligned} V(z_2,&z_3, z_4) \\ &= \left(z_2 - {z_3}^2 + z_2 Q(z_2, z_3, z_4)\right)\frac{\partial}{\partial z_2} \\ &+ \left(z_3 - {z_4}^2 + z_3 Q(z_2, z_3, z_4)\right)\frac{\partial}{\partial z_3} \\ &+ \Bigg(\left(\frac{1}{3} + \frac{8}{3}\, e_5\right) z_2 - z_3 + \left(\frac{34}{27}\, e_5 - \frac{98}{27}\, {e_5}^2 + \frac{13}{108}\right) {z_2}^2 \\ &\quad - \left(\frac{1}{12} + \frac{2}{3}\, e_5\right) z_2 z_3 + \left(\frac{4}{9}\, e_5 + \frac{1}{18}\right) z_2 z_4 \\ &\quad - \left(\frac{1}{3} + \frac{11}{3}\, e_5\right) {z_3}^2 + {z_4}^2 + z_4 Q(z_2, z_3, z_4)\Bigg)\frac{\partial}{\partial z_4}\,, \end{aligned} \tag{4.12}$$

where e_5 is some root t of (4.8), and the quadratic polynomial (4.11) is now given by

$$\begin{aligned} Q(z_2,z_3, z_4) = &\left(\frac{179}{81}\, {e_5}^2 + \frac{7}{162} + \frac{121}{162}\, e_5\right) {z_2}^2 + e_5 z_3 z_4 \\ &+ \left(\frac{5}{36} + \frac{19}{9}\, e_5\right) z_2 z_4 + \left(\frac{5}{3}\, {e_5}^2 - \frac{5}{36} - \frac{16}{9}\, e_5\right) z_3 z_2 + \left(-\frac{13}{9}\, e_5 - \frac{1}{18}\right) {z_3}^2. \end{aligned}$$

Then one checks that q_1 is the only singularity of the foliation induced by the vector field (4.12) on the exceptional divisor E. This finishes the first step of the construction.

For the second step, consider a linear multiple of the radial vector field $L(z)\cdot R(z)$ as in (2.5), and consider the quadratic homogeneous vector field Z given by

(4.13)
$$\begin{aligned}
&Z(z_1,z_2,z_3,z_4)\\
&\quad= \Bigg(\left(-\tfrac{7}{162} - \tfrac{121}{162}\,e_5 - \tfrac{179}{81}\,{e_5}^2\right) {z_2}^2 + \left(\tfrac{5}{36} - \tfrac{5}{3}\,{e_5}^2 + \tfrac{16}{9}\,e_5\right) z_2 z_3\\
&\qquad + \left(-\tfrac{5}{36} - \tfrac{19}{9}\,e_5\right) z_2 z_4 + \left(\tfrac{1}{18} + \tfrac{13}{9}\,e_5\right) {z_3}^2 - e_5 z_3 z_4 \Bigg) \tfrac{\partial}{\partial z_1}\\
&\quad + \left(z_1 z_2 - {z_3}^2 \right) \tfrac{\partial}{\partial z_2} + \left(z_1 z_3 - {z_4}^2 \right) \tfrac{\partial}{\partial z_3}\\
&\quad + \Bigg(\tfrac{(8\,e_5+1)}{3} z_1 z_2 - z_1 z_3 + \left(\tfrac{34}{27}\,e_5 - \tfrac{98}{27}\,{e_5}^2 + \tfrac{13}{108}\right) {z_2}^2 + \left(-\tfrac{1}{12} - \tfrac{2}{3}\,e_5\right) z_2 z_3\\
&\qquad + \left(\tfrac{4}{9}\,e_5 + \tfrac{1}{18}\right) z_2 z_4 + \left(-\tfrac{1}{3} - \tfrac{11}{3}\,e_5\right) {z_3}^2 + {z_4}^2 \Bigg) \tfrac{\partial}{\partial z_4}\,.
\end{aligned}$$

Both vector fields $Z(z)$ and $X(z) = Z(z) + L(z)\cdot R(z)$ (given by (4.9)) induce the same foliation as (4.12) in the exceptional divisor, but the linear part $DX_1(q_1)$ of the strict transform $X_1(z)$ of $X(z)$ at the singular point q_1 now equals

$$DX_1(q_1) = \begin{pmatrix} r_1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & \tfrac{8}{3}\,e_5 + \tfrac{1}{3} & -1 & 0 \end{pmatrix}.$$

We have chosen $r_1 = 1, r_2 = 0$ and $r_3 = 0$ in the vector field (2.5) in order to get the repeated-eigenvalue condition (b), from the definition of $\mathcal{X}_{4,\,1}$.

This remark finishes the construction of the example and the proof of part 1 of the Lemma.

2) Let

$$\mathcal{Y} \subset \mathcal{Y}_2 \subset \frac{\mathbf{m}^3_{\mathbb{C}^4,0}\cdot\Theta_{\mathbb{C}^4,0}}{\mathbf{m}^4_{\mathbb{C}^4,0}\cdot\Theta_{\mathbb{C}^4,0}}$$

be the subvariety defined as the complement of the subvarieties formed by those vector fields whose blown-up foliation has semisimple linear part at its singular point. The proof that $\mathcal{Y}$ satisfies the desired properties may be adapted from that of Lemma 7, part 2, in [5], and will not be given here. We simply want to point out that adding the cubic vector field $z_1^3\frac{\partial}{\partial z_1}$ to the vector field $X(z)$ given by (4.9), one gets a vector field

$$Y(z) = X(z) + z_1^3\frac{\partial}{\partial z_1} \tag{4.14}$$

satisfying conditions (a), (b) and (c) from Definition 4.1; the linear part at q_1 of the strict transform Y_1 of (4.14) becomes

$$DY_1(q_1) = \begin{pmatrix} 1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & \tfrac{8}{3}\,e_5 + \tfrac{1}{3} & -1 & 0 \end{pmatrix}.$$

This remark finishes the proof of Lemma 4.3. $\square$

We close this section remarking that the vector field (4.14) also satisfies condition (d) from Definition 4.1 and hence, that it belongs to the set $\mathcal{X}_{4,\,1}$ from Definition 4.1.

5. Five-dimensional examples

In this section we will simply sketch the construction of a familiy of five-dimensional germs of vector fields without separatrices, since the main ideas have been largely explained in the previous section.

Definition 5.1. Let

$$\mathcal{X}_{5,\,1} \subset \frac{\mathbf{m}^2_{\mathbb{C}^5,0}\cdot\Theta_{\mathbb{C}^5,0}}{\mathbf{m}^5_{\mathbb{C}^5,0}\cdot\Theta_{\mathbb{C}^5,0}} \tag{5.1}$$

be the set of polynomial vector fields $\{X\}$ in $\mathbb{C}^5$, with non-vanishing terms only in degrees 2 and 3, that satisfy the following conditions:

(a) X is non-dicritical at 0 and the adapted foliation $\mathcal{F}_1\mid_E$ induced by X and the blowing up at 0, restricted to the exceptional divisor E, has only three singular points q_1, q_2 and q_3, of multiplicities $22, 8$ and 1, respectively.

(b) For each singular point q_i, the linear part $L(\mathcal{F}_1\,, q_i)$ of $\mathcal{F}_1$ satisfies that the normal eigenvalue to E is non-zero and double.

(c) For each singular point q_i, the Jordan block in $L(\mathcal{F}_1\,, q_i)$ associated to the normal eigenvalue is non-semisimple.

(d) For each singular point q_i, the quotients of the normal eigenvalue with the remaining distinct eigenvalues of $L(\mathcal{F}_1\,, q)$ are not strictly positive rational numbers.

Remark 5.2. By Theorem 3.3, any vector field of algebraic multiplicity 2 whose 3-jet lies on $\mathcal{X}_{5,\,1}$ has an isolated singularity at 0 and does not have a separatrix through 0. As in the four-dimensional case, we will prove that the set $\mathcal{X}_{5,\,1}$ is not empty by computing an element of it.

The projective space Ξ_5 of lines through 0 in the vector space of quadratic vector fields in $\mathbb{C}^5$ has dimension 74. The group $GL(5,\mathbb{C})$ of linear automorphisms of $\mathbb{C}^5$ has dimension 25 and acts naturally on Ξ_5. Now we prove:

Lemma 5.3.

1) *The set of quadratic homogeneous vector fields $\{Y\}$ in $\mathbb{C}^5$ that satisfy conditions* (a) *and* (b) *from Definition 5.1 contains a quasiprojective subvariety $\mathcal{Y}_2$ of Ξ_5 of codimension at most* 47.

2) *The set of polynomial vector fields $\{Y\}$ in $\mathbb{C}^5$ with non-vanishing terms only in degrees 2 and 3 that satisfy conditions* (a), (b) *and* (c) *from Definition 5.1 contains a quasiprojective subvariety $\mathcal{Y}$ of codimension at most* 47.

Sketch of proof. **1)** We will restrict our search to the subspace $\Xi_{5,2}$ of Ξ_5 consisting of quadratic vector fields having two invariant hyperplanes. It is of codimension 20 in Ξ_5, and $GL(5,\mathbb{C})$ acts on it. Under this action, we first normalize the hyperplanes to be $(z_4 = 0)$ and $(z_5 = 0)$ and the positions of the singular points q_i, to be $p_1 = [0,1,0,0,0]$, $p_2 = [0,0,0,1,0]$ and $p_3 = [0,0,0,0,1]$, respectively. The codimension in $\Xi_{5,2}$ of the subspace $\Xi_{5,2}(\{p_i\})$ formed by quadratic vector fields whose strict transform $\mathcal{F}_1\mid_E$ vanishes on the points $\{p_i\}$ is equal to 8 (only 3 linear conditions for each one of the points p_2 and p_3, and only 2 for the point p_1, since the vector fields under consideration are tangent to the hyperplanes $(z_4 = 0)$ and $(z_5 = 0)$).

On this space, one may then write out, similarily as we did in the proof of **1)** in Lemma 4.3, the 21 equations for p_1 being singular of multiplicity at least 22; the 7 equations for p_2 being singular of multiplicity at least 8 (and the 3 open conditions for these multiplicities to be strictly equal to $22, 8$ and 1, respectively); the 3 open conditions on the linear parts $L(\mathcal{F}_1, p_i)$ to have at least one non-zero eigenvalue and finally, the 3 conditions on the normal eigenvalue to be non-zero and double.

In the *open* subset of Ξ_5 that corresponds to foliations in $\mathcal{F}\mathrm{ol}(1, \mathbb{CP}^4)$ with only isolated singularities (Remark 2.1), the above set of conditions defines a quasiprojective subvariety $\mathcal{Y}_1 \subset \Xi_{5,2} \simeq \mathbb{P}^{54}$, consisting of quadratic vector fields having $(z_4 = 0)$ and $(z_5 = 0)$ as invariant hyperplanes, and whose strict transform foliation $\mathcal{F}_1$ satisfies conditions (a) and (b) from Definition 5.1 on the points $q_i = p_i$, $i = 1, 2, 3$. If non-empty, each irreducible component has dimension at least $54-(8+21+7+3) = 15$. Let $GL(5, \mathbb{C})\cdot\mathcal{Y}_1$ be the constructible set obtained by the action of the linear group on $\mathcal{Y}_1$ and let $\mathcal{Y}_2$ be the interior of this set in its Zariski closure $(\overline{GL(5, \mathbb{C}) \cdot \mathcal{Y}_1})^O$. If non-empty, $\mathcal{Y}_2$ is a quasi-projective subvariety of $\Xi_{5,2}$ of dimension at least $15 + 12 = 27$, hence of codimension at most $54 - 27 = 27$. Then, the codimension of $\mathcal{Y}_2$ in Ξ_5 is at most 47.

To prove that $\mathcal{Y}_2$ is not empty, we give an element X in $\mathcal{Y}_1$:

The vector field X_1 given by

$$
\begin{aligned}
X_1(z_1, ..., z_5) = \\
&= \left(z_1z_2 - z_1z_4 - z_1z_5 - {z_3}^2\right)\frac{\partial}{\partial z_1} \\
&+ (z_1z_3 + z_2z_4 + z_2z_5)\frac{\partial}{\partial z_2} + \left({z_1}^2 + z_3z_5\right)\frac{\partial}{\partial z_3} + z_4z_5\frac{\partial}{\partial z_4} + 0\frac{\partial}{\partial z_5},
\end{aligned} \tag{5.2}
$$

has singularities on p_1, p_2 and p_3 of multiplicities $22, 8$ and 1, respectively and each linear part $L(\mathcal{F}_1 |_E, p_i)$ has at least on non-zero eigenvalue. Adding up to X_1 a suitable linear multiple of the radial vector field, we obtain the vector field

$$
\begin{aligned}
X(z_1, ..., z_5) \\
&= \left(z_1z_2 - z_1z_4 - z_1z_5 - {z_3}^2 + (z_2 + z_4 + z_5)\, z_1\right)\frac{\partial}{\partial z_1} \\
&+ (z_1z_3 + z_2z_4 + z_2z_5 + (z_2 + z_4 + z_5)\, z_2)\frac{\partial}{\partial z_2} \\
&+ \left({z_1}^2 + z_3z_5 + (z_2 + z_4 + z_5)\, z_3\right)\frac{\partial}{\partial z_3} \\
&+ (z_4z_5 + (z_2 + z_4 + z_5)\, z_4)\frac{\partial}{\partial z_4} + (z_2 + z_4 + z_5)\, z_5\frac{\partial}{\partial z_5}
\end{aligned} \tag{5.3}
$$

satisfying the remaining repeated-eigenvalue conditions on each $L(\mathcal{F}_1, p_i)$.

Then one checks that p_1, p_2 and p_3 are the only singularities of the foliation $\mathcal{F}_1 |_E$ induced by the vector field (5.3) on the exceptional divisor E, and hence, it belongs to $\mathcal{Y}_1$. This finishes the proof of the first part of Lemma 5.3.

2) To prove the second part of Lemma 5.3 we refer the reader to Lemma 4.3. We simply point out here that the vector field Z given by

$$
\begin{aligned}
Z(z_1, ..., z_5) \\
&= \left({z_2}^3 + z_1z_2 - z_1z_4 - z_1z_5 - {z_3}^2 + (z_2 + z_4 + z_5)\, z_1\right)\frac{\partial}{\partial z_1} \\
&+ \left({z_4}^3 + z_1z_3 + z_2z_4 + z_2z_5 + (z_2 + z_4 + z_5)\, z_2\right)\frac{\partial}{\partial z_2} \\
&+ \left({z_1}^2 + z_3z_5 + (z_2 + z_4 + z_5)\, z_3\right)\frac{\partial}{\partial z_3} \\
&+ \left({z_5}^3 + z_4z_5 + (z_2 + z_4 + z_5)\, z_4\right)\frac{\partial}{\partial z_4} \\
&+ (z_2 + z_4 + z_5)\, z_5\frac{\partial}{\partial z_5},
\end{aligned} \tag{5.4}
$$

has been obtained from the vector field (5.3) by adding a suitable cubic homogeneous vector field to obtain the non-semisimplicity conditions (c) from Definition 5.1. This finishes the proof of Lemma 5.3. □

We point out that the vector fields given by (5.3) and (5.4) do satisfy condition (d) from Definition 5.1, and hence, that (5.4) belongs to the set $\mathcal{X}_{5,\,1}$ from Definition 5.1.

We close with the following remark concerning the codimension of the family just obtained:

Remark 5.4. In the spaces of germs of holomorphic vector fields of algebraic multiplicity two, the codimension of the families of germs without separatrices in $\mathbb{C}^3$ (from [5]) and in $\mathbb{C}^4$ (from Section 4) have been shown to be equal to the sum (2.3) of the local multiplicities (respectively, 7 and 15). The expected codimension of a corresponding family in $\mathbb{C}^5$ thereby is 31, but we obtained a family $\mathcal{Y}_2$ of codimension 42. However, the codimension of $\mathcal{Y}_2$ in the subspace $\Xi_{5,2} \subset \Xi_5$ (defined in the the proof of Lemma 5.3) is equal to $54 - 27 = 27$. The difference $4 = 31 - 27$ arises from the fact that the dimension of the stabilizer of 3 points in $\mathbb{CP}^4$ is equal to 12, and the codimension of $\Xi_{5,2}(\{p_i\})$ in $\Xi_{5,2}$ is equal to 8 (again, see the proof of Lemma 5.3 for definitions).

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