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TRANSFERS OF CHERN CLASSES IN BP-COHOMOLOGY AND CHOW RINGS

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ABSTRACT. The BP^*-module structure of $BP^*(BG)$ for extraspecial 2-groups is studied using transfer and Chern classes. These give rise to p-torsion elements in the kernel of the cycle map from the Chow ring to ordinary cohomology first obtained by Totaro.

1. INTRODUCTION

Let G be a compact Lie group, e.g. a finite group, and BG its classifying space. For complex oriented cohomology theories h one can define in $h^*(BG)$ Chern classes of complex representations of G, and also transfer maps. We are interested in the Mackey closure $\overline{Ch}_h(G)$ of the ring of Chern classes in $h^*(BG)$, namely the subring of $h^*(BG)$ recursively generated by transfers of Chern classes. By [HKR], this is equal to the h^*-module generated by transfers of Euler classes.

For ordinary mod p cohomology, Green and Leary [GL] showed that the inclusion map $i : \overline{Ch}_{H\mathbb{Z}/p} \hookrightarrow H^*(BG;\mathbb{Z}/p)$ is an F-isomorphism, i.e., the induced map of varieties is a homeomorphism. Green and Minh [GM], however, noticed that $i/\sqrt{0}$ need not be an isomorphism in general. Next consider $h = BP$ or $h = K(n)$, the n-th Morava K-theory, at a fixed prime p. Following Hopkins, Kuhn and Ravenel [HKR], we shall call a group G "good" for h-theory if $h^*(BG)$ is generated (as an h^*-module) by transferred Euler classes of representations of subgroups of G. It is clear that if the Sylow p-subgroup of G is good, then so is G, and one has an isomorphism $h^*(BG) \cong \overline{Ch}_h(G)$. Furthermore, it follows from [RWY] that G is good for BP if it is good for $K(n)$ for all n. Examples for groups that are $K(n)$-good for all n are the finite symmetric groups. Another typical case are p-groups of p-rank at most 2 and $p \geq 5$: in [Y1] it is shown that the Thom map $\rho : BP^*(-) \to H^*(-)_{(p)}$ induces an isomorphism $BP^*(BG) \otimes_{BP^*} \mathbb{Z}_{(p)} \cong H^{even}(BG)$. Note however that I. Kriz claimed that $K(n)^{odd}(BG) \neq 0$ for some p-groups G.

On the other hand, B. Totaro [T1] found a way to compare BP-theory to the Chow ring. For a complex algebraic variety X, the groups $CH^i(X)$ of codimension i algebraic cycles modulo rational equivalence assemble to the Chow ring $CH^*(X) = \sum_i CH^i(X)$. Totaro constructed a map $\tilde{\rho} : CH^*(X) \to BP^*(X) \otimes_{BP^*} \mathbb{Z}_{(p)}$ such that the composition

$$\bar{\rho} : CH^i(X)_{(p)} \xrightarrow{\tilde{\rho}} BP^*(X) \otimes_{BP^*} \mathbb{Z}_{(p)} \xrightarrow{\rho} H^*(X)_{(p)}$$

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coincides with the cycle map. One of the main results of [T1] is that there exists a group G for which the kernel of $\bar{\rho}$ contains p-torsion elements. To prove this, Totaro defined the Chow ring of a classifying space BG as $\mathrm{Lim}_{m\to\infty} CH^*((\mathbb{C}^m - S)/G)$, where G acts on $\mathbb{C}^m - S$ freely and $\mathrm{codim}(S) \to \infty$ as $m \to \infty$. He then constructed a non-zero element x in $\mathrm{Ker}(\rho)$ such that

$$x \in \mathrm{Im}(\overline{Ch}_{BP}(BG) \to (BP^*(BG) \otimes_{BP^*} \mathbb{Z}_{(p)}))\,. \tag{1.1}$$

Since transfers and Chern classes also exist in the Chow ring $CH^*(BG)$, there is an element $\bar{x} \in \overline{Ch}_{CH}(G)$ that also lies in $\mathrm{Ker}(\bar{\rho})$. The group Totaro uses is $G = \mathbb{Z}/2 \times D_+^{1+4}$, where $D_+^{1+4} = D(2)$ is the extraspecial 2-group of order 32, which is isomorphic to the central product of two copies of the dihedral group D_8 of order 8. He first proves that there exists an element $x \in BP^*(BD(2))$ satisfying (1.1) but which restricts to zero under the map $\rho_{\mathbb{Z}/2} : BP^*(-) \to H^*(-;\mathbb{Z}/2)$, where he uses the computation of $BP^*(BSO(4))$ from [KY].

Let $D(n) = 2_+^{1+2n}$ denote the extraspecial 2-group of order 2^{2n+1}; it is isomorphic to the central product of n copies of D_8. In this paper, we construct non-zero elements $x \in BP^*(BD(n))$ satisfying (1.1) but with $\rho_{\mathbb{Z}/2}(x) = 0$ directly for each n.

Let $\tilde{W}$ be a maximal elementary abelian 2-subgroup and N the center of $D(n)$. For a one-dimensional real representation e of $\tilde{W}$ restricting non-trivially to the center, set $\Delta = \mathrm{Ind}_{\tilde{W}}^{D(n)}(e)$. This is the unique irreducible representation which acts non-trivially on N. Then the i-th Stiefel-Whitney class $w_i(\Delta)$ for $i < 2^n$ can be written as a polynomial in variables $w_1(e_j), 1 \le j \le 2n$, for 1-dimensional representations e_j of $D(n)/N$ ([Q], Remark 5.13), i.e. $w_i(\Delta) = w_i(w_1(e_1), \dots , w_1(e_{2n}))$. Let $e'_{\mathbb{C}}$ denote the complex representation induced from the real representation e'. Then we can prove that

$$x = c_{2^{n-1}}(\Delta_{\mathbb{C}}) - w_{2^{n-1}}(c_1(e_{1\mathbb{C}}), \dots , c_1(e_{2n\mathbb{C}})) \tag{1.2}$$

satisfies (1.1) together with $\rho_{\mathbb{Z}/2}(x) = 0$, and furthermore conclude that $\mathrm{Ker}(\rho) \neq 0$ for $G = \mathbb{Z}/2 \times D(n)$.

Second, we construct a non-nilpotent element $x \in BP^*(BG)$ which is not in $\overline{Ch}_{BP}(BG)$ and such that

$$x \in \mathrm{Ker}(\rho) \text{ and } 0 \neq x \in (BP^*(BG) \otimes_{BP^*} \mathbb{Z}_{(p)}). \tag{1.3}$$

However we do not know whether x comes from the Chow ring or not, and we only obtain the result for $n = 3, 4$. Set

$$x = [v_1 \otimes w_{2^n}(\Delta)] \tag{1.4}$$

to be the class represented by $v_1 \otimes w_{2^n}(\Delta)$ in the E_∞-page of the Atiyah-Hirzebruch spectral sequence. If this element exists, then restricting to the center of $D(n)$ we see that x is not in $\overline{Ch}_{BP}(BG)$. However, it seems difficult to prove that this cycle is permanent. For the case $n = 3, 4$, we use BP-theory of $B\mathrm{Spin}(7)$ and $B\mathrm{Spin}(9)$ computed in [KY] to see that x is a permanent cycle.

These arguments do not seem to work for other extraspecial 2-groups, nor for 2-groups that have a cyclic maximal normal subgroup [S].

In Section 2, we recall the mod 2 cohomology of extraspecial 2-groups following [Q]. In particular, $w_{2^n-2^i}(\Delta)$ is represented by the Dickson invariant D_i, and we study the action of the Milnor primitives Q_j on D_i. To see that $\rho(x) \neq 0$ in $H^*(BD(n);\mathbb{Z})$, we recall the integral cohomology in Section 3. In Section 4, we show that x satisfies (1.1). In Section 5, we study how elements in $\mathrm{Ker}(\rho)$ are represented in the Atiyah-Hirzebruch spectral sequence, assuming some technical conditions which are satisfied in the cases $n = 3$ and $n = 4$. The element x in (1.4) is proved not to be in $\overline{Ch}_{BP}(BD(n))$ in Section 6. In Section 7 the element x in (1.4) is proved to be a permanent cycle in the Atiyah-Hirzebruch spectral sequence for $n = 3, 4$ by comparing the spectral sequence to the corresponding spectral sequence for $H^*(B\mathrm{Spin}(2n+1))$. The last section gives more examples of p-torsion elements in the kernel of the cycle map, using spinor groups and the exceptional group F_4.

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2. Extraspecial 2-groups

The extraspecial 2-group $D(n) = 2_+^{1+2n}$ is the central product of n copies of the dihedral group D_8 of order 8. So there is a central extension

$$0 \to N \longrightarrow D(n) \xrightarrow{\pi} V \to 0 \tag{2.1}$$

with $N \cong \mathbb{Z}/2$ and V elementary abelian of rank $2n$. Take a set of generators $c, \tilde{a}_1, \ldots, \tilde{a}_{2n}$ of $D(n)$ such that c is a generator of N, the elements $a_i = \pi(\tilde{a}_i)$ form a $\mathbb{Z}/2$-basis of V, and

$$[\tilde{a}_j, \tilde{a}_{2i}] = \begin{cases} c & \text{if } j = 2i-1, \\ 0 & \text{else.} \end{cases}$$

Using the Hochschild-Serre spectral sequence associated to extension (2.1), Quillen [Q] determined the mod 2 cohomology of $D(n)$. Let e_i denote the real 1-dimensional representation of $D(n)$ given as the projection onto $\langle a_i \rangle$ followed by the nontrivial character $\langle a_i \rangle \to \{\pm 1\} \subset \mathbb{R}$, and $e : \tilde{V}^{odd} \to N \to \{\pm 1\} \subset \mathbb{R}$, where $\tilde{V}^{odd} = \langle c, \tilde{a}_{2i-1} \mid 1 \le i \le n \rangle$ is a maximal elementary abelian 2-subgroup of $D(n)$. Define classes $x_i \in H^1(D(n); \mathbb{Z}/2)$, $w_{2^n} \in H^{2^n}(D(n); \mathbb{Z}/2)$ as the Euler classes of the e_i and of $\Delta = \mathrm{Ind}_{\tilde{V}^{odd}}^{D(n)}(e)$, respectively. The extension (2.1) is represented by the class $f = x_1 x_2 + \cdots + x_{2n-1} x_{2n}$, and one has

$$H^*(BD(n); \mathbb{Z}2) \cong \mathbb{Z}/2[w_{2^n}] \otimes \mathbb{Z}2[x_1, \ldots, x_{2n}]/(f, Q_0 f, \ldots, Q_{n-2} f), \tag{2.2}$$

where the Q_i are Milnor's operations recursively defined by $Q_0 = Sq^1$ and $Q_i = [Sq^{2^i}, Q_{i-1}]$. The extension class f defines a quadratic form $q : V \to \mathbb{Z}/2$ on V. A subspace $W \subset V$ is said to be q-isotropic if $q(x) = 0$ for all $x \in W$. The maximal (elementary) abelian subgroups of $D(n)$ are in one-to-one correspondence with the maximal isotropic subspaces of V. Indeed, if W is maximal isotropic, then $\tilde{W} := \pi^{-1}(W) \cong N \oplus W$ is maximal (elementary) abelian. Quillen also proved that the mod 2 cohomology of $D(n)$ is detected on maximal elementary abelian subgroups, i.e. the restrictions define an injective map

$$H^*(BD(n); \mathbb{Z}/2) \hookrightarrow \prod H^*(\tilde{W}; \mathbb{Z}/2), \tag{2.3}$$

where the product ranges over conjugacy classes of maximal elementary abelian subgroups. Since the restriction of Δ to any such $\tilde{W}$ is the real regular representation (see [Q], Section 5), we have

$$\mathrm{Res}_{\tilde{W}}(w_{2^n}) = \prod_{x \in H^1(W; \mathbb{Z}/2)} (z + x), \tag{2.4}$$

where z denotes the generator of $H^*(N;\mathbb{Z}/2)$ dual to c. For simplicity, write $w' = \operatorname{Res}_{\tilde{W}} w_{2^n}$, and choose generators of $H^*(W;\mathbb{Z}/2) \cong \mathbb{Z}/2[x'_1, \dots, x'_n]$. It is well-known that the right hand side of (2.4) can be written in terms of Dickson invariants,

$$w' = z^{2^n} + D_1 z^{2^{n-1}} + \cdots + D_n z, \tag{2.5}$$

where D_i has degree $2^n - 2^{n-i}$ and $H^*(W;\mathbb{Z}/2)^{\mathrm{GL}_2(\mathbb{Z}/2)} \cong \mathbb{Z}/2[D_1, \dots, D_n]$. Using that the product of all the x'_i's is clearly invariant and that the Milnor primitives are derivations, it is easy to see that the Dickson invariants may be written in terms of the Q_i as follows:

$$\begin{array}{rcl} D_n & = & Q_0 Q_1 \dots Q_{n-2}(x'_1 \cdots x'_n), \\ D_i & = & (Q_0 \dots \hat{Q}_{n-i-1} \dots Q_{n-1}(x'_1 \cdots x'_n))/D_n. \end{array} \tag{2.6}$$

Lemma 2.1. *The Milnor operations $Q_1, \dots, Q_{n-1}$ act by*

(1) $Q_{n-1} D_i = D_n D_i$;
(2) $Q_{n-j-1} D_j = D_n$;
(3) $Q_i D_j = 0$ for $i < n-1$ and $i \neq n-j-1$.

Proof. First note that from (2.6) and $Q_k^2 = 0$ we immediately get $Q_k(D_n) = 0$ for $k \neq n-1$, and $Q_{n-1} D_n = Q_0 \dots Q_{n-1}(x'_1 \cdots x'_n) = D_n^2$. Thus, for each $1 \leq i \leq n-1$,

$$\begin{array}{rcl} 0 & = & Q_{n-1}(Q_0 \dots \hat{Q}_{n-i-1} \dots Q_{n-1})(x'_1 \cdots x'_n) = Q_{n-1}(D_i D_n) \\ & = & (Q_{n-1} D_i) D_n + D_i Q_{n-1} D_n = (Q_{n-1} D_i) D_n + D_i D_n^2, \end{array}$$

whence (1). Similarly, (2) is implied by

$$\begin{array}{rcl} D_n^2 & = & Q_{n-1} \dots Q_0(x'_1 \cdots x'_n) = Q_{n-i-1}(D_i D_n) \\ & = & (Q_{n-i-1} D_i) D_n + D_i Q_{n-i-1} D_n = (Q_{n-i-1} D_i) D_n\,. \end{array}$$

Finally, for $k \neq n-i-1$ we get $0 = Q_k(D_i D_n) = (Q_k D_i) D_n + D_i Q_k D_n = (Q_k D_i) D_n$. $\square$

Corollary 2.2. *$Q_{n-1} w' = D_n w'$ and $Q_k w' = 0$ for $k < n-1$.*

Proof. For $j \neq n-1$, we have $Q_j w' = \sum_{i=1}^{n-1} (Q_j D_i) z^{2^{n-i}} + Q_j(D_n z) = D_n z^{2^{j+1}} + D_n z^{2^{j+1}} = 0$. For $j = n-1$, we get $Q_{n-1} w' = 0 + D_n D_1 z^{2^{n-1}} + \cdots + D_n D_{n-1} z^2 + Q_{n-1}(D_n z)$. The last term equals $D_n^2 z + D_n z^{2^n}$; the claim follows. $\square$

Corollary 2.3. *$Q_k w_{2^n} = 0$ for $0 \leq k \leq n-2$, but $Q_{n-1} w_{2^n} \neq 0$.* $\square$

For future reference we note:

Lemma 2.4. *$Q_n(D_n w') = D_n^2 {w'}^2$, $Q_n(D_{n-1} D_n) = D_n^4$, $Q_n(D_{n-1} w') = D_n^3 w' + D_n D_{n-1} {w'}^2$, $Q_{n+1} Q_n(D_{n-1} w') = D_n^4 {w'}^4$.*

Proof. From (2.5) we see that $Q_n(D_n w') = (Q_n D_n) z^{2^n} + \sum_{i=1}^{n-1} Q_n(D_n D_i) z^{2^{n-i}} + D_n^2 z^{2^{n+1}}$. The coefficients of $z^{2^{n+1}}$ and z tell us that this is equal to $D_n^2 {w'}^2$. Comparing coefficients further shows that $Q_n(D_n) = D_1^2 D_n^2$ and $Q_n(D_n D_i) = D_n^2 D_{i+1}^2$; in particular, $Q_n(D_n D_{n-1}) = D_n^4$. Thus we have

$$\begin{array}{rcl} Q_n(D_n D_{n-1} D_n w') & = & Q_n(D_n D_{n-1}) D_n w' + D_n D_{n-1} Q_n(D_n w') \\ & = & D_n^5 w' + D_n^3 D_{n-1} {w'}^2\,. \end{array}$$

Hence we get $Q_n(D_{n-1}w') = D_n^3 w' + D_n D_{n-1} {w'}^2$. Next consider

$$Q_{n+1}(D_n w') = (Q_{n+1}D_n)z^{2^n} + \sum_{i=1}^{n-1} Q_{n+1}(D_n D_i) z^{2^{n-i}} + D_n^2 z^{2^{n+2}}.$$

From the coefficients of $z^{2^{n+2}}$ and $z^{2^{n+1}}$, we see that $Q_{n+1}(D_n w') = D_n^2 {w'}^4 + D_n^2 D_1^4 {w'}^2$ and hence $Q_{n+1}(D_n D_{n-1}) = D_n^4 D_1^4$. Therefore

$$Q_{n+1}(D_n D_{n-1} D_n w') = D_n^4 D_1^4 D_n w' + D_n D_{n-1}(D_n^2 {w'}^4 + D_n^2 D_1^4 {w'}^2).$$

Thus $Q_{n+1}(D_{n-1}w') = D_n^3 w' D_1^4 + D_n D_{n-1} {w'}^4 + D_n D_{n-1} D_1^4 {w'}^2$. Hence

$$\begin{aligned} Q_{n+1}Q_n(D_{n-1}w') &= Q_n Q_{n+1}(D_{n-1}w') \\ &= D_n^4 {w'}^2 D_1^4 + D_n^4 {w'}^4 + D_n^4 D_1^4 {w'}^2 = D_n^4 {w'}^4. \end{aligned}$$

□

3. The integral cohomology

The integral cohomology of $D(n)$ is studied by Harada and Kono ([HK]; also see [BC]) by means of the Bockstein spectral sequence

(3.1) $$E_1 = H^*(BG; \mathbb{Z}/2) \Longrightarrow \mathbb{Z}/2 \otimes H^*(BG)/(2\text{-torsion}).$$

The E_2-page of this spectral sequence is the Q_0-homology of $H^*(BG;\mathbb{Z}/2)$, and $E_\infty \cong \mathbb{Z}/2$ for a finite group G. For $0 \le i \le n-2$, let

$$R(i) = H^*(BV;\mathbb{Z}/2)/(f, Q_0 f, \dots, Q_i f).$$

Using the long exact sequence associated to the short exact sequence

(3.2) $$0 \to R(i-1) \xrightarrow{Q_i f} R(i-1) \longrightarrow R(i) \to 0,$$

Harada and Kono computed the E_2-page for $D(n)$ as follows:

(3.3) $$H(H^*(BD(n);\mathbb{Z}/2); Q_0) \cong \Lambda(a, b_1, \dots, b_{n-1}) \otimes \mathbb{Z}/2[w_{2^n}],$$

where $|a| = 3$ and $|b_i| = 2^i$. Since $E_\infty \cong \mathbb{Z}/2$, the first non-trivial differential must be $da = b_1$, and there have to be subsequent differentials $d(ab_i) = b_{i+1}$. Thus there appear exactly n non-zero differentials in this spectral sequence. On the other hand, using corestriction arguments it is easy to see that the exponent of $H^*(BD(n))$ is at most $n+1$. Based on these facts, Harada and Kono proved the following.

Theorem 3.1 ([HK])**.** *Let $C(n)^* = H^*(BD(n))/J_V$, where J_V is the ideal generated by the image of $H^*(BV)$ in $H^*(BD(n))$. Then $C(n)^* \subset H^*(BD(n))$, and there is an additive isomorphism*

$$C(n)^k = \begin{cases} \mathbb{Z}/2^{\nu_2(k)} & \text{if } \nu_2(k) \le n-1, \\ \mathbb{Z}/2^{n+1} & \text{if } \nu_2(k) = n, \end{cases}$$

where $\nu_2(k)$ denotes the 2-adic valuation of k. □

Let $c_k(n)$ denote a $\mathbb{Z}_{(2)}$-module generator of $C(n)^{2^k}$. Then $c_n(n)$ reduces to w_{2^n} modulo $H^*(BV;\mathbb{Z}/2)$. Consider the restriction map $i : C(n)^* \to C(n-1)^*$. Now $c_{n-1}(n-1) = w_{2^{n-1}} \mod H^*(BV;\mathbb{Z}/2)$ implies $i^* c_n(n) = c_{n-1}(n-1)^2$. Since the order of $c_{n-1}(n)$ is 2^{n-1} and the order of $c_{n-1}(n-1)$ is 2^n, we know that $i^* c_{n-1}(n) = 2^s c_{n-1}(n-1)$ for some $s > 0$. A corestriction argument now implies $s = 1$, since the index of $D(n-1)$ in $D(n)$ is 2.

The elements a and b_j are natural in the sense that $i^*(a) = a$ and $i^*(b_j) = b_j$ for $1 \leq j \leq n-2$, abusing notation. Thus $i^*c_j(n) = c_j(n-1)$ for $j < n-1$, and we obtain

Corollary 3.2. *If $n \geq 2$, there is an additive isomorphism*

$$C(n)^* \cong \mathbb{Z}\{1, 2\bar{w}_{2^2}^{\epsilon_2} \cdots \bar{w}_{2^{n-1}}^{\epsilon_{n-1}} \mid \epsilon_i = 0 \text{ or } 1\}[\bar{w}_{2^n}]/(2^{i+1}\bar{w}_{2^i} = 0 \mid 2 \leq i \leq n),$$

where the $\bar{w}_{2^i}$ are the reductions of the elements w_{2^i} in $H^{2^i}(BD(i))$. □

Remark. When $n = 1$, the element $w_2 \in H^*(BD_8; \mathbb{Z}/2)$ does not lift to the integral cohomology and $C(1)^* \cong \mathbb{Z}[\bar{w}_2^2]/(4\bar{w}_2^2)$.

4. Brown-Peterson cohomology of $BD(n)$

Let $BP^*(-; \mathbb{Z}/2)$ denote BP-theory mod 2 with coefficients

$$BP^*/(2) = \mathbb{Z}/2[v_1, v_2, \dots].$$

We consider the Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*} = H^*(BD(n); \mathbb{Z}/2) \otimes BP^* \Longrightarrow BP^*(BD(n); \mathbb{Z}/2)\,. \tag{4.1}$$

Lemma 4.1. *The elements x_i^2 and $w_{2^n}^2$ are permanent cycles in the spectral sequence* (4.1).

Proof. These elements are the top Chern classes of the representations $e_{i\mathbb{C}}$ and $\Delta_{\mathbb{C}}$, respectively. □

It is well-known that some of the differentials of (4.1) are given by

$$d_{2^{i+1}-1}(x) = v_i \otimes Q_i x \mod (v_1, \dots, v_{i-1})\,. \tag{4.2}$$

Since $Q_{n-1}w_{2^n} \neq 0$ by Corollary 2.3, we know that w_{2^n} cannot be a permanent cycle, which implies $w_{2^n} \notin \mathrm{Im}[\rho_{\mathbb{Z}/2} : BP^*(BD(n)) \to H^*(BD(n); \mathbb{Z}/2)]$. Thus the integral lift $\bar{w}_{2^n}$ of w_{2^n} does not lie in the image of $\rho : BP^*(BD(n)) \to H^*(BD(n))$, either.

As above, let $\tilde{W}$ denote a maximal elementary abelian subgroup of $D(n)$, and $w(\Delta)$ the total Stiefel-Whitney class of Δ. Then

$$\begin{aligned} \mathrm{Res}_{\tilde{W}}^{D(n)}(w(\Delta)) &= \prod(1+x+z) = (1+z)^{2^n} + D_1(1+z)^{2^{n-1}} + \cdots + D_n(1+z) \\ &= 1 + D_1 + \cdots + D_n + \mathrm{Res}_{\tilde{W}}^{D(n)}(w_{2^n})\,; \end{aligned}$$

in particular,

$$\mathrm{Res}_{\tilde{W}}^{D(n)}(w_{2^n-2^{n-i}}(\Delta)) = D_i\,. \tag{4.3}$$

Hence, by (2.2), we can choose polynomials $\tilde{D}_i \in \mathbb{Z}/2[x_1, \dots, x_{2n}] \cong H^*(BV; \mathbb{Z}/2)$ with $w_{2^n-2^{n-i}} = \tilde{D}_i$. Recall that J_V denotes the image of $H^*(BV)$ in $H^*(BD(n))$.

Theorem 4.2. *There is an element in $BP^*(BD(n))$,*

$$x = c_{2^{n-1}}(\Delta_{\mathbb{C}}) - \tilde{D}_1(c_1(e_{1\mathbb{C}}), \dots, c_1(e_{2n\mathbb{C}})),$$

which is non-zero in $BP^*(BD(n)) \otimes_{BP^*} \mathbb{Z}/2$, and such that

(1) $\rho(x) = 2\bar{w}_{2^n} \mod J_V$,
(2) $\rho_{\mathbb{Z}/2}(x) = 0$ *in $H^*(BD(n); \mathbb{Z}/2)$.*

Proof. Since x is defined via Chern classes, it is an element of $BP^*(BD(n))$. Assertion (2) is immediate from (4.3). Since $\bar{w}_{2^n} \notin \mathrm{Im}(\rho)$, to prove (1) it suffices to show that x is a BP^*-module generator. Let $F = \langle a_1 a_2 \rangle \subset D(n)$; this is cyclic of order 4. By the double coset formula,

$$\begin{aligned}
\mathrm{Res}_F^{D(n)} \mathrm{Ind}_{\tilde{V}^{odd}}^{D(n)}(e_{\mathbb{C}}) &= \bigoplus_{Fg\tilde{V}^{odd}} \mathrm{Ind}^F_{F\cap g^{-1}\tilde{V}^{odd}g} \mathrm{Res}^{g^{-1}\tilde{V}^{odd}g}_{F\cap g^{-1}\tilde{V}^{odd}g}(g^* e_{\mathbb{C}}) \\
&= \bigoplus^{2^{n-1}} \mathrm{Ind}_N^F(e_{\mathbb{C}}),
\end{aligned}$$

since the elements $g = a_2^{\epsilon_2} \cdots a_{2n}^{\epsilon_{2n}}$, $\epsilon_i = 0$ or 1, form a complete set of double coset representatives. Notice that $\mathrm{Ind}_N^F(e_{\mathbb{C}})$ decomposes as $e_F \oplus -e_F$, where e_F is a faithful 1-dimensional complex representation of $\mathbb{Z}/4$. Thus the total Chern class of $\Delta_{\mathbb{C}}$ restricts to F as

$$\mathrm{Res}_F(c(\Delta_{\mathbb{C}})) = ((1+u)(1-u))^{2^{n-1}} = (1-u^2)^{2^{n-1}} \quad \text{with } H^*(BF) \cong \mathbb{Z}[u]/(4u)\,.$$

Consequently, we have $\mathrm{Res}_F(c_{2^{n-1}}(\Delta_{\mathbb{C}})) = 2u^{2^{n-1}}$ in $H^*(F)$. Since $\mathrm{Res}_F(c_1(e_{i\mathbb{C}})) = 2\lambda_i u$ for some $\lambda_i \in \mathbb{Z}/4$, we immediately obtain $\mathrm{Res}_F(\tilde{D}_i) = 0$, and therefore (1). □

Now recall the following lemma of Totaro.

Lemma 4.3 ([T1])**.** *Let p be a prime and X any space. If $\rho_{\mathbb{Z}/p} : BP^*(X) \otimes_{BP^*} \mathbb{Z}_{(p)} \to H^*(X;\mathbb{Z}/p)$ is not injective, then $\rho : BP^{*+2}(X \times B\mathbb{Z}/p) \otimes_{BP^*} \mathbb{Z}_{(p)} \to H^{*+2}(X \times B\mathbb{Z}/p)$ is also not injective.* □

Proof. We have $BP^*(B\mathbb{Z}/p) \otimes_{BP^*} \mathbb{Z}_{(p)} \cong H^*(B\mathbb{Z}/p)_{(p)} \cong \mathbb{Z}_{(p)}[u]/(pu)$ with u in degree two. If $\rho_{\mathbb{Z}/p}(x) = 0$, then $\rho(x \otimes u) = 0$ in $H^*(X \times B\mathbb{Z}/p)_{(p)}$. On the other hand, it is well-known that $BP^*(B\mathbb{Z}/p)$ is BP^*-flat and thus $BP^*(X \times B\mathbb{Z}/p) \cong BP^*(X) \otimes_{BP^*} BP^*(B\mathbb{Z}/p)$. Hence if $0 \neq x \in BP^*(X)_{BP^*}\mathbb{Z}_{(p)}$, then $x \otimes u$ is also non-zero in $BP^*(X \times B\mathbb{Z}/p) \otimes_{BP^*} \mathbb{Z}_{(p)}$. □

Let $\rho' : CH^*(-) \to H^*(-)$ denote the cycle map, and $\rho'_{\mathbb{Z}/2}$ the cycle map followed by reduction modulo 2. Since Chow rings have Chern classes, we easily deduce

Corollary 4.4. *There is a non-zero element x' in $CH^{2^n}(BD(n))$ satisfying*

1. $\rho'(x') = 2\bar{w}_{2^n} \mod J_V$*;*
2. $\rho'_{\mathbb{Z}/2}(x') = 0$*.*

Hence $\rho' : CH^{2^n+2}(B(D(n) \times \mathbb{Z}/2)) \to H^{2^n+2}(B(D(n) \times \mathbb{Z}/2))$ is not injective. □

Remark. First note that the above argument does not hold for $n = 1$. Indeed, in that case $H^*(BD_8) \subset \mathrm{Im}(\rho)$ modulo $H^*(BV)$. Similar facts hold for 2-groups G which have a cyclic maximal normal subgroup [S], i.e. dihedral, semidihedral, quasidihedral, and generalized quaternion groups of order a power of 2. Moreover, $BP^*(BG)$ is generated by Chern classes for these groups. The extraspecial 2-groups of order 2^{2n+1} are of two types. Quillen calls them the real and the quaternionic type, where the real type corresonds to the groups $D(n)$ considered above, and the quaternionic group of order 2^{n+1} is the central product of $D(n-1)$ with the quaternion group Q_8 of order 8. Consider now this second case, and denote this group by $D'(n)$; it also has center $\mathbb{Z}/2$ with quotient $V \cong (\mathbb{Z}/2)^{2n}$. In Quillen's notation [Q], this corresponds to $h = n+1$ and $r = 2$. The quadratic form (extension class) is $f = x_1^2 + x_1x_2 + x_2^2 + \sum_{i=2}^n x_{2i-1}x_{2i}$, and the cohomology is given by

$$H^*(BD'(n);\mathbb{Z}/2) \cong \mathbb{Z}/2[w_{2^{n+1}}] \otimes \mathbb{Z}/2[x_1 \dots , x_{2n}]/(f, Q_0f, \dots , Q_{n-1}f)\,.$$

Here the x_i are as before the generators of $H^*(BV;\mathbb{Z}/2)$ inflated to $D'(n)$, and $w_{2^{n+1}}$ is the Euler class of the 2^{n+1}-dimensional irreducible representation Δ. The cohomology of $D'(n)$ is also detected on subgroups $\tilde{W} \cong Q_8 \times W$ in one-to-one correspondence with maximal isotropic subspaces, i.e. there is an injection

$$H^*(BD'(n);\mathbb{Z}/2) \hookrightarrow \prod_W H^*(B(Q_8 \times W);\mathbb{Z}/2),$$

where W ranges over the maximal isotropic subspaces of V (which have dimension $n-1$). The Stiefel-Whitney classes $w_j(\Delta)$ are zero except for the following values of j ([Q], (5.6)):

$$\mathrm{Res}_{Q_8\times W}(w_j(\Delta)) = \begin{cases} (D_i')^4 & \text{for } j = 2^h - 2^{h-i},\ 1 \le i \le n-1, \\ \sum_{i=0}^{n-2} e^{2^i}(D'_{n-i-1})^4 & \text{for } j = 2^{n+1}, \end{cases}$$

where $e \in H^4(Q_8;\mathbb{Z}/2)$ is the Euler class of the obvious 4-dimensional irreducible representation of Q_8, and D_i' is the degree $(2^{n-1} - 2^{n-1-i})$ Dickson invariant for rank $n-1$. Thus almost all arguments for $D(n)$ work in this case, too, except for $Q_m w_j(\Delta) = 0$. For example, we can define $x = c_{2^n}(e_{\mathbb{C}}) - (\tilde{D}_1')^4$ in $BP^*(BD'(n))$; this class satisfies $\rho(x) = 2\bar{w}_{2^{n+1}}$ and $\rho_{\mathbb{Z}/2}(x) = 0$. However, it seems that we cannot prove that x is a BP^*-module generator of $BP^*(BD'(n))$, because $\mathrm{Res}_N(c_{2^n}(\mathrm{Ind}_{\mathbb{Z}/4\oplus W}^{D'(n)}(e_F))) = u^{2^n}$ and $w_{2^{n+1}}(\Delta) \in \mathrm{Im}(\rho) \mod (H^*(BV))$.

5. Permanent cycles

This section deals with the Atiyah-Hirzebruch spectral sequence converging to $BP^*(BD(n))$. In the course of the section, we shall make several technical assumptions on the behaviour of this spectral sequence. These will be verified for $n = 3, 4$ in Section 7.

Given a space X, each non-zero element $x \in BP^*(X)$ with $\rho(x) = 0 \in H^*(X)_{(p)}$ is represented by a non-zero element in $E_\infty^{*;a}$ with $a < 0$ in the Atiyah-Hirzebruch spectral sequence converging to $BP^*(X)$.

Assumption 5.1. *Let $n \ge 3$. In the Atiyah-Hirzebruch spectral sequence converging to $BP^*(BD(n))$, every nonzero element in the ideal $(2, v_1, \dots , v_{n-2}) \otimes \bar{w}_{2^n}$ is a nonzero permanent cycle.*

The outer automorphism group of $D(n)$ is the orthogonal group $O(V)$ of V associated to the quadratic form q ([BC], p. 216). Since Δ is the unique irreducible representation which acts non-trivially on the center, the element w_{2^n} is invariant under the orthogonal group ([Q], Remark 4.7). Moreover, the invariant ring generated by the Stiefel-Whitney classes of Δ ([Q], Corollary 5.12 and Remark 5.14),

$$H^*(BD(n);\mathbb{Z}/2)^{O(V)} = \mathbb{Z}/2[\tilde{D}_1, \dots , \tilde{D}_n, w_{2^n}] \quad \text{with} \quad \tilde{D}_i = w_{2^n-2^{n-i}}(\Delta)\,.$$

To consider the Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*}(X) = H^*(X) \otimes BP^* \Longrightarrow BP^*(X) \tag{5.1}$$

for the spaces $X = BD(n)$ or $B\mathrm{Spin}(m)$, we need the integral version of the above invariant ring. Note that $\beta(\tilde{D}_{n-1}) = \tilde{D}_n$; let

$$W(\Delta) = \mathbb{Z}_{(2)}[\tilde{D}_1, \dots, \tilde{D}_{n-2}, \tilde{D}_{n-1}^2, \tilde{D}_n, w_{2^n}]/(2\tilde{D}_n)\,.$$

Suppose X is a space such that

$$\begin{array}{l}\text{there is a map } f : W(\Delta) \to H^*(X)_{(2)} \text{ such that} \\ W(\Delta)/(2) \subset H^*(X;\mathbb{Z}/2) \text{ as } \Lambda(Q_0, \dots, Q_n)\text{-algebras.}\end{array} \tag{5.2}$$

In Section 7 we shall see that we may take $X = B\mathrm{Spin}(7)$ and $X = B\mathrm{Spin}(9)$ for the cases $n = 3$ and $n = 4$, respectively. In the spectral sequence for $BD(n)$, let $W(\Delta)_r$ be the subalgebra of $E_r^{*,*}$ which is the subquotient algebra of $BP^* \otimes f(W(\Delta))$. In general, the invariants $(E_r^{*,*})^{O(V)}$ are not equal to $W(\Delta)_r$. Below we consider the case were $W(\Delta)_r$ is nevertheless closed under the differentials.

Lemma 5.2. *Let $n \geq 3$. Suppose that X satisfies* (5.2) *and $d_r(W(\Delta)_r) \subset W(\Delta)_r$ for all $r \geq 2$ in the spectral sequence* (5.1). *Then each element in the ideal $(2, v_1, \dots, v_{n-2}) \otimes w_{2^n}$ and the ideal $(2, v_1, \dots, v_{n-2-i}) \otimes \tilde{D}_i$, $1 \leq i \leq n-2$, is a permanent cycle. Moreover, if w_{2^n} (resp. $\tilde{D}_i$) is not in the image of Q_k, then $[v_k \otimes w_{2^n}]$ (resp.$[v_k \otimes \tilde{D}_i], i \leq n-k-2$) is a non-zero element in $E_\infty^{*,*}$.*

Before beginning with the proof, recall the cohomology theory $P(m)^*(-)$ with coefficients $BP^*/(2, v_1, \dots, v_{m-1}) \cong \mathbb{Z}/2[v_m, v_{m+1}, \dots]$. In particular, the theory $P(1)^*(-)$ is mod 2 BP-theory $BP^*(-;\mathbb{Z}/2)$.

Proof. First note that $|\tilde{D}_i| = 2^n - 2^{n-i}$, which is even except for the case $i = n$. Hence

$$W(\Delta)_2^{odd} = P(1)^*[w_{2^n}, \tilde{D}_1, \dots, \tilde{D}_{n-2}, \tilde{D}_{n-1}^2, \tilde{D}_n^2]\{\tilde{D}_n\}\,.$$

By induction, we assume that for $2^r \leq i \leq 2^{r+1} - 1$

$$\begin{array}{c} W(\Delta)_i^{odd} = P(r)^* \otimes A \otimes B_r \otimes C_r\{\tilde{D}_n\} \quad \text{with} \\ A = \mathbb{Z}/2[w_{2^n}]\,, B_r = \mathbb{Z}/2[\tilde{D}_1, \dots, \tilde{D}_{n-r-1}]\,, C_r = \mathbb{Z}/2[\tilde{D}_{n-r}^2, \dots, \tilde{D}_n^2]\,.\end{array} \tag{5.3}$$

Let $E(P(m))_i^{*,*}$ denote the Atiyah-Hirzebruch spectral sequence converging to $P(m)^*(X)$, and let $\rho_i : E_i^{*,*} \to E(P(m))_i^{*,*}$ be the map of spectral sequences induced from the natual transformation $\rho : BP^*(X) \to P(m)^*(X)$ of cohomology theories. Since $|v_r| = -2^{r+1}+2$, we see that $E(P(r))_{2^{r+1}-1}^{*,*} \cong E(P(r))_2^{*,*}$ for degree reasons. Now $W(\Delta)_i^{odd}$ is $P(r)^*$-free, so the restriction map $\rho_i|W(\Delta)_i$ is injective for $i < 2^{r+1}$. Hence there is no element x with $0 \neq d_i(x) \in W(\Delta)_i^{odd}$, and we have $W(\Delta)_{2^r}^{odd} \cong W(\Delta)_{2^{r+1}-1}^{odd}$. Except for $\tilde{D}_{n-r-1}$, generators in $W(\Delta)_i$ are annihilated by Q_r and thus by $d_{2^{r+1}-1}$. The non-zero differential is

$$d_{2^{r+1}-1}(\tilde{D}_{n-r-1}) = v_r \otimes Q_r(\tilde{D}_{n-r-1}) = v_r \otimes \tilde{D}_n.$$

Therefore (5.3) also holds for $i = 2^{r+1}$. Since $W(\Delta)_i$ is a $P(r)^*$-module, every element in the ideal $(2, \dots, v_{r-1}) \otimes \tilde{D}_{n-r-1}$ is a cycle in $E_{2^{r+1}}^{*,*}$. Consequently we get

$$W(\Delta)_{2^n-1}^{odd} = P(n-1)^* \otimes A \otimes C_{n-1}\{\tilde{D}_n\}\,.$$

The next differential is

$$d_{2^n-1}(w_{2^n}) = v_{n-1} \otimes w_{2^n}\tilde{D}_n \quad \text{and} \quad d_{2^n-1}(\tilde{D}_n) = v_{n-1} \otimes \tilde{D}_n^2\,.$$

Hence we have

$$W(\Delta)^{odd}_{2^{n+1}-1} = P(n)^* \otimes C_n\{\tilde{D}_n w_{2^n}\}, \quad \text{where} \quad C_n = \mathbb{Z}/2[w_{2^n}^2] \otimes C_{n-1}\,.$$

Here we note that each element in the ideal $(2, v_1, \dots, v_{n-2}) \otimes w_{2^n}$ is a cycle in $E^{*,*}_{2^n}$, because $w_{2^n}\tilde{D}_n$ generates a $P(n-1)^*$-module in $E^{*,*}_{2^n-1}$.

Finally, we consider the differential

$$d_{2^{n+1}-1}(w_{2^n}\tilde{D}_n) = v_n \otimes Q_{n+1}(w_{2^n}\tilde{D}_n) = v_n \otimes (w_{2^n}^2\tilde{D}_n^2)\,.$$

Thus $W(\Delta)^{odd}_{2^{n+1}} = 0$, and each element in the above ideals is a permanent cycle. If $w_{2^r} \notin \mathrm{Im}(Q_r)$, then, considering the map ρ_{2^r-1}, we see that $[v_r \otimes w_{2^n}]$ is nonzero. □

Next we consider the mod 2 version of the above arguments. We study the Atiyah-Hirzebruch spectral sequence

(5.4) $$E_2 = H^*(X; \mathbb{Z}/2) \otimes P(1)^* \Longrightarrow P(1)^*(X)\,.$$

Denote the invariant ring by

$$W(\Delta; \mathbb{Z}/2) = H^*(BD(n); \mathbb{Z}/2)^{O(V)} \cong W(\Delta)/(2) \otimes \Lambda(D_{n-1})\,.$$

We consider the following situation:

(5.5) there is an injection $W(\Delta; \mathbb{Z}/2) \subset H^*(X; \mathbb{Z}/2)$ as $\Lambda(Q_0, \dots Q_{n+1})$-algebras.

In the spectral sequence (5.4), let $W(\Delta; \mathbb{Z}/2)_r$ be the subalgebra of $E_r^{*,*}$ which is the subquotient algebra of $P(1)^* \otimes (W(\Delta); \mathbb{Z}/2)$. Of course the mod 2 reductions of the permanent cycles in (5.1) are also permanent cycles in (5.4). Moreover we have

Lemma 5.3. *Let $n \geq 3$. Suppose that X satisfies* (5.5) *and $d_r(W(\Delta; \mathbb{Z}/2)_r) \subset W(\Delta; \mathbb{Z}/2)_r$ for all $r \geq 2$ in the spectral sequence (5.4). Then every element in the ideal $(v_1, \dots, v_{n-1}) \otimes w_{2^n}\tilde{D}_{n-1}$ or $(v_1, \dots, v_{n-2}) \otimes \tilde{D}_{n-1}$ is a permanent cycle.*

Proof. The proof is similar to the BP^*-case. In particular, for $i \leq 2^n - 1$, we have $W(\Delta; \mathbb{Z}/2)^{odd}_i = W(\Delta)^{odd}_i/(2) \otimes \Lambda(D_{n-1})$. The difference starts with

$$d_{2^n-1}(x) = v_{n-1} \otimes \tilde{D}_n x \quad \text{for } x = w_{2^n}, \tilde{D}_{n-1}, \tilde{D}_n.$$

Hence we get

$$W(\Delta; \mathbb{Z}/2)^{odd}_{2^{n+1}-1} = P(n)^* \otimes C_n\{\tilde{D}_n w_{2^n}, \tilde{D}_n\tilde{D}_{n-1}\}\,.$$

Here note that $\tilde{D}_{n-1}w_{2^n}$ is a cycle in $E^{*,*}_{2^{n+1}-1}$. We also know that each element in the ideal $(v_1, \dots, v_{n-2}) \otimes \tilde{D}_{n-1}$ is a cycle in $E^{*,*}_{2^{n+1}-1}$.

From Lemma 2.4 and (2.3), the image of $w_{2^n}\tilde{D}_n$ (resp. $\tilde{D}_{n-1}\tilde{D}_n$) under the differential $d_{2^{n+1}}$ is $v_n \otimes (w_{2^n}^2\tilde{D}_n^2)$ (resp. $\tilde{D}_n^4$). Hence we see that

$$\mathrm{Ker}(W(\Delta; \mathbb{Z}/2)^{odd}_{2^{n+1}-1}) = P(n)^* \otimes C_n\{a\} \quad \text{where } a = \tilde{D}_{n-1}\tilde{D}_n w_{2^n}^2 + \tilde{D}_n^3 w_{2^n}\,.$$

Since $d_{2^{n+1}-1}(w_{2^n}\tilde{D}_{n-1}) = v_n \otimes a$, we get $W(\Delta; \mathbb{Z}/2)^{odd}_{2^{n+2}-1} = P(n+1)^* \otimes C_n\{a\}$. Here note that $v_{n-1} \otimes w_{2^n}\tilde{D}_n$ is a cycle in $E^{odd}_{2^{n+2}-1}$. The last nonzero differential is, again by Lemma 2.4,

$$d_{2^{n+2}-1}(a) = v_{n+1} \otimes \tilde{D}_n^4 w_{2^n}^4\,.$$

Thus $W(\Delta; \mathbb{Z}/2)^{odd}_{2^{n+2}} = 0$. Hence we get the permanency of elements in the lemma. □

If $BP^*(X)$ is 2-torsion free, e.g., $BP^*(X)$ and $P(1)^*(X)$ are generated by even dimensional elements, then the Bockstein exact sequence induces an isomorphism $BP^*(X)/(2) \cong P(1)^*(X)$. In particular, $BP^*(X) \otimes_{BP^*} \mathbb{Z}/2 \cong P(1)^*(X) \otimes_{P(1)^*} \mathbb{Z}/2$. These facts hold for $X = B\mathrm{Spin}(m)$ for $m = 7, 9$ (see Section 7 below). The following assertion seems reasonable for dimensional reasons.

Assumption 5.4. *Suppose that* (5.2) *holds and* $BP^*(X)/(2) \cong P(1)^*(X)$. *The element* $[2 \otimes \tilde{D}_{n-i-1}]$ *(resp.* $[2w_{2^n}]$*) in the spectral sequence* (5.1) *corresponds to the element* $[v_i \otimes \tilde{D}_{n-1}]$ *(resp.* $[v_{n-1} \otimes w_{2^n} D_{n-1}]$*) in the spectral sequence* (5.4)*, e.g. the element* x *with* $\rho_{\mathbb{Z}/2}(x) = 0$ *in Theorem 4.2 is represented by* $[v_{n-1} \otimes w_{2^n} D_{n-1}]$.

Remark. Let M_p be the Moore space such that $H^2(M_p) \cong \mathbb{Z}/p$. Then there is an isomorphism $P(1)^*(X) \cong BP^{*+2}(X \wedge M_p)$ if $BP^*(X)$ is p-torsion free. Hence we can deduce the behaviour of the Atiyah-Hirzebruch spectral sequence converging to $BP^*(D(n) \times B\mathbb{Z}/2)$ from that converging to $P(1)^*(BD(n))$.

6. Transfers of Chern classes

To study Chern classes, we consider the restriction to the center $N \cong \mathbb{Z}/2$ of $D(n)$. Let I denote the ideal $(2, v_1, v_2, \dots)$ in BP^*. Then

$$\rho_{\mathbb{Z}/2} : BP^*(BN)/I \cong \mathbb{Z}/2[z^2] \subset H^*(BN; \mathbb{Z}/2)\,.$$

Since the image of the restriction $H^*(BD(n); \mathbb{Z}/2) \to H^*(BN; \mathbb{Z}/2)$ is generated by $w_{2^n} \notin \mathrm{Im}(\rho_{\mathbb{Z}/2})$, we see that

$$\mathrm{Im}[BP^*(BD(n)) \to BP^*(BN)/I] = \mathbb{Z}/2[u^{2^n}]\,, \tag{6.1}$$

where u denotes the obvious generator in degree 2. Let ξ be a complex representation of $D(n)$; it restricts to N as the sum of m copies (say) of the nontrivial character $e_{\mathbb{C}}$ plus some trivial representations. Then there is an element $u' \equiv u$ mod I in $BP^*(BN)$ with

$$\mathrm{Res}_N(c(\xi)) = (1 + u')^m, \tag{6.2}$$

where $c(\xi)$ denotes as usual the total Chern class of ξ. Then u^m lies in the image of $BP^*(BD(n)) \to BP^*(BN)/I$, and so m has to be divisible by 2^n.

Proposition 6.1. *Suppose Assumption 5.1 holds and* $n \geq 3$. *Then the permanent cycles* $[v_1 \bar{w}_{2^n}], \dots, [v_{n-1} \bar{w}_{2^n}]$ *are not represented by* BP^**-linear combinations of products of Chern classes.*

Proof. Let ξ be a representation satifying (6.2) for some $m = 2^n m'$. The restriction of the total Chern class of ξ is given by

$$\begin{aligned} \mathrm{Res}_N(c(\xi)) &= 1 + 2m'(u')^{2^{n-1}} \quad \text{mod } (I^2, u^{2^n}) \\ &= 1 + m'(v_1 u^{2^{n-1}+1} + \cdots + v_i u^{2^{n-1}+2^i-1} + \cdots) \quad \text{mod } (I^2, u^{2^n}), \end{aligned}$$

which does not contain the term $v_i u^{2^{n-1}}$. But

$$\mathrm{Res}_N([v_i \bar{w}_{2^n}]) = v_i u^{2^{n-1}} \quad \text{mod } (u^{2^{n-1}+1})\,.$$

Hence no BP^*-linear combination of products of Chern classes can represent $[v_i \bar{w}_{2^n}]$. $\square$

Theorem 6.2. *Suppose Assumption 5.1 holds and* $n \geq 3$. *Then the permanent cycles* $[v_1 \bar{w}_{2^n}], \dots, [v_{n-2} \bar{w}_{2^n}]$ *are not represented by transfers of* BP^**-linear combinations of products of Chern classes.*

Proof. Let H be a subgroup of $D(n)$, and suppose $[v_j\bar{w}_{2^n}] = \mathrm{Tr}_H^{D(n)}(x)$ for some $x \in BP^*(BH)$. By the double coset formula,

$$\mathrm{Res}_N^{D(n)} \mathrm{Tr}_H^{D(n)}(x) = \sum_{HgN} \mathrm{Tr}^N_{g^{-1}Hg\cap N} \mathrm{Res}^{g^{-1}Hg}_{g^{-1}Hg\cap N}(g^*x), \tag{6.3}$$

where the sum ranges over double coset representatives g of $H\backslash D(n)/N$. If H intersects N trivially, then so does any conjugate of H. Hence we need only consider subgroups H containing the center, and the double coset formula evaluates to $|D(n)/H| \cdot \mathrm{Res}_N(x)$. Since this element is represented by

$$\mathrm{Res}_N[v_i\bar{w}_{2^n}] = v_i u^{2^{n-1}} \not\equiv 0 \mod I^2\,,$$

we get $|D(n)/H| = 2$, and thus $H \cong D(n-1) \times \mathbb{Z}/2$ or $H \cong D(n-1) \times_N \mathbb{Z}/4$.

The total Chern class $c(\zeta)$ of any representation ζ of $D(n-1)$ restricts as

$$\mathrm{Res}_N(c(\zeta)) = (1+u')^{2^{n-1}m} = 1 + mu^{2^{n-1}} \mod (I, u^{2^n})\,.$$

Hence we have

$$\begin{aligned}\mathrm{Res}_N(2c(\zeta)) = 2 + 2mu^{2^{n-1}} &= (v_1u^2 + \cdots + v_iu^{2^i} + \cdots)\\ &\quad + m(v_1u^{2^{n-1}+1} + \cdots + v_iu^{2^{n-1}+2^{i-1}} + \cdots) \mod (I^2, u^{2^n}),\end{aligned}$$

which does not contain $v_iu^{2^{n-1}}$. Thus $[v_j\bar{w}_{2^n}]$ is not represented by any BP^*-linear combination of products of Chern classes. □

7. $BP^*(B\mathrm{Spin}(7))$ and $BP^*(B\mathrm{Spin}(9))$

The mod 2 cohomology of $B\mathrm{Spin}(n)$ was computed by Quillen [Q]:

(7.1)

$$H^*(B\mathrm{Spin}(n);\mathbb{Z}/2) \cong \mathbb{Z}/2[w_{2^h}(\Delta)] \otimes \mathbb{Z}/2[w_2,\ldots,w_n]/(w_2, Q_0w_2,\ldots,Q_{h-1}w_2),$$

where Δ is a spin representation of $\mathrm{Spin}(n)$ and 2^h the Radon-Hurwitz number (see [Q], §6). This is proved by calculating the Serre spectral sequence of the fibration

$$B\mathbb{Z}/2 \longrightarrow B\mathrm{Spin}(n) \longrightarrow BSO(n)\,. \tag{7.2}$$

We consider the case $n = 7$. Then $h = 3$, and the mod 2 cohomology of $B\mathrm{Spin}(n)$ is a polynomial algebra on the Stiefel-Whitney classes w_4, w_6, w_7, w_8 of a spin representation, i.e.

$$H^*(B\mathrm{Spin}(7);\mathbb{Z}/2) \cong \mathbb{Z}/2[w_4, w_6, w_7, w_8]\,. \tag{7.3}$$

Recall that $\mathrm{Spin}(7)$ has the exceptional Lie group G_2 as a subgroup. G_2 contains a rank three elementary abelian 2-subgroup, and its mod 2 cohomology is isomorphic to the rank three Dickson invariants, i.e. $H^*(BG_2;\mathbb{Z}/2) \cong \mathbb{Z}/2[D_1, D_2, D_3]$. Here we may identify the Dickson invariants with the Stiefel-Whitney classes of the restriction of the spin representation to G_2, namely $D_1 = w_4$, $D_2 = w_6$, and $D_3 = w_7$. In particular, we have $H^*(B\mathrm{Spin}(7);\mathbb{Z}/2) \cong \mathbb{Z}/2[D_1, D_2, D_3] \otimes \mathbb{Z}/2[w_8]$.

Thus $H^*(B\mathrm{Spin}(7)) = W(\Delta)$, and the technical assumptions of Section 5 are satified with $X = B\mathrm{Spin}(7))$. Hence all results from that section hold in this case.

Indeed, the Brown-Peterson cohomology of $B\mathrm{Spin}(7)$ is given in [KY]. In the Atiyah-Hirzebruch spectral sequence converging to $BP^*(B\mathrm{Spin}(7))$, all non-zero differentials are of the form $d_{2^m-1} = v_{m-1} \otimes Q_{m-1}$:

$$d_3w_4 = v_1w_7\,,\quad d_7w_7 = v_2w_7^2\,,\quad d_7w_8 = v_2w_7w_8\,,\quad d_{15}(w_7w_8) = v_3w_7^2w_8^2\,.$$

Thus

$$\begin{aligned}E_\infty^{*,*} &= E_{16}^{*,*}\\ &\cong BP^*\{1, 2w_4, 2w_8, 2w_4w_8, v_1w_8\} \otimes A \oplus (P(3)^*[w_7^2]\{w_7^2\} \otimes A)/(v_3w_7^2w_8^2)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{with } A = \mathbb{Z}_{(2)}[w_4^2, w_6^2, w_8^2]\,.\end{aligned}$$

For the spectral sequence converging to $P(1)^*(B\mathrm{Spin}(7))$, the arguments from Section 5 give

$$\begin{aligned}E_\infty^{*,*} = E_{32}^{*,*} &\cong P(1)^*\{1, v_1w_6, v_1w_8, v_1w_6w_8, v_2w_6w_8\} \otimes A\\ &\oplus P(3)^*[w_7^2]\{w_7^2\} \otimes A/(v_3w_7^4, v_3w_7^2w_8^2, v_4w_7^4w_8^4)\,.\end{aligned}$$

Note that $BP^*(B\mathrm{Spin}(7))/(2)$ is isomorphic to $P(1)^*(B\mathrm{Spin}(7))$, which is also implied by the relations

$$2[w_7^2] + v_3[w_7^4] + \cdots = 0 \quad \text{and} \quad 2[w_7^2w_8^2] + v_4[w_7^4w_8^4] + \cdots = 0,$$

which follow from the fact that if $\sum v_ix_i = 0$ in $BP^*(X)$, then there exist classes $y \in H^*(X;\mathbb{Z}/p)$ with $\rho_{\mathbb{Z}/p}(x_i) = Q_iy$ ([Y1]).

Theorem 7.1. *The element $[v_1w_8]$ is not represented by a transfer of a BP^*-linear combination of products of Chern classes.*

Proof. This follows from Proposition 6.1 by looking at the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & \mathrm{Spin}(7) \arrow[r] & SO(7) \arrow[r] & 1\\ 0 \arrow[r] & \mathbb{Z}/2 \arrow[r] \arrow[u, "="] & D(3) \arrow[r] \arrow[u] & (\mathbb{Z}/2)^6 \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

whose rows are central extensions. $\square$

Similar arguments work for $\mathrm{Spin}(9)$ and $D(4)$; in this case the Radon-Hurwitz number is 16. The mod 2 cohomology is

$$H^*(B\mathrm{Spin}(9);\mathbb{Z}/2) \cong H^*(B\mathrm{Spin}(7);\mathbb{Z}/2) \otimes \mathbb{Z}/2[w_{16}]\,.$$

Since $D(4) \subset \mathrm{Spin}(9)$, the above cohomology ring contains the rank four Dickson algebra $\mathbb{Z}/2[D_1, \ldots, D_4]$. The invariant D_1 is equal to $w_8 + w_4^2$, from (2.5). Since $Sq^4D_1 = D_2$, we get $D_2 = w_8w_4 + w_6^2$. Similarly $D_3 = w_8w_6 + w_7^2$ and $D_4 = w_8w_7$. We consider the Atiyah-Hirzebruch spectral sequence converging to $P(1)^*(B\mathrm{Spin}(9))$. The odd degree part of $E_2^{*,*}$-page is additively

$$E_2^{*,odd} \cong P(1)^*[w_{16}] \otimes B \otimes \Lambda(w_4, w_6, w_8)\{w_7\} \quad \text{with} \quad B = \mathbb{Z}/2[w_4^2, w_6^2, w_7^2, w_8^2]\,.$$

Using the calculations in Section 5, we can compute

$$\begin{aligned}E_4^{*,odd} &\cong P(2)^*[w_{16}] \otimes B \otimes \Lambda(w_6, w_8)\{w_7\},\\ E_8^{*,odd} &\cong P(3)^*[w_{16}] \otimes B \otimes \{w_6w_7, w_8w_7\},\\ E_{16}^{*,odd} &\cong P(4)^*[w_{16}^2] \otimes B \otimes \{w_8w_7^3 + w_7w_6w_8^2 = D_3D_4, w_8w_7w_{16} = D_4w_{16}\}\,.\end{aligned}$$

The next term is $E_{32}^{*,odd} \cong P(5)^*[w_{16}^2] \otimes B\{a\}$, and finally $E_{64}^{*,odd} = 0$. Therefore all differentials have the form $d_{2^{r+1}}(x) = v_r \otimes Q_r(x)$ and the assumptions needed in the lemmas of Section 5 hold. The integral case can also be proved to satisfy these assumptions by similar but easier arguments. Indeed, $BP^*(B\mathrm{Spin}(9))$ is also computed in [KY].

Theorem 7.2. *In $BP^*(BD(4))$, the elements $[v_1 \otimes w_{16}]$ and $[v_2 \otimes w_{16}]$ are not transfers of BP^*-linear combinations of products of Chern classes.*

8. A 4-dimensional permanent cycle

In this section, we show that the class $2w_4$ in $H^*(B\mathrm{Spin}(n))_{(2)}$ is represented by a Chern class. A similar statement holds for the exceptional group F_4 and $p = 3$.

Suppose that G is a simply connected simple Lie group having p-torsion in $H^*(G)$. Then it is known that G is 2-connected and there is an element $x_3 \in H^3(G;\mathbb{Z}/p)$ with $Q_1 x_3 \neq 0$. Consider the classifying space BG and its cohomology. Denote by y_4 the transgression of x_3 in $H^*(BG;\mathbb{Z}/p)$, so that $Q_1(y_4) \neq 0$. We shall denote the integral lift of y_4 to $H^4(BG)_{(p)}$ also by y_4. Then y_4 is not in the image from $BP^*(BG)$, and the following lemma is immediate.

Lemma 8.1. *If $py_4 \in H^4(BG)_{(p)}$ is represented by a Chern class, then the kernel of the map $\bar{\rho} : CH^2(BG)/p \to H^4(BG;\mathbb{Z}/p)$ is not injective.*

First, we consider the case $G = \mathrm{Spin}(2n+1)$ and $p = 2$. The complex representation ring is

$$R(\mathrm{Spin}(2n+1)) \cong \mathbb{Z}[\lambda_1, ..., \lambda_{n-1}, \Delta_C],$$

where λ_i is the i-th elementary symmetric function in variables $z_1^2 + z_1^{-2}, \dots, z_n^2 + z_n^{-2}$ in $R(T) \cong \mathbb{Z}[z_1, z_1^{-1}, \dots, z_n, z_n^{-1}]$ for the maximal torus T in $\mathrm{Spin}(2n+1)$. Consider the restriction to $R(S^1) \cong \mathbb{Z}[z_1, z_1^{-1}]$. Since

$$Res_{S^1}(\lambda_1) = z_1^2 + z_1^{-2} + 2(n-1),$$

the total Chern class of this representation is

$$\mathrm{Res}_{BS^1}(c(\lambda_1)) = (1+2u)(1-2u)\,.$$

Therefore $4u^2 \in H^*(BS^1)$ is the restriction of a Chern class in $H^*(B\mathrm{Spin}(n+1))_{(2)}$.

On the other hand, consider the diagram:

$$\begin{tikzcd}
H^*(BT;\mathbb{Z}/2) & H^*(B\mathrm{Spin}(2n+1);\mathbb{Z}/2) \arrow[l] \\
H^*(BT;\mathbb{Z}/2) \arrow[u, "p_T^*"] & H^*(BSO(2n+1);\mathbb{Z}/2) \arrow[l] \arrow[u, "p^*"']
\end{tikzcd}$$

Here $p_T^*(u_i) = 2u_i$, and we see that $\mathrm{Res}_{BT}(w_4) = 0$ in $H^*(BT;\mathbb{Z}/2)$. Thus

$$\mathrm{Res}_{BS^1}(H^4(B\mathrm{Spin}(2n+1)_{(2)}) \subset \mathbb{Z}_{(2)}\{2u^2\}\,.$$

Therefore we see that for $G = \mathrm{Spin}(2n+1)$ and $p = 2$ the assumptions of Lemma 8.1 are satisfied. By naturality, all the groups $\mathrm{Spin}(n)$ for $n \geq 7$ satisfy the assumptions.

Next consider the case $G = F_4$ and $p = 3$. The exceptional Lie group F_4 contains $\mathrm{Spin}(8)$ as a subgroup, and

$$R(F_4) \cong R(\mathrm{Spin}(8))^{\Sigma_3}$$

in $R(T) \cong \mathbb{Z}[z_1, z_1^{-1}, ..., z_4, z_4^{-1}]$ (for the details of the action of Σ_3, see [A], Chapter 14). There is a 26-dimensional irreducible representation U of F_4, whose restriction to $\mathrm{Spin}(8)$ is $2 + \lambda_1 + \Delta^+ + \Delta^-$, where $\Delta^{\pm}$ are the half spin representations of dimension 8. The weight of $\Delta^{\pm}$ is $\frac{1}{2}(\pm x_1 \pm x_2 \pm x_3 \pm x_4)$, with an even number of minus signs for Δ^+ and an odd number for Δ^-. Thus

$$\mathrm{Res}_{S^1}(\Delta^+) = \sum_{\epsilon_1\epsilon_2\epsilon_3\epsilon_4=1} z_1^{\epsilon_1} z_2^{\epsilon_2} z_3^{\epsilon_3} z_4^{\epsilon_4},$$

and similarly for Δ^-. Restricting further to S^1, we obtain

$$\mathrm{Res}_{S^1}(U) = z_1^2 + z_2^2 + 8 + 8z_1 + 8z_1^{-1}.$$

Therefore its total Chern class is

$$\begin{aligned}\mathrm{Res}_{BS^1}(c(U)) &= (1+2u)(1-2u)(1+u)^8(1-u)^8\\ &= (1-4u^2)(1-u^2)^8 = 1 - 12u^2 + \cdots .\end{aligned}$$

Hence $3y_4 \in H^4(BF_4)_{(3)} \cong \mathbb{Z}_{(3)}$ is represented by a Chern class. Thus we get the following therorem.

Theorem 8.2. *Let $G = \mathrm{Spin}(n)$, $n \geq 7$ and $p = 2$, or $G = F_4$ and $p = 3$. The kernels of the maps*

$$CH^2(BG)/(p) \to H^4(BG; \mathbb{Z}/p),$$
$$CH^3(BG \times B\mathbb{Z}/p)_{(p)} \to H^6(BG \times B\mathbb{Z}/p)_{(p)}$$

are both non-zero.

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