

DISCRETE SERIES CHARACTERS AS LIFTS FROM TWO-STRUCTURE GROUPS

REBECCA A. HERB

ABSTRACT. Let G be a connected reductive Lie group with a relatively compact Cartan subgroup. Then it has relative discrete series representations. The main result of this paper is a formula expressing relative discrete series characters on G as “lifts” of relative discrete series characters on smaller groups called two-structure groups for G . The two-structure groups are connected reductive Lie groups which are locally isomorphic to the direct product of an abelian group and simple groups which are real forms of $SL(2, \mathbf{C})$ or $SO(5, \mathbf{C})$. They are not necessarily subgroups of G , but they “share” the relatively compact Cartan subgroup and certain other Cartan subgroups with G . The character identity is similar to formulas coming from endoscopic lifting, but the two-structure groups are not necessarily endoscopic groups, and the characters lifted are not stable. Finally, the formulas are valid for non-linear as well as linear groups.

1. INTRODUCTION

Let G be a connected reductive Lie group with a relatively compact Cartan subgroup. Then it has relative discrete series representations. The main result of this paper is a formula expressing relative discrete series characters on G as “lifts” of relative discrete series characters on smaller groups called two-structure groups for G . The two-structure groups are connected reductive Lie groups which are locally isomorphic to the direct product of an abelian group and simple groups which are real forms of $SL(2, \mathbf{C})$ or $SO(5, \mathbf{C})$. They are not necessarily subgroups of G , but they “share” the relatively compact Cartan subgroup and certain other Cartan subgroups with G . The character identity is similar to formulas coming from endoscopic lifting, but the two-structure groups are not necessarily endoscopic groups, and the characters lifted are not stable. Finally, the formulas are valid for non-linear as well as linear groups.

Two-structures were used in [5] to prove identities for the constants occurring in Harish-Chandra’s discrete series character formulas. In these formulas the character is restricted to the connected component of a Cartan subgroup, and the two-structures used are those corresponding to a set of real roots determined by the connected component. Thus there is a different formula for each connected component of each Cartan subgroup. These formulas are used in this paper to prove the lifting formula. The main technical difficulty is to see that all these formulas can

Received by the editors June 1, 1999.

2000 *Mathematics Subject Classification.* Primary 22E30, 22E45.

Supported in part by NSF Grant DMS 9705645.

be combined into a formula which is uniform on all Cartan subgroups and given in terms of the two-structures of the full root system of G .

Two-structures for G are defined as follows. Let B be a relatively compact Cartan subgroup of G , that is a Cartan subgroup that is compact modulo the center of G . Let $\underline{g}$ and $\underline{b}$ denote the real Lie algebras of G and B respectively, and let $\Phi = \Phi(\underline{g}_{\mathbb{C}}, \underline{b}_{\mathbb{C}})$ denote the set of roots of $\underline{g}_{\mathbb{C}}$ with respect to $\underline{b}_{\mathbb{C}}$. Let φ be a root subsystem of Φ , that is a subset of Φ which is closed under its own reflections, and hence is a root system with the same inner product as that of Φ . Then φ is called a two-structure for Φ if it satisfies the following two properties.

- (i) Every irreducible factor of φ is of type A_1 or $B_2 \simeq C_2$.
- (ii) Let φ^+ be any choice of positive roots for φ . Then if w is in the Weyl group of Φ with $w\varphi^+ = \varphi^+$ we have $\det w = 1$.

We let $\mathcal{T}(\Phi)$ denote the set of all two-structures for Φ .

The sets $\mathcal{T}(\Phi)$ for irreducible Φ can be described as follows. If Φ has one root length or is of type G_2 , then $\mathcal{T}(\Phi)$ consists of all root subsystems of Φ of type A_1^k where k is the size of a maximal set of orthogonal roots in Φ . If Φ is of type $B_{2k}, C_{2k}, k \geq 1$, or $F_{2k}, k = 2$, then $\mathcal{T}(\Phi)$ consists of all root subsystems of Φ of type B_2^k . Finally, if Φ is of type $B_{2k+1}, C_{2k+1}, k \geq 1$, then $\mathcal{T}(\Phi)$ consists of all root subsystems of Φ of type $B_2^k \times A_1$.

The groups G_φ are defined as follows. Fix $\varphi \in \mathcal{T}(\Phi)$. It is a root subsystem of Φ . However it need not be closed in Φ under addition, and so does not necessarily correspond to a subalgebra of $\underline{g}$. Write $\varphi = \varphi_1 \cup \dots \cup \varphi_k$ for its decomposition into irreducible factors. Then each $\varphi_i, 1 \leq i \leq k$, is closed under addition in Φ , and so corresponds naturally to a simple Lie subalgebra $\underline{g}_i$ of $\underline{g}$. This subalgebra has the property that if $\underline{b}_i = \underline{b} \cap \underline{g}_i$, then $\underline{b}_i$ is a Cartan subalgebra of $\underline{g}_i$ and $\Phi((\underline{g}_i)_{\mathbb{C}}, (\underline{b}_i)_{\mathbb{C}}) = \varphi_i, 1 \leq i \leq k$. We also define

$$\underline{g}_0 = \underline{b}_0 = \{H \in \underline{b} : \alpha(H) = 0 \ \forall \ \alpha \in \varphi\}.$$

Then

$$\sum_{i=0}^k \underline{b}_i = \underline{b}.$$

Let G_i be the connected subgroup of G corresponding to $\underline{g}_i$, $B_i = \exp(\underline{b}_i) = B \cap G_i, 0 \leq i \leq k$. Let $G_0 \times G_1 \times \dots \times G_k$ denote the abstract direct product of the groups $G_i, 0 \leq i \leq k$. Since $B_i \subset B, 0 \leq i \leq k$, and B is abelian, the mapping $f : B_0 \times \dots \times B_k \rightarrow B$ given by

$$f(b_0, \dots, b_k) = b_0 \dots b_k, \quad b_i \in B_i, 0 \leq i \leq k,$$

is a group homomorphism. Let Z_B denote the kernel of this homomorphism. It is easy to prove that Z_B is a central subgroup of $G_0 \times \dots \times G_k$. Define

$$G_\varphi = (G_0 \times \dots \times G_k)/Z_B \quad \text{and} \quad B_\varphi = (B_0 \times \dots \times B_k)/Z_B.$$

Then G_φ is a connected reductive Lie group with relatively compact Cartan subgroup B_φ , and the mapping $f_B : B_\varphi \rightarrow B$ induced by f is an isomorphism.

Let $\underline{g}_\varphi$ and $\underline{b}_\varphi$ denote the Lie algebras of G_φ and B_φ respectively. Then

$$\Phi((\underline{g}_\varphi)_{\mathbb{C}}, (\underline{b}_\varphi)_{\mathbb{C}}) = \varphi.$$

Since all irreducible factors of φ are of type A_1 or B_2 , G_φ is locally isomorphic to the direct product of an abelian group and simple groups which are real forms of $SL(2, \mathbb{C})$ or $SO(5, \mathbb{C})$. Note that

$$\underline{g}_\varphi = \bigoplus_{i=0}^k \underline{g}_i$$

can be identified with a subspace, but not necessarily a subalgebra, of $\underline{g}$, because elements from the different subalgebras $\underline{g}_i$ do not necessarily commute with each other as elements of $\underline{g}$. This is because, although roots in different irreducible factors of φ are orthogonal to each other, they need not be strongly orthogonal as elements of Φ . Thus G_φ can not necessarily be embedded as a subgroup of G .

If Φ is spanned by orthogonal roots, then each $\varphi \in \mathcal{T}(\Phi)$ has the same rank as Φ . Thus if Φ is spanned by orthogonal roots and G is semisimple, $\underline{g}_0 = \underline{b}_0 = \{0\}$, so that G_φ is also semisimple. If Φ is not spanned by orthogonal roots, then $\text{rank } \varphi < \text{rank } \Phi$, and so in this case $\underline{g}_0 = \underline{b}_0$ is always non-trivial. Thus even if G is semisimple, it is possible that G_φ is not. This is why we work with the class of reductive groups.

Let $\mathcal{SO}(\Phi)$ denote the set of all subsets of Φ consisting of strongly orthogonal noncompact roots. Each $S \in \mathcal{SO}(\Phi)$ corresponds to a Cartan subgroup H_S of G . Its Lie algebra satisfies $(\underline{h}_S)_{\mathbb{C}} = c_S(\underline{b}_{\mathbb{C}})$ where c_S is the Cayley transform associated to S . Further, H_S can be decomposed as $H_S = T_S A_S$ where A_S is a split group and T_S is compact modulo the center of G and has identity component $T_S^0 \subset B$. While T_S need not be abelian when G is not linear, $T_S^1 = T_S \cap B$ is always abelian, since B is. We define $H_S^1 = T_S^1 A_S$. If $\gamma \in G'$ is any regular semisimple element of G , then there are $S \in \mathcal{SO}(\Phi)$ and $h \in H_S^1$ (not unique) such that $\gamma = xhx^{-1}$ for some $x \in G$.

Now for $\varphi \in \mathcal{T}(\Phi)$, we let $\mathcal{SO}(\varphi)$ denote the set of all subsets of φ consisting of noncompact roots which are strongly orthogonal in φ . Each $S \in \mathcal{SO}(\varphi)$ corresponds as above to a Cartan subgroup $H_{\varphi,S} = T_{\varphi,S} A_{\varphi,S}$ of G_φ . In this case we always have $T_{\varphi,S} \subset B_\varphi$, so that $H_{\varphi,S}^1 = H_{\varphi,S}$.

Let $\mathcal{SO}(\Phi, \varphi) = \mathcal{SO}(\Phi) \cap \mathcal{SO}(\varphi)$. Then for $S \in \mathcal{SO}(\Phi, \varphi)$ we have Cartan subgroups H_S of G and $H_{\varphi,S}$ of G_φ . If $S_i = S \cap \varphi_i$, $1 \leq i \leq k$, then

$$H_{\varphi,S} = (H_{0,S_0} \times H_{1,S_1} \times \dots \times H_{k,S_k}) / Z_B$$

where $H_{0,S_0} = B_0$ and H_{i,S_i} is the Cartan subgroup of G_i corresponding to S_i , $1 \leq i \leq k$. Now $H_{i,S_i} \subset H_S^1$, $0 \leq i \leq k$, and $(h_0, h_1, \dots, h_k) \mapsto h_0 h_1 \dots h_k, h_i \in H_{i,S_i}$, $0 \leq i \leq k$, factors through $H_{\varphi,S}$ to give an isomorphism f_S of $H_{\varphi,S}$ onto H_S^1 .

The isomorphisms $f_S : H_{\varphi,S} \rightarrow H_S^1$, $S \in \mathcal{SO}(\Phi, \varphi)$, can be used to define a correspondence of orbits between G_φ and G . For $x \in G_\varphi$ and $g \in G$, let $\mathcal{O}_\varphi(x)$ and $\mathcal{O}_G(g)$ denote the orbits of x in G_φ and g in G respectively. Let $x \in G'_\varphi$, the set of regular semisimple elements in G_φ . Then there exist $S \in \mathcal{SO}(\varphi)$ and $h \in H_{\varphi,S}$ (not unique) such that $\mathcal{O}_\varphi(x) = \mathcal{O}_\varphi(h)$. We will say that $\mathcal{O}_\varphi(x)$ is a good orbit in G'_φ if $S \in \mathcal{SO}(\Phi, \varphi)$, that is the roots in S are strongly orthogonal, not only in φ , but also in Φ . In the case that $\mathcal{O}_\varphi(x) = \mathcal{O}_\varphi(h)$, $h \in H_{\varphi,S}$, is a good orbit, we have the Cartan subgroup H_S of G and the isomorphism $f_S : H_{\varphi,S} \rightarrow H_S^1$. We define

$$F_\varphi(\mathcal{O}_\varphi(x)) = \mathcal{O}_G(f_S(h)).$$

We prove in §4 that this definition is independent of the choice of $S \in \mathcal{SO}(\varphi)$ and $h \in H_{\varphi,S}$ with $\mathcal{O}_\varphi(x) = \mathcal{O}_\varphi(h)$.

Now that we have a correspondence of orbits between G_φ and G we can define lifting from G_φ to G . For $x \in G$, write $\det(t - 1 + \text{Ad}_G(x)) = D_G(x)t^n + \text{terms of higher degree}$, where t is an indeterminate. Then x is regular just in case $D_G(x) \neq 0$, and we write G' for the set of regular elements. We also write $D_\varphi(x) = D_{G_\varphi}(x)$, $x \in G_\varphi$, for the corresponding function on G_φ . Let $S \in \text{SO}(\Phi, \varphi)$ and $h' \in H_{\varphi, S}$ such that $h = f_S(h') \in H_S^1 \cap G'$. Then we define

$$D_\varphi^G(h') = |D_\varphi(h')|^{\frac{1}{2}} |D_G(h)|^{-\frac{1}{2}}.$$

Then D_φ^G extends uniquely to a class function on the union of good orbits $\mathcal{O}_\varphi$ of G'_φ such that $F_\varphi(\mathcal{O}_\varphi)$ is a regular orbit in G .

We can now define a lifting from G_φ to G as follows. For $g \in G'$, let $X_\varphi(g)$ denote a complete set of representatives for the good G_φ orbits which map to $\mathcal{O}_G(g)$ under the orbit correspondence F_φ . Let Θ be a class function defined on G'_φ . Now for $g \in G'$, we define

$$(\text{Lift}_\varphi^G \Theta)(g) = \sum_{h' \in X_\varphi(g)} D_\varphi^G(h') \Theta(h').$$

Since D_φ^G and Θ are constant on G_φ orbits, $(\text{Lift}_\varphi^G \Theta)(g)$ does not depend on the choice of $X_\varphi(g)$. It defines a class function on G' .

We now turn to discrete series characters. Let L_B denote the set of all $\tau \in i\mathfrak{b}^*$ such that e^τ gives a well-defined character of B , and let ρ_G denote the half-sum of positive roots in Φ . Then $\lambda \in i\mathfrak{b}^*$ is a discrete series parameter for G if $\langle \lambda, \alpha \rangle \neq 0$ for all $\alpha \in \Phi$ and $\lambda - \rho_G \in L_B$. Let λ be a discrete series parameter for G and let $\varphi \in \mathcal{T}(\Phi)$. When we identify $\mathfrak{g}_\varphi$ with a subspace of $\mathfrak{g}$, we have $\mathfrak{b}_\varphi = \mathfrak{b}$. Thus λ is a potential candidate for discrete series parameter of G_φ . Since $\varphi \subset \Phi$, we have $\langle \lambda, \alpha \rangle \neq 0$ for all $\alpha \in \varphi$. Let ρ_φ denote the half-sum of positive roots in φ . Then $\lambda - \rho_\varphi = (\lambda - \rho_G) + (\rho_G - \rho_\varphi)$. Thus $\lambda - \rho_\varphi \in L_{B_\varphi} = L_B$ if and only if $\rho_G - \rho_\varphi \in L_B$. If Φ is of type A_{2n} , $n \geq 1$, then there is always $\varphi \in \mathcal{T}(\Phi)$ such that $\rho_G - \rho_\varphi \notin L_B$. However we prove in §5 that if Φ contains no irreducible factors of type A_{2n} , $n \geq 1$, then $\rho_G - \rho_\varphi$ is in the root lattice of Φ , and hence always is an element of L_B . As a consequence of this result we have the following theorem.

Theorem 1.1. *Suppose that Φ contains no irreducible factors of type A_{2n} , $n \geq 1$. Let $\lambda \in i\mathfrak{b}^*$ be a discrete series parameter for G . Then for every $\varphi \in \mathcal{T}(\Phi)$, λ is also a discrete series parameter for G_φ .*

Because of Theorem 1.1, we will assume that Φ contains no irreducible factors of type A_{2n} , $n \geq 1$. For any discrete series parameter λ for G , we let Θ_λ denote the character of the discrete series representation (actually relative discrete series if G has noncompact center) of G corresponding to λ . Similarly, we let Θ_λ^φ denote the character of the (relative) discrete series representation of G_φ corresponding to λ .

In §6 we will define (and give an explicit formula for) a positive integer $c(g)$ for each $g \in G'$. It depends only on the orbit of g in G and is constant on connected components of Cartan subgroups. We also define signs $\epsilon_\varphi^G(\lambda) = \pm 1$, $\varphi \in \mathcal{T}(\Phi)$, corresponding to discrete series parameters of G . The main result of this paper is the following character formula.

Theorem 1.2. *There is a dense open subset G'' of G' so that*

$$\Theta_\lambda(g) = c(g) \sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon_\varphi^G(\lambda) (\text{Lift}_\varphi^G \Theta_\lambda^\varphi)(g), \quad g \in G''.$$

Since all $\varphi \in \mathcal{T}(\Phi)$ are conjugate via $W(\Phi)$, the complex Lie algebras $(\underline{g}_\varphi)_{\mathbb{C}}$ are all isomorphic for different φ . However, the real forms $\underline{g}_\varphi$ and hence the groups G_φ , can vary with φ depending on which roots of φ are compact and noncompact. For a fixed $\varphi \in \mathcal{T}(\Phi)$, the orbit mapping F_φ need not be surjective, and $\text{Lift}_\varphi^G \Theta_\lambda^\varphi$ will be zero on any orbit not in the image of F_φ . We need the contributions from all $\varphi \in \mathcal{T}(\Phi)$ in order to recover Θ_λ on G .

Theorem 1.2 is stated more precisely as Theorem 6.4. Although for simplicity we have only stated it for discrete series characters here, it is also valid for limits of discrete series. It is proven by first directly computing the lifts of the discrete series characters Θ_λ^φ using an explicit description of the sets $X_\varphi(g), g \in G''$, given in Theorem 4.13. Then the machinery of two-structures and the formula from [5] expressing discrete series constants on G in terms of two-structures are used to show that the sum of the lifts is Θ_λ .

Suppose that Φ contains an irreducible factor of type $A_{2n}, n \geq 1$. Then there is an invariant neighborhood Ω of the identity in G with the following properties. Let $\varphi \in \mathcal{T}(\Phi)$ and let Ω'_φ denote the union of all good orbits in G_φ which map into $\Omega \cap G'$ via the orbit correspondence F_φ . Then for any discrete series parameter λ of G we can define a class function Θ_λ^φ on Ω'_φ which is a discrete series character of a two-fold cover of G_φ . Further, we can define $\text{Lift}_\varphi^G \Theta_\lambda^\varphi$ in $\Omega \cap G'$, and the formula of Theorem 1.2 is valid for $g \in \Omega \cap G''$.

In the case that G is linear, the work of Shelstad [10, 11, 12] on endoscopy gives a formula for discrete series characters in terms of the lifts of stable discrete series characters on endoscopic groups. The formula in Theorem 1.2 is independent of this theory. The group G is not required to be linear, two-structure groups are not necessarily endoscopic groups, and no stable discrete series characters are used. In the linear case, a lifting formula similar to Theorem 1.2 expressing stable discrete series characters on G in terms of stable discrete series characters on two-structure groups is given in [6]. For an expository account of the results on discrete series characters in [5, 6] and this paper, see [7].

The organization of the paper is as follows. In §2 we review the definition of and well-known results about the Cartan subgroups $H_S, S \in \mathcal{SO}(\Phi)$. The main result is Theorem 2.5 which characterizes which elements $h \in H_S^1, h' \in H_{S'}^1, S, S' \in \mathcal{SO}(\Phi)$, are conjugate. In §3 we review discrete series character formulas for G , and rewrite them in a form convenient for lifting. The main theorem from [5] expanding discrete series constants in terms of two-structures is stated as Theorem 3.4. The definition of the groups G_φ and the orbit correspondence are in §4. The main results are Lemma 4.8 which shows that the orbit map is well-defined, and Theorem 4.13 which gives an explicit description of the sets $X_\varphi(g), g \in G''$, occurring in the definition of lifting. In §5 we review the theory of two-structures and prove results about two-structures which will be needed. In particular, we prove in Theorem 5.5 that $\rho_G - \rho_\varphi$ is in the root lattice of Φ in the case that Φ contains no irreducible factors of type A_{2n} . In §6 we restate and prove Theorem 1.2 as Theorem 6.4, modulo a technical result about two-structures, Lemma 6.8. This technical lemma is proven in §7.

2. REGULAR SEMISIMPLE ORBITS IN G

In this section we restate well-known results about regular semisimple orbits in G in a form that is convenient for discrete series character formulas and the orbit

correspondence. The main result is Theorem 2.5. Proofs of routine lemmas are omitted.

Let G be a connected reductive Lie group. Given any subgroup H of G we will use the corresponding lower case underlined letter $\underline{h}$ for the real Lie algebra of H , and $\underline{h}_{\mathbf{C}}$ for its complexification. For any root system Φ , we let $W(\Phi)$ denote the Weyl group of Φ .

Let θ be a Cartan involution of G as in [13] and let K be the fixed point set of θ . In the case that G has compact center, K is a maximal compact subgroup of G . In general, K contains the center Z_G of G and is compact modulo Z_G . Now G has relative discrete series representations just in case $\text{rank } G = \text{rank } K$ so that G has a Cartan subgroup $B \subset K$. We will assume that this is the case. Then B is a relatively compact Cartan subgroup of G , that is B is compact modulo Z_G , and is unique up to conjugacy. Let $\Phi = \Phi(\underline{g}_{\mathbf{C}}, \underline{b}_{\mathbf{C}})$ denote the roots of $\underline{b}_{\mathbf{C}}$ in $\underline{g}_{\mathbf{C}}$ and $\Phi_K = \Phi(\underline{k}_{\mathbf{C}}, \underline{b}_{\mathbf{C}})$ denote the roots of $\underline{b}_{\mathbf{C}}$ in $\underline{k}_{\mathbf{C}}$. Roots in Φ_K are called compact roots.

Since G is reductive, $\underline{g} = \underline{z} + \underline{g}_s$ where $\underline{z}$ is the center of $\underline{g}$ and $\underline{g}_s$ is semisimple. We also have $\underline{b} = \underline{z} + \underline{b}_s$ where $\underline{b}_s = \underline{b} \cap \underline{g}_s$. For each $\alpha \in \Phi$ we let $(\underline{g}_{\alpha})_{\mathbf{C}}$ denote the root space corresponding to α in $(\underline{g}_s)_{\mathbf{C}}$ and write H_{α}^* for the element of $(\underline{b}_s)_{\mathbf{C}}$ satisfying

$$\beta(H_{\alpha}^*) = 2\langle\beta, \alpha\rangle/\langle\alpha, \alpha\rangle, \quad \forall \beta \in \Phi.$$

For each noncompact $\alpha \in \Phi$ fix a Cayley transform c_{α} . Recall from page 418 of [9] that we can choose $X_{\alpha} \in (\underline{g}_{\alpha})_{\mathbf{C}}, Y_{\alpha} \in (\underline{g}_{-\alpha})_{\mathbf{C}}$ so that

$$c_{\alpha} = \text{Ad} \exp(\pi/4)(Y_{\alpha} - X_{\alpha}).$$

Let $SO(\Phi)$ denote the set of all subsets of Φ consisting of strongly orthogonal noncompact roots. That is, $S \in SO(\Phi)$ if $S \subset \Phi \setminus \Phi_K$ and for any $\alpha, \beta \in S$ with $\alpha \neq \beta$, $\langle\alpha, \beta\rangle = 0$ and $\alpha \pm \beta \notin \Phi$. Each $S \in SO(\Phi)$ corresponds to a Cartan subgroup H_S of G as follows. Define

$$\underline{t}_S = \{H \in \underline{b} : \alpha(H) = 0 \quad \forall \alpha \in S\}, \quad \underline{b}_S = \sum_{\alpha \in S} i\mathbf{R}H_{\alpha}^*.$$

Let $c_S = \prod_{\alpha \in S} c_{\alpha}$. Then H_S is the Cartan subgroup of G with Lie algebra

$$\underline{h}_S = \underline{t}_S \oplus \underline{a}_S$$

where

$$\underline{a}_S = c_S(i\underline{b}_S) = \sum_{\alpha \in S} \mathbf{R}(X_{\alpha} + Y_{\alpha}).$$

It satisfies $(\underline{h}_S)_{\mathbf{C}} = c_S(\underline{b}_{\mathbf{C}})$ since c_S is the identity on $\underline{t}_S$. Define $T_S = H_S \cap K$ and $A_S = \exp(\underline{a}_S)$. Then $H_S = T_S A_S$. Note that T_S need not be connected. The identity component T_S^0 of T_S is contained in B , but in general not every connected component of T_S will lie in B . Write $T_S^1 = T_S \cap B, H_S^1 = T_S^1 A_S$. While T_S need not be abelian when G is not linear, T_S^1 is always abelian, since B is. Thus H_S^1 is always abelian. When we write $h = ta \in H_S^1$ we always mean that $t \in T_S^1$ and $a \in A_S$.

Fix $t \in T_S^1$. Since $tT_S^0 \subset B$ and every $\alpha \in \Phi$ determines a character e^{α} of B , we can define

$$\Phi_{t,S} = \{\alpha \in \Phi : e^{\alpha}(tt_0) = 1 \quad \forall t_0 \in T_S^0\}.$$

When S is fixed we will also write $\Phi_t = \Phi_{t,S}, t \in T_S^1$.

Lemma 2.1. *Fix $S \in \mathcal{SO}(\Phi)$. Then we have the following.*

- (i) $T_S^1 = (T_S \cap B_S)T_S^0$ where $B_S = \exp(\underline{b}_S)$.
- (ii) $T_S \cap B_S$ consists of all elements of the form $\exp(\pi i \sum_{\alpha \in S} n_\alpha H_\alpha^*)$ where $n_\alpha \in \mathbf{Z}$ for all $\alpha \in S$.
- (iii) $T_S^1 = \{b \in B : e^\alpha(b) = 1 \ \forall \ \alpha \in S\}$.
- (iv) For all $t \in T_S^1$, $S \subset \Phi_{t,S}$ and spans $\Phi_{t,S}$.
- (v) For $b \in T_S^1$, $e^\beta(b) = e^{c_S \beta}(b)$ for all $\beta \in \Phi$.

For $S, S' \in \mathcal{SO}(\Phi)$, we define $S \equiv S'$ if $\underline{t}_S = \underline{t}_{S'}$. Let $v \in W(\Phi_K), H \in \underline{b}$, and $b = \exp(H) \in B$. Then we write $vb = \exp(vH)$. This is well-defined and gives an identification of $W(\Phi_K)$ with $W(G, B) = N_G(B)/B$. Let G' denote the set of regular semisimple elements of G .

Lemma 2.2. (i) *Let $S, S' \in \mathcal{SO}(\Phi)$. Then H_S is conjugate to $H_{S'}$ in G if and only if there is $v \in W(\Phi_K)$ such that $S' \equiv vS$.*
(ii) *Given any $x \in G'$ there are $S \in \mathcal{SO}(\Phi)$ and $h \in H_S^1$ such that $\mathcal{O}_G(x) = \mathcal{O}_G(h)$.*

Lemma 2.3. *Let $S, S' \in \mathcal{SO}(\Phi)$ with Cayley transforms $c_S, c_{S'}$ respectively. Let $h = ta \in H_S^1 \cap G', h' = t'a' \in H_{S'}^1 \cap G'$.*

- (i) *Suppose there is $x \in G$ such that $xhx^{-1} = h'$. Then there are $v \in W(\Phi_K), s \in W(\Phi_t)$, such that $t' = vt$, $S' \equiv vS$, $\log a' = c_{S'} v s c_S^{-1} \log a$, and $\text{Ad}(x)H = c_{S'} v s c_S^{-1} H$ for all $H \in \underline{h}_S$.*
- (ii) *Suppose that there are $v \in W(\Phi_K), s \in W(\Phi_t)$, such that $t' = vt$, $S' \equiv vS$, and $\log a' = c_{S'} v s c_S^{-1} \log a$. Then there is $x \in G$ such that $h' = xhx^{-1}$ and $\text{Ad}(x)H = c_{S'} v s c_S^{-1} H$ for all $H \in \underline{h}_S$.*

For $S, S' \in \mathcal{SO}(\Phi), t \in T_S^1$, we will write $S' \equiv_t S$ if $S' \equiv S$ and $t \in T_{S'}^1$. In this case we have $tT_S^0 = tT_{S'}^0$, so that $\Phi_{t,S} = \Phi_{t,S'}$. By Lemma 2.1, $S' \equiv_t S$ if and only if $S' \subset \Phi_{t,S}$ and spans $\Phi_{t,S}$.

Lemma 2.4. *Fix $h = ta \in H_S^1, v \in W(\Phi_K), s \in W(\Phi_t)$, and $S' \in \mathcal{SO}(\Phi)$ such that $v^{-1}S' \equiv_t S$. Then there is an element $h' = t'a' \in H_{S'}^1$ given by*

$$t' = vt, \quad \log a' = c_{S'} v s c_S^{-1} \log a.$$

For $S \in \mathcal{SO}(\Phi), h = ta \in H_S^1$, we will let $W_0(h)$ denote the set of all $u \in W(\Phi_K) \cap W(\Phi_t)$ such that $ut = t$. Then for $(v, s), (v', s') \in W(\Phi_K) \times W(\Phi_t)$, we will say $(v, s) \sim (v', s')$ if there is $u \in W_0(h)$ such that $v' = vu, s' = u^{-1}s$. This gives an equivalence relation on $W(\Phi_K) \times W(\Phi_t)$ and we write $[v, s]$ for the equivalence class containing (v, s) and $W(\Phi_K, h)$ for the set of equivalence classes. We write $w = 1$ for the equivalence class $[1, 1]$.

Remark 2.1. In Lemma 2.7 of [7] we prove that $W_0(h) = W(\Phi_K) \cap W(\Phi_t)$ if and only if h is central in H_S . Of course if all Cartan subgroups of G are abelian, as is the case when G is linear, then this implies that $W_0(h) = W(\Phi_K) \cap W(\Phi_t)$ always. Thus in this case we can identify $W(\Phi_K, h)$ with the set of all $w \in W(\Phi)$ such that there are $v \in W(\Phi_K), s \in W(\Phi_t)$ such that $w = vs$. However when $W_0(h)$ is a proper subgroup of $W(\Phi_K) \cap W(\Phi_t)$, $W(\Phi_K, h)$ cannot be identified with a subset of $W(\Phi)$.

Let $v \in W(\Phi_K)$ and $S' \in SO(\Phi)$ with $v^{-1}S' \equiv_t S$. Then for any $u \in W_0(h)$, we also have $(vu)^{-1}S' \equiv_t S$ since $t = u^{-1}t \in u^{-1}T_{v^{-1}S'}^1 = T_{u^{-1}v^{-1}S'}^1$. Thus we can write $\Lambda(h)$ for the set of all pairs (w, S') such that $w = [v, s] \in W(\Phi_K, h)$, $S' \in SO(\Phi)$, and $v^{-1}S' \equiv_t S$. Note that for any $w = [v, s] \in W(\Phi_K, h)$, $(w, vS) \in \Lambda(h)$. However there may be other $S' \in SO(\Phi)$ with $(w, S') \in \Lambda(h)$.

Let $(w, S') \in \Lambda(h)$, and let $(v, s) \in W(\Phi_K) \times W(\Phi_t)$ such that $w = [v, s]$. We will write $h' = (w, S') \cdot h$ for the element $h' = t'a' \in H_{S'}^1$ given as in Lemma 2.4 by

$$t' = vt, \quad \log a' = c_{S'} v s c_S^{-1} \log a.$$

This definition clearly does not depend on the representative (v, s) for w .

Note that $\Lambda(h)$ depends only on the connected component hH_S^0 of h in H_S^1 . In fact, as a consequence of Lemma 2.3 (ii), we see that for $(w, S') \in \Lambda(h)$ there is $x \in G$ such that $(w, S') \cdot h_1 = xh_1x^{-1}$ for all $h_1 \in hH_S^0$.

For any $x \in G$, we let $\mathcal{O}_G(x) = \{g x g^{-1} : g \in G\}$ denote the orbit of x in G . Using the above notation, we can rewrite Lemma 2.3 as follows.

Theorem 2.5. *Let $S \in SO(\Phi)$, $h \in H_S^1 \cap G'$. Then*

$$\mathcal{O}_G(h) \cap \left(\bigcup_{S' \in SO(\Phi)} H_{S'}^1 \right) = \{(w, S') \cdot h : (w, S') \in \Lambda(h)\}.$$

Lemma 2.6. *Let $S \in SO(\Phi)$, $h = ta \in H_S^1 \cap G'$. Fix $(w, S') \in \Lambda(h)$, and let $h' = t'a' = (w, S') \cdot h$. Let $(v, s) \in W(\Phi_K) \times W(\Phi_t)$ such that $w = [v, s]$. Then $w^{-1} = [v^{-1}, v s^{-1} v^{-1}] \in W(\Phi_K, h')$, $(w^{-1}, S) \in \Lambda(h')$ and $h = (w^{-1}, S) \cdot h'$. Further, suppose $(w', S'') \in \Lambda(h')$ and let $(v', s') \in W(\Phi_K) \times W(\Phi_{t'})$ such that $w' = [v', s']$. Then $w'w = [v'v, v^{-1}s'vs] \in W(\Phi_K, h)$, $(w'w, S'') \in \Lambda(h)$ and $(w', S'') \cdot h' = (w'w, S'') \cdot h$.*

In order to prove the lifting theorem we will need to restrict to a dense open subset of $H_S^1 \cap G'$. Define

$$(2.1) \quad H_S'' = \{h \in H_S^1 \cap G' : \text{if } (w, S) \in \Lambda(h) \text{ and } (w, S) \cdot h = h, \text{ then } w = 1\}.$$

Remark 2.2. H_S'' can be a proper subset of $H_S^1 \cap G'$. For example, suppose that G is a simple group of adjoint type with $\Phi = \{\pm\alpha\}$ of type A_1 . Then $B = H_\emptyset$ consists of all elements of the form $b(s) = \exp(isH_\alpha^*)$ and $b(s) = 1$ just in case $s \in \pi\mathbb{Z}$. Further, $W(\Phi_K, b) = W(\Phi_K)$ and $\Lambda(b) = \{(w, \emptyset) : w \in W(\Phi_K)\}$ for all $b \in B$. Suppose that G is noncompact. Then $\Phi_K = \emptyset$, so that $B'' = B \cap G'$. However, if G is compact, then $\Phi_K = \Phi$ and the reflection $s_\alpha \in W(\Phi_K)$. Now $b = b(\pi/2)$ is regular since $\exp(\alpha)(b) = -1$, but $(s_\alpha, \emptyset) \cdot b = b(-\pi/2) = b$. Thus $b \notin B''$.

Lemma 2.7. *Let $S \in SO(\Phi)$. Then H_S'' is a dense open subset of $H_S^1 \cap G'$. It is the set of all $h \in H_S^1$ such that $C_G(h) \subset H_S$.*

Proof. The first statement is routine. The second follows easily from Lemma 2.3. $\square$

Remark 2.3. Suppose that G has abelian Cartan subgroups. Then $H_S \subset C_G(h)$ for all $h \in H_S^1$. Thus in this case $h \in H_S''$ if and only if $C_G(h) = H_S$. When G is the set of real points of a linear algebraic group $\mathbf{G}$, an element $g \in G$ is called strongly regular if its centralizer in $\mathbf{G}$ is a (connected) torus. Suppose that $h \in H_S^1$ is strongly regular. Then its centralizer in G is the set of real points of a torus of $\mathbf{G}$, and hence must be equal to H_S . Thus $h \in H_S''$.

3. DISCRETE SERIES ON G

In this section we review discrete series character formulas and rewrite them using the notation of §2. In the case that G has noncompact center these are really relative discrete series, but this will play no role in the arguments, so we will just use the term discrete series. For more details see [5, 7, 9].

Fix a choice Φ^+ of positive roots for Φ , and define

$$\rho_G = \rho(\Phi^+) = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.$$

Let L_B denote the set of all $\tau \in E = i\mathfrak{h}^*$ such that $\exp(\tau)$ is well-defined on B . Recall that $\lambda \in i\mathfrak{h}^*$ is called a discrete series parameter for G if

$$\lambda - \rho_G \in L_B \text{ and } \langle \lambda, \alpha \rangle \neq 0 \ \forall \alpha \in \Phi.$$

Further, $\lambda \in i\mathfrak{h}^*$ is called a limit of discrete series parameter for G if it is not a discrete series parameter, but $\lambda - \rho_G \in L_B$. Let

$$E'(\Phi) = \{\tau \in E : \langle \tau, \alpha \rangle \neq 0 \ \forall \alpha \in \Phi\}.$$

For any connected component E^* of $E'(\Phi)$, define

$$\epsilon_G(\Phi^+ : E^*) = \epsilon_G(E^*) = \text{sign} \prod_{\alpha \in \Phi^+} \langle \alpha, \tau \rangle, \quad \tau \in E^*.$$

Fix a discrete series or limit of discrete series parameter λ of G . If λ is a discrete series parameter, let $E^* = E^*(\lambda)$ denote the connected component of $E'(\Phi)$ containing λ . If λ is a limit of discrete series parameter, let $E^* = E^*(\lambda)$ be a fixed connected component of $E'(\Phi)$ such that λ is in the closure of $E^*(\lambda)$.

Write

$$\Delta'_G(\Phi^+ : b) = \Delta'_G(b) = \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}(b)), \quad b \in B,$$

and define

$$q_G = \frac{1}{2} \dim(G/K).$$

Then as in [1, 2, 8], when λ is a discrete series parameter there is a unique (relative) discrete series representation of G with character Θ_λ given on regular elements $b \in B$ by

$$\Theta_\lambda(b) = \epsilon_G(E^*) (-1)^{q_G} \Delta'_G(b)^{-1} \sum_{v \in W(\Phi_K)} \det v \ e^{v\lambda - \rho_G}(b).$$

Note that although $\epsilon_G(E^*)$, $\Delta'_G(b)$, and ρ_G all depend on the choice Φ^+ of positive roots, the character Θ_λ is independent of this choice.

Now let $S \in \mathcal{SO}(\Phi)$ with corresponding Cayley transform c_S and Cartan subgroup H_S . Then $c_S \Phi^+$ gives a choice of positive roots for $\Phi(\underline{g}_\mathbb{C}, (\underline{h}_S)_\mathbb{C})$. For $h \in H_S$, define

$$\Delta'_G(\Phi^+ : h) = \Delta'_G(h) = \prod_{\alpha \in c_S \Phi^+} (1 - e^{-\alpha}(h)).$$

For $h = ta \in H_S^1$, let

$$\Phi_R^+(h) = \{\alpha \in \Phi_t : \alpha(c_S^{-1} \log a) > 0\}.$$

Define $\Phi_{K,t} = \Phi_K \cap \Phi_t$. Then $W(\Phi_{K,t}) \subset W_0(h) = \{u \in W(\Phi_K) \cap W(\Phi_t) : ut = t\}$. It is possible that $W(\Phi_{K,t})$ is a proper subgroup of $W_0(h)$. Define

$$c(K, h) = [W_0(h)/W(\Phi_{K,t})].$$

Lemma 3.1. *For all $h = ta \in H_S^1 \cap G'$ we have*

$$\begin{aligned} \epsilon_G(E^*) (-1)^{q_G} \Delta'_G(h) \Theta_\lambda(h) &= c(K, h) \sum_{v \in W_0(h) \setminus W(\Phi_K)} \det v e^{v\lambda - \rho_G}(t) \\ &\times \sum_{s \in W(\Phi_t)} \det s c(s : vE^* : \Phi_R^+(h)) \exp((sv\lambda - \rho_G)(c_S^{-1} \log a)). \end{aligned}$$

Here the constants $c(s : vE^* : \Phi_R^+(h))$ are defined as in [5].

Proof. We will use formula (2.6) from [5], which is a restatement of Harish-Chandra's formulas from §23 of [1]. In [5] the results are only stated for the semisimple finite center case, but as in [8] the formulas are also valid for the general case. Note that in [5] Θ_λ was defined without the terms $\epsilon_G(E^*) (-1)^{q_G}$ and was only a character up to sign. The root system Φ in [5] is our Φ_t . Thus in our notation we can rewrite formula (2.6) of [5] as

$$\begin{aligned} \epsilon_G(E^*) (-1)^{q_G} \Delta'_G(h) \Theta_\lambda(h) &= \sum_{v \in W(\Phi_{K,t}) \setminus W(\Phi_K)} \det v e^{v\lambda - \rho_G}(t) \\ &\times \sum_{s \in W(\Phi_t)} \det s c(s : vE^* : \Phi_R^+(h)) \exp((sv\lambda - \rho_G)(c_S^{-1} \log a)) \\ &= \sum_{v \in W_0(h) \setminus W(\Phi_K)} \sum_{u \in W(\Phi_{K,t}) \setminus W_0(h)} \det(uv) e^{uv\lambda - \rho_G}(t) \\ &\times \sum_{s \in W(\Phi_t)} \det s c(s : uvE^* : \Phi_R^+(h)) \exp((suv\lambda - \rho_G)(c_S^{-1} \log a)) \\ &= \sum_{v \in W_0(h) \setminus W(\Phi_K)} \sum_{u \in W(\Phi_{K,t}) \setminus W_0(h)} \det v e^{uv\lambda - \rho_G}(t) \\ &\times \sum_{s \in W(\Phi_t)} \det s c(su^{-1} : uvE^* : \Phi_R^+(h)) \exp((sv\lambda - \rho_G)(c_S^{-1} \log a)). \end{aligned}$$

Fix $u \in W(\Phi_K) \cap W(\Phi_t)$ and define $\psi : \Phi_t \rightarrow \Phi_t$ by $\psi(\alpha) = u\alpha$. Since $u\Phi_K = \Phi_K$ and $u\Phi_t = \Phi_t$, $u\Phi_{K,t} = \Phi_{K,t}$. Further, for $s \in W(\Phi_t)$, $\psi s \psi^{-1} = usu^{-1}$. Thus using (3.1), (3.6) of [5] we have

$$c(s : vE^* : \Phi_R^+(h)) = c(usu^{-1} : uvE^* : u\Phi_R^+(h)) = c(su^{-1} : uvE^* : \Phi_R^+(h)).$$

Further,

$$e^{uv\lambda - \rho_G}(t) = e^{v\lambda - u^{-1}\rho_G}(u^{-1}t) = e^{v\lambda - \rho_G}(u^{-1}t) e^{u\rho_G - \rho_G}(t).$$

But ρ_G is a weight for Φ and hence Φ_t , so that $u\rho_G - \rho_G$ is in the root lattice of Φ_t . Thus $e^{u\rho_G - \rho_G}(t) = 1$. Now if $u \in W_0(h) \subset W(\Phi_K) \cap W(\Phi_t)$, we have $u^{-1}t = t$ so that

$$e^{uv\lambda - \rho_G}(t) = e^{v\lambda - \rho_G}(t).$$

□

When λ is a limit of discrete series parameter, we define Θ_λ using the formula in Lemma 3.1, noting that it depends on the fixed choice of $E^* = E^*(\lambda)$. We can simplify the notation of Lemma 3.1 as follows. Let $\tau \in L_B$. Then for any $h = ta \in H_S^1$ we write

$$\xi_\tau(h) = e^\tau(t) \exp(\tau(c_S^{-1} \log a)).$$

Let $w = [v, s] \in W(\Phi_K, h)$, and pick $S' \in \mathcal{SO}(\Phi)$ such that $(w, S') \in \Lambda(h)$. (Recall that such an S' always exists since we can always take $S' = vS$.) Then we define

$$\xi_\tau(w \cdot h) = \xi_\tau((w, S') \cdot h), \quad \Delta'_G(w \cdot h) = \Delta'_G((w, S') \cdot h),$$

$$\Phi_R^+(w \cdot h) = \Phi_R^+((w, S') \cdot h).$$

It is easy to check that $\xi_\tau((w, S') \cdot h)$, $\Delta'_G((w, S') \cdot h)$, and $\Phi_R^+((w, S') \cdot h)$ are independent of the choice of S' . Thus $\xi_\tau(w \cdot h)$, $\Delta'_G(w \cdot h)$, and $\Phi_R^+(w \cdot h)$ are well-defined. We also define $\det w = \det v \det s$, and for $\lambda \in i\mathfrak{b}^*$, we write $w\lambda = vs\lambda$. These definitions are clearly independent of the choice of representative (v, s) .

Lemma 3.2. *Let $w = [v, s] \in W(\Phi_K, h)$, $h = ta \in H_S^1$.*

- (i) *For any $\tau \in L_B$, $\xi_\tau(w \cdot h) = e^{v^{-1}\tau}(t) \exp((s^{-1}v^{-1}\tau)(c_S^{-1} \log a))$.*
- (ii) *For any $\alpha \in \Phi$, $\xi_\alpha(w \cdot h) = \xi_{s^{-1}v^{-1}\alpha}(h)$.*
- (iii) *$\Delta'_G(w \cdot h) = \det w \xi_{\rho_G - w^{-1}\rho_G}(h) \Delta'_G(h)$.*
- (iv) *$\Phi_R^+(w \cdot h) = w\Phi_R^+(h)$.*
- (v) *$c(s^{-1} : v^{-1}E^* : \Phi_R^+(h)) = c(1 : E^* : \Phi_R^+(w \cdot h))$.*

Proof. Parts (i)-(iv) are routine. Using part (iv),

$$c(1 : E^* : \Phi_R^+(w \cdot h)) = c(1 : E^* : vs\Phi_R^+(h)).$$

Now define $\psi : vs\Phi_t \rightarrow s\Phi_t$ by $\psi(\alpha) = v^{-1}\alpha$. Then $\psi(vs\Phi_t \cap \Phi_K) = s\Phi_t \cap \Phi_K$ so that using (3.1), (3.6) of [5],

$$c(1 : E^* : vs\Phi_R^+(h)) = c(1 : v^{-1}E^* : s\Phi_R^+(h)) = c(s^{-1} : v^{-1}E^* : \Phi_R^+(h)).$$

□

Theorem 3.3. *Let $h \in H_S^1 \cap G'$. Then*

$$\begin{aligned} \epsilon_G(E^*) (-1)^{q_G} \Theta_\lambda(h) \\ = c(K, h) \sum_{w \in W(\Phi_K, h)} \Delta'_G(w \cdot h)^{-1} \xi_{\lambda - \rho_G}(w \cdot h) c(1 : E^* : \Phi_R^+(w \cdot h)). \end{aligned}$$

Proof. Using Lemma 3.1 and the change of variables $s \mapsto s^{-1}, v \mapsto v^{-1}$ we can write

$$\begin{aligned} \epsilon_G(E^*) (-1)^{q_G} \Theta_\lambda(h) &= c(K, h) \Delta'_G(h)^{-1} \sum_{w=[v,s] \in W(\Phi_K, h)} \det w \\ &\times e^{v^{-1}\lambda - \rho_G}(t) c(s^{-1} : v^{-1}E^* : \Phi_R^+(h)) \exp((s^{-1}v^{-1}\lambda - \rho_G)(c_S^{-1} \log a)). \end{aligned}$$

Let $w = [v, s] \in W(\Phi_K, h)$. Then by Lemma 3.2(iii) we have

$$\Delta'_G(w \cdot h)^{-1} = \Delta'_G(h)^{-1} \det w \xi_{w^{-1}\rho_G - \rho_G}(h).$$

Since $w^{-1}\rho_G - \rho_G$ is in the root lattice of Φ , we can use Lemma 3.2(i) and (ii) to write

$$\begin{aligned}\xi_{w^{-1}\rho_G - \rho_G}(h)\xi_{\lambda - \rho_G}(w \cdot h) &= \xi_{\lambda - w\rho_G}(w \cdot h) \\ &= e^{v^{-1}\lambda - s\rho_G}(t) \exp((s^{-1}v^{-1}\lambda - \rho_G)(c_S^{-1} \log a)).\end{aligned}$$

But $\rho_G - s\rho_G$ is in the root lattice of Φ_t so that

$$e^{v^{-1}\lambda - s\rho_G}(t) = e^{v^{-1}\lambda - \rho_G}(t).$$

Combining this with Lemma 3.2(v), we have

$$\begin{aligned}& \sum_{w \in W(\Phi_K, h)} \Delta'_G(w \cdot h)^{-1} \xi_{\lambda - \rho_G}(w \cdot h) c(1 : E^* : \Phi_R^+(w \cdot h)) \\ &= \Delta'_G(h)^{-1} \sum_{w=[v, s] \in W(\Phi_K, h)} \det w e^{v^{-1}\lambda - \rho_G}(t) \\ & \quad \times c(s^{-1} : v^{-1}E^* : \Phi_R^+(h)) \exp((s^{-1}v^{-1}\lambda - \rho_G)(c_S^{-1} \log a)).\end{aligned}$$

□

The constants appearing in the discrete series character formula of Theorem 3.3 are of the form $c(1 : E^* : \Phi_t^+)$ where Φ_t is a root system spanned by strongly orthogonal noncompact roots, Φ_t^+ is a choice of positive roots for Φ_t , and E^* is a connected component of $E'(\Phi)$. These constants satisfy the following identity. See [5] for details.

Let $\mathcal{T}(\Phi_t)$ denote the set of all two-structures for Φ_t . We will say $\psi \in \mathcal{T}(\Phi_t)$ is of noncompact type if ψ is spanned by a collection of strongly orthogonal noncompact roots of Φ_t , that is there is $S' \in \mathcal{SO}(\Phi)$ such that $S' \subset \psi$ and spans ψ . This is equivalent to the condition that all long roots of ψ are noncompact in Φ_t , where by a long root of ψ we mean any root in an A_1 factor and any long root in a $B_2 \simeq C_2$ factor. Write $\mathcal{T}_n(\Phi_t)$ for the two-structures of noncompact type for Φ_t .

The following theorem is proven in [5]. It contains signs $\epsilon(\psi : \Phi_t^+) = \pm 1$ corresponding to a choice of positive roots Φ_t^+ for Φ_t and $\psi \in \mathcal{T}(\Phi_t)$. The definition of these signs is given in (5.1). It also contains a constant $c(\Phi_t)$ which is defined as follows. For any $\psi \in \mathcal{T}(\Phi_t)$, $L(\Phi_t) \subset L(\psi)$ where $L(\Phi_t), L(\psi)$ denote the weight lattices of Φ_t and ψ respectively. Let

$$(3.1) \quad c(\Phi_t) = [L(\psi) : L(\Phi_t)].$$

It is independent of the choice of ψ since all $\psi \in \mathcal{T}(\Phi_t)$ are conjugate via $W(\Phi_t)$. Its values for Φ_t irreducible are given by the following table, which also gives the type of ψ in each case.

Φ_t	B_{2n}	B_{2n+1}	C_{2n}	C_{2n+1}	D_{2n}	E_7	E_8	F_4	G_2
ψ	B_2^n	$B_2^n \times B_1$	C_2^n	$C_2^n \times C_1$	A_1^{2n}	A_1^7	A_1^8	B_2^2	A_1^2
$c(\Phi_t)$	2^{n-1}	2^n	1	1	2^{n-1}	2^3	2^4	2	2

Theorem 3.4. *Let E^* be any connected component of $E'(\Phi)$ and let Φ_t^+ be any choice of positive roots for Φ_t . Then*

$$c(1 : E^* : \Phi_t^+) = c(\Phi_t) \sum_{\psi \in \mathcal{T}_n(\Phi_t)} \epsilon(\psi : \Phi_t^+) c(1 : E^* : \psi \cap \Phi_t^+).$$

4. DEFINITION OF G_φ AND THE ORBIT CORRESPONDENCE

Let Φ be any root system. Then a root subsystem $\varphi \subset \Phi$ is called a two-structure for Φ if it satisfies the following two properties.

- (i) Every irreducible factor of φ is of type A_1 or $B_2 \simeq C_2$.
- (ii) Let φ^+ be any choice of positive roots for φ . Then if $w \in W(\Phi)$ with $w\varphi^+ = \varphi^+$ we have $\det w = 1$.

Let $\mathcal{T}(\Phi)$ denote the set of all two-structures for Φ .

Note that $\varphi \in \mathcal{T}(\Phi)$ is a root subsystem of Φ , that is a subset of Φ which is closed under its own reflections. It is not necessarily a subroot system. That is, there may be $\alpha, \beta \in \varphi$ such that $\alpha + \beta \in \Phi$, but $\alpha + \beta \notin \varphi$. However, let φ_0 be an irreducible factor of φ , and let Φ_0 denote the intersection of Φ with the linear subspace of $i\mathbb{R}\varphi_0^*$ spanned by φ_0 . Then Φ_0 is a subroot system of Φ with the same rank as φ_0 . Since there are no root systems of the same rank properly containing a root system of type A_1 or B_2 , we must have $\varphi_0 = \Phi_0$. Thus φ_0 is a subroot system of Φ .

We want to associate to every $\varphi \in \mathcal{T}(\Phi)$ a group G_φ which shares certain Cartan subgroups with G . Fix $\varphi \in \mathcal{T}(\Phi)$, and write $\varphi = \varphi_1 \cup \dots \cup \varphi_k$ for its decomposition into irreducible factors. Each $\varphi_i, 1 \leq i \leq k$, is a subroot system of Φ and corresponds to a Lie subalgebra $\underline{g}_i$ of $\underline{g}$ as follows.

As in §2, for each $\alpha \in \Phi$ we have the root space $(\underline{g}_\alpha)_\mathbb{C}$ and the normalized root vector $H_\alpha^* \in (\underline{b}_s)_\mathbb{C}$. Now define

$$(\underline{b}_i)_\mathbb{C} = \sum_{\alpha \in \varphi_i} \mathbb{C}H_\alpha^*, \quad \underline{b}_i = \underline{b} \cap (\underline{b}_i)_\mathbb{C} = \sum_{\alpha \in \varphi_i} \mathbb{R}iH_\alpha^*;$$

$$(\underline{g}_i)_\mathbb{C} = (\underline{b}_i)_\mathbb{C} + \sum_{\alpha \in \varphi_i} (\underline{g}_\alpha)_\mathbb{C}, \quad \underline{g}_i = \underline{g} \cap (\underline{g}_i)_\mathbb{C}.$$

Let σ be the involution of $\underline{g}_\mathbb{C}$ with fixed points $\underline{g}$. For each $\alpha \in \Phi$ we have $\alpha^\sigma \in \Phi$ defined by

$$\alpha^\sigma(H) = \overline{\alpha(\sigma H)}, \quad H \in \underline{b}_\mathbb{C}.$$

Since α takes pure imaginary values on $\underline{b}$, we have $\alpha^\sigma = -\alpha$ for all $\alpha \in \Phi$. Thus for all $\alpha \in \Phi$,

$$\sigma(H_\alpha^*) = H_{\alpha^\sigma}^* = H_{-\alpha}^*, \quad \sigma((\underline{g}_\alpha)_\mathbb{C}) = (\underline{g}_{\alpha^\sigma})_\mathbb{C} = (\underline{g}_{-\alpha})_\mathbb{C}.$$

Thus $(\underline{g}_i)_\mathbb{C}$ is stable under σ . Now the restriction of σ to $(\underline{g}_i)_\mathbb{C}$ is an involution with fixed points $\underline{g}_i$. Thus $\underline{g}_i$ is a real form of $(\underline{g}_i)_\mathbb{C}$.

Since φ_i is of type A_1 or $B_2 \simeq C_2$, $(\underline{g}_i)_\mathbb{C}$ is isomorphic to either $\underline{sl}(2, \mathbb{C})$ or $\underline{so}(5, \mathbb{C}) \simeq \underline{sp}(4, \mathbb{C})$. Thus $\underline{g}_i$ is isomorphic to one of

$$\underline{sl}(2, \mathbb{R}), \quad \underline{su}(2), \quad \underline{so}(3, 2), \quad \underline{so}(4, 1), \quad \text{or} \quad \underline{so}(5).$$

We also define

$$\underline{g}_0 = \underline{b}_0 = \{H \in \underline{b} : \alpha(H) = 0 \ \forall \ \alpha \in \varphi\} \quad \text{and} \quad \varphi_0 = \emptyset.$$

Let G_i be the connected subgroup of G corresponding to $\underline{g}_i$, $K_i = G_i \cap K$, $B_i = \exp(\underline{b}_i) = G_i \cap B$. Let $G_0 \times G_1 \times \dots \times G_k$ denote the abstract direct product of the groups $G_i, 0 \leq i \leq k$, and define $f : G_0 \times \dots \times G_k \rightarrow G$ by

$$f(g_0, \dots, g_k) = g_0 \dots g_k, \quad g_i \in G_i, 0 \leq i \leq k.$$

Here $g_0 \dots g_k$ denotes the product in G of the elements $g_i \in G_i \subset G$. Note that f depends on the ordering of the irreducible factors $\varphi_i, 1 \leq i \leq k$, and is not necessarily a group homomorphism, since elements from the different subgroups G_i do not necessarily commute with each other as elements of G . This is because, although roots in different irreducible factors of φ are orthogonal to each other, they need not be strongly orthogonal as elements of Φ .

However, since $B_i \subset B, 0 \leq i \leq k$, and B is abelian, $f : B_0 \times \dots \times B_k \rightarrow B$ is a group homomorphism. Let Z_B denote the kernel of this homomorphism, and let Z_i denote the center of $G_i, 0 \leq i \leq k$. The following lemma is easy to check

Lemma 4.1. *$f : B_0 \times \dots \times B_k \rightarrow B$ is surjective and $Z_B \subset Z_0 \times \dots \times Z_k$ is a central subgroup of $G_0 \times \dots \times G_k$.*

Define

$$(4.1) \quad G_\varphi = (G_0 \times \dots \times G_k)/Z_B, \quad K_\varphi = (K_0 \times \dots \times K_k)/Z_B,$$

$$(4.2) \quad B_\varphi = (B_0 \times \dots \times B_k)/Z_B, \quad Z_\varphi = (Z_0 \times \dots \times Z_k)/Z_B.$$

Then G_φ and K_φ are connected reductive Lie groups, B_φ is a Cartan subgroup of both G_φ and K_φ , and Z_φ is the center of G_φ . The Lie algebra $\underline{g}_\varphi = \sum_{i=0}^k \underline{g}_i$ of G_φ can be identified with a subset, but not necessarily a subalgebra, of $\underline{g}$. Note that each $\underline{g}_i$ is stable under the Cartan involution θ of $\underline{g}$ so that the restriction of θ to G_i is a Cartan involution of G_i . Further, K_i is the set of fixed points of θ in G_i . Thus θ induces a Cartan involution of G_φ with fixed points K_φ .

Let $\exp$ denote the exponential mapping from $\underline{g}$ into G , and $\exp_\varphi$ denote the exponential mapping of $\underline{g}_\varphi$ into G_φ . The Cartan subalgebra $\underline{b} = \sum_{i=0}^k \underline{b}_i$ can be identified with the Lie algebra of B_φ and is a Cartan subalgebra of $\underline{g}_\varphi$ and $\underline{k}_\varphi$. Further,

$$\Phi((\underline{g}_\varphi)_\mathbb{C}, \underline{b}_\mathbb{C}) = \varphi, \quad \Phi((\underline{k}_\varphi)_\mathbb{C}, \underline{b}_\mathbb{C}) = \varphi_K = \varphi \cap \Phi_K.$$

The Weyl group $W(\varphi_K)$ generated by reflections in the compact roots $\varphi_K = \varphi \cap \Phi_K$ of φ is naturally a subgroup of $W(\Phi_K)$. For $H \in \underline{b}, b = \exp_\varphi(H), v \in W(\varphi_K)$, define $vb = \exp_\varphi(vH)$. The following lemma is also routine.

Lemma 4.2. (i) *The mapping $f_B : B_\varphi \rightarrow B$ induced by f is an isomorphism.*

(ii) *$f_B(\exp_\varphi(H)) = \exp(H)$ for all $H \in \underline{b}$.*

(iii) *$f_B(vb) = vf_B(b)$ for all $b \in B_\varphi, v \in W(\varphi_K)$.*

(iv) *$Z_G \subset f_B(Z_\varphi)$.*

(v) *K_φ/Z_φ and B_φ/Z_φ are compact.*

We have seen that G_φ together with the subgroups K_φ and B_φ satisfies the same hypotheses as our original group G with its subgroups K and B . Thus the results of §2 and §3 can be applied to the group G_φ .

Remark 4.1. Suppose that Φ contains no irreducible factors of type $B_n, n \geq 3$, or F_4 . Then for any $\varphi \in \mathcal{T}(\Phi)$, roots from different irreducible factors of φ are strongly orthogonal. Thus for $0 \leq i \neq j \leq k$, G_i and G_j commute inside G . Hence $f : G_0 \times \dots \times G_k \rightarrow G$ is a group homomorphism, and it is easy to see that the kernel of f is Z_B . Thus G_φ is embedded as a subgroup of G in this case.

Now for $\varphi \in \mathcal{T}(\Phi)$ as above, we let $\mathcal{SO}(\varphi)$ denote the set of all subsets of φ consisting of noncompact roots which are strongly orthogonal in φ . Since $\varphi_K = \varphi \cap \Phi_K$, noncompact roots in φ are also noncompact in Φ . Further, the inner product on φ is the restriction of the inner product in Φ . However, for $\alpha, \beta \in \varphi$ with $\langle \alpha, \beta \rangle = 0$, α strongly orthogonal to β in φ means that $\alpha \pm \beta \notin \varphi$, while α, β strongly orthogonal in Φ means $\alpha \pm \beta \notin \Phi$. Thus $\mathcal{SO}(\varphi)$ need not be a subset of $\mathcal{SO}(\Phi)$.

For each noncompact root $\alpha \in \varphi \subset \Phi$ we have the fixed root vectors X_α, Y_α used to define the Cayley transform c_α . Then since $X_\alpha, Y_\alpha \in (\underline{g}_\varphi)_{\mathbb{C}}$,

$$c_{\varphi, \alpha} = \text{Ad exp}_\varphi(\pi/4)(Y_\alpha - X_\alpha)$$

is a Cayley transform defined on $(\underline{g}_\varphi)_{\mathbb{C}}$. Now each $S \in \mathcal{SO}(\varphi)$ corresponds as in §2 to a Cartan subgroup $H_{\varphi, S} = T_{\varphi, S} A_{\varphi, S}$ of G_φ with Cartan subalgebra $\underline{h}_{\varphi, S}$. Let $S_i = S \cap \varphi_i, 0 \leq i \leq k$. Then $S = \bigcup_i S_i$, and each S_i is a set of strongly orthogonal noncompact roots in φ_i . Thus S_i corresponds to a Cartan subgroup $H_{i, S_i} = T_{i, S_i} A_{S_i}$ of G_i . Since $Z_B \subset Z_0 \times \dots \times Z_k$, it is contained in every Cartan subgroup of $G_0 \times \dots \times G_k$, and

$$H_{\varphi, S} = (H_{0, S_0} \times \dots \times H_{k, S_k}) / Z_B.$$

Lemma 4.3. (i) For any $S \in \mathcal{SO}(\varphi)$ and $0 \leq i \leq k$, $T_{i, S_i} \subset B_i$. Thus $T_{\varphi, S} \subset B_\varphi$.
(ii) Let $S \in \mathcal{SO}(\varphi)$ and $h = ta \in H_{\varphi, S}$. Then

$$W(\varphi_K) \cap W(\varphi_t) = W(\varphi_{K, t}).$$

Proof. For $i = 0$, $G_0 = B_0$ is abelian, and $S_0 = \emptyset, T_{0, S_0} = B_0$. For $1 \leq i \leq k$, G_i is locally isomorphic to one of $SL(2, \mathbf{R}), SU(2), SO(3, 2), SO(4, 1)$, or $SO(5)$. It is trivial that these properties holds for the compact groups and easy to check that they hold in the other three cases. They are also independent of local isomorphism class. $\square$

Every Cartan subgroup H_φ of G_φ is conjugate to one of the form $H_{\varphi, S}, S \in \mathcal{SO}(\varphi)$. However if the roots in S are not strongly orthogonal as elements of Φ , $H_{\varphi, S}$ will not correspond to a Cartan subgroup H_S of G . This is because the groups A_{i, S_i} will not commute with each other, and so the restriction of f to $(H_{1, S_1} \times \dots \times H_{k, S_k})$ is not a group homomorphism. Thus we restrict our attention to strongly orthogonal sets in

$$\mathcal{SO}(\Phi, \varphi) = \{S \in \mathcal{SO}(\Phi) : S \subset \varphi\} = \mathcal{SO}(\Phi) \cap \mathcal{SO}(\varphi).$$

Lemma 4.4. Let $S, S' \in \mathcal{SO}(\varphi)$ such that $H_{\varphi, S}$ and $H_{\varphi, S'}$ are conjugate in G_φ . Then $S \in \mathcal{SO}(\Phi, \varphi)$ if and only if $S' \in \mathcal{SO}(\Phi, \varphi)$.

Proof. Since $H_{\varphi, S}$ and $H_{\varphi, S'}$ are conjugate in G_φ , by Lemma 2.2 there is $v \in W(\varphi_K)$ such that $vS \equiv S'$. Since $W(\varphi_K) \subset W(\Phi_K)$, we know that $S \in \mathcal{SO}(\Phi)$ if and only if $vS \in \mathcal{SO}(\Phi)$. Thus it suffices to show that for $S, S' \in \mathcal{SO}(\varphi)$ with $S' \equiv S$, we have $S \in \mathcal{SO}(\Phi)$ if and only if $S' \in \mathcal{SO}(\Phi)$. But φ is the union of irreducible factors φ_i of type A_1 or B_2 , and it is easy to see for these two cases that if $S_i, S'_i \in \mathcal{SO}(\varphi_i)$ with $S_i \equiv S'_i$, then $S_i \cup -S_i = S'_i \cup -S'_i$. Thus if $S \equiv S'$, we have $S \cup -S = S' \cup -S'$. Thus $S \in \mathcal{SO}(\Phi)$ if and only if $S' \in \mathcal{SO}(\Phi)$. $\square$

Recall that $\underline{g}_\varphi$ is a subset, although not necessarily a subalgebra, of $\underline{g}$. However, we have the following easy lemma.

Lemma 4.5. For $S \in SO(\Phi, \varphi)$, $\underline{h}_{\varphi, S} = \underline{h}_S$, and $c_{\varphi, S}H = c_S H$ for all $H \in \underline{b}_C$.

Because of Lemma 4.5 we will drop the subscripts φ and write $\underline{h}_S$ and c_S for the Cartan subalgebra and Cayley transform corresponding to $S \in SO(\Phi, \varphi)$ in either $\underline{g}$ or $\underline{g}_{\varphi}$.

Lemma 4.6. Let $S \in SO(\Phi, \varphi)$. For each $0 \leq i \leq k$, H_{i, S_i} is a subgroup of H_S^1 . Further, the homomorphism $f_S : H_{\varphi, S} \rightarrow H_S^1$ induced by f is an isomorphism. It satisfies $f_S(\exp_{\varphi}(H)) = \exp H$ for all $H \in \underline{h}_S$.

Proof. By definition, $H_S = C_G(\underline{h}_S)$. Fix $0 \leq i \leq k$. Then $H_{i, S_i} = T_{i, S_i} A_{i, S_i} = C_{G_i}(\underline{h}_{i, S_i})$. Clearly $A_{i, S_i} = \exp \underline{a}_{i, S_i} \subset \exp \underline{a}_S = A_S$. Let $t_i \in T_{i, S_i}$. We know t_i centralizes $\underline{h}_{i, S_i}$. We must show that it centralizes $\underline{h}_{j, S_j} = \underline{b}$ for all $j \neq i$. But by Lemma 4.3, $t_i \in B_i$. Thus it centralizes $\underline{t}_{j, S_j} \subset \underline{b}$. Now $\underline{a}_{j, S_j}$ is generated by elements of the form $X_{\beta} + Y_{\beta}, \beta \in S_j$. Now $t_i \in B_i = \exp(\sum_{\alpha \in \varphi_i} i \mathbf{R} H_{\alpha}^*)$, and $\beta(H_{\alpha}^*) = 0$ for all $\beta \in S_j, \alpha \in \varphi_i$. Thus t_i centralizes $\underline{a}_{j, S_j}$. We have seen that H_{i, S_i} centralizes all of $\underline{h}_S$ so that $H_{i, S_i} = C_{G_i}(\underline{h}_S) = G_i \cap H_S \subset H_S$. By Lemma 4.3, we in fact have $H_{i, S_i} = T_{i, S_i} A_{i, S_i} \subset (T_S \cap B) A_S = H_S^1$.

Since H_S^1 is abelian we know that

$$f : H_{0, S_0} \times \dots \times H_{k, S_k} \rightarrow H_S^1$$

is a group homomorphism. Suppose that $(h_0, \dots, h_k)$ is in the kernel of f . Write $h_i = t_i a_i$ where $t_i \in T_{i, S_i}, a_i \in A_{i, S_i}$. Then $f(h_0, \dots, h_k) = t_0 \dots t_k a_0 \dots a_k = 1$ implies that $t_0 \dots t_k = 1$ and $a_0 \dots a_k = 1$. Since $t_i \in B_i$, this implies that $(t_0, \dots, t_k) \in Z_B$. But since $A_S = A_{0, S_0} \dots A_{k, S_k}$ is a direct product, $a_0 \dots a_k = 1$ implies that $a_i = 1, 0 \leq i \leq k$. Thus $(h_0, \dots, h_k) = (t_0, \dots, t_k) \in Z_B$.

We have shown that $f_S : H_{\varphi, S} \rightarrow H_S^1$ is injective. Now suppose that $h = ta \in H_S^1$ where $t \in T_S \cap B$ and $a \in A_S$. Since $A_S = A_{0, S_0} \dots A_{k, S_k}$ is a direct product, there are unique $a_i \in A_{i, S_i}, 0 \leq i \leq k$, such that $a = a_0 \dots a_k$. Since $t \in B = B_0 \dots B_k$ there are (not necessarily unique) $b_i \in B_i, 0 \leq i \leq k$, such that $t = b_0 \dots b_k$. Let $0 \leq i \leq k$. Then $t = b_0 \dots b_k \in T_S$ centralizes $\underline{h}_S$ and hence $\underline{h}_{i, S_i}$. But as above, for each $0 \leq j \neq i \leq k$, every $b_j \in B_j$ centralizes $\underline{h}_{i, S_i}$. Thus b_i centralizes $\underline{h}_{i, S_i}$ and hence is an element of $H_{i, S_i} \cap B_i = T_{i, S_i}$. Thus $h_i = b_i a_i \in H_{i, S_i}, 0 \leq i \leq k$, and $f(h_0, \dots, h_k) = ta = h$. Thus f_S is surjective.

Finally, for any $H = \sum_{0 \leq i \leq k} H_i \in \underline{h}_S$,

$$f_S(\exp_{\varphi}(H)) = f(\exp(H_0), \dots, \exp(H_k)) = \exp(H_0) \dots \exp(H_k) = \exp(H).$$

□

Let $S \in SO(\varphi), h = ta \in H_{\varphi, S}$. By Lemma 4.3 (ii) we have $W(\varphi_K) \cap W(\varphi_t) = W(\varphi_{K, t})$. Thus $ut = t$ for all $u \in W(\varphi_K) \cap W(\varphi_t)$ so that $W_0(h) = W(\varphi_K) \cap W(\varphi_t) = W(\varphi_{K, t})$. Thus if we define

$$W(\varphi_K, h) = \{[v, s] : (v, s) \in W(\varphi_K) \times W(\varphi_t)\}$$

as in §2, we have $[v, s] = [v', s']$ if and only if $vs = v's'$. Thus we can identify $W(\varphi_K, h)$ with the set of all elements $w \in W(\varphi)$ such that $w = vs$ for some $v \in W(\varphi_K), s \in W(\varphi_t)$. As in §2, for $S, S' \in SO(\varphi), h = ta \in H_{\varphi, S}$, we will write $S' \equiv_t S$ if $S' \equiv S$ and $t \in T_{\varphi, S'}$. We let $\Lambda(\varphi, h)$ denote the set of all pairs (w, S') such that $w = vs \in W(\varphi_K, h)$ and $v^{-1}S' \equiv_t S$. For $h = ta \in H_{\varphi, S}, (w, S') \in \Lambda(\varphi, h), w = vs$, we define $(w, S') \cdot h = h' = t'a' \in H_{\varphi, S'}$ where $t' = vt, \log_{\varphi} a' = c_{S'} w c_S^{-1} \log_{\varphi}(a)$. It is easy to check the following.

Lemma 4.7. *Let $S \in SO(\Phi, \varphi)$, $h' = t'a' \in H_{\varphi, S}$, $h = ta = f_S(h')$.*

- (i) $\varphi_{t'} = \varphi \cap \Phi_t$.
- (ii) *Let $(w, S') \in \Lambda(\varphi, h')$ and write $w = vs$, $v \in W(\varphi_K)$, $s \in W(\varphi_{t'})$. Then $S' \in SO(\Phi, \varphi)$ and $v^{-1}S' \equiv_t S$.*

Let $S \in SO(\Phi, \varphi)$, $h' = t'a' \in H_{\varphi, S}$, $h = ta = f_S(h')$. Then by Lemma 4.7 (i), $W(\varphi_{t'}) \subset W(\Phi_t)$. Thus for every $v \in W(\varphi_K)$, $s \in W(\varphi_{t'})$, we have $(v, s) \in W(\Phi_K) \times W(\Phi_t)$. Suppose that $v, v' \in W(\varphi_K)$, $s, s' \in W(\varphi_{t'})$, such that $vs = v's'$. Then there is $u \in W(\varphi_K) \cap W(\varphi_{t'}) = W(\varphi_{K, t'}) \subset W(\Phi_{K, t}) \subset W_0(h)$ such that $v' = vu$, $s' = u^{-1}s$, and so $[v', s'] = [v, s]$ as elements of $W(\Phi_K, h)$. Thus $vs \mapsto [v, s]$ gives a well-defined mapping from $W(\varphi_K, h')$ into $W(\Phi_K, h)$. Suppose that $[v, s] = 1$. Then $v = s^{-1} \in W_0(h)$. In particular, $vs = 1$. Thus the mapping from $W(\varphi_K, h')$ into $W(\Phi_K, h)$ defined by $vs \mapsto [v, s]$ is injective. We will use this mapping to identify $W(\varphi_K, h')$ with a subset of $W(\Phi_K, h)$. Now let $(w, S') \in \Lambda(\varphi, h')$, $w = vs$. Then by Lemma 4.7 (ii) we have $S' \in SO(\Phi, \varphi)$, and $v^{-1}S' \equiv_t S$. Thus $([v, s], S') \in \Lambda(h)$. Thus we can also identify $\Lambda(\varphi, h')$ with a subset of $\Lambda(h)$.

Let G'_φ denote the set of regular semisimple elements of G_φ . For any $x \in G_\varphi$, let $\mathcal{O}_\varphi(x)$ denote the orbit of x in G_φ .

Lemma 4.8. *Let $S, S' \in SO(\varphi)$, $h \in H_{\varphi, S} \cap G'_\varphi$, $h' \in H_{\varphi, S'} \cap G'_\varphi$, such that $\mathcal{O}_\varphi(h) = \mathcal{O}_\varphi(h')$. Then $S \in SO(\Phi, \varphi)$ if and only if $S' \in SO(\Phi, \varphi)$. Further, in the case that $S, S' \in SO(\Phi, \varphi)$, then $\mathcal{O}_G(f_S(h)) = \mathcal{O}_G(f_{S'}(h'))$. Finally, let $S \in SO(\Phi, \varphi)$, $h \in H_{\varphi, S}$, $(w, S') \in \Lambda(\varphi, h)$. Then $f_{S'}((w, S') \cdot h) = (w, S') \cdot f_S(h)$.*

Proof. Since $\mathcal{O}_\varphi(h) = \mathcal{O}_\varphi(h')$, $H_{\varphi, S}$ and $H_{\varphi, S'}$ are conjugate in G_φ . Thus by Lemma 4.4, $S \in SO(\Phi)$ if and only if $S' \in SO(\Phi)$.

Assume that $S, S' \in SO(\Phi, \varphi)$. Write $h = ta \in H_{\varphi, S}$, $h' = t'a' \in H_{\varphi, S'}$. Since $\mathcal{O}_\varphi(h) = \mathcal{O}_\varphi(h')$ we can apply Theorem 2.5 to the group G_φ to conclude that there are $v \in W(\varphi_K)$, $s \in W(\varphi_t)$ such that $(vs, S') \in \Lambda(\varphi, h)$ and $h' = (vs, S') \cdot h$. Then

$$t' = vt, \quad S' \equiv vS, \quad \log_\varphi a' = c_{S'} v s c_S^{-1} \log_\varphi a.$$

Now we can write $f_S(h) = f_B(t)f_S(a)$ where $f_B(t) \in T_S^1$ is the image of $t \in T_{\varphi, S} \subset B_\varphi$ under the isomorphism f_B , $f_S(a) = \exp(\log_\varphi a) \in A_S$ and $f_{S'}(h') = f_B(t')f_{S'}(a')$ where $f_B(t') \in T_{S'}^1$, $f_{S'}(a') = \exp(\log_\varphi a') \in A_{S'}$. Now by Lemma 4.2, when we consider $W(\varphi_K)$ as a subgroup of $W(\Phi_K)$, $f_B(t') = f_B(vt) = v f_B(t)$. Further

$$\log(f_{S'}(a')) = \log_\varphi a' = c_{S'} v s c_S^{-1} \log_\varphi a = c_{S'} v s c_S^{-1} \log(f_S(a)).$$

Thus $f_{S'}(h') = ([v, s], S') \cdot f_S(h)$, and by Theorem 2.5, $f_S(h)$ and $f_{S'}(h')$ are conjugate in G .

Finally, let $S \in SO(\Phi, \varphi)$, $h \in H_{\varphi, S}$, $(w, S') \in \Lambda(\varphi, h)$. Write $w = vs$ and $(vs, S') \cdot h = t'a'$. Then by the above argument, $f_{S'}(h') = ([v, s], S') \cdot f_S(h)$. $\square$

We will use the isomorphisms $f_S : H_{\varphi, S} \rightarrow H_S^1$, $S \in SO(\Phi, \varphi)$, to define a correspondence of orbits between G_φ and G . Let $\gamma \in G'_\varphi$. Then there exist $S \in SO(\varphi)$ and $h \in H_{\varphi, S}$ (not unique) such that $\mathcal{O}_\varphi(\gamma) = \mathcal{O}_\varphi(h)$. We will say that $\mathcal{O}_\varphi(\gamma)$ is a good orbit in G'_φ if $S \in SO(\Phi, \varphi)$. In the case that $\mathcal{O}_\varphi(\gamma) = \mathcal{O}_\varphi(h)$, $h \in H_{\varphi, S}$, is a good orbit, we have the Cartan subgroup H_S of G and the isomorphism $f_S : H_{\varphi, S} \rightarrow H_S$ given by Lemma 4.6. We define

$$(4.3) \quad F_\varphi(\mathcal{O}_\varphi(\gamma)) = \mathcal{O}_G(f_S(h)).$$

By Lemma 4.8, the definitions of good orbit and the mapping F_φ are independent of the choice of $S \in \mathcal{SO}(\varphi)$ and $h \in H_{\varphi,S}$ with $\mathcal{O}_\varphi(\gamma) = \mathcal{O}_\varphi(h)$.

Fix $S \in \mathcal{SO}(\Phi)$ and $h \in H_S^1 \cap G'$. Define

$$W(\Phi_K, \varphi, h) = \{w \in W(\Phi_K, h) : \exists S' \in \mathcal{SO}(\Phi, \varphi) \text{ such that } (w, S') \in \Lambda(h)\}.$$

The following two lemmas are an easy consequence of Theorem 2.5 applied to G and G_φ .

Lemma 4.9. *Let $w \in W(\Phi_K, \varphi, h)$ and suppose that $S', S'' \in \mathcal{SO}(\Phi, \varphi)$ such that $(w, S'), (w, S'') \in \Lambda(h)$. Then*

$$\mathcal{O}_\varphi(f_{S'}^{-1}((w, S') \cdot h)) = \mathcal{O}_\varphi(f_{S''}^{-1}((w, S'') \cdot h)).$$

For $w \in W(\Phi_K, \varphi, h)$, we will write

$$\mathcal{O}(\varphi, w \cdot h) = \mathcal{O}_\varphi(f_{S'}^{-1}((w, S') \cdot h))$$

where $S' \in \mathcal{SO}(\Phi, \varphi)$ such that $(w, S') \in \Lambda(h)$. By Lemma 4.9, it is independent of the choice of S' .

Lemma 4.10. *Let $S \in \mathcal{SO}(\Phi)$ and $h \in H_S^1 \cap G'$. Then*

$$F_\varphi^{-1}(\mathcal{O}_G(h)) = \{\mathcal{O}(\varphi, w \cdot h) : w \in W(\Phi_K, \varphi, h)\}.$$

For $w \in W(\Phi_K, \varphi, h)$, let $S' \in \mathcal{SO}(\Phi, \varphi)$ such that $(w, S') \in \Lambda(h)$. We will write $W(\varphi_K, w \cdot h) = W(\varphi_K, h')$ where $h' = f_{S'}^{-1}((w, S') \cdot h) \in H_{\varphi, S'}$. By Lemma 4.7 and the remark preceding Lemma 2.4, $\varphi_{S', t'} = \varphi \cap \Phi_{S', t'}$ is independent of the choices of S' , so that $W(\varphi_K, w \cdot h)$ is also independent of this choice.

Let $w_1 \in W(\Phi_K, \varphi, h)$, $w \in W(\varphi_K, w_1 \cdot h)$. Then there is an element $ww_1 \in W(\Phi_K, h)$ defined as in Lemma 2.6.

Lemma 4.11. *Let $w_1 \in W(\Phi_K, \varphi, h)$. Then for any $w \in W(\varphi_K, w_1 \cdot h)$, $ww_1 \in W(\Phi_K, \varphi, h)$.*

Proof. Write $w_1 = [v_1, s_1]$ where $v_1 \in W(\Phi_K)$ and $s_1 \in W(\Phi_t)$ and $w = [v, s]$ where $v \in W(\varphi_K)$, $s \in W(\varphi \cap \Phi_{v_1 t})$. Then as in Lemma 2.6 we have

$$ww_1 = [vv_1, v_1^{-1}sv_1s_1]$$

where $vv_1 \in W(\Phi_K)$ and $v_1^{-1}sv_1s_1 \in W(\Phi_t)$. Let $S_1 \in \mathcal{SO}(\Phi, \varphi)$ such that $(w_1, S_1) \in \Lambda(h)$. Since $S_1 \in \mathcal{SO}(\Phi, \varphi)$ and $v \in W(\varphi_K)$, $vS_1 \in \mathcal{SO}(\Phi, \varphi)$. Since $S_1 \equiv v_1S$ and $v_1t \in T_{S_1}^1$, then $vS_1 \equiv vv_1S$ and $vv_1t \in T_{vS_1}^1$. Thus $(ww_1, vS_1) \in \Lambda(h)$ and so $ww_1 \in W(\Phi_K, \varphi, h)$. $\square$

We want to characterize exactly which of the orbits $\mathcal{O}(\varphi, w \cdot h)$, $w \in W(\Phi, \varphi, h)$, are equal. For this we need an extra regularity condition. Recall that in (2.1) we defined

$$H_S'' = \{h \in H_S^1 \cap G' : \text{if } (w, S) \in \Lambda(h) \text{ and } (w, S) \cdot h = h, \text{ then } w = 1\}.$$

Let $w_1, w_2 \in W(\Phi_K, \varphi, h)$. Then we write $w_2 \sim_{\varphi, h} w_1$, or more simply $w_2 \sim w_1$, if there is $w \in W(\varphi_K, w_1 \cdot h)$ such that $w_2 = ww_1$.

Lemma 4.12. *Let $h \in H_S''$.*

(i) *Let $w_1, w_2 \in W(\Phi_K, \varphi, h)$. Then*

$$\mathcal{O}(\varphi, w_1 \cdot h) = \mathcal{O}(\varphi, w_2 \cdot h)$$

if and only if $w_2 \sim_{\varphi, h} w_1$. In particular, $w_2 \sim_{\varphi, h} w_1$ is an equivalence relation on $W(\Phi, \varphi, h)$.

- (ii) Let $w_i, 1 \leq i \leq p$, be a complete set of representatives for the equivalence classes in $W(\Phi_K, \varphi, h)$ given by $\sim_{\varphi, h}$. Let $\sigma \in W(\Phi_K, \varphi, h)$. Then there are unique $1 \leq i \leq p$ and $w \in W(\varphi_K, w_i \cdot h)$ such that $\sigma = ww_i$.

Proof. (i) Let $S_1, S_2 \in SO(\Phi, \varphi)$ such that $(w_i, S_i) \in \Lambda(h), i = 1, 2$. Write

$$h'_i = f_{S_i}^{-1}((w_i, S_i) \cdot h), \quad i = 1, 2.$$

Then by definition

$$\mathcal{O}(\varphi, w_i \cdot h) = \mathcal{O}_\varphi(f_{S_i}^{-1}((w_i, S_i) \cdot h)) = \mathcal{O}_\varphi(h'_i), \quad i = 1, 2.$$

By Theorem 2.5, $\mathcal{O}_\varphi(h'_1) = \mathcal{O}_\varphi(h'_2)$ if and only if there is $w \in W(\varphi_K, h'_1) = W(\varphi_K, w_1 \cdot h)$ such that $(w, S_2) \in \Lambda(\varphi, h'_1)$ and $h'_2 = (w, S_2) \cdot h'_1$. Now by Lemmas 2.7 and 4.8, $h'_2 = (w, S_2) \cdot h'_1$ if and only if

$$\begin{aligned} h &= (w_2^{-1}, S) \cdot (w_2, S_2) \cdot h = (w_2^{-1}, S) \cdot f_{S_2}(h'_2) \\ &= (w_2^{-1}, S) \cdot (w, S_2) \cdot f_{S_1}(h'_1) = (w_2^{-1}w, S) \cdot (w_1, S_1) \cdot h = (w_2^{-1}ww_1, S) \cdot h. \end{aligned}$$

Thus $\mathcal{O}_\varphi(h'_1) = \mathcal{O}_\varphi(h'_2)$ if and only if there is $w \in W(\varphi_K, w_1 \cdot h)$ such that

$$h = (w_2^{-1}ww_1, S) \cdot h.$$

Now since $h \in H''_S$, this is if and only if $w_2^{-1}ww_1 = 1$ so that $w_2 = ww_1$.

(ii) Let $\sigma \in W(\Phi_K, \varphi, h)$. Since $\sim$ is an equivalence relation, there is a unique $1 \leq i \leq p$ so that $\sigma \sim w_i$. Thus there is $w \in W(\varphi_K, w_i \cdot h)$ such that $\sigma = ww_i$. Suppose that $w' \in W(\varphi_K, w_i \cdot h)$ such that $\sigma = w'w_i$. Write $w_i = [v_i, s_i], w = [v, s], w' = [v', s'], v_i \in W(\Phi_K), s_i \in W(\Phi_t), v, v' \in W(\varphi_K), s, s' \in W(\varphi_{v_i t})$. Then if $ww_i = w'w_i$ we must have $vsv_i s_i = v's'v_i s_i$ as elements of $W(\Phi)$ so that $vs = v's'$. But recall that Lemma 4.3 (ii) implies that $w = [v, s] = w' = [v', s']$ as elements of $W(\varphi_K, w_i \cdot h)$ if and only if $vs = v's'$. $\square$

For $\gamma \in G'$, let $X_\varphi(\gamma)$ denote a complete set of representatives for the good G_φ orbits which map to $\mathcal{O}_G(\gamma)$ under the orbit correspondence F_φ . As an immediate consequence of Lemmas 4.10 and 4.12 we have the following.

Theorem 4.13. *Let $h \in H''_S$, and let $w_i, 1 \leq i \leq p$, be a complete set of representatives for the equivalence classes in $W(\Phi_K, \varphi, h)$ given by $\sim_{\varphi, h}$. For each $1 \leq i \leq p$, fix $S_i \in SO(\Phi, \varphi)$ such that $(w_i, S_i) \in \Lambda(h)$. Then we can take*

$$X_\varphi(h) = \{f_{S_i}^{-1}((w_i, S_i) \cdot h) : 1 \leq i \leq p\}.$$

5. TWO-STRUCTURES

In this section we review basic facts about two-structures and prove results which will be needed for the lifting of discrete series characters. As in §4 we let $\mathcal{T}(\Phi)$ denote the set of all two-structures for Φ .

Fix a choice Φ^+ of positive roots, and let $\varphi \in \mathcal{T}(\Phi)$. Let $\mathcal{B}$ be the set of positive long roots of φ , where by a long root of φ we mean any root in an A_1 factor or any long root in a $B_2 \simeq C_2$ factor. Then $\mathcal{B}$ is an orthogonal basis for φ . Suppose that $\beta \in \Phi$ is orthogonal to every $\alpha \in \mathcal{B}$. Then β is orthogonal to φ and the reflection s_β in β is an element of $W(\Phi)$ with $\det s_\beta = -1$ and $s_\beta \varphi^+ = \varphi^+$. This contradicts the definition of two-structures. Thus no root of Φ is orthogonal to every $\alpha \in \mathcal{B}$.

Choose an ordering $\{\alpha_1, \dots, \alpha_n\}$ of $\mathcal{B}$ which satisfies the following condition. Suppose that two roots of $\mathcal{B}$ are in the same irreducible factor of type B_2 of φ . Then they should be adjacent in the ordering so that they are α_i, α_{i+1} for some $1 \leq i \leq n-1$.

Now since they are long roots in a factor of type B_2 , $\frac{1}{2}(\pm\alpha_i \pm \alpha_{i+1})$ are roots in φ . We order α_i, α_{i+1} so that $\frac{1}{2}(\alpha_i - \alpha_{i+1}) \in \Phi^+$. Since no root of Φ is orthogonal to every $\alpha \in \mathcal{B}$, we can use the lexicographic ordering with respect to $\mathcal{B}$ to define a new choice of positive roots for Φ which we will denote by $\Phi^+(\mathcal{B})$. Now there is a unique $w \in W(\Phi)$ such that $\Phi^+(\mathcal{B}) = w\Phi^+$. As in page 501 of [9] we define

$$(5.1) \quad \epsilon(\varphi : \Phi^+) = \det w.$$

Lemma 5.1. *The definition of $\epsilon(\varphi : \Phi^+)$ is independent of the choice of ordering of $\mathcal{B}$ used, so long as the ordering satisfies the above condition for each irreducible factor of φ of type B_2 .*

Proof. In the case that Φ is spanned by strongly orthogonal roots so that $\text{rank } \Phi = \text{rank } \varphi$, problems in page 513 of [9] show how to prove that $\epsilon(\varphi : \Phi^+)$ is well-defined. We will modify this proof to prove the result in the cases that Φ is not spanned by strongly orthogonal roots. We may as well assume that Φ is irreducible. In this case all roots of Φ have the same length, so there are no factors of type B_2 in φ and hence no restrictions on the ordering of $\mathcal{B}$.

Let $\{\alpha_1, \dots, \alpha_n\}$ be an ordering of $\mathcal{B}$. Let Φ_1^+ be the positive roots corresponding to this ordering, and let Φ_2^+ be the positive roots corresponding to the ordering where α_i and α_{i+1} are interchanged for some $1 \leq i \leq n-1$. If $\Phi_1^+ = w_1\Phi^+$ and $\Phi_2^+ = w_2\Phi^+$, then $\Phi_2^+ = w_2w_1^{-1}\Phi_1^+$. Thus to show $\det w_1 = \det w_2$ it suffices to show that $\Phi_1^+ \cap \Phi_2^-$ contains an even number of roots.

For any $\gamma \in \Phi$, define $c_i = c_i(\gamma) = \langle \alpha_i, \gamma \rangle / \langle \alpha_i, \alpha_i \rangle$, $1 \leq i \leq n$. Then $\Phi_1^+ \cap \Phi_2^-$ consists of all roots γ such that $c_1 = \dots = c_{i-1} = 0$, $c_i > 0$, and $c_{i+1} < 0$. Thus γ has the form

$$\gamma = c_i\alpha_i + c_{i+1}\alpha_{i+1} + \lambda$$

where λ is in the real vector space spanned by Φ , is orthogonal to $\alpha_1, \dots, \alpha_i, \alpha_{i+1}$, but is not necessarily a root. Further, $\lambda \neq 0$ since α_i and α_{i+1} are strongly orthogonal, and all roots of Φ have the same length. Now

$$\gamma' = -s_{\alpha_i}s_{\alpha_{i+1}}\gamma = c_i\alpha_i + c_{i+1}\alpha_{i+1} - \lambda$$

is also an element of $\Phi_1^+ \cap \Phi_2^-$. Thus the roots in $\Phi_1^+ \cap \Phi_2^-$ can be grouped in pairs. $\square$

Define

$$W(\varphi : \Phi^+) = \{w \in W(\Phi) : w(\varphi \cap \Phi^+) \subset \Phi^+\}.$$

The following lemma is easy to check from the definitions.

Lemma 5.2. *For any $\sigma \in W(\varphi : \Phi^+)$,*

$$\epsilon(\sigma\varphi : \Phi^+) = \det \sigma \epsilon(\varphi : \Phi^+).$$

Further, for any $s \in W(\Phi)$,

$$\epsilon(s\varphi : s\Phi^+) = \epsilon(\varphi : \Phi^+).$$

Theorem 5.3. *Suppose that Φ is irreducible and let Φ^+ be any choice of positive roots for Φ . Then*

$$\sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(\varphi : \Phi^+) = \begin{cases} (-1)^n & \text{if } \Phi \text{ is of type } A_{2n}; \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Suppose that Φ_1^+ and Φ_2^+ are two choices of positive roots. Then there is $w \in W(\Phi)$ such that $\Phi_2^+ = w\Phi_1^+$, and using Lemma 5.2,

$$\sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(\varphi : \Phi_2^+) = \sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(\varphi : w\Phi_1^+) = \sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(w^{-1}\varphi : \Phi_1^+) = \sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(\varphi : \Phi_1^+)$$

since w permutes the elements of $\mathcal{T}(\Phi)$. Thus it is enough to prove the result for one choice of positive roots.

We will prove the result by induction on the rank of Φ . It is obvious when Φ has rank 0 or 1. Assume that the rank of Φ is greater than one, and that the theorem is true for root systems of rank less than that of Φ .

Fix $\varphi_0 \in \mathcal{T}(\Phi)$ and pick a basis $\mathcal{B}$ of long roots of φ_0 . Pick an ordering $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ so that if two roots of $\mathcal{B}$ are in the same irreducible factor of type B_2 of φ_0 , then they are adjacent in the ordering. Let $\Phi^+ = \Phi^+(\mathcal{B})$.

Define $\Phi_1 = \{\beta \in \Phi : \langle \beta, \alpha_i \rangle = 0, 1 \leq i \leq n-1\}$, $\Phi_1^+ = \Phi_1 \cap \Phi^+$. Then $\alpha_n \in \Phi_1^+$ so that $\Phi_1^+ \neq \emptyset$. Let α be a simple root for Φ_1^+ . Then because Φ^+ is the lexicographic ordering with respect to $\alpha_1, \dots, \alpha_n$, it is easy to see that α is also a simple root for Φ^+ .

If Φ is spanned by strongly orthogonal roots, then $\alpha_1, \dots, \alpha_n$ is an orthogonal basis for the vector space spanned by Φ , so we must have $\Phi_1^+ = \{\alpha_n\}$ and $\alpha = \alpha_n$. Suppose Φ is not spanned by strongly orthogonal roots. Then Φ is of type A_{2n} , $n \geq 1$, A_{2n-1} , $n \geq 2$, D_{2n+1} , $n \geq 1$, or E_6 , in which case $n = 4$. For any $\beta \in \Phi$, let Φ_β denote the set of roots in Φ orthogonal to β . If Φ is of type A_{2n} , A_{2n-1} , D_{2n+1} , or E_6 , and $\beta \in \Phi$, then Φ_β is of type A_{2n-2} , A_{2n-3} , $D_{2n-1} \times A_1$, or A_5 respectively. Using this we can see that Φ_1 is of type A_1 except when Φ is of type A_{2n} , in which case it is of type A_2 . Thus $\alpha = \alpha_n$ except when Φ is of type A_{2n} . In this case α_n is the unique nonsimple root in Φ_1^+ .

Suppose that $\varphi \in \mathcal{T}(\Phi)$ and $\alpha \notin \varphi$. Then since α is simple for Φ^+ , $s_\alpha(\varphi \cap \Phi^+) \subset \Phi^+$ so that by Lemma 5.2,

$$\epsilon(s_\alpha \varphi : \Phi^+) = -\epsilon(\varphi : \Phi^+).$$

Thus such two-structures appear in pairs which cancel in the sum, and so

$$\sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(\varphi : \Phi^+) = \sum_{\varphi \in \mathcal{T}(\Phi, \alpha)} \epsilon(\varphi : \Phi^+)$$

where $\mathcal{T}(\Phi, \alpha) = \{\varphi \in \mathcal{T}(\Phi) : \alpha \in \varphi\}$.

Let $\varphi \in \mathcal{T}(\Phi, \alpha)$ and suppose that $\varphi_\alpha = \varphi \cap \Phi_\alpha \notin \mathcal{T}(\Phi_\alpha)$. (Here as above, Φ_α denotes the roots in Φ orthogonal to α .) Using the techniques of Lemmas 4.8 and 4.9 of [5] we see that such two-structures appear in pairs which cancel in the sum. Thus in any case we have

$$\sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(\varphi : \Phi^+) = \sum_{\varphi \in \mathcal{T}(\Phi, \alpha)'} \epsilon(\varphi : \Phi^+)$$

where $\mathcal{T}(\Phi, \alpha)' = \{\varphi \in \mathcal{T}(\Phi, \alpha) : \varphi_\alpha \in \mathcal{T}(\Phi_\alpha)\}$.

Suppose that Φ is spanned by strongly orthogonal roots. Then it is proven in Lemmas 3.3 and 4.10 of [5] that $\varphi \mapsto \varphi_\alpha$ gives a bijection between $\mathcal{T}(\Phi, \alpha)'$ and $\mathcal{T}(\Phi_\alpha)$, and that

$$\epsilon(\varphi : \Phi^+) = \epsilon(\varphi_\alpha : \Phi_\alpha^+)$$

for all $\varphi \in \mathcal{T}(\Phi, \alpha)'$, where $\Phi_\alpha^+ = \Phi^+ \cap \Phi_\alpha$. Thus the theorem follows from our induction hypothesis applied to the root system Φ_α since Φ_α is spanned by strongly orthogonal roots and hence contains no irreducible factors of type A_{2n} .

Thus we may as well assume that all roots of Φ are of the same length so that every $\varphi \in \mathcal{T}(\Phi)$ is of type A_1^n . In this case, when $\alpha \in \varphi$ we have $\varphi = \varphi_\alpha \cup \{\pm\alpha\}$. Clearly φ_α is always in $\mathcal{T}(\Phi_\alpha)$ in this case, and any $\psi_\alpha \in \mathcal{T}(\Phi_\alpha)$ can be expanded by adding $\pm\alpha$ to become an element of $\mathcal{T}(\Phi)$. Thus the mapping $\varphi \mapsto \varphi_\alpha$ gives a bijection between $\mathcal{T}(\Phi, \alpha)$ and $\mathcal{T}(\Phi_\alpha)$.

Let φ_0 be the two-structure used above to define Φ^+ . Then by definition $\epsilon(\varphi_0 : \Phi^+) = 1$. If Φ is not of type A_{2n} , then $\alpha = \alpha_n$ so that $\varphi_0 \in \mathcal{T}(\Phi, \alpha)$. Write $\varphi_1 = \varphi_0$ in this case. Now suppose that Φ is of type A_{2n} . Recall Φ_1 is of type A_2 and that α is a simple root for Φ_1^+ while α_n is the unique nonsimple root in Φ_1^+ . Let α' denote the other simple root in Φ_1^+ , and define $\varphi_1 = s_{\alpha'}\varphi_0$. Then $s_{\alpha'}\alpha_i = \alpha_i$, $1 \leq i \leq n-1$, while $s_{\alpha'}\alpha_n = \alpha$. Thus $\varphi_1 \in \mathcal{T}(\Phi, \alpha)$ and $s_{\alpha'}\varphi_0^+ = \varphi_1^+$ so that $\epsilon(\varphi_1 : \Phi^+) = -1$.

In either case, we let $\psi_1 = (\varphi_1)_\alpha$. Then $\epsilon(\psi_1 : \Phi_\alpha^+) = 1$ since Φ_α^+ is the lexicographic ordering with respect to the basis $\{\alpha_1, \dots, \alpha_{n-1}\}$ of ψ_1 . Now let $\varphi \in \mathcal{T}(\Phi, \alpha)$. Since $\varphi_\alpha \in \mathcal{T}(\Phi_\alpha)$, there is $\sigma \in W(\Phi_\alpha)$ such that $(\varphi_\alpha \cap \Phi^+) = \sigma(\psi_1 \cap \Phi^+)$. Now $(\varphi_1 \cap \Phi^+) = (\psi_1 \cap \Phi^+) \cup \{\alpha\}$ and $\sigma\alpha = \alpha$, so that $\sigma(\varphi_1 \cap \Phi^+) = (\varphi_\alpha \cap \Phi^+) \cup \{\alpha\} = \varphi \cap \Phi^+$. Thus

$$\epsilon(\varphi : \Phi^+) = \det \sigma \epsilon(\varphi_1 : \Phi^+) \quad \text{and} \quad \epsilon(\varphi_\alpha : \Phi_\alpha^+) = \det \sigma \epsilon(\psi_1 : \Phi_\alpha^+).$$

Now if Φ is not of type A_{2n} , then $\epsilon(\varphi_1 : \Phi^+) = \epsilon(\psi_1 : \Phi_\alpha^+) = 1$ so that $\epsilon(\varphi : \Phi^+) = \epsilon(\varphi_\alpha : \Phi_\alpha^+)$ for all $\varphi \in \mathcal{T}(\Phi, \alpha)$. If Φ is of type A_{2n} , then

$$\epsilon(\varphi_1 : \Phi^+) = -1 = -\epsilon(\psi_1 : \Phi_\alpha^+)$$

so that $\epsilon(\varphi : \Phi^+) = -\epsilon(\varphi_\alpha : \Phi_\alpha^+)$ for all $\varphi \in \mathcal{T}(\Phi, \alpha)$. The theorem now follows from our induction hypothesis applied to the root system Φ_α since Φ_α contains no irreducible factors of type A_{2n} unless Φ is of type A_{2n} , while in this case Φ_α is of type $A_{2(n-1)}$. $\square$

Remark 5.1. A different definition of the signs $\epsilon(\varphi : \Phi^+)$ was given in [4]. This definition satisfies Lemma 5.2, but $\sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(\varphi : \Phi^+) = 1$ for every root system Φ . Thus the two definitions agree unless Φ contains an irreducible factor of type A_{2k} , $k \equiv 1 \pmod{2}$, in which case they give opposite signs.

Let Φ^+ be any choice of positive roots for Φ , and for $\varphi \in \mathcal{T}(\Phi)$, let $\varphi^+ = \varphi \cap \Phi^+$. Then we define

$$\rho(\Phi^+) = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha, \quad \rho(\varphi^+) = \frac{1}{2} \sum_{\alpha \in \varphi^+} \alpha, \quad \rho(\Phi^+, \varphi) = \rho(\Phi^+) - \rho(\varphi^+).$$

Let $\mathcal{B}$ be a basis of long roots for φ , and let $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ be an ordering so that if two roots of $\mathcal{B}$ are in the same irreducible factor of type B_2 of φ , then they are adjacent in the ordering. Let $\Phi^+ = \Phi^+(\mathcal{B})$, $\varphi^+ = \varphi \cap \Phi^+$.

Lemma 5.4. *Assume that Φ is irreducible and let $\varphi \in \mathcal{T}(\Phi)$. Then*

$$[\Phi^+ \setminus \varphi^+] \equiv \begin{cases} 2k \pmod{4} & \text{if } \Phi \text{ is of type } A_{2k}; \\ 0 \pmod{4} & \text{otherwise.} \end{cases}$$

Further, if Φ is not of type A_{2k} or C_n, n odd, then

$$\rho(\Phi^+, \varphi) = \sum_{i=1}^n n_i \alpha_i$$

where $n_i \in \mathbf{Z}, 1 \leq i \leq n$. Suppose that Φ is of type $C_{2k+1}, k \geq 0$. Then φ has a unique irreducible factor of type A_1 . If the root from this irreducible factor is first in the ordering, then again we have

$$\rho(\Phi^+, \varphi) = \sum_{i=1}^n n_i \alpha_i$$

where $n_i \in \mathbf{Z}, 1 \leq i \leq n = 2k + 1$. If this root is last in the ordering, then

$$\rho(\Phi^+, \varphi) = \frac{1}{2} \left(\sum_{i=1}^{2k} \alpha_i \right) + \sum_{i=1}^n n_i \alpha_i$$

where $n_i \in \mathbf{Z}, 1 \leq i \leq n = 2k + 1$.

Proof. Suppose first that Φ is of type $A_{2k}, k \geq 1$. Then $[\Phi^+] = k(2k + 1)$ and φ is of type A_1^k so that $[\varphi^+] = k$. Thus $[\Phi^+ \setminus \varphi^+] = 2k^2 \equiv 2k \pmod{4}$.

Suppose that Φ is of type G_2 . Then $\varphi = \{\pm\alpha_1, \pm\alpha_2\}$ where α_1 is long, α_2 is short, and $\langle \alpha_1, \alpha_2 \rangle = 0$. Now

$$\Phi \setminus \varphi = \left\{ \pm \frac{1}{2} \alpha_1 \pm \frac{1}{2} \alpha_2, \pm \frac{1}{2} \alpha_1 \pm \frac{3}{2} \alpha_2 \right\}.$$

Thus $[\Phi^+ \setminus \varphi^+] = 4$, and if $\mathcal{B} = \{\alpha_1, \alpha_2\}$, then $\rho(\Phi^+) - \rho(\varphi^+) = \alpha_1$, while if $\mathcal{B} = \{\alpha_2, \alpha_1\}$, then $\rho(\Phi^+) - \rho(\varphi^+) = 2\alpha_2$.

Now assume that Φ is not of type A_{2k} or G_2 . For any $\gamma \in \Phi$, define

$$c_i(\gamma) = \langle \alpha_i, \gamma \rangle / \langle \alpha_i, \alpha_i \rangle, \quad 1 \leq i \leq n.$$

Since Φ is not of type G_2 we have $c_i \in \{0, \pm 1/2, \pm 1\}, 1 \leq i \leq n$.

Fix $\gamma \in \Phi^+$. Since no root of Φ is orthogonal to every element of $\mathcal{B}$, there is at least one $i, 1 \leq i \leq n$, such that $c_i(\gamma) \neq 0$. Suppose that there is only one such index. That is, there is $1 \leq r \leq n$ such that $c_r(\gamma) \neq 0, c_i(\gamma) = 0, i \neq r$. Then as in the proof of Theorem 5.3, since Φ is not of type A_{2n} , the set of roots in Φ orthogonal to $(n - 1)$ orthogonal roots must be of type A_1 . Thus $\gamma = \alpha_r \in \varphi^+$.

Fix $\gamma \in \Phi^+ \setminus \varphi^+$. Then by the above, there are at least two indices $i, 1 \leq i \leq n$, such that $c_i = c_i(\gamma) \neq 0$. Let $r < s$ be the two smallest such indices. Then we can write

$$\gamma = c_r \alpha_r + c_s \alpha_s + \gamma'$$

where $c_r > 0, c_s \neq 0$, and γ' is in the real vector space spanned by Φ , is orthogonal to $\alpha_i, 1 \leq i \leq s$, but is not necessarily a root. Since $c_r > 0$, we must have $c_r \in \{1/2, 1\}$. Now define

$$\gamma_1 = \gamma = c_r \alpha_r + c_s \alpha_s + \gamma', \quad \gamma_2 = -s_{\alpha_r} \gamma = c_r \alpha_r - c_s \alpha_s - \gamma',$$

$$\gamma_3 = s_{\alpha_s} \gamma = c_r \alpha_r - c_s \alpha_s + \gamma', \quad \gamma_4 = -s_{\alpha_s} s_{\alpha_r} \gamma = c_r \alpha_r + c_s \alpha_s - \gamma'.$$

Since $\alpha_r, \alpha_s \in \varphi$ and $\gamma \notin \varphi$, we have $\gamma_i \notin \varphi, 1 \leq i \leq 4$. Thus $\gamma_i \in \Phi^+ \setminus \varphi^+$ for all $1 \leq i \leq 4$.

Suppose that $\gamma' \neq 0$. Then the roots $\gamma_i, 1 \leq i \leq 4$ are distinct. Further,

$$\frac{1}{2} \sum_{i=1}^4 \gamma_i = 2c_r \alpha_r, \quad 2c_r \in \{1, 2\},$$

and so is an integral multiple of α_r . Thus roots of this type in $\Phi^+ \setminus \varphi^+$ can be grouped into subsets of four roots so that the half sum of each group is an integral multiple of α_i for some $1 \leq i \leq n$.

Now assume that $\gamma' = 0$. Then $\gamma = c_r \alpha_r + c_s \alpha_s$ is in the linear span of the orthogonal roots α_r, α_s . This cannot happen if all roots of Φ are of the same length. Thus the lemma is proven in the equal length case.

Now assume that Φ is of type B_n, C_n , or F_4 , and $\gamma \in \Phi^+ \setminus \varphi^+$ such that $\gamma = c_r \alpha_r + c_s \alpha_s$. Now since α_r, α_s are strongly orthogonal, we see that γ can only be a root if α_r, α_s are both long, γ is short, and $c_r = 1/2, c_s = \pm 1/2$. But if Φ is of type B_n , all short roots are elements of φ . Thus we may as well assume that Φ is of type C_n or F_4 .

Assume that Φ is of type $C_n, n \geq 1$, or $F_n, n = 4$. Then the short roots of Φ^+ are all roots of the form $\frac{1}{2}(\alpha_i \pm \alpha_j), 1 \leq i < j \leq n$. Let k be the greatest integer in $n/2$. Then there are $2k$ short roots in φ^+ . Thus if n is even, there are $4k(k-1)$ short roots in $\Phi^+ \setminus \varphi^+$, while if n is odd, there are $4k^2$ short roots in $\Phi^+ \setminus \varphi^+$.

Suppose that n is even, or n is odd and the root from the irreducible factor of type A_1 of φ is last in the ordering. Then $\gamma \in \varphi^+$ just in case $\gamma = \frac{1}{2}(\alpha_{2i-1} \pm \alpha_{2i})$ for some $1 \leq i \leq k$. Thus the half sum of short roots in $\Phi^+ \setminus \varphi^+$ is

$$\frac{1}{2} \sum_{1 \leq j \leq n} (n-j) \alpha_j - \frac{1}{2} \sum_{1 \leq i \leq k} \alpha_{2i-1} = \sum_{1 \leq i \leq k} (n-2i) \left(\frac{1}{2} \right) (\alpha_{2i-1} + \alpha_{2i}).$$

If $n = 2k$ is even, then $\frac{1}{2}(n-2i) = k-i \in \mathbf{Z}, 1 \leq i \leq k$, while if $n = 2k+1$ is odd, then $\frac{1}{2}(n-2i) = k-i+1/2$.

Now suppose that n is odd and the root from the irreducible factor of type A_1 of φ_0 is first in the ordering. Then $\gamma \in \varphi^+$ just in case $\gamma = \frac{1}{2}(\alpha_{n-2i-1} \pm \alpha_{n-2i})$ for some $0 \leq i \leq k-1$, where k is the greatest integer in $n/2$. Thus the half sum of short roots in $\Phi^+ \setminus \varphi^+$ is

$$\frac{1}{2} \sum_{1 \leq j \leq n} (n-j) \alpha_j - \frac{1}{2} \sum_{0 \leq i \leq k-1} \alpha_{n-2i-1} = \sum_{0 \leq j \leq k-1} j \alpha_{n-2j-1} + \sum_{1 \leq j \leq k} j \alpha_{n-2j}.$$

□

Write $r(\Phi) = ([\Phi] - [\varphi])/4$ where $\varphi \in \mathcal{T}(\Phi)$. Then $r(\Phi)$ is independent of φ since all $\varphi \in \mathcal{T}(\Phi)$ are conjugate via $W(\Phi)$, and is an integer by Lemma 5.4.

Theorem 5.5. *Let Φ^+ be any choice of positive roots for Φ . Then*

$$\sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(\varphi : \Phi^+) = (-1)^{r(\Phi)}.$$

Let $\varphi \in \mathcal{T}(\Phi), \varphi^+ = \varphi \cap \Phi^+$.

- (i) *Suppose that Φ is of type $A_{2n}, n \geq 1$. Then $\rho(\Phi^+) - \rho(\varphi^+)$ is not in the weight lattice of Φ .*
- (ii) *Suppose that Φ contains no irreducible factors of type $A_{2n}, n \geq 1$. Then $\rho(\varphi^+)$ is in the weight lattice of Φ and $\rho(\Phi^+) - \rho(\varphi^+)$ is in the root lattice of Φ .*

Proof. The first result follows directly from combining Theorem 5.3 and Lemma 5.4, and (i) is easy to check. It remains to prove (ii). Since $\rho(\Phi^+)$ is in the weight lattice of Φ , the first statement will follow from the second. But by Lemma 5.4 there is a choice Φ_1^+ of positive roots for Φ so that $\rho(\Phi_1^+) - \rho(\varphi \cap \Phi_1^+)$ is in the root lattice of φ , hence in the root lattice of Φ . But $\rho(\Phi^+) - \rho(\varphi^+)$ differs from $\rho(\Phi_1^+) - \rho(\varphi \cap \Phi_1^+)$ by elements of the root lattice of Φ . $\square$

Define rank $\mathcal{T}(\Phi)$ to be the common rank of every $\varphi \in \mathcal{T}(\Phi)$. Let $\mathcal{T}_{\text{aug}}(\Phi)$ denote the set of all root subsystems $\varphi \subset \Phi$ such that every irreducible factor of φ is of type A_1 or B_2 and rank $\varphi = \text{rank } \mathcal{T}(\Phi)$. Thus $\mathcal{T}(\Phi) \subset \mathcal{T}_{\text{aug}}(\Phi)$.

Lemma 5.6. *Let $\varphi \in \mathcal{T}_{\text{aug}}(\Phi)$. Then there is at least one $\varphi_0 \in \mathcal{T}(\Phi)$ such that $\varphi \subset \varphi_0$. Moreover, when Φ is irreducible, φ_0 is unique unless:*

- (i) Φ is of type B_n and φ contains at least 3 irreducible factors of type A_1 generated by short roots;
- (ii) Φ is of type C_n and φ contains at least 3 irreducible factors of type A_1 generated by long roots;
- (iii) Φ is of type F_4 and φ contains 4 irreducible factors of type A_1 generated by roots of the same length.

Finally, if $\varphi \notin \mathcal{T}(\Phi)$ then there are two irreducible factors of φ of type A_1 which generate a root system of type B_2 in Φ .

Proof. We may as well assume that Φ is irreducible. If Φ contains no root systems of type B_2 , then augmented two-structures φ of Φ must be of type A_1^n where $n = \text{rank } \mathcal{T}(\Phi)$. But all such φ are two-structures. Thus in this case $\mathcal{T}_{\text{aug}}(\Phi) = \mathcal{T}(\Phi)$ and the lemma is trivial. If Φ contains root systems of type B_2 , then Φ is of type B_n, C_n , or F_4 . In these cases it is easy to check the lemma using the following facts. If Φ is of type B_n , then any long root of Φ is contained in a unique irreducible factor of Φ of type B_2 , while any pair of orthogonal short roots span an irreducible factor of type B_2 . If Φ is of type C_n , then any short root of Φ is contained in a unique irreducible factor of Φ of type B_2 , while any pair of orthogonal long roots span an irreducible factor of type B_2 . Finally, when Φ is of type F_4 , any two orthogonal roots of the same length determined an irreducible factor of type B_2 . $\square$

The following lemma is elementary.

- Lemma 5.7.** (i) *Let φ be a root subsystem of Φ such that every irreducible factor is of type A_1 or B_2 . Then $\varphi \in \mathcal{T}_{\text{aug}}(\Phi)$ if and only if no root of Φ is orthogonal to every root in φ .*
- (ii) *Let S be any set of orthogonal roots in Φ . Then there is $\varphi \in \mathcal{T}(\Phi)$ such that $S \subset \varphi$.*

6. LIFTING OF DISCRETE SERIES

In this section we will define lifting from the two-structure groups G_φ to G and restate Theorem 1.2 more precisely as Theorem 6.4. We then prove Theorem 6.4, modulo the proof of Lemma 6.8, which is deferred to the next section.

As in §1, for $S \in SO(\Phi, \varphi)$, $h \in H_{\varphi, S}$, such that $f_S(h) \in G'$, we define

$$D_\varphi^G(h) = |D_G(f_S(h))|^{-\frac{1}{2}} |D_\varphi(h)|^{\frac{1}{2}}.$$

Let $\mathcal{O}_\varphi$ be a good orbit in G'_φ such that $F_\varphi(\mathcal{O}_\varphi)$ is a regular semisimple orbit in G . Then there are $S \in \mathcal{SO}(\Phi, \varphi)$, $h \in H_{\varphi, S}$, such that $h \in \mathcal{O}_\varphi$. Now for any $\gamma \in \mathcal{O}_\varphi$ we can define

$$D_\varphi^G(\gamma) = D_\varphi^G(h).$$

It is easy to see that this is independent of the choice of S and h .

As in §4, for $g \in G'$, we let $X_\varphi(g)$ denote a complete set of representatives for the good G_φ orbits which map to $\mathcal{O}_G(g)$ under the orbit correspondence F_φ . It is possible that $X_\varphi(g) = \emptyset$.

Let Θ be a class function defined on G'_φ . Now for $g \in G'$, we define

$$(6.1) \quad (\text{Lift}_\varphi^G \Theta)(g) = \sum_{h' \in X_\varphi(g)} D_\varphi^G(h') \Theta(h').$$

In the case that $X_\varphi(g) = \emptyset$, we have $(\text{Lift}_\varphi^G \Theta)(g) = 0$. The definition does not depend on the choice of $X_\varphi(g)$. If $g, g' \in G'$ with $\mathcal{O}_G(g) = \mathcal{O}_G(g')$ we can take $X_\varphi(g) = X_\varphi(g')$. Thus $\text{Lift}_\varphi^G \Theta$ is a class function on G' .

- Lemma 6.1.** (i) Suppose that Φ contains no irreducible factors of type A_{2k} , $k \geq 1$. Let $\lambda \in \mathfrak{ib}^*$ be a discrete series parameter for G . Then λ is also a discrete series parameter for G_φ for every $\varphi \in \mathcal{T}(\Phi)$. Further, if λ is a limit of discrete series parameter for G , then it is either a discrete series or limit of discrete series parameter for G_φ , $\varphi \in \mathcal{T}(\Phi)$.
- (ii) Suppose that Φ is irreducible of type A_{2k} , $k \geq 1$. Then there is $\varphi \in \mathcal{T}(\Phi)$ such that no discrete series parameter or limit of discrete series parameter for G is a discrete series or limit of discrete series parameter for G_φ .

Proof. (i) follows easily from Theorem 5.5.

(ii) Using the classification in [9] for example, we see that Φ_K is of type $A_{p-1} \times A_{q-1}$ for some p and q with $p + q = 2k + 1$. Suppose that λ is a discrete series or limit of discrete series parameter for G . Then $\lambda - \rho_G \in L_B$. It is easy to check that when Φ is of type A_{2k} , ρ_G is in the root lattice of Φ , and hence is an element of L_B . Thus $\lambda \in L_B$. Now $\lambda - \rho_\varphi \in L_{B_\varphi} = L_B$ if and only if $\rho_\varphi \in L_B$. Thus it suffices to find $\varphi \in \mathcal{T}(\Phi)$ such that $\rho_\varphi \notin L_B$.

Now $\Phi_K = \emptyset$ only if $p \leq 1$ and $q \leq 1$. But $p + q \geq 3$, so this cannot occur. Thus $\Phi_K \neq \emptyset$. Fix $\alpha \in \Phi_K$, $\varphi \in \mathcal{T}(\Phi)$. Since by Theorem 5.5(ii) ρ_φ is not a weight of Φ , there is $\beta \in \Phi$ such that $2\langle \rho_\varphi, \beta \rangle / \langle \beta, \beta \rangle \notin \mathbf{Z}$. Now since Φ is irreducible, there is $w \in W(\Phi)$ such that $\alpha = w\beta$. Now $w\varphi \in \mathcal{T}(\Phi)$ and $\rho_{w\varphi}$ is not a weight for Φ_K , and so $\rho_{w\varphi} \notin L_B$. $\square$

Fix a discrete series or limit of discrete series parameter λ for G and fix $E^* = E^*(\lambda)$ as in §3 so that λ is in the closure of E^* . As in §3 we have the discrete series or limit of discrete series character Θ_λ corresponding to λ and E^* . For any $\varphi \in \mathcal{T}(\Phi)$, E^* is contained in a unique connected component E_φ^* of

$$E'(\varphi) = \{\tau \in E : \langle \tau, \alpha \rangle \neq 0 \ \forall \ \alpha \in \varphi\}.$$

Suppose first that Φ contains no irreducible factors of type A_{2k} . Since λ is a discrete series or limit of discrete series parameter for G_φ and λ is in the closure of E_φ^* , λ and E_φ^* also determine a discrete series or limit of discrete series character Θ_λ^φ for G_φ .

Now suppose that Φ contains irreducible factors of type A_{2k} , $k \geq 1$. Let ω denote the set of all $X \in \underline{g}$ such that $|\text{Im } \lambda| < \pi$ for every eigenvalue λ of adX . Define

$\Omega = \exp(\omega) \subset G$. Then ω is an invariant neighborhood of the identity in $\mathfrak{g}$ and Ω is an invariant neighborhood of the identity in G . Let $\Omega' = \Omega \cap G'$. Now for any $\varphi \in \mathcal{T}(\Phi)$, we let Ω'_φ denote the union of all good regular G_φ -orbits which map into Ω' via the orbit map F_φ . If Θ is any class function on Ω'_φ , we can define a class function on Ω' as in (6.1) by

$$(\text{Lift}_\varphi^G \Theta)(g) = \sum_{x \in X_\varphi(g)} D_\varphi^G(x) \Theta(x), \quad g \in \Omega'.$$

Note that by definition, for any $g \in \Omega'$, $X_\varphi(g) \subset \Omega'_\varphi$.

We will define a class function Θ_λ^φ on Ω'_φ . The only obstruction to defining Θ_λ^φ on G'_φ is that $\xi_{\lambda-\rho_\varphi}$ is not well-defined in the case that $e^{\rho(\Phi^+, \varphi)}$ is not well-defined on B . Let E_Φ denote the subspace of $E = i\mathfrak{b}^*$ spanned by Φ .

Lemma 6.2. *Let $S \in SO(\Phi)$, and $h \in H_S \cap \Omega'$. Then there is $H \in \underline{h}_S \cap \omega$ such that $h = \exp H$. For all $\tau \in E_\Phi$,*

$$h \mapsto \xi_\tau(h) = e^{\tau(c_S^{-1}H)}$$

gives a well defined function on $H_S \cap \Omega'$. Finally, if $(w, S') \in \Lambda(h)$, then

$$\xi_\tau((w, S') \cdot h) = \xi_{w^{-1}\tau}(h).$$

Proof. Since $h \in \Omega$ there is $H \in \omega$ such that $h = \exp H$. Then $H \in C_{\underline{g}}(h) = \underline{h}_S$, so that $H \in \underline{h}_S \cap \omega$. Now suppose that $H, H' \in \underline{h}_S \cap \omega$ such that $h = \exp H = \exp H'$. For any $\alpha \in \Phi$, $\alpha(c_S^{-1}H - c_S^{-1}H') \in 2\pi i\mathbf{Z}$. But $|\text{Im } \alpha(c_S^{-1}H - c_S^{-1}H')| \leq |\text{Im } c_S \alpha(H)| + |\text{Im } c_S \alpha(H')| < 2\pi$ because $H, H' \in \omega$. Thus $\alpha(c_S^{-1}H - c_S^{-1}H') = 0$ for all $\alpha \in \Phi$ so that $\tau(c_S^{-1}H) = \tau(c_S^{-1}H')$ for all $\tau \in E_\Phi$. Finally, if $h = \exp H, H \in \underline{h}_S \cap \omega$, then for $(w, S') \in \Lambda(h)$, $(w, S') \cdot h = \exp c_{S'} w c_S^{-1} H$ where $c_{S'} w c_S^{-1} H \in \underline{h}_{S'} \cap \omega$. Thus by definition,

$$\xi_\tau((w, S') \cdot h) = e^{\tau(w c_S^{-1} H)} = e^{w^{-1} \tau(c_S^{-1} H)} = \xi_{w^{-1} \tau}(h).$$

□

Let $S \in SO(\Phi, \varphi)$, $h' \in H_{\varphi, S} \cap \Omega'_\varphi$, and let $h = f_S(h') \in H_S \cap \Omega'$. Let Φ^+ be a choice of positive roots for Φ . Then $\rho(\Phi^+, \varphi) \in E_\Phi$, so using Lemma 6.2 we can define

$$\xi_{\rho(\Phi^+, \varphi)}(h') = \xi_{\rho(\Phi^+, \varphi)}(h), \quad \xi_{\lambda-\rho_\varphi}(h') = \xi_{\lambda-\rho_\varphi}(h) = \xi_{\lambda-\rho}(h) \xi_{\rho(\Phi^+, \varphi)}(h).$$

Now we can define Θ_λ^φ on Ω'_φ using the formula in Theorem 3.3 applied to the group G_φ with the discrete series constants corresponding to E_φ^* . (See Lemma 6.5 for details.) It corresponds to a discrete series character on a two-fold cover of G_φ .

In the case that Φ contains no irreducible factors of type $A_{2n}, n \geq 1$, we define $\Omega = G, \Omega' = G'$, and $\Omega'_\varphi = G'_\varphi$. This will allow us to handle the two cases simultaneously from now on.

Let Φ^+ be a choice of positive roots for Φ and let $\varphi^+ = \varphi \cap \Phi^+$. Recall that

$$q_G = \frac{1}{2} \dim(G/K), \quad \epsilon_G(\Phi^+ : E^*) = \epsilon_G(E^*) = \text{sign} \prod_{\alpha \in \Phi^+} \langle \alpha, \tau \rangle, \quad \tau \in E^*,$$

and let

$$q_\varphi = \frac{1}{2} \dim(G_\varphi/K_\varphi), \quad \epsilon_\varphi(\varphi^+ : E^*) = \epsilon_\varphi(E^*) = \text{sign} \prod_{\alpha \in \varphi^+} \langle \alpha, \tau \rangle, \quad \tau \in E^*.$$

Define

$$(6.2) \quad \epsilon_{\varphi}^G(E^*) = \epsilon_{\varphi}^G(\Phi^+ : E^*) = \epsilon(\varphi : \Phi^+)(-1)^{q_G}(-1)^{q_{\varphi}} \epsilon_{\varphi}(\varphi^+ : E^*) \epsilon_G(\Phi^+ : E^*).$$

Lemma 6.3. $\epsilon_{\varphi}^G(E^*)$ is independent of the choice Φ^+ of positive roots.

Proof. Fix one choice Φ^+ of positive roots and write $\varphi^+ = \varphi \cap \Phi^+$ as above. Let $w \in W(\Phi)$ so that $w\Phi^+$ is another choice of positive roots. Then there is a unique $u \in W(\varphi)$ so that $w\Phi^+ \cap \varphi = u\varphi^+$. Further, $w^{-1}u\varphi^+ \subset \Phi^+$ so that $w^{-1}u \in W(\varphi : \Phi^+)$. Thus there is $v \in W(\varphi : \Phi^+)$ so that $w = uv^{-1}$. Now by Lemma 5.2,

$$\epsilon(\varphi : w\Phi^+) = \epsilon(vu^{-1}\varphi : \Phi^+) = \epsilon(v\varphi : \Phi^+) = \det v \epsilon(\varphi : \Phi^+).$$

But

$$\epsilon_G(w\Phi^+ : E^*) = \det w \epsilon_G(\Phi^+ : E^*)$$

and

$$\epsilon_{\varphi}(u\varphi^+ : E^*) = \det u \epsilon_{\varphi}(\varphi^+ : E^*).$$

□

For $S \in \mathcal{SO}(\Phi)$, $h = ta \in H_S^1$, define

$$(6.3) \quad c(h) = c(\Phi_t)c(K, h).$$

Here $c(K, h) = [W_0(h)/W(\Phi_{K,t})]$ is the constant appearing in the discrete series character formula of Theorem 3.3, and $c(\Phi_t)$ is the constant occurring in Theorem 3.4. Note that $c(h)$ depends only on the connected component of h in H_S^1 . Further, if $S, S' \in \mathcal{SO}(\Phi)$, $h = ta \in H_S^1 \cap G'$, $h' = t'a' \in H_{S'}^1 \cap G'$ such that $\mathcal{O}_G(h) = \mathcal{O}_G(h')$, then Φ_t and $\Phi_{t'}$ are conjugate via $W(\Phi_K)$, so that $c(h) = c(h')$. Thus if $g \in G'$ and $S \in \mathcal{SO}(\Phi)$, $h \in H_S^1$, such that $\mathcal{O}_G(g) = \mathcal{O}_G(h)$, we can define $c(g) = c(h)$, and this is independent of the choice of S and h .

Recall that in §4, for any $S \in \mathcal{SO}(\Phi)$ we defined H_S'' to be the set of all $h \in H_S^1 \cap G'$ such that if $(w, S) \in \Lambda(h)$ and $(w, S) \cdot h = h$, then $w = 1$. Let G'' be the set of all $g \in G$ such that there are $S \in \mathcal{SO}(\Phi)$, $h \in H_S''$, such that $\mathcal{O}_G(g) = \mathcal{O}_G(h)$. Using Theorem 2.5 and Lemma 2.6 it is easy to see that this definition is also independent of the choice of S and h .

Theorem 6.4. Let λ be any discrete series or limit of discrete series parameter for G . Then for all $g \in \Omega \cap G''$,

$$\Theta_{\lambda}(g) = c(g) \sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon_{\varphi}^G(E^*) (\text{Lift}_{\varphi}^G \Theta_{\lambda}^{\varphi})(g).$$

The remainder of this section is devoted to the proof of Theorem 6.4. We will reduce the proof to a technical result, Lemma 6.8, which will be proven in §7.

Fix a set of positive roots Φ^+ and let $\varphi^+ = \Phi^+ \cap \varphi$, $\varphi \in \mathcal{T}(\Phi)$. Define

$$\rho_G = \rho(\Phi^+) = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha, \quad \rho_{\varphi} = \rho(\varphi^+) = \frac{1}{2} \sum_{\alpha \in \varphi^+} \alpha, \quad \rho(\Phi^+, \varphi) = \rho(\Phi^+) - \rho(\varphi^+).$$

Now let $S \in \mathcal{SO}(\Phi)$ with corresponding Cayley transform c_S . When Φ contains no irreducible factors of type A_{2n} , $n \geq 1$, $\rho(\Phi^+, \varphi)$ is in the root lattice of Φ by Theorem 5.5. Thus for $h = ta \in H_S^1$, as in §3 we can define

$$\xi_{\rho(\Phi^+, \varphi)}(h) = e^{\rho(\Phi^+, \varphi)}(t) \exp(\rho(\Phi^+, \varphi)(c_S^{-1} \log a)).$$

Otherwise, we define $\xi_{\rho(\Phi^+, \varphi)}(h)$, $h \in H_S^1 \cap \Omega'$, using Lemma 6.2. Now for $h \in H_S^1 \cap \Omega'$, we can define

$$(6.4) \quad \Delta_\varphi^G(\Phi^+ : h) = \prod_{\alpha \in c_S(\Phi^+ \setminus \varphi^+)} (1 - e^{-\alpha}(h)) \xi_{-\rho(\Phi^+, \varphi)}(h).$$

We also define

$$(6.5) \quad \delta(\Phi^+, \varphi, h) = |\Delta_\varphi^G(\Phi^+ : h)| \Delta_\varphi^G(\Phi^+ : h)^{-1}.$$

Suppose that $S \in \mathcal{SO}(\Phi, \varphi)$. As in §3, define

$$\Delta'_G(\Phi^+ : h) = \Delta'_G(h) = \prod_{\alpha \in c_S \Phi^+} (1 - e^{-\alpha}(h)), \quad h \in H_S;$$

$$\Delta'_\varphi(\varphi^+ : h') = \Delta'_\varphi(h') = \prod_{\alpha \in c_S \varphi^+} (1 - e^{-\alpha}(h')), \quad h' \in H_{\varphi, S}.$$

Then for $h' \in H_{\varphi, S}$ such that $f_S(h') = h \in \Omega'$, we have

$$\Delta_\varphi^G(\Phi^+ : h) = \Delta'_G(\Phi^+ : h)^{-1} \Delta'_\varphi(\varphi^+ : h') \xi_{-\rho(\Phi^+, \varphi)}(h),$$

and it is easy to see that

$$(6.6) \quad D_\varphi^G(h') = |\Delta_\varphi^G(\Phi^+ : h)| = \delta(\Phi^+, \varphi, h) \Delta_\varphi^G(\Phi^+ : h).$$

Fix a choice Φ^+ of positive roots, and for $\varphi \in \mathcal{T}(\Phi)$, let $\varphi^+ = \Phi^+ \cap \varphi$. We use these choices of positive roots to define $\Delta'_G, \Delta'_\varphi, \rho_G, \rho_\varphi, \epsilon_G(E^*)$, and $\epsilon_\varphi(E^*)$ for all $\varphi \in \mathcal{T}(\Phi)$.

Fix $\varphi \in \mathcal{T}(\Phi), S \in \mathcal{SO}(\Phi, \varphi)$. We use the notation of §3 for G_φ . Then Θ_λ^φ is given on $H_{\varphi, S} \cap \Omega'_\varphi$ by the following formula.

Lemma 6.5. *Let $h' = t'a' \in H_{\varphi, S} \cap \Omega'_\varphi$. Then*

$$\begin{aligned} & (-1)^{q_\varphi} \epsilon_\varphi(E^*) \Theta_\lambda^\varphi(h') \\ &= \sum_{w \in W(\varphi_K, h')} \Delta'_\varphi(w \cdot h')^{-1} \xi_{\lambda - \rho_\varphi}(w \cdot h') c(1 : E_\varphi^* : \varphi_R^+(w \cdot h')). \end{aligned}$$

Proof. In the case that λ is a discrete series or limit of discrete series parameter for G_φ , this follows directly from Theorem 3.3 since by Lemma 4.3(ii), $W(\varphi_K) \cap W(\varphi_{t'}) = W(\varphi_K, t')$, so that the constant $c(K_\varphi, h') = 1$. Otherwise, this formula can be taken as the definition of Θ_λ^φ . $\square$

For $h \in H_S^1 \cap \Omega'$ with $f_S^{-1}(h) = h' \in H_{\varphi, S}$, write $W(\varphi_K, h) = W(\varphi_K, h')$. We also write $\varphi_R^+(h) = \varphi_R^+(h')$ and $c(1 : E^* : \varphi_R^+(h)) = c(1 : E_\varphi^* : \varphi_R^+(h'))$. For $w \in W(\Phi_K, \varphi, h)$, we will write

$$\delta(\Phi^+, \varphi, w \cdot h) = \delta(\Phi^+, \varphi, (w, S') \cdot h)$$

where $S' \in \mathcal{SO}(\Phi, \varphi)$ such that $(w, S') \in \Lambda(h)$. It is easy to see that it is independent of the choice of S' .

Lemma 6.6. *Let $h \in H_S^1 \cap \Omega', S \in \mathcal{SO}(\Phi, \varphi)$. Then*

$$\begin{aligned} & \epsilon_\varphi^G(E^*) D_\varphi^G(f_S^{-1}(h)) \Theta_\lambda^\varphi(f_S^{-1}(h)) = \epsilon(\varphi : \Phi^+) \epsilon_G(E^*) (-1)^{q_G} \\ & \times \sum_{w \in W(\varphi_K, h)} \Delta'_G(w \cdot h)^{-1} \xi_{\lambda - \rho_G}(w \cdot h) \delta(\Phi^+, \varphi, w \cdot h) c(1 : E^* : \varphi_R^+(w \cdot h)). \end{aligned}$$

Proof. Let $f_S^{-1}(h) = h' = t'a' \in H_{\varphi,S}$. Then $D_\varphi^G(h') = D_\varphi^G(w \cdot h')$ for all $w \in W(\varphi_K, h')$. Thus using (6.6), Lemma 6.5, and the definition of $\epsilon_\varphi^G(E^*)$,

$$\begin{aligned} \epsilon_\varphi^G(E^*) D_\varphi^G(h') \Theta_\lambda^\varphi(h') &= \epsilon(\varphi : \Phi^+) \epsilon_G(E^*) (-1)^{q_G} \\ &\times \sum_{w \in W(\varphi_K, h')} D_\varphi^G(w \cdot h') \Delta'_\varphi(w \cdot h')^{-1} \xi_{\lambda-\rho_\varphi}(w \cdot h') c(1 : E^* : \varphi_R^+(w \cdot h')) \\ &= \epsilon(\varphi : \Phi^+) \epsilon_G(E^*) (-1)^{q_G} \\ &\times \sum_{w \in W(\varphi_K, h')} \Delta'_G(w \cdot h)^{-1} \xi_{\lambda-\rho_G}(w \cdot h') \delta(\Phi^+, \varphi, w \cdot h) c(1 : E^* : \varphi_R^+(w \cdot h')). \end{aligned}$$

But

$$\xi_{\lambda-\rho_G}(w \cdot h') = \xi_{\lambda-\rho_G}(w \cdot h).$$

□

Lemma 6.7. *Let $\varphi \in \mathcal{T}(\Phi)$, $S \in \mathcal{SO}(\Phi)$, $h \in H_S'' \cap \Omega$. Then*

$$\begin{aligned} \epsilon_\varphi^G(E^*) (\text{Lift}_\varphi^G \Theta_\lambda^\varphi)(h) \\ &= \epsilon_G(E^*) (-1)^{q_G} \sum_{w \in W(\Phi_K, \varphi, h)} \epsilon(\varphi : \Phi^+) \delta(\Phi^+, \varphi, w \cdot h) \\ &\times \Delta'_G(w \cdot h)^{-1} \xi_{\lambda-\rho_G}(w \cdot h) c(1 : E^* : \varphi_R^+(w \cdot h)). \end{aligned}$$

Proof. By Theorem 4.13, we can take

$$X_\varphi(h) = \{f_{S_i}^{-1}((w_i, S_i) \cdot h) : 1 \leq i \leq p\}$$

where the $w_i = [v_i, s_i]$, $1 \leq i \leq p$, are a complete set of representatives of the equivalence classes in $W(\Phi_K, \varphi, h)$ for the equivalence relation $\sim_{\varphi, h}$, and $S_i \in \mathcal{SO}(\Phi, \varphi)$, $1 \leq i \leq p$. Write

$$h'_i = f_{S_i}^{-1}((w_i, S_i) \cdot h), \quad h_i = (w_i, S_i) \cdot h, \quad 1 \leq i \leq p.$$

Then we have

$$\epsilon_\varphi^G(E^*) (\text{Lift}_\varphi^G \Theta_\lambda^\varphi)(h) = \epsilon_\varphi^G(E^*) \sum_{1 \leq i \leq p} D_\varphi^G(h'_i) \Theta_\lambda^\varphi(h'_i).$$

Fix $1 \leq i \leq p$. Then using Lemma 6.6,

$$\begin{aligned} \epsilon_\varphi^G(E^*) D_\varphi^G(h'_i) \Theta_\lambda^\varphi(h'_i) &= \epsilon(\varphi : \Phi^+) \epsilon_G(E^*) (-1)^{q_G} \\ &\times \sum_{w \in W(\varphi_K, h_i)} \Delta'_G(w \cdot h_i)^{-1} \xi_{\lambda-\rho_G}(w \cdot h_i) \delta(\Phi^+, \varphi, w \cdot h_i) c(1 : E^* : \varphi_R^+(w \cdot h_i)). \end{aligned}$$

Let $w \in W(\varphi_K, h_i)$. Then there are $v \in W(\varphi_K)$, $u \in W(\varphi_{v_i t})$ such that $w = vu$, and $vS_i \in \mathcal{SO}(\Phi, \varphi)$. Then $(w, vS_i) \in \Lambda(\varphi, h'_i) \subset \Lambda(h_i)$ and $(w, vS_i) \cdot h_i = (w, vS_i) \cdot (w_i, S_i) \cdot h = (ww_i, vS_i) \cdot h$ where $ww_i \in W(\Phi_K, \varphi, h)$ by Lemma 4.11. Thus we can write

$$\begin{aligned} \epsilon_\varphi^G(E^*) D_\varphi^G(h'_i) \Theta_\lambda^\varphi(h'_i) \\ &= \epsilon_G(E^*) (-1)^{q_G} \sum_{w \in W(\varphi_K, h_i)} \delta(\Phi^+, \varphi, ww_i \cdot h) \epsilon(\varphi : \Phi^+) \\ &\times \Delta'_G(ww_i \cdot h)^{-1} \xi_{\lambda-\rho_G}(ww_i \cdot h) c(1 : E^* : \varphi_R^+(ww_i \cdot h)). \end{aligned}$$

By Lemma 4.12 (ii), for each $\sigma \in W(\Phi_K, \varphi, h)$ there are unique $1 \leq i \leq p$ and $w \in W(\varphi_K, h_i)$ such that $\sigma = ww_i$. Thus

$$\begin{aligned} & \epsilon_\varphi^G(E^*) (\text{Lift}_\varphi^G \Theta_\lambda^\varphi)(h) \\ &= \epsilon_G(E^*) (-1)^{q_G} \sum_{w \in W(\Phi_K, \varphi, h)} \delta(\Phi^+, \varphi, w \cdot h) \epsilon(\varphi : \Phi^+) \\ & \quad \times \Delta'_G(w \cdot h)^{-1} \xi_{\lambda - \rho_G}(w \cdot h) c(1 : E^* : \varphi_R^+(w \cdot h)). \end{aligned}$$

□

We will prove the following technical lemma in §7. Define $\mathcal{T}_{\text{aug}}(\Phi_t)$ as in §5. For $\psi \in \mathcal{T}_{\text{aug}}(\Phi_t)$, let

$$\mathcal{T}(\Phi, \psi) = \{\varphi \in \mathcal{T}(\Phi) : \varphi \cap \Phi_t = \psi\}.$$

Lemma 6.8. *Let $S \in \mathcal{SO}(\Phi)$, $h = ta \in H_S^1 \cap \Omega'$. Let Φ^+ be any choice of positive roots for Φ and let $\psi \in \mathcal{T}_{\text{aug}}(\Phi_t)$. Then*

$$\sum_{\varphi \in \mathcal{T}(\Phi, \psi)} \epsilon(\varphi : \Phi^+) \delta(\Phi^+, \varphi, h) = \begin{cases} \epsilon(\psi : \Phi_R^+(h)), & \text{if } \psi \in \mathcal{T}(\Phi_t); \\ 0, & \text{otherwise.} \end{cases}$$

Assuming Lemma 6.8, we can complete the proof of Theorem 6.4 as follows.

Proof of Theorem 6.4. Since both sides of the equation are class functions on $G'' \cap \Omega$, it suffices to prove the equation for every $h \in H_S'' \cap \Omega$, $S \in \mathcal{SO}(\Phi)$.

Fix $h \in H_S'' \cap \Omega$, $S \in \mathcal{SO}(\Phi)$. Recall that $c(h) = c(\Phi_t)c(K, h)$. Thus by Lemma 6.7 we have

$$\begin{aligned} & c(h) \sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon_\varphi^G(E^*) (\text{Lift}_\varphi^G \Theta_\lambda^\varphi)(h) \\ &= \epsilon_G(E^*) (-1)^{q_G} c(K, h) \sum_{w \in W(\Phi_K, h)} \Delta'_G(w \cdot h)^{-1} \xi_{\lambda - \rho_G}(w \cdot h) \\ & \quad \times c(\Phi_t) \sum_{\varphi \in \mathcal{T}(\Phi, h, w)} \delta(\Phi^+, \varphi, w \cdot h) \epsilon(\varphi : \Phi^+) c(1 : E^* : \varphi_R^+(w \cdot h)), \end{aligned}$$

where $\mathcal{T}(\Phi, h, w)$ denotes the set of all $\varphi \in \mathcal{T}(\Phi)$ such that $w \in W(\Phi_K, \varphi, h)$. Using the formula for $\Theta_\lambda(h)$ in Theorem 3.3, we see that it suffices to prove that for all $w \in W(\Phi_K, h)$,

$$\begin{aligned} & c(\Phi_t) \sum_{\varphi \in \mathcal{T}(\Phi, h, w)} \delta(\Phi^+, \varphi, w \cdot h) \epsilon(\varphi : \Phi^+) c(1 : E^* : \varphi_R^+(w \cdot h)) \\ &= c(1 : E^* : \Phi_R^+(w \cdot h)). \end{aligned}$$

Define $\mathcal{T}(\Phi, h) = \{\varphi \in \mathcal{T}(\Phi) : \exists S' \in \mathcal{SO}(\Phi, \varphi) \text{ with } (1, S') \in \Lambda(h)\}$. Let $(w, S') \in \Lambda(h)$. Then it is easy to see using Lemma 2.6 that

$$\mathcal{T}(\Phi, h, w) = \mathcal{T}(\Phi, (w, S') \cdot h).$$

For $w = [v, s] \in W(\Phi_K, h)$, $c(\Phi_t) = c(\Phi_{vt})$. Thus,

$$c(\Phi_t) \sum_{\varphi \in \mathcal{T}(\Phi, h, w)} \delta(\Phi^+, \varphi, w \cdot h) \epsilon(\varphi : \Phi^+) c(1 : E^* : \varphi_R^+(w \cdot h))$$

depends only on $(w, S') \cdot h$, not on w and h separately, and so we see that it suffices to prove that for all $S \in \mathcal{SO}(\Phi)$, $h \in H_S^1 \cap \Omega'$,

$$c(\Phi_t) \sum_{\varphi \in \mathcal{T}(\Phi, h)} \delta(\Phi^+, \varphi, h) \epsilon(\varphi : \Phi^+) c(1 : E^* : \varphi_R^+(h)) = c(1 : E^* : \Phi_R^+(h)).$$

Let $\varphi \in \mathcal{T}(\Phi, h)$, and let $\psi = \varphi \cap \Phi_t$. Then $\varphi_R^+(h) = \psi \cap \Phi_R^+(h)$ depends only on ψ . Further, by definition there is $S' \in \mathcal{SO}(\Phi, \varphi)$ such that $(1, S') \in \Lambda(h)$. Thus $S' \subset \Phi_t$ and spans Φ_t . Now $S' \subset \psi$ and spans ψ , so that $\text{rank } \psi = \text{rank } \Phi_t$. Clearly every irreducible factor of ψ is of type A_1 or B_2 . Thus $\psi \in \mathcal{T}_{\text{aug}}(\Phi_t)$. Further, since $S' \subset \psi$, we have $\psi \in \mathcal{T}_n(\Phi_t)$ if and only if $\psi \in \mathcal{T}(\Phi_t)$.

Thus we can write

$$\begin{aligned} c(\Phi_t) \sum_{\varphi \in \mathcal{T}(\Phi, h)} \epsilon(\varphi : \Phi^+) \delta(\Phi^+, \varphi, h) c(1 : E^* : \varphi_R^+(h)) \\ = c(\Phi_t) \sum_{\psi \in \mathcal{T}_{\text{aug}}(\Phi_t)} c(1 : E^* : \psi \cap \Phi_R^+(h)) \sum_{\varphi \in \mathcal{T}(\Phi, h, \psi)} \epsilon(\varphi : \Phi^+) \delta(\Phi^+, \varphi, h), \end{aligned}$$

where for $\psi \in \mathcal{T}_{\text{aug}}(\Phi_t)$, $\mathcal{T}(\Phi, h, \psi)$ is the set of all $\varphi \in \mathcal{T}(\Phi, h)$ such that $\varphi \cap \Phi_t = \psi$.

Let $\psi \in \mathcal{T}_{\text{aug}}(\Phi_t)$. If there is no $S' \in \mathcal{SO}(\Phi)$ such that $(1, S') \in \Lambda(h)$ and $S' \subset \psi$, then by the above $\mathcal{T}(\Phi, \psi, h) = \emptyset$. Suppose that there is $S' \in \mathcal{SO}(\Phi)$ such that $(1, S') \in \Lambda(h)$ and $S' \subset \psi$. Then for all $\varphi \in \mathcal{T}(\Phi, \psi)$, $S' \subset \varphi$ so that $\mathcal{T}(\Phi, \psi) = \mathcal{T}(\Phi, \psi, h)$. Now if $\psi \notin \mathcal{T}(\Phi)$,

$$\sum_{\varphi \in \mathcal{T}(\Phi, h, \psi)} \epsilon(\varphi : \Phi^+) \delta(\Phi^+, \varphi, h) = \sum_{\varphi \in \mathcal{T}(\Phi, \psi)} \epsilon(\varphi : \Phi^+) \delta(\Phi^+, \varphi, h) = 0$$

by Lemma 6.8. If $\psi \in \mathcal{T}(\Phi_t)$, then $\psi \in \mathcal{T}_n(\Phi_t)$, and again using Lemma 6.8,

$$\sum_{\varphi \in \mathcal{T}(\Phi, h, \psi)} \epsilon(\varphi : \Phi^+) \delta(\Phi^+, \varphi, h) = \sum_{\varphi \in \mathcal{T}(\Phi, \psi)} \epsilon(\varphi : \Phi^+) \delta(\Phi^+, \varphi, h) = \epsilon(\psi : \Phi_R^+(h)).$$

Thus by Theorem 3.4 we have

$$\begin{aligned} c(\Phi_t) \sum_{\varphi \in \mathcal{T}(\Phi, h)} \epsilon(\varphi : \Phi^+) \delta(\Phi^+, \varphi, h) c(1 : E^* : \varphi_R^+(h)) \\ = c(\Phi_t) \sum_{\psi \in \mathcal{T}_n(\Phi_t)} \epsilon(\psi : \Phi_R^+(h)) c(1 : E^* : \psi \cap \Phi_R^+(h)) = c(1 : E^* : \Phi_R^+(h)). \end{aligned}$$

□

7. PROOF OF LEMMA 6.8

We keep the notation of §6. We may as well assume that Φ is irreducible. Throughout this section we fix $S \in \mathcal{SO}(\Phi)$ and $h = ta \in H_S^1 \cap \Omega'$. Further, we fix $\gamma \in T_S \cap B_S$ such that $h \in \gamma H_S^0$, and $H_1 \in \underline{t}_S$ such that $\gamma^{-1}t = \exp(H_1)$. Let $H_2 = c_S^{-1} \log a \in i\underline{b}_S$. Thus $h = \gamma \exp(c_S H)$ where $H = H_1 + H_2$. In the case that Φ is of type A_{2n} , $n \geq 1$, by Lemma 6.2 we can assume that $\gamma = 1$, and $H_1 + c_S H_2 \in \omega$. Thus in this case for any $\tau \in E_\Phi$ we define $e^\tau(\gamma) = 1$. For any $\alpha \in \Phi$, write

$$\Delta(\alpha) = e^{\frac{\alpha(H)}{2}} - e^{-\alpha}(\gamma) e^{-\frac{\alpha(H)}{2}};$$

$$\delta(\alpha) = \Delta(\alpha) |\Delta(\alpha)|^{-1}.$$

The following lemma is an easy consequence of the definitions.

Lemma 7.1. For any $\varphi \in \mathcal{T}(\Phi)$ and any choice Φ^+ of positive roots,

$$\delta(\Phi^+, \varphi, h) = e^{\rho(\Phi^+, \varphi)}(\gamma) \prod_{\alpha \in \Phi^+ \setminus (\varphi \cap \Phi^+)} \delta(\alpha).$$

Lemma 7.2. For any $\varphi \in \mathcal{T}(\Phi)$, $\epsilon(\varphi : \Phi^+) \delta(\Phi^+, \varphi, h)$ is independent of the choice Φ^+ of positive roots.

Proof. Fix one choice Φ^+ of positive roots and write $\varphi^+ = \varphi \cap \Phi^+$. Let $w \in W(\Phi)$ so that $w\Phi^+$ is another choice of positive roots. Then as in Lemma 6.3 we can write $w = uv^{-1}$ where $u \in W(\varphi)$ with $w\Phi^+ \cap \varphi = u\varphi^+$ and $v \in W(\varphi : \Phi^+)$. Then as in Lemma 6.3,

$$\epsilon(\varphi : w\Phi^+) = \det v \epsilon(\varphi : \Phi^+).$$

By Lemma 7.1 we have

$$\delta(\Phi^+, \varphi, h) = e^{\rho(\Phi^+) - \rho(\varphi^+)}(\gamma) \prod_{\alpha \in \Phi^+ \setminus (\varphi^+)} \delta(\alpha).$$

It is easy to check from the definition that for any $\alpha \in \Phi$,

$$\delta(-\alpha) = -e^\alpha(\gamma) \delta(\alpha).$$

Thus we have

$$\begin{aligned} \prod_{\alpha \in w\Phi^+} \delta(\alpha) &= \det w e^{\rho(\Phi^+) - \rho(w\Phi^+)}(\gamma) \prod_{\alpha \in \Phi^+} \delta(\alpha), \\ \prod_{\alpha \in u\varphi^+} \delta(\alpha) &= \det u e^{\rho(\varphi^+) - \rho(u\varphi^+)}(\gamma) \prod_{\alpha \in \varphi^+} \delta(\alpha). \end{aligned}$$

Thus

$$\delta(w\Phi^+, \varphi, h) = \det v \delta(\Phi^+, \varphi, h).$$

□

Let $\Phi_S = c_S \Phi = \Phi(\underline{g}_C, \underline{h}_{S,C})$. Then we can define

$$\Phi_{R,S} = \{\alpha \in \Phi_S : \alpha(H) \in \mathbf{R} \ \forall H \in \underline{h}_S\};$$

$$\Phi_{I,S} = \{\alpha \in \Phi_S : \alpha(H) \in i\mathbf{R} \ \forall H \in \underline{h}_S\};$$

$$\Phi_{CPX,S} = \Phi_S \setminus (\Phi_{R,S} \cup \Phi_{I,S}).$$

As in §4 we let σ be the involution of $\underline{g}_C$ with fixed points $\underline{g}$. For each $\alpha \in \Phi_S$ we have $\alpha^\sigma \in \Phi_S$ defined by

$$\alpha^\sigma(H) = \overline{\alpha(\sigma(H))}, \quad H \in \underline{h}_{S,C}.$$

Then $\alpha^\sigma = \alpha$ if and only if $\alpha \in \Phi_{R,S}$ and $\alpha^\sigma = -\alpha$ if and only if $\alpha \in \Phi_{I,S}$.

Define

$$\Phi_R = c_S^{-1} \Phi_{R,S}, \quad \Phi_I = c_S^{-1} \Phi_{I,S}, \quad \Phi_{CPX} = c_S^{-1} \Phi_{CPX,S}.$$

For any $\alpha \in \Phi$ we will write $\alpha^{\sigma,S} = c_S^{-1}(c_S \alpha)^\sigma$. Let Φ^+ be a choice of positive roots for Φ . We say that $\Phi_{CPX}^+ = \Phi^+ \cap \Phi_{CPX}$ is σ -stable if $\alpha \in \Phi_{CPX}^+$ if and only if $\alpha^{\sigma,S} \in \Phi_{CPX}^+$.

Lemma 7.3. Let $\varphi \in \mathcal{T}(\Phi)$ such that $\varphi \cap \Phi_t \in \mathcal{T}_{\text{aug}}(\Phi_t)$. Then for any $\alpha \in \Phi$, $\alpha \in \varphi$ if and only if $\alpha^{\sigma,S} \in \varphi$.

Proof. Suppose $\alpha \in \Phi$, $H_1 \in \underline{t}_S$, $H_2 \in \underline{b}_S$. Then $\alpha^{\sigma, S}(H_1 + H_2) = \alpha(-H_1 + H_2)$. Since $\varphi \cap \Phi_t \in \mathcal{T}_{\text{aug}}(\Phi_t)$, there is an orthogonal set $S' \subset \varphi \cap \Phi_t$ such that $[S'] = \text{rank } \Phi_t$. Let $w_{S'}$ denote the product of the reflections corresponding to the roots in S' . Then $w_{S'} \in W(\varphi)$ and $w_{S'}(H_1 + H_2) = H_1 - H_2$ for all $H_1 \in \underline{t}_S$, $H_2 \in \underline{b}_S$. Thus for all $\alpha \in \Phi$, $\alpha^{\sigma, S} = -w_{S'}\alpha$. Thus $\alpha \in \varphi$ if and only if $\alpha^{\sigma, S} \in \varphi$. $\square$

Note that

$$\Phi_t = \{\alpha \in \Phi_R : e^\alpha(\gamma) = 1\}.$$

Define

$$\Psi = \Phi_I \cup \Phi_t.$$

Lemma 7.4. (i) $\text{rank } \mathcal{T}(\Psi) = \text{rank } \mathcal{T}(\Phi)$.

(ii) For every $\varphi \in \mathcal{T}(\Phi)$ such that $\varphi \cap \Phi_t \in \mathcal{T}_{\text{aug}}(\Phi_t)$, $\varphi \cap \Phi_I \in \mathcal{T}_{\text{aug}}(\Phi_I)$, and $\varphi \cap \Psi \in \mathcal{T}_{\text{aug}}(\Psi)$.

(iii) For every $\psi \in \mathcal{T}(\Psi)$ there is $\varphi \in \mathcal{T}(\Phi)$ such that $\varphi \cap \Psi = \psi$. Moreover, φ is unique unless Φ is of type C_n , n odd, and Ψ has three irreducible factors of type C_r , r odd.

(iv) Suppose that Φ is of type A_{2n} , $n \geq 1$. Then Φ_I is of type A_{2s} for some $0 \leq s \leq n$. Otherwise, Ψ contains no irreducible factors of type A_{2k} , $k \geq 1$.

Proof. (i) and (ii) are elementary.

(iii) Let $\psi \in \mathcal{T}(\Psi)$. Then every irreducible factor of ψ is of type A_1 or B_2 , and $\text{rank } \psi = \text{rank } \mathcal{T}(\Psi) = \text{rank } \mathcal{T}(\Phi)$, so $\psi \in \mathcal{T}_{\text{aug}}(\Phi)$. By Lemma 5.6 there is $\varphi \in \mathcal{T}(\Phi)$ such that $\psi \subset \varphi$. Then $\varphi \cap \Psi \in \mathcal{T}_{\text{aug}}(\Psi)$ and contains the two-structure ψ . Thus $\varphi \cap \Psi = \psi$.

Now suppose that φ is not uniquely determined by ψ . By Lemma 5.6, Φ is of type B_n , C_n , or F_4 . When Φ is of type B_n , Ψ will be of the form $D_2^k \times D_{2p} \times B_r \times B_s$ where $k, p, r, s \geq 0$ with $2k + 2p + r + s = n$, and the D_2, D_{2p} factors consist of long roots in Φ . Thus ψ contains at most two irreducible factors of type A_1 generated by short roots coming from the B_r and B_s factors where r or s is odd. Thus by Lemma 5.6, φ is uniquely determined by ψ .

When Φ is of type F_4 , Ψ will be of the form $F_4, B_4, B_3 \times B_1, B_2 \times B_2, B_2 \times A_1^2, B_1^2 \times A_1^2$, or $C_3 \times A_1$, where we use A_1 to denote an irreducible factor generated by one long root and B_1 to denote an irreducible factor generated by one short root. Thus ψ will be of the form $B_2^2, B_2 \times B_1^2, B_2 \times A_1^2$, or $B_1^2 \times A_1^2$. Again, by Lemma 5.6, $\mathcal{T}(\Phi, \psi)$ will contain exactly one element.

When Φ is of type C_n , Ψ will be of the form $D_2^k \times C_p \times C_r \times C_s$ where $k, p, r, s \geq 0$ with $2k + p + r + s = n$, and the D_2 factors are generated by short roots of Φ . Thus ψ has less than three irreducible factors of type A_1 generated by long roots, and hence φ is unique by Lemma 5.6, unless p, r, s are all odd. In this case n must also be odd.

(iv) Since Φ_t is spanned by strongly orthogonal roots, it contains no irreducible factors of type A_{2k} , $k \geq 1$. Further, Φ_I is the orthogonal complement in Φ of S . Now if Φ is spanned by orthogonal roots, so is Φ_I , so it contains no A_{2k} factors. Suppose that Φ is not spanned by orthogonal roots. Then it is of type $A_{2n}, A_{2n+1}, D_{2n+1}$, or E_6 , and as in the proof of Theorem 5.3, the orthogonal complement of one root is of type $A_{2n-2}, A_{2n-1}, D_{2n-1} \times A_1$, or A_5 respectively. Thus the orthogonal complement of an arbitrary orthogonal set S is of type A_{2s} when Φ is of type A_{2n} , and contains only simple factors of type A_{2k+1}, D_{2k+1} , or E_6 otherwise. $\square$

Let Ψ^+ be any choice of positive roots for Ψ . Then for any $\psi \in \mathcal{T}_{\text{aug}}(\Psi)$, we define

$$\delta_0(\Psi^+, \psi, h) = \prod_{\alpha \in \Psi^+ \setminus \psi^+} \delta(\alpha)$$

where $\psi^+ = \psi \cap \Psi^+$. For $\psi \in \mathcal{T}_{\text{aug}}(\Psi)$, let $\mathcal{T}(\Phi, \psi) = \{\varphi \in \mathcal{T}(\Phi) : \varphi \cap \Psi = \psi\}$.

Lemma 7.5. *Let Φ^+ be any choice of positive roots for Φ and let $\psi \in \mathcal{T}_{\text{aug}}(\Psi)$, $\varphi \in \mathcal{T}(\Phi, \psi)$. Then*

$$\delta(\Phi^+, \varphi, h) = e^{\rho(\Phi^+, \varphi)}(\gamma) \delta_0(\Psi \cap \Phi^+, \psi, h) \prod_{\alpha \in \Phi_{CPX}^+ \setminus (\varphi \cap \Phi_{CPX}^+)} \delta(\alpha).$$

If in addition, Φ_{CPX}^+ is σ -stable, then

$$\prod_{\alpha \in \Phi_{CPX}^+ \setminus (\varphi \cap \Phi_{CPX}^+)} \delta(\alpha) = 1.$$

Proof. Let $\varphi \in \mathcal{T}(\Phi, \psi)$. By Lemma 7.1,

$$\delta(\Phi^+, \varphi, h) = e^{\rho(\Phi^+, \varphi)}(\gamma) \prod_{\alpha \in \Phi^+ \setminus \varphi^+} \delta(\alpha).$$

Let $\alpha \in \Phi^+ \setminus \Psi^+$. Then $\alpha \in (\Phi_R \setminus \Phi_t) \cup \Phi_{CPX}$. If $\alpha \in \Phi_R \setminus \Phi_t$, then $e^{-\alpha}(\gamma) = -1$ so that

$$\Delta(\alpha) = e^{\frac{\alpha(H)}{2}} + e^{-\frac{\alpha(H)}{2}}$$

is real and positive since $\alpha(H)$ is real. Thus $\delta(\alpha) = 1$ in this case. Thus

$$\prod_{\alpha \in \Phi^+ \setminus \varphi^+} \delta(\alpha) = \delta_0(\Psi \cap \Phi^+, \Psi \cap \varphi, h) \prod_{\alpha \in \Phi_{CPX}^+ \setminus \varphi \cap \Phi_{CPX}^+} \delta(\alpha).$$

Suppose that Φ_{CPX}^+ is σ -stable. Then Φ_{CPX}^+ can be divided into pairs $\alpha, \alpha^{\sigma, S}$. Since $e^{-\alpha}(\gamma) = \pm 1$ is real, we see that

$$\Delta(\alpha^{\sigma, S}) = \overline{\Delta(\alpha)}.$$

Thus

$$\delta(\alpha^{\sigma, S})\delta(\alpha) = 1.$$

Further, by Lemma 7.3, $\alpha \in \varphi$ if and only if $\alpha^{\sigma, S} \in \varphi$. Thus

$$\prod_{\alpha \in \Phi_{CPX}^+ \setminus \varphi \cap \Phi_{CPX}^+} \delta(\alpha) = 1.$$

□

Lemma 7.6. *Let Φ^+ be any choice of positive roots for Φ and let $\psi \in \mathcal{T}_{\text{aug}}(\Psi)$, $\psi \notin \mathcal{T}(\Psi)$. Then*

$$\sum_{\varphi \in \mathcal{T}(\Phi, \psi)} \epsilon(\varphi : \Phi^+) \delta(\Phi^+, \varphi, h) = 0.$$

Proof. The statement is vacuous when Φ is of type A_{2n} since in this case Ψ has one root length so that $\mathcal{T}_{\text{aug}}(\Psi) = \mathcal{T}(\Psi)$. Thus we can assume that Φ is not of type A_{2n} . Further, since both sides of the equation are independent of Φ^+ by Lemma 7.2, it is enough to prove the lemma for one choice of Φ^+ .

Since $\psi \notin \mathcal{T}(\Psi)$, by Lemma 5.6 there are $\alpha_1, \alpha_2 \in \psi$ occurring in A_1 factors of ψ such that α_1, α_2 are an orthogonal basis for root system of type B_2 in Ψ . Note that this implies that α_1, α_2 are either both in Φ_t or both in Φ_I . Now $\alpha_0 = c(\alpha_1 - \alpha_2) \in \Psi$ where $c \in \{1, 1/2\}$. Let s denote the reflection corresponding to α_0 . Then s interchanges α_1 and α_2 , and fixes every root of Ψ orthogonal to both α_1 and α_2 . In particular, $s\psi = \psi$.

Let $\mathcal{B} = \{\beta_1, \dots, \beta_n\}$ be an ordered orthogonal basis of ψ such that $\beta_1, \dots, \beta_r$ is a basis for $\psi \cap \Phi_t$ and $\beta_{r+1}, \dots, \beta_n$ is a basis for $\psi \cap \Phi_I$. Further, if $\alpha_1, \alpha_2 \in \Phi_t$, then we take $\beta_i = \alpha_i, i = 1, 2$, while if $\alpha_1, \alpha_2 \in \Phi_I$, we take $\beta_{n-1} = \alpha_1, \beta_n = \alpha_2$. Let $\Phi^+ = \Phi^+(\mathcal{B})$. Then Φ_{CPX}^+ is σ -stable. Let $\varphi \in \mathcal{T}(\Phi, \psi)$, and suppose that there is $\alpha \in \varphi^+$ such that $s\alpha \in \Phi^-$. Since s interchanges α_1 and α_2 and fixes every other $\beta_i \in \mathcal{B}$, we must have $\langle \alpha, \alpha_1 \rangle > 0, \langle \alpha, \alpha_2 \rangle < 0$. In particular, we must have α_1, α_2 in the same simple factor of φ . But then $\alpha_0 \in \varphi \cap \Psi = \psi$. This contradicts the assumption that α_1, α_2 occur in A_1 factors of ψ . Thus $s\varphi^+ \subset \Phi^+$ for all $\varphi \in \mathcal{T}(\Phi, \psi)$.

Let $\varphi \in \mathcal{T}(\Phi, \psi)$. Then $s\varphi \cap \Psi = s(\varphi \cap \Psi) = s\psi = \psi$, so that $s\varphi \in \mathcal{T}(\Phi, \psi)$ also. Further, since $s^2 = 1$, $s(s\varphi) = \varphi$. Thus we can divide $\mathcal{T}(\Phi, \psi)$ into pairs, $\varphi, s\varphi$. But by Lemma 5.2,

$$\epsilon(s\varphi : \Phi^+) = \det s \epsilon(\varphi : \Phi^+) = -\epsilon(\varphi : \Phi^+).$$

Further

$$\rho(\Phi^+, s\varphi) = \rho(\Phi^+, \varphi) + \rho(\varphi^+) - s\rho(\varphi^+).$$

Since $\rho(\varphi^+)$ is a weight for Φ and hence for Ψ , $\rho(\varphi^+) - s\rho(\varphi^+)$ is in the root lattice of Ψ , so that

$$e^{\rho(\Phi^+, s\varphi)}(\gamma) = e^{\rho(\Phi^+, \varphi)}(\gamma) e^{\rho(\varphi^+) - s\rho(\varphi^+)}(\gamma) = e^{\rho(\Phi^+, \varphi^+)}(\gamma).$$

Then, by Lemma 7.5, since Φ_{CPX}^+ is σ -stable,

$$\begin{aligned} \delta(\Phi^+, s\varphi, h) &= e^{\rho(\Phi^+, s\varphi)}(\gamma) \delta_0(\Psi \cap \Phi^+, \psi, h) \\ &= e^{\rho(\Phi^+, \varphi)}(\gamma) \delta_0(\Psi \cap \Phi^+, \psi, h) = \delta(\Phi^+, \varphi, h). \end{aligned}$$

Thus

$$\epsilon(s\varphi : \Phi^+) \delta(\Phi^+, s\varphi, h) + \epsilon(\varphi : \Phi^+) \delta(\Phi^+, \varphi, h) = 0$$

for all $\varphi \in \mathcal{T}(\Phi, \psi)$. □

Fix $\psi \in \mathcal{T}(\Psi)$, and $\varphi \in \mathcal{T}(\Phi, \psi)$. Let $\mathcal{B}$ be a basis for ψ which contains two long orthogonal roots from every irreducible factor of ψ of type B_2 .

Lemma 7.7. *There is an ordering $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ with the following properties.*

- (i) *For every irreducible factor φ_j of φ , the roots in $\mathcal{B} \cap \varphi_j$ are adjacent in the ordering.*
- (ii) *Let Φ be of type C_n, n odd, and let α_s be the root in $\mathcal{B}$ which generates the unique irreducible factor of φ of type C_1 . Then if $\alpha_s \in \Phi_t$, it is the first root in $\mathcal{B}$, while if $\alpha_s \in \Phi_I$, then it is the last root in $\mathcal{B}$.*

(iii) Let $\Phi^+ = \Phi^+(\mathcal{B})$. Then

$$\prod_{\alpha \in \Phi_{CPX}^+ \setminus \varphi \cap \Phi_{CPX}^+} \delta(\alpha) = 1.$$

Proof. Let Ψ_i be an irreducible factor of Ψ . We will call Ψ_i imaginary, respectively real, if it is contained in Φ_I , respectively Φ_t . Since $\Psi = \Phi_I \cup \Phi_t$ every Ψ_i is either real or imaginary. Similarly, we call every element of Φ_t a real root of Ψ and every element of Φ_I an imaginary root of Ψ .

Suppose first that φ contains no irreducible factors of type B_2 . Then conditions (i) and (ii) are vacuous, so that it suffices to produce an ordering of $\mathcal{B}$ which satisfies (iii). For this we just need to order $\mathcal{B}$ so that the real roots precede the imaginary roots, so that Φ_{CPX}^+ is σ -stable.

Now suppose that φ contains an irreducible factor of type B_2 , so that Φ is of type B_n, C_n , or F_4 . Let $\Psi_i, i = 1, 2$, be distinct irreducible factors of Ψ . We will say they are linked if there is an irreducible factor φ_j of φ of type B_2 which has nonempty intersection with both. In this case $\varphi_j \cap \Psi_i$ is an irreducible factor of $\psi_i = \psi \cap \Psi_i$ of type A_1 , and the roots in $\varphi_j \cap \Psi_1$ and $\varphi_j \cap \Psi_2$ have the same length in Φ .

Suppose that Φ is of type B_n . Then Φ_I is of type $B_r \times A_1^k$ and Φ_t is of type $A_1^k \times D_{2p} \times B_q$ where $r + 2k + 2p + q = n$. Let $\Psi_{2i-1}, 1 \leq i \leq k$, denote the irreducible factors of Φ_t of type A_1 . Then for each $1 \leq i \leq k$ there is a unique irreducible factor Ψ_{2i} of Φ_I of type A_1 which is linked with Ψ_{2i-1} . The irreducible factor Ψ_{2k+1} of Φ_t of type D_{2p} cannot be linked to another factor. Finally, the factors Ψ_{2k+2} of Φ_t of type B_q and Ψ_{2k+3} of Φ_I of type B_r are linked just in case q and r are both odd. Now $\Psi_1, \dots, \Psi_{2k+3}$ is an ordering of the irreducible factors of Ψ with the property that linked factors are adjacent.

Now we order $\mathcal{B}$ as follows. If Ψ_i is not linked to any other irreducible factors, then we order $\mathcal{B}_i = \Psi_i \cap \mathcal{B}$ so that if two roots are in the same irreducible factor of φ they are adjacent in the ordering. If Ψ_i, Ψ_{i+1} are linked, we order $\mathcal{B}_i$ so that if two roots are in the same irreducible factor of φ they are adjacent in the ordering, and the linked root is last. We order $\mathcal{B}_{i+1}$ so that if two roots are in the same irreducible factor of φ they are adjacent in the ordering, and the linked root is first. Now order $\mathcal{B}$ by taking $\mathcal{B} = \mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_{2k+3}$. This produces an ordering for $\mathcal{B}$ which satisfies (i).

Let $\alpha \in \Phi_{CPX}$ and let α_i be the first element of $\mathcal{B}$ such that $\langle \alpha, \alpha_i \rangle \neq 0$. Suppose that $\alpha_i \in \Phi_I$. If $\alpha_i \in \Psi_{2k+3}$, then α is orthogonal to every real root of $\mathcal{B}$, so that $\alpha \in \Phi_I$. Suppose that $\alpha_i \in \Psi_{2j}$ for some $1 \leq j \leq k$. Write $\mathcal{B}_{2j-1} = \{\beta\}$. Now

$$\alpha \in \Phi_\beta = \{\gamma \in \Phi : \langle \gamma, \beta \rangle = 0\}.$$

Since β is a long root and Φ is of type B_n , Φ_β is of type $A_1 \times B_{n-2}$. Here the factor of type A_1 is Ψ_{2j} and the factor of type B_{n-2} is the set of all $\gamma \in \Phi$ which are orthogonal to both β and α_i . Since $\langle \alpha, \alpha_i \rangle \neq 0$, we must have $\alpha = \pm \alpha_i \in \Phi_I$. These contradictions show that $\alpha_i \in \Phi_t$. Thus Φ_{CPX}^+ is σ -stable so that (iii) is satisfied.

Suppose that Φ is of type C_n . Then Φ_I is of type $C_r \times A_1^k$ and Φ_t is of type $A_1^k \times C_p \times C_q$ where $r + 2k + p + q = n$. Define $\Psi_i, 1 \leq i \leq 2k$, as in the B_n case. When n is odd we let α_s denote the root of $\mathcal{B}$ which generates the unique irreducible factor of φ_0 of type A_1 .

Suppose that n is even, or that n is odd and $\alpha_s \in \Phi_I$. Let Ψ_{2k+3} be the irreducible factor of Φ_I of type C_r . The factors of Φ_t of type C_p and C_q should be labeled Ψ_{2k+1} and Ψ_{2k+2} . If Ψ_{2k+3} is linked to one of these two factors, then it should be labeled Ψ_{2k+2} , while the remaining factor is Ψ_{2k+1} . Now $\Psi_1, \dots, \Psi_{2k+3}$ is an ordering of the irreducible factors of Ψ with the property that linked factors are adjacent. We now order $\mathcal{B}$ as in the B_n case. In addition, if n is odd, then $\alpha_s \in \Psi_{2k+3}$ and this factor is not linked to any other. Thus we can order the roots in $\mathcal{B} \cap \Psi_{2k+3}$ so that α_s is last. Using the argument in the B_n case, with the roles of long and short roots reversed we see that this ordering satisfies (iii).

Suppose that n is odd, and $\alpha_s \in \Phi_t$. Then it is in one of the factors of Φ_t of type C_p, C_q . This factor should be labelled Ψ_0 . It is not linked to any other factor. The other factor of this type should be labelled Ψ_{2k+1} and the irreducible factor of Φ_I of type C_r should be labelled Ψ_{2k+2} . Now $\Psi_0, \dots, \Psi_{2k+2}$ is an ordering of the irreducible factors of Ψ with the property that linked factors are adjacent. Order $\mathcal{B}$ as usual. Since Ψ_0 is not linked, we can order the roots in $\mathcal{B} \cap \Psi_0$ so that α_s is first, and as above this ordering satisfies (iii).

Finally, suppose that Φ is of type F_4 . If Φ_I consists of only one irreducible factor, then we can easily order the irreducible factors of Ψ so that any linked irreducible factors are adjacent, and the imaginary factor is last. As above this yields an ordering of $\mathcal{B}$ satisfying (i) and (iii).

Now suppose that Φ is of type F_4 and Φ_I is reducible. This occurs only when S consists of two roots, one short and one long. In this case $\Psi = \{\pm\alpha_i, 1 \leq i \leq 4\}$ is of type $B_1^2 \times A_1^2$ and we can write $\Phi_t = \Psi_1 \cup \Psi_3$ and $\Phi_I = \Psi_2 \cup \Psi_4$ where $\Psi_i = \{\pm\alpha_i\}, 1 \leq i \leq 4$. We will order the roots so that α_1 and α_2 have the same length and α_3 and α_4 have the same length. Then Ψ_{2i-1} and Ψ_{2i} must be linked, $1 \leq i \leq 2$, so this ordering satisfies (i).

Let $\beta \in \Phi_{CPX}^+$, and assume that $-\beta^{\sigma,S} \in \Phi^+$. Then we must have $\langle \beta, \alpha_1 \rangle = 0$ and $\langle \beta, \alpha_2 \rangle \neq 0$. There are four roots of this type given by

$$\beta_1 = c_2\alpha_2 + c_3(\alpha_3 + \alpha_4), \quad \beta_2 = c_2\alpha_2 + c_3(\alpha_3 - \alpha_4),$$

$$\beta_3 = c_2\alpha_2 + c_3(-\alpha_3 + \alpha_4), \quad \beta_4 = c_2\alpha_2 + c_3(-\alpha_3 - \alpha_4)$$

where $c_2, c_3 \in \{1/2, 1\}$. Further, $\beta_3 = -\beta_1^{\sigma,S}$ and $\beta_4 = -\beta_2^{\sigma,S}$. Now, since for $i = 1, 2$,

$$\delta(\beta_i)\delta(-\beta_i^{\sigma,S}) = -e^{\beta_i}(\gamma), \quad \prod_{i=1}^4 \delta(\beta_i) = e^{\beta_1+\beta_2}(\gamma).$$

But $\beta_1 + \beta_2 = 2c_2\alpha_2 + 2c_3\alpha_3$ and $\gamma = \exp(H_0)$ where $H_0 = \pi i(n_1 H_{\alpha_1}^* + n_3 H_{\alpha_3}^*)$ with $n_i \in \mathbf{Z}, i = 1, 3$. Thus

$$(\beta_1 + \beta_2)(H_0) = 2\pi i 2c_3 n_3 \in 2\pi i \mathbf{Z}$$

so that

$$\prod_{i=1}^4 \delta(\beta_i) = 1.$$

Now $\varphi \cap \Psi = \psi = \Psi$ since all irreducible factors of Ψ are of type A_1 . Since the orthogonal roots in a factor of type B_2 must have the same length, $\varphi = \varphi_1 \cup \varphi_2$

where φ_i is the unique factor of type B_2 containing α_{2i-1} and α_{2i} , $i = 1, 2$. Thus $\beta_i \notin \varphi$, $1 \leq i \leq 4$. Since for all other $\alpha \in \Phi_{CPX}^+$, $\alpha^{\sigma, S} \in \Phi^+$ we have

$$\prod_{\alpha \in \Phi_{CPX}^+ \setminus (\varphi \cap \Phi_{CPX}^+)} \delta(\alpha) = 1.$$

□

For any $\psi \in \mathcal{T}(\Psi)$ and choice Ψ^+ of positive roots for Ψ , we have defined

$$\delta_0(\Psi^+, \psi, h) = \prod_{\alpha \in \Psi^+ \setminus (\psi \cap \Psi^+)} \delta(\alpha).$$

If Φ is not of type A_{2n} , then by Theorem 5.5 and Lemma 7.4(iv), $\rho(\Psi^+, \psi)$ is in the root lattice of Ψ so that

$$e^{\rho(\Psi^+, \psi)}(\gamma) = 1.$$

When Φ is of type A_{2n} , $e^{\rho(\Psi^+, \psi)}(\gamma) = 1$ by definition. Thus we have

$$\delta_0(\Psi^+, \psi, h) = \delta(\Psi^+, \psi, h) = e^{\rho(\Psi^+, \psi)}(\gamma) \prod_{\alpha \in \Psi^+ \setminus (\psi \cap \Psi^+)} \delta(\alpha).$$

By Lemma 7.2 applied to Ψ , we see that $\epsilon(\psi : \Psi^+) \delta(\Psi^+, \psi, h)$ is independent of the choice of Ψ^+ .

Lemma 7.8. *Let Φ^+ and Ψ^+ be any choices of positive roots and let $\psi \in \mathcal{T}(\Psi)$. Then*

$$\sum_{\varphi \in \mathcal{T}(\Phi, \psi)} \epsilon(\varphi : \Phi^+) \delta(\Phi^+, \varphi, h) = \epsilon(\psi : \Psi^+) \delta(\Psi^+, \psi, h).$$

Proof. Fix $\varphi \in \mathcal{T}(\Phi, \psi)$, and let $\mathcal{B}(\varphi) = \{\alpha_1, \dots, \alpha_n\}$ be an ordered basis for ψ satisfying the conditions in Lemma 7.7 with respect to φ . Let

$$\Phi^+(\varphi) = \Phi^+(\mathcal{B}(\varphi)), \quad \Psi^+(\varphi) = \Psi^+(\mathcal{B}(\varphi)).$$

Then $\epsilon(\psi : \Psi^+(\varphi)) = 1$ by definition. Further,

$$\delta(\Phi^+(\varphi), \varphi, h) = e^{\rho(\Phi^+(\varphi), \varphi)}(\gamma) \delta(\Psi^+(\varphi), \psi, h)$$

by Lemma 7.5.

Modify $\mathcal{B} = \mathcal{B}(\varphi)$ as follows. If φ_i is an irreducible factor of φ of type A_1 , then $\mathcal{B} \cap \varphi_i = \{\alpha_p\}$ for some $1 \leq p \leq n$, and we set $\beta_p = \alpha_p$. If φ_i is an irreducible factor of φ of type B_2 , $\mathcal{B} \cap \varphi_i = \{\alpha_p, \alpha_{p+1}\}$ for some $1 \leq p \leq n-1$. The two roots have consecutive indices by definition, and since they are orthogonal in a root system of type B_2 , either both are short or both are long. If both are long, then we define $\beta_p = \alpha_p$ and $\beta_{p+1} = \alpha_{p+1}$. However if both are short, we replace them by $\beta_p = \alpha_p + \alpha_{p+1}$ and $\beta_{p+1} = \alpha_p - \alpha_{p+1}$. This produces an ordered basis $\mathcal{B}' = \{\beta_1, \dots, \beta_n\}$, for φ which contains two orthogonal long roots from every irreducible factor of φ of type B_2 , and they are adjacent in the ordering. By definition, $\epsilon(\varphi : \Phi^+(\mathcal{B}')) = 1$. But it is easy to check that $\Phi^+(\mathcal{B}) = \Phi^+(\mathcal{B}')$. Thus $\epsilon(\varphi : \Phi^+(\varphi)) = 1$.

Now by Lemma 7.2, for any Φ^+ and Ψ^+ we have

$$\begin{aligned} \sum_{\varphi \in \mathcal{T}(\Phi, \psi)} \epsilon(\varphi : \Phi^+) \delta(\Phi^+, \varphi, h) &= \sum_{\varphi} \epsilon(\varphi : \Phi^+(\varphi)) \delta(\Phi^+(\varphi), \varphi, h) \\ &= \sum_{\varphi} e^{\rho(\Phi^+(\varphi), \varphi)}(\gamma) \delta(\Psi^+(\varphi), \psi, h) \epsilon(\psi : \Psi^+(\varphi)) \\ &= \epsilon(\psi : \Psi^+) \delta(\Psi^+, \psi, h) \sum_{\varphi} e^{\rho(\Phi^+(\varphi), \varphi)}(\gamma). \end{aligned}$$

Thus it suffices to prove that

$$\sum_{\varphi \in \mathcal{T}(\Phi, \psi)} e^{\rho(\Phi^+(\varphi), \varphi)}(\gamma) = 1.$$

Suppose that Φ is not of type C_{2m+1} , $m \geq 1$. Then by Lemma 7.4, $\mathcal{T}(\Phi, \psi) = \{\varphi\}$ has only one element. Further, by Lemma 5.4, if Φ is not of type A_{2n} , $\rho(\Phi^+(\varphi), \varphi)$ is in the root lattice of Ψ since each $\beta_i \in \mathcal{B}'$ is. Thus $e^{\rho(\Phi^+(\varphi), \varphi)}(\gamma) = 1$. If Φ is of type A_{2n} , then $e^{\rho(\Phi^+(\varphi), \varphi)}(\gamma) = 1$ by definition.

Suppose that Φ is of type C_{2m+1} , and fix $\varphi \in \mathcal{T}(\Phi, \psi)$. Let α_0 denote the element of $\mathcal{B}$ which generates the irreducible factor of φ of type C_1 . If $\alpha_0 \in \Phi_t$, then it is first in $\mathcal{B}'$. Thus again by Lemma 5.4 $\rho(\Phi^+, \varphi)$ is in the root lattice of Ψ so that as above

$$e^{\rho(\Phi^+(\varphi), \varphi)}(\gamma) = 1.$$

Suppose that $\alpha_0 \in \Phi_I$. Then it is last in $\mathcal{B}'$ and by Lemma 5.4,

$$\rho(\Phi^+(\varphi), \varphi) = \frac{1}{2} \sum_{i=1}^{2m} \beta_i + \lambda$$

where λ is in the root lattice of Ψ . Thus

$$e^{\rho(\Phi^+(\varphi), \varphi)}(\gamma) = e^{(1/2) \sum_{i=1}^{2m} \beta_i}(\gamma).$$

Now Φ_I is of type $A_1^k \times C_r$ and Φ_t is of type $A_1^k \times C_p \times C_q$, and by the choice of $\mathcal{B}$ in Lemma 7.7, $\delta_i = \frac{1}{2}(\beta_{2i-1} + \beta_{2i}) \in \Psi$ for all $1 \leq i \leq m$, except in the case that p and q are both odd, in which case exactly one $\delta_i \in \Phi_R \setminus \Phi_t$. Thus

$$e^{\rho(\Phi^+(\varphi), \varphi)}(\gamma) = \begin{cases} -1, & \text{if } p \text{ and } q \text{ are both odd;} \\ 1, & \text{otherwise.} \end{cases}$$

Now if exactly one of p, q, r is odd, then $\mathcal{T}(\Phi, \psi) = \{\varphi\}$ has only one element and $e^{\rho(\Phi^+(\varphi), \varphi)}(\gamma) = 1$. If all three of p, q, r are odd, then $\mathcal{T}(\Phi, \psi) = \{\varphi_i, 1 \leq i \leq 3\}$. For two of these elements, say φ_1 and φ_2 , the unique irreducible factor of type C_1 is contained in Φ_t . For the other, φ_3 , the unique irreducible factor of type C_1 is contained in Φ_I . Thus by the above,

$$\sum_{i=1}^3 e^{\rho(\Phi^+(\varphi_i), \varphi_i)}(\gamma) = 1 + 1 - 1 = 1.$$

□

Lemma 7.9. *There is a choice $\Psi^+(h)$ of positive roots for Ψ such that $\Psi^+(h) \cap \Phi_t = \Phi_R^+(h)$ and*

$$\delta(\Psi^+(h), \psi, h) = (-1)^{r(\Phi_I)}$$

for every $\psi \in \mathcal{T}(\Psi)$. Here $r(\Phi_I)$ is defined as in Theorem 5.5.

Proof. Recall that for any choice of $\Psi^+, \psi \in \mathcal{T}(\Psi)$, $\psi^+ = \psi \cap \Psi^+$,

$$\delta(\Psi^+, \psi, h) = \delta_0(\Psi^+, \psi, h) = \prod_{\alpha \in \Psi^+ \setminus \psi^+} \delta(\alpha).$$

We have $h = \gamma \exp(c_S H)$ where $H = H_1 + H_2$, $H_1 \in \underline{t}_S$, $H_2 = c_S^{-1} \log a$. Now for all $\alpha \in \Psi$, $e^\alpha(\gamma) = 1$, so that

$$\Delta(\alpha) = \exp(\alpha(H)/2) - \exp(-\alpha(H)/2).$$

Suppose that $\alpha \in \Phi_R^+(h)$. Then $\alpha(H) = \alpha(H_2) = \alpha(c_S^{-1} \log a)$ is real and positive. Thus $\Delta(\alpha) = 2 \sinh(\alpha(H_2)/2)$ is real and positive, so that $\delta(\alpha) = 1$. Let Φ_I^+ be any choice of positive roots for Φ_I , and write $\psi_I^+ = \psi \cap \Phi_I^+$. Then if $\Psi^+ = \Phi_R^+(h) \cup \Phi_I^+$, we have

$$\delta(\Psi^+, \psi, h) = \prod_{\alpha \in \Phi_I^+ \setminus \psi_I^+} \delta(\alpha).$$

Define

$$L_S = \{H \in \underline{t}_S : \alpha(H) \in 2\pi i \mathbf{Z} \ \forall \ \alpha \in \Phi_I\}.$$

Then we can write $H_1 = H'_1 + H'_0$ where $H'_0 \in L_S$ and $H'_1 \in \underline{t}_S$ satisfies $|\alpha(H'_1)| \leq 2\pi$ for all $\alpha \in \Phi_I$. When Φ is of type A_{2n} , we have $H_1 + c_S H_2 \in \omega$, so that for $\alpha \in \Phi_I$, $|\alpha(H_1)| = |\operatorname{Im} \alpha(H_1 + c_S H_2)| < \pi$. Thus in this case we can take $H_1 = H'_1$, $H'_0 = 0$. For $\alpha \in \Phi_I$, write

$$\Delta'(\alpha) = \exp(\alpha(H'_1)/2) - \exp(-\alpha(H'_1)/2), \quad \delta'(\alpha) = \Delta'(\alpha) |\Delta'(\alpha)|^{-1}.$$

For every $\alpha \in \Phi_I(S)$, $\alpha(H'_0) \in 2\pi i \mathbf{Z}$, so that

$$\exp(\alpha(H'_0)/2) = \exp(-\alpha(H'_0)/2).$$

Thus

$$\Delta(\alpha) = \exp(\alpha(H_1)/2) - \exp(-\alpha(H_1)/2) = \exp(\alpha(H'_0)/2) \Delta'(\alpha);$$

$$\prod_{\alpha \in \Phi_I^+ \setminus \psi_I^+} \Delta(\alpha) = \exp((\rho(\Phi_I^+) - \rho(\psi_I^+))(H'_0)) \prod_{\alpha \in \Phi_I^+ \setminus \psi_I^+} \Delta'(\alpha).$$

When Φ is of type A_{2n} , $H'_0 = 0$. Otherwise, for every $\psi \in \mathcal{T}(\Psi)$, $\psi_I \in \mathcal{T}(\Phi_I)$ so that by Theorem 5.5 and Lemma 7.4(iv), $\rho(\Phi_I^+) - \rho(\psi_I^+)$ is in the root lattice of Φ_I . Thus in any case we have

$$\exp((\rho(\Phi_I^+) - \rho(\psi_I^+))(H'_0)) = 1; \quad \delta(\Psi^+, \psi, h) = \prod_{\alpha \in \Phi_I^+ \setminus \psi_I^+} \delta'(\alpha).$$

Let $\alpha \in \Phi_I$. Then $t_\alpha = \alpha(-iH'_1)/2$ is real, and $\Delta'(\alpha) = 2i \sin t_\alpha$. By our choice of H'_1 , we have $-\pi < t_\alpha < \pi$. Further, since $\Delta(\alpha) \neq 0$, we have $t_\alpha \neq 0$, so we can define a choice $\Phi_I^+(h)$ of positive roots for Φ_I by $\alpha \in \Phi_I^+(h)$ if and only if $t_\alpha > 0$. Now for $\alpha \in \Phi_I^+(h)$, $0 < t_\alpha < \pi$ so that $\sin t_\alpha > 0$ and $\delta'(\alpha) = i$.

Let $\Psi^+(h) = \Phi_I^+(h) \cup \Phi_R^+(h)$. For $\psi \in \mathcal{T}(\Psi)$, $2r(\Phi_I) = [\Phi_I^+(h) \setminus (\psi \cap \Phi_I^+(h))]$ by definition. Thus by the above,

$$\delta(\Psi^+(h), \psi, h) = i^{2r(\Phi_I)} = (-1)^{r(\Phi_I)}.$$

□

Proof of Lemma 6.8. Let $\psi_t \in \mathcal{T}_{\text{aug}}(\Phi_t)$. Then we must show that for any choice of Φ^+ ,

$$\sum_{\varphi \in \mathcal{T}(\Phi, \psi_t)} \epsilon(\varphi : \Phi^+) \delta(\Phi^+, \varphi, h) = \begin{cases} \epsilon(\psi_t : \Phi_R^+(h)), & \text{if } \psi_t \in \mathcal{T}(\Phi_t); \\ 0, & \text{otherwise.} \end{cases}$$

Fix $\psi_t \in \mathcal{T}_{\text{aug}}(\Phi_t)$. For each $\varphi \in \mathcal{T}(\Phi, \psi_t)$, $\varphi \cap \Psi = \psi_I \cup \psi_t$ where $\psi_I \in \mathcal{T}_{\text{aug}}(\Phi_I)$ by Lemma 7.4. We can write

$$\sum_{\varphi \in \mathcal{T}(\Phi, \psi_t)} \epsilon(\varphi : \Phi^+) \delta(\Phi^+, \varphi, h) = \sum_{\psi_I \in \mathcal{T}_{\text{aug}}(\Phi_I)} \sum_{\varphi \in \mathcal{T}(\Phi, \psi_I \cup \psi_t)} \epsilon(\varphi : \Phi^+) \delta(\Phi^+, \varphi, h).$$

But by Lemma 7.6,

$$\sum_{\varphi \in \mathcal{T}(\Phi, \psi_I \cup \psi_t)} \epsilon(\varphi : \Phi^+) \delta(\Phi^+, \varphi, h) = 0$$

unless $\psi_I \cup \psi_t \in \mathcal{T}(\Psi)$. In particular, if $\psi_t \notin \mathcal{T}(\Phi_t)$, we have

$$\sum_{\varphi \in \mathcal{T}(\Phi, \psi_t)} \epsilon(\varphi : \Phi^+) \delta(\Phi^+, \varphi, h) = 0,$$

and if $\psi_t \in \mathcal{T}(\Phi_t)$, we only get a nonzero contribution from $\psi_I \in \mathcal{T}(\Phi_I)$.

Suppose that $\psi_t \in \mathcal{T}(\Phi_t)$. Then by the above,

$$\sum_{\varphi \in \mathcal{T}(\Phi, \psi_t)} \epsilon(\varphi : \Phi^+) \delta(\Phi^+, \varphi, h) = \sum_{\psi_I \in \mathcal{T}(\Phi_I)} \sum_{\varphi \in \mathcal{T}(\Phi, \psi_I \cup \psi_t)} \epsilon(\varphi : \Phi^+) \delta(\Phi^+, \varphi, h).$$

Thus by Lemma 7.8, for any choice Ψ^+ of positive roots for Ψ ,

$$\sum_{\varphi \in \mathcal{T}(\Phi, \psi_t)} \epsilon(\varphi : \Phi^+) \delta(\Phi^+, \varphi, h) = \sum_{\psi_I \in \mathcal{T}(\Phi_I)} \epsilon(\psi_I \cup \psi_t : \Psi^+) \delta(\Psi^+, \psi_I \cup \psi_t, h).$$

By Lemma 7.9, we can choose $\Psi^+(h)$ so that $\Psi^+(h) \cap \Phi_t = \Phi_R^+(h)$ and for all $\psi_I \in \mathcal{T}(\Phi_I)$,

$$\delta(\Psi^+(h), \psi_I \cup \psi_t, h) = (-1)^{r(\Phi_I)}.$$

Write $\Phi_I^+(h) = \Phi_I \cap \Psi^+(h)$. Using this choice of $\Psi^+(h)$,

$$\begin{aligned} \sum_{\varphi \in \mathcal{T}(\Phi, \psi_t)} \epsilon(\varphi : \Phi^+) \delta(\Phi^+, \varphi, h) &= (-1)^{r(\Phi_I)} \sum_{\psi_I \in \mathcal{T}(\Phi_I)} \epsilon(\psi_I \cup \psi_t : \Psi^+(h)) \\ &= (-1)^{r(\Phi_I)} \epsilon(\psi_t : \Phi_R^+(h)) \sum_{\psi_I \in \mathcal{T}(\Phi_I)} \epsilon(\psi_I : \Phi_I^+(h)) = \epsilon(\psi_t : \Phi_R^+(h)) \end{aligned}$$

by Theorem 5.5. □

REFERENCES

- [1] Harish-Chandra, *Discrete series for semisimple Lie groups I*, Acta Math., **113** (1965), 241–318. MR **36**:2744
- [2] Harish-Chandra, *Harmonic analysis on real reductive groups I*, J. Funct. Anal., **19** (1975), 104–204. MR **53**:3201
- [3] R. Herb, *Fourier inversion and the Plancherel theorem*, (Proc. Marseille Conf. 1980), Lecture Notes in Math. Vol. 880, Springer-Verlag, Berlin and New York, 1981, 197–210. MR **83f**:22013
- [4] R. Herb, *The Plancherel theorem for semisimple Lie groups without compact Cartan subgroups*, (Proc. Marseille Conf. 1982), Lecture Notes in Math. Vol. 1020, Springer-Verlag, Berlin and New York, 1983, 73–79. MR **85a**:22001

- [5] R. Herb, *Discrete series characters and two-structures*, Trans. AMS, **350** (1998), 3341–3369. MR **98k**:22058
- [6] R. Herb, *Stable discrete series characters as lifts from complex two-structure groups*, Pacific J. Math. **196** (2000), 187–212. CMP 2001:04
- [7] R. Herb, *Two-structures and discrete series character formulas*, Proceedings of Symposia in Pure Mathematics, Vol. 68, A.M.S. Providence RI, 2000, 285–319. CMP 2000:16
- [8] R. Herb and J.A. Wolf, *The Plancherel theorem for general semisimple groups*, Compositio Math., **57** (1986), 271–355. MR **87h**:22020
- [9] A.W. Knap, *Representation Theory of Semisimple Groups, An Overview Based on Examples*, Princeton U. Press, Princeton, N.J., 1986. MR **87j**:22022
- [10] D. Shelstad, *Orbital integrals and a family of groups attached to a real reductive group*, Ann. Sci. Ecole Norm. Sup. **12** (1979), 1–31. MR **81k**:22014
- [11] D. Shelstad, *Embeddings of L-groups*, Canad. J. Math. **33** (1981), 513–558. MR **83e**:22022
- [12] D. Shelstad, *L-indistinguishability for real groups*, Math. Ann. **259**(1982), 385–430. MR **84c**:22017
- [13] J.A. Wolf, *Unitary representations on partially holomorphic cohomology spaces*, Memoirs AMS, **138** (1974). MR **56**:3194

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MARYLAND 20742
E-mail address: rah@math.umd.edu