

COHOMOLOGY OF PROJECTIVE SPACE SEEN BY RESIDUAL COMPLEX

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ABSTRACT. A subcomplex of a residual complex on projective space is constructed for computing the cohomology modules of locally free sheaves. A constructive new proof of the Bott formula is given by explicitly exhibiting bases for the cohomology modules.

1. INTRODUCTION

This paper is about the cohomology structure of locally free sheaves on a projective space. It is a part of the program for rendering concrete realizations of Grothendieck duality and finding applications of such realizations. In the paper, we propose a new method to organize cohomology data using residual complexes.

Let κ be a field. The cohomology modules of quasi-coherent sheaves on the n -dimensional projective space \mathbb{P}^n over κ are usually calculated using Čech cohomology. The need for Čech cohomology is supported by the common belief that injective resolutions of a given sheaf of modules are inaccessible from the computational point of view. Our philosophy about this belief is that, for sheaves of modules related to duality theory, some injective complexes can be explicitly constructed and give rise directly to cohomology information. For instance, the Gorenstein property of the ring $\kappa[[X^{t_1}, \dots, X^{t_n}]]$ is characterized by a value semigroup via a complex of injective modules

$$\cdots \rightarrow 0 \rightarrow \kappa((X)) \rightarrow \mathrm{Hom}_{\kappa}^c(\kappa[[X^{t_1}, \dots, X^{t_n}]], \kappa) \rightarrow 0 \rightarrow \cdots,$$

see [7]. In [9], some classical theorems on the projective plane, such as the Newton theorem and the Cayley-Bacharach theorem, are obtained from a complex of injective sheaves of modules. The complexes used in the above-cited works are called residual complexes, and they become subtle in higher dimensional projective spaces. In this paper, our philosophy on injective resolutions is emphasized by illustrating how the cohomology modules of locally free sheaves on \mathbb{P}^n are determined by the structure of a residual complex on \mathbb{P}^n .

We recall the definition of residual complexes. Let \mathcal{X} be a locally Noetherian scheme. A residual complex \mathcal{M}^\bullet on \mathcal{X} is a complex of quasi-coherent injective $\mathcal{O}_{\mathcal{X}}$ -modules, bounded below, with coherent cohomology sheaves, and such that there

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is an isomorphism

$$\bigoplus_{i \in \mathbb{Z}} \mathcal{M}^i \simeq \bigoplus_{\mathfrak{p} \in \mathcal{X}} J(\mathfrak{p}),$$

where $J(\mathfrak{p})$ is the quasi-coherent $\mathcal{O}_{\mathcal{X}}$ -module which is the constant sheaf $M(\mathfrak{p})$, a given injective hull of the residue field $\kappa(\mathfrak{p})$ over the local ring $\mathcal{O}_{\mathcal{X}, \mathfrak{p}}$, on $\{\mathfrak{p}\}^-$, and 0 elsewhere [3, page 304]. The importance of residual complexes is shown in [3], which lays the foundation of Grothendieck duality using the language of derived categories and derived functors. However, the subtlety of the coboundary maps

$$\mathcal{M}^i \rightarrow \mathcal{M}^{i+1} \quad (i \in \mathbb{Z})$$

of residual complexes, already mentioned by Grothendieck [2, page 113], is not fully treated in [3]. Fortunately, residual complexes and their properties are now better understood in concrete terms, thanks to [5], [6], [8], [10], [11], [12], [13], [14], [15], [16] and others. With success in concrete realizations of Grothendieck duality, it is desirable to see what information on schemes one gets from these concrete aspects of duality.

This paper focuses on cohomology modules of locally free sheaves of modules on \mathbb{P}^n . The next section sketches a residual complex J^\bullet on \mathbb{P}^n which is a simplified version of the residual complex constructed in [5]. As J^\bullet is an injective resolution of $\mathcal{O}_{\mathbb{P}^n}(-n-1)$, the complex $J^\bullet \otimes \mathcal{F}(n+1)$ provides an injective resolution for any locally free sheaf of modules \mathcal{F} on \mathbb{P}^n . In section 3, we investigate the structure of $J^\bullet \otimes \mathcal{F}(n+1)$. We construct a subcomplex $\mathcal{F}^{(\bullet)}$ of $J^\bullet \otimes \mathcal{F}(n+1)$ which gives rise to the cohomology modules of \mathcal{F} . In the last section, we explicitly give a basis for $H^q(\mathbb{P}^n, \Omega_{\mathbb{P}^n/\kappa}^p(m))$ by working on the complex $\Omega_{\mathbb{P}^n/\kappa}^p(m)^{(\bullet)}$ to illustrate that $\mathcal{F}^{(\bullet)}$ is indeed suitable for computing the cohomology modules of \mathcal{F} . More precisely, we give a basis $\{\omega_{t_0 \dots t_{n-q} \dots t_n}^{j_1 \dots j_p}\}$ for $\Omega_{\mathbb{P}^n/\kappa}^p(m)^{(q)}$ and explicitly describe coboundary maps

$$d^{(q)} : \Omega_{\mathbb{P}^n/\kappa}^p(m)^{(q)} \rightarrow \Omega_{\mathbb{P}^n/\kappa}^p(m)^{(q+1)}$$

in terms of operators $\frac{X_\ell}{X_{n-q}}$ on $\omega_{t_0 \dots t_{n-q} \dots t_n}^{j_1 \dots j_p}$. We will prove the following facts:

1. $H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n/\kappa}^p(m))$ has a basis consisting of

$$\sum_{\ell=0}^p (-1)^{\ell+1} \frac{X_{j_\ell}}{X_n} \omega_{t_0 \dots t_{n-i} 0 \dots 0}^{j_0 \dots \hat{j}_\ell \dots j_p}$$

satisfying

- $0 \leq i \leq n - p,$
- $0 \leq j_0 < \dots < j_p = n - i,$
- $t_0, \dots, t_{n-i} \geq 0,$
- $t_0 + \dots + t_{n-i} = m - p - 1.$

(Note that, to satisfy these conditions, p has to be less than m .)

2. For $0 < q < n$, $H^q(\mathbb{P}^n, \Omega_{\mathbb{P}^n/\kappa}^p(m))$ has a basis consisting of

$$\omega_{0 \dots 0 \underline{(-1)} \dots (-1)}^{(n-q+1)(n-q+2) \dots n}.$$

(For this element to be well-defined, p has to equal q and m has to vanish.)

3. $H^n(\mathbb{P}^n, \Omega_{\mathbb{P}^n/\kappa}^p(m))$ has a basis consisting of

$$\omega_{(-1) \dots (-1) t_i \dots t_n}^{12 \dots i j_{i+1} \dots j_p}$$

satisfying

- $0 \leq i \leq p,$
- $i < j_{i+1} < \dots < j_p \leq n,$
- $t_i < -1,$
- $t_{i+1}, \dots, t_n < 0,$
- $t_i + \dots + t_n = m - p + i - 1.$

(Note that, to satisfy these conditions, m has to be less than $p - n$.)

Counting the cardinality of these bases (see Corollary 4.4 and Corollary 4.9), the Bott formula is recovered.

2. RESIDUAL COMPLEX ON PROJECTIVE SPACE

As we work on a projective space $\mathbb{P}^n = \text{Proj}(\kappa[X_0, X_1, \dots, X_n])$ over κ , the approach in [5] to residual complexes can be simplified. In particular, a canonical injective hull of the residue field of a point on \mathbb{P}^n can be constructed without the identifying process in [5, section 2].

For each point $\mathfrak{p} \in \mathbb{P}^n$, the $\mathcal{O}_{\mathbb{P}^n, \mathfrak{p}}$ -module

$$M_{\mathbb{P}^n}(\mathfrak{p}) := H_{\mathfrak{m}_{\mathfrak{p}}}^d(\Omega_{\mathcal{O}_{\mathbb{P}^n, \mathfrak{p}}/\kappa}^n),$$

where d is the height of \mathfrak{p} and $\mathfrak{m}_{\mathfrak{p}}$ is the maximal ideal of $\mathcal{O}_{\mathbb{P}^n, \mathfrak{p}}$, is an injective hull of the residue field of

$$\kappa \left[\frac{X_0}{X_i}, \dots, \frac{X_{i-1}}{X_i}, \frac{X_{i+1}}{X_i}, \dots, \frac{X_n}{X_i} \right]_{\mathfrak{p}}.$$

Let $J_{\mathbb{P}^n}(\mathfrak{p})$ be the constant sheaf which is $M_{\mathbb{P}^n}(\mathfrak{p})$ on $\{\mathfrak{p}\}^-$, and 0 elsewhere. We write $M_{\mathbb{P}^n}(\mathfrak{p})$ and $J_{\mathbb{P}^n}(\mathfrak{p})$ simply as $M(\mathfrak{p})$ and $J(\mathfrak{p})$, if it is clear from the context that we are working on \mathbb{P}^n . Cover \mathbb{P}^n by affine open subsets

$$V_i = \text{Spec } \kappa \left[\frac{X_0}{X_i}, \dots, \frac{X_{i-1}}{X_i}, \frac{X_{i+1}}{X_i}, \dots, \frac{X_n}{X_i} \right].$$

If \mathfrak{p} is contained in some V_i , then there is a canonical isomorphism

$$M(\mathfrak{p}) \simeq H_{\mathfrak{m}_{\mathfrak{p}}}^d(\Omega_{\kappa[\frac{X_0}{X_i}, \dots, \frac{X_{i-1}}{X_i}, \frac{X_{i+1}}{X_i}, \dots, \frac{X_n}{X_i}]_{\mathfrak{p}}/\kappa}^n).$$

By identifying the above two modules, elements in $M(\mathfrak{p})$ can be described by generalized fractions

$$\left[\begin{array}{c} \omega \\ f_1^{t_1}, \dots, f_d^{t_d} \end{array} \right],$$

where

$$\omega \in \Omega_{\kappa[\frac{X_0}{X_i}, \dots, \frac{X_{i-1}}{X_i}, \frac{X_{i+1}}{X_i}, \dots, \frac{X_n}{X_i}]_{\mathfrak{p}}/\kappa}^n, \quad t_1, \dots, t_d \geq 1,$$

and f_1, \dots, f_d is a system of parameters of $\kappa[\frac{X_0}{X_i}, \dots, \frac{X_{i-1}}{X_i}, \frac{X_{i+1}}{X_i}, \dots, \frac{X_n}{X_i}]_{\mathfrak{p}}$. The reader is referred to [6, Chapter 2] for the definition and properties of generalized fractions. In this paper, we allow t_ℓ to be any integer: By convention,

$$\left[\begin{array}{c} \omega \\ f_1^{t_1}, \dots, f_d^{t_d} \end{array} \right] = 0,$$

if some $t_\ell \leq 0$.

Example 1. For each $1 \leq i \leq n$, let \mathfrak{p} be the generic point of the closed subscheme $\mathbb{P}^i := \text{Proj}(\kappa[X_0, \dots, X_i])$ of \mathbb{P}^n . The elements of $M(\mathfrak{p})$ are sums of the elements of the form

$$\left[\begin{array}{c} \frac{h}{g} d_{X_0}^{X_1} \wedge \dots \wedge d_{X_0}^{X_n} \\ (\frac{X_{i+1}}{X_0})^{t_{i+1}}, \dots, (\frac{X_n}{X_0})^{t_n} \end{array} \right],$$

where $g, h \in \kappa[\frac{X_1}{X_0}, \dots, \frac{X_i}{X_0}]$ and $t_{i+1}, \dots, t_n \geq 1$.

Example 2. Let \mathfrak{p} be a point in $\mathbb{P}^i \setminus \mathbb{P}^{i-1}$ for some $1 \leq i \leq n$. As a prime ideal of $\kappa[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}]$, \mathfrak{p} is generated by a prime ideal \mathfrak{p}' of $\kappa[\frac{X_0}{X_i}, \dots, \frac{X_{i-1}}{X_i}]$ and $\frac{X_{i+1}}{X_i}, \dots, \frac{X_n}{X_i}$. The elements of $M(\mathfrak{p})$ are sums of the elements of the form

$$\left[\begin{array}{c} \frac{h}{g} d_{X_i}^{X_0} \wedge \dots \wedge d_{X_i}^{X_{i-1}} \wedge d_{X_i}^{X_{i+1}} \wedge \dots \wedge d_{X_i}^{X_n} \\ f_1, \dots, f_\ell, (\frac{X_{i+1}}{X_i})^{t_{i+1}}, \dots, (\frac{X_n}{X_i})^{t_n} \end{array} \right],$$

where

$$h \in \kappa \left[\frac{X_0}{X_i}, \dots, \frac{X_{i-1}}{X_i} \right], g \in \kappa \left[\frac{X_0}{X_i}, \dots, \frac{X_{i-1}}{X_i} \right] \setminus \mathfrak{p}', f_1, \dots, f_\ell \in \kappa \left[\frac{X_0}{X_i}, \dots, \frac{X_{i-1}}{X_i} \right]$$

form a system of parameters of $\kappa[\frac{X_0}{X_i}, \dots, \frac{X_{i-1}}{X_i}]_{\mathfrak{p}'}$, and $t_{i+1}, \dots, t_n \geq 1$.

Let

$$J_{\mathbb{P}^n}^j := \bigoplus_{\text{ht } \mathfrak{p}=j} J(\mathfrak{p}).$$

To define coboundary maps in order to make a residual complex

$$\dots \rightarrow 0 \rightarrow J_{\mathbb{P}^n}^0 \rightarrow J_{\mathbb{P}^n}^1 \rightarrow \dots \rightarrow J_{\mathbb{P}^n}^n \rightarrow 0 \rightarrow \dots$$

on \mathbb{P}^n , it suffices to define

$$d_{\mathbb{P}^n; \mathfrak{p}, \mathfrak{q}}: M(\mathfrak{p}) \rightarrow M(\mathfrak{q})$$

for each pair of points \mathfrak{p} and \mathfrak{q} with $\text{ht } \mathfrak{q} = 1 + \text{ht } \mathfrak{p}$. We write $d_{\mathbb{P}^n; \mathfrak{p}, \mathfrak{q}}$ simply as $d_{\mathfrak{p}, \mathfrak{q}}$, if it is clear from the context that we are working on \mathbb{P}^n . A description of the maps $d_{\mathfrak{p}, \mathfrak{q}}$ can be found in [5]. Here we spell out two special cases of $d_{\mathfrak{p}, \mathfrak{q}}$ that we use in this paper: Let

$$\omega_i^{j_1 \dots j_p} = d_{X_i}^{X_{j_1}} \wedge \dots \wedge d_{X_i}^{X_{j_p}}$$

and

$$\omega_i = \omega_i^{0 \dots \hat{i} \dots n},$$

where $0 \leq i, j_1, \dots, j_p \leq n$ and $0 \dots \hat{i} \dots n$ is the sequence obtained by removing i from the sequence $0 \dots n$. The symbols $\omega_i^{j_1 \dots j_p}$ and ω_i are just shorthand for elements in some modules depending on the context.

Formula 1. For each $1 \leq i \leq n$, let \mathfrak{p} (resp. \mathfrak{q}) be the generic point of \mathbb{P}^i (resp. \mathbb{P}^{i-1}). The map $d_{\mathfrak{p}, \mathfrak{q}}$ is given by

$$d_{\mathfrak{p}, \mathfrak{q}} \left(\left[\begin{array}{c} (\frac{X_i}{X_0})^{-t_i} \frac{h}{g} \omega_0 \\ (\frac{X_{i+1}}{X_0})^{t_{i+1}}, \dots, (\frac{X_n}{X_0})^{t_n} \end{array} \right] \right) = \left[\begin{array}{c} \frac{h}{g} \omega_0 \\ (\frac{X_i}{X_0})^{t_i}, (\frac{X_{i+1}}{X_0})^{t_{i+1}}, \dots, (\frac{X_n}{X_0})^{t_n} \end{array} \right],$$

where $t_{i+1}, \dots, t_n \geq 1$, $t_i \in \mathbb{Z}$, and $g, h \in \kappa[\frac{X_1}{X_0}, \dots, \frac{X_i}{X_0}]$ have no factors $\frac{X_i}{X_0}$.

Formula 2. Let \mathfrak{p} and \mathfrak{q} be two points in $\mathbb{P}^i \setminus \mathbb{P}^{i-1}$ with $\text{ht } \mathfrak{q} = 1 + \text{ht } \mathfrak{p}$ for some $1 \leq i \leq n$. As a prime ideal of $\kappa[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}]$, \mathfrak{p} (resp. \mathfrak{q}) is generated by a prime ideal \mathfrak{p}' (resp. \mathfrak{q}') of $\kappa[\frac{X_0}{X_i}, \dots, \frac{X_{i-1}}{X_i}]$ and $\frac{X_{i+1}}{X_i}, \dots, \frac{X_n}{X_i}$. The map $d_{\mathbb{P}^n; \mathfrak{p}, \mathfrak{q}}$ is given by

$$\begin{aligned} d_{\mathbb{P}^n; \mathfrak{p}, \mathfrak{q}} & \left(\left[\begin{array}{c} \frac{h}{g} \omega_i \\ f_1, \dots, f_\ell, \left(\frac{X_{i+1}}{X_i}\right)^{t_{i+1}}, \dots, \left(\frac{X_n}{X_i}\right)^{t_n} \end{array} \right] \right) \\ & = \left[\begin{array}{c} \frac{h'}{g'} \omega_i \\ f'_1, \dots, f'_{\ell+1}, \left(\frac{X_{i+1}}{X_i}\right)^{t_{i+1}}, \dots, \left(\frac{X_n}{X_i}\right)^{t_n} \end{array} \right], \end{aligned}$$

where

- $t_{i+1}, \dots, t_n \geq 1$,
- $h, h', f_1, \dots, f_\ell, f'_1, \dots, f'_{\ell+1} \in \kappa[\frac{X_0}{X_i}, \dots, \frac{X_{i-1}}{X_i}]$,
- $g \in \kappa[\frac{X_0}{X_i}, \dots, \frac{X_{i-1}}{X_i}] \setminus \mathfrak{p}'$,
- $g' \in \kappa[\frac{X_0}{X_i}, \dots, \frac{X_{i-1}}{X_i}] \setminus \mathfrak{q}'$,

satisfy the following conditions:

- f_1, \dots, f_ℓ form a system of parameters of $\kappa[\frac{X_0}{X_i}, \dots, \frac{X_{i-1}}{X_i}]_{\mathfrak{p}'}$,
- $f'_1, \dots, f'_{\ell+1}$ form a system of parameters of $\kappa[\frac{X_0}{X_i}, \dots, \frac{X_{i-1}}{X_i}]_{\mathfrak{q}'}$,
-

$$d_{\mathbb{P}^i; \mathfrak{p}', \mathfrak{q}'} \left(\left[\begin{array}{c} \frac{h}{g} \omega_i^{0 \cdots (i-1)} \\ f_1, \dots, f_\ell \end{array} \right] \right) = \left[\begin{array}{c} \frac{h'}{g'} \omega_i^{0 \cdots (i-1)} \\ f'_1, \dots, f'_{\ell+1} \end{array} \right].$$

We remark that it is a subtle problem to give an explicit formula for h' , g' , and $f'_1, \dots, f'_{\ell+1}$, see [9, Example 3].

The complex $J_{\mathbb{P}^n}^\bullet$ gives an injective resolution of $\omega_{\mathbb{P}^n} := \Omega_{\mathbb{P}^n/\kappa}^n$. To describe a short exact sequence

$$0 \rightarrow \omega_{\mathbb{P}^n} \rightarrow J_{\mathbb{P}^n}^0 \rightarrow J_{\mathbb{P}^n}^1,$$

it suffices to describe a map

$$\Gamma(V_i, \omega_{\mathbb{P}^n}) \rightarrow \Gamma(V_i, J_{\mathbb{P}^n}^0)$$

for each $0 \leq i \leq n$. The map we need is trivial in formalism: it has the form

$$h\omega_i \mapsto \frac{h}{1}\omega_i,$$

where $h \in \kappa[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}]$.

3. COMPLEXES GIVING RISE TO COHOMOLOGY

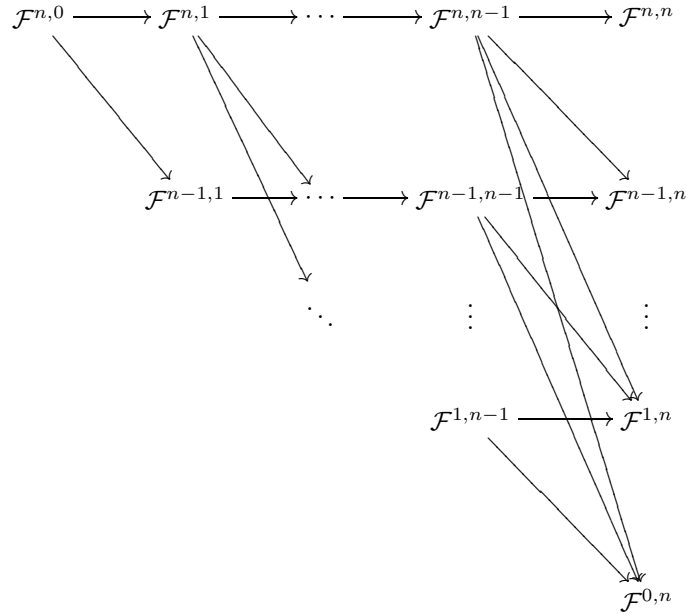
The residual complex $J_{\mathbb{P}^n}^\bullet$ can be organized to reveal information about the cohomology modules of locally free sheaves of \mathbb{P}^n : Let $\mathbb{P}^{-1} = \emptyset$. For any locally free sheaf \mathcal{F} of \mathbb{P}^n and $0 \leq i, j \leq n$, we define

$$\mathcal{F}^{i,j} := \bigoplus_{\mathfrak{p}} M(\mathfrak{p}) \otimes_{\mathcal{O}_{\mathbb{P}^n, \mathfrak{p}}} \mathcal{F}(n+1)_{\mathfrak{p}},$$

where \mathfrak{p} ranges over all points in $\mathbb{P}^i \setminus \mathbb{P}^{i-1}$ with height j . Then $\mathcal{F}^{i,j} = 0$ for $j < n-i$, and

$$\Gamma(\mathbb{P}^n, J^j \otimes \mathcal{F}(n+1)) \simeq \bigoplus_{i=n-j}^n \mathcal{F}^{i,j}.$$

As $d_{\mathfrak{p},\mathfrak{q}}$ vanishes unless \mathfrak{q} is an immediate specialization of \mathfrak{p} and the immediate specializations of the points in $\mathbb{P}^i \setminus \mathbb{P}^{i-1}$ lies in \mathbb{P}^i , the non-trivial coboundary maps of the complex $\Gamma(\mathbb{P}^n, J_{\mathbb{P}^n}^\bullet \otimes \mathcal{F}(n+1))$ consist of at most those $\mathcal{F}^{i,j} \rightarrow \mathcal{F}^{k,j+1}$ with $i \geq k$. We indicate these maps in the following diagram:



Let $\mathcal{F}^{(n)} = \mathcal{F}^{0,n}$, and let $\mathcal{F}^{(i)}$ be the kernel of the map $\mathcal{F}^{n-i,i} \rightarrow \mathcal{F}^{n-i,i+1}$ for $0 \leq i \leq n-1$.

Lemma 3.1. *For each $1 \leq i \leq n$, the subcomplex*

$$\dots \rightarrow 0 \rightarrow \omega_{\mathbb{P}^n}^{n-i,i} \rightarrow \omega_{\mathbb{P}^n}^{n-i,i+1} \rightarrow \dots \rightarrow \omega_{\mathbb{P}^n}^{n-i,n} \rightarrow 0 \rightarrow \dots$$

of $\Gamma(\mathbb{P}^n, J_{\mathbb{P}^n}^\bullet)$ has only the i -th cohomology non-trivial, which as a κ -vector space has a basis consisting of the elements of the form

$$\left[\begin{array}{l} (\frac{X_0}{X_i})^{t_0} \dots (\frac{X_{i-1}}{X_i})^{t_{i-1}} \omega_i \\ (\frac{X_{i+1}}{X_i})^{t_{i+1}}, \dots, (\frac{X_n}{X_i})^{t_n} \end{array} \right],$$

where $t_0, \dots, t_{i-1} \geq 0$ and $t_{i+1}, \dots, t_n \geq 1$.

Proof. From the construction, $J_{\mathbb{P}^i}^\bullet$ is a resolution of $\omega_{\mathbb{P}^i}$. Thanks to Formula 2, the complex $\omega_{\mathbb{P}^n}^{n-i,\bullet}$ as a κ -vector space is a direct sum of \mathbb{N}^{n-i} copies (indexed by $(\frac{X_{i+1}}{X_i})^{t_{i+1}}, \dots, (\frac{X_n}{X_i})^{t_n}$, where $t_{i+1}, \dots, t_n \geq 1$) of $\Gamma(\mathbb{P}^i \setminus \mathbb{P}^{i-1}, J_{\mathbb{P}^i}^\bullet)$. Hence $\omega_{\mathbb{P}^n}^{n-i,\bullet}$ has only the $(n-i)$ -th cohomology non-trivial. Since the kernel of $\omega_{\mathbb{P}^i}^{i,0} \rightarrow \omega_{\mathbb{P}^i}^{i,1}$ consists of elements of the form

$$hd \frac{X_0}{X_i} \wedge \dots \wedge d \frac{X_{i-1}}{X_i},$$

where $h \in \kappa[\frac{X_0}{X_i}, \dots, \frac{X_{i-1}}{X_i}]$, the $(n-i)$ -th cohomology of $\omega_{\mathbb{P}^n}^{n-i,\bullet}$ is generated by the elements of the form

$$\left[\begin{array}{l} (\frac{X_0}{X_i})^{t_0} \dots (\frac{X_{i-1}}{X_i})^{t_{i-1}} \omega_i \\ (\frac{X_{i+1}}{X_i})^{t_{i+1}}, \dots, (\frac{X_n}{X_i})^{t_n} \end{array} \right],$$

where $t_0, \dots, t_{i-1} \geq 0$ and $t_{i+1}, \dots, t_n \geq 1$. These elements are linearly independent. \square

Theorem 3.2. *For any locally free sheaf of modules \mathcal{F} on \mathbb{P}^n , $H^i(\mathbb{P}^n, \mathcal{F})$ is isomorphic to the i -th cohomology module of the subcomplex*

$$\dots \rightarrow 0 \rightarrow \mathcal{F}^{(0)} \rightarrow \mathcal{F}^{(1)} \rightarrow \dots \rightarrow \mathcal{F}^{(n-1)} \rightarrow \mathcal{F}^{(n)} \rightarrow 0 \rightarrow \dots$$

of the complex $\Gamma(\mathbb{P}^n, J_{\mathbb{P}^n}^\bullet \otimes \mathcal{F}(n+1))$.

Proof. First we compare the complexes $\mathcal{F}^{i,\bullet}$ and $\omega_{\mathbb{P}^n}^{i,\bullet}$ for $1 \leq i \leq n$. As modules, $\omega_{\mathbb{P}^n}^{i,\bullet}$ is the direct sum of those injective hulls $M(\mathfrak{p})$ with $\mathfrak{p} \in V_i \cap \mathbb{P}^i$, and $\mathcal{F}^{i,\bullet}$ is the direct sum of rank \mathcal{F} copies of $\omega_{\mathbb{P}^n}^{i,\bullet}$. Since the coboundary maps of $\mathcal{F}^{i,\bullet}$ and $\omega_{\mathbb{P}^n}^{i,\bullet}$ are given by the coboundary maps of $\Gamma(V_i, J_{\mathbb{P}^n}^\bullet \otimes \mathcal{F}(n+1))$ and $\Gamma(V_i, J_{\mathbb{P}^n}^\bullet)$, respectively, as complexes, $\mathcal{F}^{i,\bullet}$ is also isomorphic to the direct sum of rank \mathcal{F} copies of $\omega_{\mathbb{P}^n}^{i,\bullet}$. Therefore $\mathcal{F}^{i,\bullet}$ has only one non-vanishing cohomology $H^{n-i}(\mathcal{F}^{i,\bullet})$, which is $\mathcal{F}^{(n-i)}$. With this fact, it is a standard trick to derive the theorem by diagram chasing. The details are left to the reader. \square

Note that $\mathcal{F}^{(\bullet)}$ is also a subcomplex of

$$\dots \rightarrow 0 \rightarrow \mathcal{F}^{n,0} \rightarrow \mathcal{F}^{n-1,1} \rightarrow \dots \rightarrow \mathcal{F}^{1,n-1} \rightarrow \mathcal{F}^{0,n} \rightarrow 0 \rightarrow \dots,$$

which is isomorphic to the sum of rank \mathcal{F} copies of

$$(1) \quad \dots \rightarrow 0 \rightarrow \omega_{\mathbb{P}^n}^{n,0} \rightarrow \omega_{\mathbb{P}^n}^{n-1,1} \rightarrow \dots \rightarrow \omega_{\mathbb{P}^n}^{1,n-1} \rightarrow \omega_{\mathbb{P}^n}^{0,n} \rightarrow 0 \rightarrow \dots.$$

As the coboundary map of the complex (1) is explicitly described in Formula 1, the dimension $h^i(\mathbb{P}^n, \mathcal{F})$ of the κ -vector space $H^i(\mathbb{P}^n, \mathcal{F})$ can be calculated directly.

4. BOTT FORMULA

The following formula is well-known.

Bott Formula ([1]).

$$h^q(\mathbb{P}^n, \Omega_{\mathbb{P}^n/\kappa}^p(m)) = \begin{cases} \binom{m-1}{p} \binom{m+n-p}{m}, & \text{for } q = 0, 0 \leq p \leq n, p < m; \\ 1, & \text{for } m = 0, 0 \leq p = q \leq n; \\ \binom{-m-1}{n-p} \binom{-m+p}{-m}, & \text{for } q = n, 0 \leq p \leq n, m < p - n; \\ 0, & \text{otherwise.} \end{cases}$$

It is interesting to know what the combinatorial numbers that occur in this formula mean. In this section, we interpret these numbers by exhibiting a basis for the κ -vector space $H^q(\mathbb{P}^n, \Omega_{\mathbb{P}^n/\kappa}^p(m))$.

For $i, j_1, \dots, j_p, t_0, \dots, t_n \in \mathbb{Z}$ satisfying

- $0 \leq i, j_1, \dots, j_p \leq n$,
- $t_0 + \dots + t_n = m - p - 1$,

we define an element $\omega_{t_0 \dots t_n}^{j_1 \dots j_p}$ in $\Omega_{\mathbb{P}^n/\kappa}^p(m)^{n-i,i}$: If $p > 0$,

$$\omega_{t_0 \dots t_n}^{j_1 \dots j_p} := \left[\begin{matrix} \left(\frac{X_0}{X_i}\right)^{t_0} \dots \left(\frac{X_{i-1}}{X_i}\right)^{t_{i-1}} \omega_i \\ \left(\frac{X_{i+1}}{X_i}\right)^{-t_{i+1}}, \dots, \left(\frac{X_n}{X_i}\right)^{-t_n} \end{matrix} \right] \otimes \omega_i^{j_1 \dots j_p} \otimes X_i^{m+n+1}.$$

If $p = 0$, the symbol $\omega_{t_0 \dots \underline{t_i} \dots t_n}^{j_1 \dots j_p}$ is understood as $\omega_{t_0 \dots \underline{t_i} \dots t_n}$ and defined by

$$\omega_{t_0 \dots \underline{t_i} \dots t_n} := \left[\begin{array}{c} (\frac{X_0}{X_i})^{t_0} \dots (\frac{X_{i-1}}{X_i})^{t_{i-1}} \omega_i \\ (\frac{X_{i+1}}{X_i})^{-t_{i+1}}, \dots, (\frac{X_n}{X_i})^{-t_n} \end{array} \right] \otimes X_i^{m+n+1}.$$

These elements satisfy the following properties:

1. $\omega_{t_0 \dots \underline{t_i} \dots t_n}^{j_1 \dots j_p} = 0$ if and only if one of the following conditions holds:
 - $j_\ell = i$ for some ℓ .
 - $j_k = j_\ell$ for some $k \neq \ell$.
 - $t_\ell \geq 0$ for some $\ell > i$.
2. If $j_k \neq j_\ell$ for $k \neq \ell$ and σ is a permutation of $j_1 \dots j_p$, then

$$\omega_{t_0 \dots \underline{t_i} \dots t_n}^{j_1 \dots j_p} = \epsilon(\sigma) \omega_{t_0 \dots \underline{t_i} \dots t_n}^{\sigma(j_1) \dots \sigma(j_p)},$$

where $\epsilon(\sigma)$ is the sign of the permutation σ .

3. For $0 \leq k \leq n$ not equal i ,

$$\frac{X_k}{X_i} \omega_{t_0 \dots \underline{t_i} \dots t_n}^{j_1 \dots j_p} = \omega_{t'_0 \dots \underline{t'_i} \dots t'_n}^{j_1 \dots j_p},$$

where $t'_i = t_i - 1$ and $t'_\ell = t_\ell + \delta_{k\ell}$ if $\ell \neq i$.

Note that, for $0 \leq i \leq n$, $\Omega_{\mathbb{P}^n/\kappa}^p(m)^{(n-i)}$ as a κ -vector space has a basis consisting of those $\omega_{t_0 \dots \underline{t_i} \dots t_n}^{j_1 \dots j_p}$ satisfying

- $t_0, \dots, t_{i-1} \geq 0$,
- $t_{i+1}, \dots, t_n < 0$,
- $0 \leq j_1 < \dots < j_p \leq n$,
- $j_\ell \neq i$ for all ℓ .

Given an element

$$\Psi = \sum a_{t_0 \dots \underline{t_i} \dots t_n}^{j_1 \dots j_p} \omega_{t_0 \dots \underline{t_i} \dots t_n}^{j_1 \dots j_p} \quad (a_{t_0 \dots \underline{t_i} \dots t_n}^{j_1 \dots j_p} \in \kappa)$$

in $\Omega_{\mathbb{P}^n/\kappa}^p(m)^{(n-i)}$ satisfying the above conditions, we call $a_{t_0 \dots \underline{t_i} \dots t_n}^{j_1 \dots j_p}$ the coefficient of Ψ at $\omega_{t_0 \dots \underline{t_i} \dots t_n}^{j_1 \dots j_p}$.

Proposition 4.1. *For $1 \leq i \leq n$, the map $\Omega_{\mathbb{P}^n/\kappa}^p(m)^{(n-i)} \rightarrow \Omega_{\mathbb{P}^n/\kappa}^p(m)^{(n-i+1)}$, denoted by $d^{(n-i)}$, is given by*

$$\omega_{t_0 \dots \underline{t_i} \dots t_n}^{j_1 \dots j_p} \mapsto \sum_{\ell=0}^p (-1)^{\ell+1} \frac{X_{j_\ell}}{X_{i-1}} \omega_{t_0 \dots \underline{t_{i-1}} \dots t_n}^{j_0 \dots \hat{j}_\ell \dots j_p},$$

where $j_0 = i$.

Proof. Assume first that $p = 0$. Write $\omega_{t_0 \dots \underline{t_i} \dots t_n}$ in terms of $\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}$ and X_0 :

$$\omega_{t_0 \dots \underline{t_i} \dots t_n} = (-1)^i \left[\begin{array}{c} (\frac{X_1}{X_0})^{t_1} \dots (\frac{X_{i-1}}{X_0})^{t_{i-1}} (\frac{X_i}{X_0})^{t_i+1} \omega_0 \\ (\frac{X_{i+1}}{X_0})^{-t_{i+1}}, \dots, (\frac{X_n}{X_0})^{-t_n} \end{array} \right] \otimes X_0^{m+n+1}$$

By Formula 1, $\omega_{t_0 \dots \underline{t_i} \dots t_n}$ maps to

$$(-1)^i \left[\begin{array}{c} (\frac{X_1}{X_0})^{t_1} \dots (\frac{X_{i-1}}{X_0})^{t_{i-1}} \omega_0 \\ (\frac{X_i}{X_0})^{-t_i-1}, (\frac{X_{i+1}}{X_0})^{-t_{i+1}}, \dots, (\frac{X_n}{X_0})^{-t_n} \end{array} \right] \otimes X_0^{m+n+1},$$

which equals $-\frac{X_i}{X_{i-1}} \omega_{t_0 \dots \underline{t_{i-1}} \dots t_n}$.

Now assume $p > 0$. If we switch j_k and j_{k+1} for some $1 \leq k < p$, then it is easy to see that both $\omega_{t_0 \dots \underline{t_i} \dots t_n}^{j_1 \dots j_p}$ and $\sum_{\ell=0}^p (-1)^{\ell+1} \frac{X_{j_\ell}}{X_{i-1}} \omega_{t_0 \dots \underline{t_{i-1}} \dots t_n}^{j_0 \dots \hat{j}_\ell \dots j_p}$ change signs. If $j_k = j_{k+1}$ for some $1 \leq k < p$, then it is easy to see that both $\omega_{t_0 \dots \underline{t_i} \dots t_n}^{j_1 \dots j_p}$ and $\sum_{\ell=0}^p (-1)^{\ell+1} \frac{X_{j_\ell}}{X_{i-1}} \omega_{t_0 \dots \underline{t_{i-1}} \dots t_n}^{j_0 \dots \hat{j}_\ell \dots j_p}$ vanish. Therefore we may assume that $0 \leq j_1 < \dots < j_p \leq n$. If $j_k = i$ for some $1 \leq k \leq p$, it is also easy to see that both $\omega_{t_0 \dots \underline{t_i} \dots t_n}^{j_1 \dots j_p}$ and $\sum_{\ell=0}^p (-1)^{\ell+1} \frac{X_{j_\ell}}{X_{i-1}} \omega_{t_0 \dots \underline{t_{i-1}} \dots t_n}^{j_0 \dots \hat{j}_\ell \dots j_p}$ vanish. Therefore we may assume furthermore that $j_k \neq i$ for $1 \leq k \leq p$.

Write $\omega_{t_0 \dots \underline{t_i} \dots t_n}^{j_1 \dots j_p}$ in terms of $\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}$ and X_0 :

$$\omega_{t_0 \dots \underline{t_i} \dots t_n}^{j_1 \dots j_p} = \left[\begin{array}{c} \left(\frac{X_1}{X_0}\right)^{t_1} \dots \left(\frac{X_i}{X_0}\right)^{t_i} \omega_0 \\ \left(\frac{X_{i+1}}{X_0}\right)^{-t_{i+1}}, \dots, \left(\frac{X_n}{X_0}\right)^{-t_n} \end{array} \right] \otimes \sum_{\ell=0}^p (-1)^{i+\ell} \frac{X_{j_\ell}}{X_0} \omega_0^{j_0 \dots \hat{j}_\ell \dots j_p} \otimes X_0^{m+n+1}$$

By Formula 1, $\omega_{t_0 \dots \underline{t_i} \dots t_n}^{j_1 \dots j_p}$ maps to

$$\left[\begin{array}{c} \left(\frac{X_1}{X_0}\right)^{t_1} \dots \left(\frac{X_{i-1}}{X_0}\right)^{t_{i-1}} \omega_0 \\ \left(\frac{X_i}{X_0}\right)^{-t_i}, \dots, \left(\frac{X_n}{X_0}\right)^{-t_n} \end{array} \right] \otimes \sum_{\ell=0}^p (-1)^{i+\ell} \frac{X_{j_\ell}}{X_0} \omega_0^{j_0 \dots \hat{j}_\ell \dots j_p} \otimes X_0^{m+n+1},$$

which equals

$$\left[\begin{array}{c} \left(\frac{X_0}{X_{i-1}}\right)^{t_0} \dots \left(\frac{X_{i-2}}{X_{i-1}}\right)^{t_{i-2}} \omega_{i-1} \\ \left(\frac{X_i}{X_{i-1}}\right)^{-t_i}, \dots, \left(\frac{X_n}{X_{i-1}}\right)^{-t_n} \end{array} \right] \otimes \sum_{\ell=0}^p (-1)^{\ell+1} \frac{X_{j_\ell}}{X_0} \cdot \left(\frac{X_0}{X_{i-1}} \omega_{i-1}^{j_0 \dots \hat{j}_\ell \dots j_p} - \frac{X_{j_0}}{X_{i-1}} \omega_{i-1}^{0j_1 \dots \hat{j}_\ell \dots j_p} + \frac{X_{j_1}}{X_{i-1}} \omega_{i-1}^{0j_0 j_2 \dots \hat{j}_\ell \dots j_p} - \dots \right) \otimes X_{i-1}^{m+n+1}.$$

If $j_1 > 0$, the element $\omega_{i-1}^{0j_0 \dots \hat{j}_k \dots \hat{j}_\ell \dots j_p}$ appears twice in the above summation with opposite signs for each $0 \leq k < \ell \leq p$. If $j_1 = 0$, the above summation has only one non-trivial term, namely the term $\ell = 1$. In either cases, the above element equals

$$\sum_{\ell=0}^p (-1)^{\ell+1} \frac{X_{j_\ell}}{X_{i-1}} \omega_{t_0 \dots \underline{t_{i-1}} \dots t_n}^{j_0 \dots \hat{j}_\ell \dots j_p}.$$

□

Definition 4.2. For $0 \leq i \leq n-p$, $V_{p,m}^i$ is defined to be the vector space generated by those

$$\sum_{\ell=0}^p (-1)^{\ell+1} \frac{X_{j_\ell}}{X_n} \omega_{t_0 \dots \underline{t_{n-i}} 0 \dots 0}^{j_0 \dots \hat{j}_\ell \dots j_p}$$

satisfying the conditions

- $t_0, \dots, t_{n-i} \geq 0$,
- $j_0 < \dots < j_p = n - i$.

Theorem 4.3.

$$(2) \quad \text{Kernel}(d^{(0)}) = V_{p,m}^0 \oplus \dots \oplus V_{p,m}^{n-p}$$

Proof. If $p = 0$, then

$$\begin{aligned} \text{Kernel}(d^{(0)}) &= \langle \omega_{t_0 \dots t_n} | t_0, \dots, t_{n-1} \geq 0 \text{ and } t_n \geq -1 \rangle, \\ V_{p,m}^0 &= \langle \omega_{t_0 \dots t_n} | t_0, \dots, t_n \geq 0 \rangle, \\ V_{p,m}^n &= \langle \omega_{m0 \dots 0(-1)} \rangle \\ V_{p,m}^i &= \langle \omega_{t_0 \dots t_{n-i}0 \dots 0(-1)} | t_0, \dots, t_{n-i-1} \geq 0 \text{ and } t_{n-i} \geq 1 \rangle \end{aligned}$$

for $0 < i < n$. It is easy to see (2) in such case.

Now we assume that $p > 0$. It is straightforward to check that the vector spaces $V_{p,m}^0, \dots, V_{p,m}^{n-p}$ are contained in the kernel of the map $d^{(0)}$. For $-1 \leq i \leq n - p$, we define

$$W_{n-i} := \text{Kernel}(d^{(0)}) \cap \langle \omega_{t_0 \dots t_{n-i-1}0 \dots 0(-1)}^{j_1 \dots j_p} | j_1 < \dots < j_p < n - i \rangle.$$

Note that $W_{n+1} = \text{Kernel}(d^{(0)})$ and $W_p = 0$. We will show that

$$W_{n-i+1} = W_{n-i} \oplus V_{p,m}^i$$

for $0 \leq i \leq n - p$, from which the theorem follows easily.

We first show that $W_{n-i+1} = W_{n-i} + V_{p,m}^i$. Let

$$(3) \quad \Psi = \sum a_{t_0 \dots t_{n-i}}^{j_1 \dots j_p} \omega_{t_0 \dots t_{n-i}0 \dots 0(-1)}^{j_1 \dots j_p} \quad (a_{t_0 \dots t_{n-i}}^{j_1 \dots j_p} \in \kappa)$$

be an element in W_{n-i+1} , where the summation runs over those t_0, \dots, t_{n-i} and j_1, \dots, j_p satisfying the conditions

- $t_0, \dots, t_{n-i} \geq 0$,
- $j_1 < \dots < j_p < n - i + 1$.

By subtracting from Ψ the element

$$\begin{aligned} \sum a_{t_0 \dots t_{n-i-1}(t_{n-i}+1)}^{j_1 \dots j_p} &\left(\omega_{t_0 \dots t_{n-i-1}(t_{n-i}+1)0 \dots 0(-1)}^{j_1 \dots j_p} \right. \\ &\left. + \sum_{\ell=1}^p (-1)^{\ell+p+1} \frac{X_{j_\ell}}{X_n} \omega_{t_0 \dots t_{n-i}0 \dots 0}^{j_1 \dots \hat{j}_\ell \dots j_p(n-i)} \right) \end{aligned}$$

in $V_{p,m}^i$, where the summation runs over those $t_0, \dots, t_{n-i}, j_1, \dots, j_p$ satisfying the conditions

- $t_0, \dots, t_{n-i} \geq 0$,
- $j_1 < \dots < j_p < n - i$,

we may assume that the representation (3) of Ψ satisfies the additional condition

- $j_p < n - i, t_{n-i} = 0$

or the additional condition

- $j_p = n - i$.

In other words, we may assume that

$$\Psi = \Psi_1 + \Psi_2,$$

where

$$\Psi_1 = \sum a_{t_0 \dots t_{n-i-1}0}^{j_1 \dots j_p} \omega_{t_0 \dots t_{n-i-1}0 \dots 0(-1)}^{j_1 \dots j_p}$$

with indices running over those $t_0, \dots, t_{n-i-1}, j_1, \dots, j_p$ satisfying the conditions

- $t_0, \dots, t_{n-i-1} \geq 0$,

- $j_1 < \cdots < j_p < n - i$;

and

$$\Psi_2 = \sum a_{t_0 \cdots t_{n-i}}^{j_1 \cdots j_{p-1}(n-i)} \omega_{t_0 \cdots t_{n-i} 0 \cdots 0}^{j_1 \cdots j_{p-1}(n-i)} \underline{(-1)}$$

with indices running over those $t_0, \dots, t_{n-i}, j_1, \dots, j_{p-1}$ satisfying the conditions

- $t_0, \dots, t_{n-i} \geq 0$,
- $j_1 < \cdots < j_{p-1} < n - i$.

The coefficient of $d^{(0)}(\Psi)$ at $\omega_{t_0 \cdots t_{n-i-1}(t_{n-i}+1)0 \cdots 0}^{nj_1 \cdots j_{p-1}} \underline{(-1)} \underline{(-1)}$ is

$$(-1)^{p+1} a_{t_0 \cdots t_{n-i}}^{j_1 \cdots j_{p-1}(n-i)},$$

which has to be zero. Hence $\Psi_2 = 0$, and it follows that $W_{n-i+1} = W_{n-i} + V_{p,m}^i$.

Now we show that $W_{n-i} \cap V_{p,m}^i = 0$. For $i = 0$, this is trivial. So we assume that $i > 0$. Let

$$\Psi = \sum b_{t_0 \cdots t_{n-i}}^{j_0 \cdots j_p} \left(\sum_{\ell=0}^p (-1)^{\ell+1} \frac{X_{j_\ell}}{X_n} \omega_{t_0 \cdots t_{n-i} 0 \cdots \underline{0}}^{j_0 \cdots \hat{j}_\ell \cdots j_p} \right)$$

be an element in $W_{n-i} \cap V_{p,m}^i$, where the (first) summation runs over those t_0, \dots, t_{n-i} and j_0, \dots, j_p satisfying the conditions

- $t_0, \dots, t_{n-i} \geq 0$,
- $j_0 < \cdots < j_p = n - i$.

As an element in $V_{p,m}^i$, the coefficient of Ψ at $\omega_{t_0 \cdots t_{n-i-1}(t_{n-i}+1)0 \cdots 0}^{j_0 \cdots j_{p-1}} \underline{(-1)}$ is

$$(-1)^{p+1} b_{t_0 \cdots t_{n-i}}^{j_0 \cdots j_p}.$$

However, as an element in W_{n-i} , the coefficient of Ψ at $\omega_{t_0 \cdots t_{n-i-1}(t_{n-i}+1)0 \cdots 0}^{j_0 \cdots j_{p-1}} \underline{(-1)}$ is zero. Therefore all $b_{t_0 \cdots t_{n-i}}^{j_0 \cdots j_p}$ vanish, and hence $\Psi = 0$. \square

Corollary 4.4.

$$h^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n/\kappa}^p(m)) = \binom{m-1}{p} \binom{m+n-p}{m}.$$

Proof. Modulo the subspace generated by the elements of the form $\omega_{t'_0 \cdots t'_n}^{j'_1 \cdots j'_{p-1}(n-i)}$, the elements

$$\sum_{\ell=0}^p (-1)^{\ell+1} \frac{X_{j_\ell}}{X_n} \omega_{t_0 \cdots t_{n-i} 0 \cdots \underline{0}}^{j_0 \cdots \hat{j}_\ell \cdots j_p} \quad \text{and} \quad (-1)^{p+1} \omega_{t_0 \cdots t_{n-i-1}(t_{n-i}+1)0 \cdots 0}^{j_0 \cdots j_{p-1}} \underline{(-1)}$$

($j_0 < \cdots < j_p = n - i$) are the same. Hence, for a fixed i , the elements

$$\sum_{\ell=0}^p (-1)^{\ell+1} \frac{X_{j_\ell}}{X_n} \omega_{t_0 \cdots t_{n-i} 0 \cdots \underline{0}}^{j_0 \cdots \hat{j}_\ell \cdots j_p}$$

are linearly independent. Therefore

$$\begin{aligned} h^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n/\kappa}^p(m)) &= \sum_{i=0}^{n-p} \dim V_{p,m}^i \\ &= \sum_{i=0}^{n-p} \binom{n-i}{p} \binom{m+n-p-i-1}{n-i} \\ &= \binom{m-1}{p} \sum_{i=0}^{n-p} \binom{m+n-p-i-1}{m-1} \\ &= \binom{m-1}{p} \binom{m+n-p}{m}. \end{aligned}$$

□

Theorem 4.5. For $0 < q < n$, $H^q(\mathbb{P}^n, \Omega_{\mathbb{P}^n/\kappa}^p(m))$ vanishes unless $m = 0$ and $p = q$; $H^q(\mathbb{P}^n, \Omega_{\mathbb{P}^n/\kappa}^q)$ is non-trivial and generated (after identifying with the q -th cohomology of $\Omega_{\mathbb{P}^n/\kappa}^q(\bullet)$) by the equivalence class containing $\omega_{0 \dots 0 \underline{(-1)} \dots (-1)}^{(n-q+1)(n-q+2) \dots n}$.

Proof. If $p = 0$, then

$$\text{Kernel}(d^{(q)}) = \langle \omega_{t_0 \dots t_{n-q} \dots t_n} \mid t_{n-q} \geq -1 \rangle = \text{Image}(d^{(q-1)}).$$

This implies $H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) = 0$. So, to prove the theorem, we may assume that $p > 0$.

Let Ψ be an element in the kernel of the map

$$d^{(q)}: \Omega_{\mathbb{P}^n/\kappa}^p(m)^{(q)} \rightarrow \Omega_{\mathbb{P}^n/\kappa}^p(m)^{(q+1)}.$$

We claim that, by adding a suitable element in the image of $d^{(q-1)}$, Ψ becomes

$$\begin{cases} \omega_{0 \dots 0 \underline{(-1)} \dots (-1)}^{(n-q+1)(n-q+2) \dots n}, & \text{if } m = 0 \text{ and } p = q; \\ 0, & \text{otherwise.} \end{cases}$$

Recall that Ψ can be written as

$$(4) \quad \Psi = \sum a_{t_0 \dots t_n}^{j_1 \dots j_p} \omega_{t_0 \dots t_{n-q} \dots t_n}^{j_1 \dots j_p},$$

where $a_{t_0 \dots t_n}^{j_1 \dots j_p} \in \kappa$ and the summation runs over those $t_0, \dots, t_n, j_1, \dots, j_p \in \mathbb{Z}$ satisfying

- $t_0, \dots, t_{n-q-1} \geq 0$,
- $t_{n-q+1}, \dots, t_n < 0$,
- $t_0 + \dots + t_n = m - p - 1$,
- $0 \leq j_1 < \dots < j_p \leq n$,
- $j_\ell \neq n - q$ for all ℓ .

Step 1. We may assume $t_{n-q} = -1$.

If $t_{n-q} \leq -2$ and $j_\ell \neq n - q - 1$ for all ℓ , then the coefficient of $d^{(q)}(\Psi)$ at $\frac{X_{n-q}}{X_{n-q-1}} \omega_{t_0 \dots t_{n-q-1} \dots t_n}^{j_1 \dots j_p}$ is $-a_{t_0 \dots t_n}^{j_1 \dots j_p}$, which has to be zero. Using this fact, one sees that if $t_{n-q} \leq -2$ and $j_\ell = n - q - 1$ for some ℓ , then the coefficient of $d^{(q)}(\Psi)$ at $\omega_{t_0 \dots t_{n-q-1} \dots t_n}^{(n-q)j_1 \dots j_\ell \dots j_p}$ is $(-1)^{\ell+1} a_{t_0 \dots t_n}^{j_1 \dots j_p}$, which is thus also zero. Therefore $a_{t_0 \dots t_n}^{j_1 \dots j_p} = 0$ for $t_{n-q} \leq -2$.

For $t_{n-q} \geq 0$, if $j_\ell \neq n - q + 1$ for all ℓ , then

$$\omega_{t_0 \dots \underline{t_{n-q}} \dots t_n}^{j_1 \dots j_p} + d^{(q-1)} \left(\frac{X_{n-q}}{X_{n-q+1}} \omega_{t_0 \dots \underline{t_{n-q+1}} \dots t_n}^{j_1 \dots j_p} \right)$$

is contained in the κ -vector space generated by those $\omega_{t'_0 \dots \underline{t'_{n-q}} \dots t'_n}^{(n-q+1)j'_2 \dots j'_p}$ satisfying $t'_{n-q} \geq 0$. So we may assume that $j_\ell = n - q + 1$ for some ℓ . In this case,

$$\omega_{t_0 \dots \underline{t_{n-q}} \dots t_n}^{j_1 \dots j_p} = d^{(q-1)} \left((-1)^{\ell-1} \omega_{t_0 \dots \underline{t_{n-q+1}} \dots t_n}^{(n-q)j_1 \dots \widehat{j_\ell} \dots j_p} \right).$$

Therefore, to prove the claim, we may assume that the representation (4) of Ψ satisfies the additional condition:

- $t_{n-q} = -1$.

Step 2. We may assume $j_{\ell_0} = n - q + 1$ for some ℓ_0 and $t_{n-q+1} = -1$.

If $j_\ell \neq n - q + 1$ for all ℓ , then

$$\omega_{t_0 \dots \underline{t_{n-q-1}} \underline{(-1)t_{n-q+1}} \dots t_n}^{j_1 \dots j_p} + d^{(q-1)} \left(\omega_{t_0 \dots \underline{t_{n-q-1}} 0 \underline{(t_{n-q+1}-1)t_{n-q+2}} \dots t_n}^{j_1 \dots j_p} \right)$$

is contained in the κ -vector space generated by those $\omega_{t'_0 \dots \underline{t'_{n-q-1}} \underline{(-1)t'_{n-q+1}} \dots t'_n}^{(n-q+1)j'_2 \dots j'_p}$. Therefore, by adding an element of the image of $d^{(q-1)}$ to Ψ , we may assume furthermore that the representation (4) of Ψ satisfies one more condition:

- $j_{\ell_0} = n - q + 1$ for some ℓ_0 .

For $t_{n-q+1} \leq -2$, the coefficient of $d^{(q)}(\Psi)$ at $\frac{X_{n-q+1}}{X_{n-q-1}} \omega_{t_0 \dots \underline{t_{n-q-1}} \underline{(-1)t_{n-q+1}} \dots t_n}^{(n-q)j_1 \dots \widehat{j_{\ell_0}} \dots j_p}$ is $(-1)^{\ell_0+1} a_{t_0 \dots \underline{t_{n-q-1}} \underline{(-1)t_{n-q+1}} \dots t_n}^{j_1 \dots j_p}$, which has to be zero. Therefore the representation (4) of Ψ also satisfies the condition

- $t_{n-q+1} = -1$.

Step 3. We may assume $q \leq p$, $t_{n-q} = \dots = t_n = -1$, and $j_{p-q+i} = n - q + i$ for $1 \leq i \leq q$.

If $q > p$, repeating the process in Step 2, we may assume that the representation (4) is of the form

$$\Phi = \sum a_{t_0 \dots \underline{t_{n-q-1}} \underline{(-1) \dots (-1)t_{n-q+p+1}} \dots t_n}^{(n-q+1) \dots (n-q+p)} \omega_{t_0 \dots \underline{t_{n-q-1}} \underline{(-1) \dots (-1)t_{n-q+p+1}} \dots t_n}^{(n-q+1) \dots (n-q+p)}$$

Since

$$\begin{aligned} & d^{(q-1)} \left(\omega_{t_0 \dots \underline{t_{n-q-1}} 0 \underline{(-1) \dots (-1)(t_{n-q+p+1}-1)t_{n-q+p+2}} \dots t_n}^{(n-q+2) \dots (n-q+p+1)} \right) \\ &= (-1)^{p+1} \omega_{t_0 \dots \underline{t_{n-q-1}} \underline{(-1) \dots (-1)t_{n-q+p+1}} \dots t_n}^{(n-q+1) \dots (n-q+p)} \end{aligned}$$

by adding to Ψ an element of the image of $d^{(q-1)}$, Ψ becomes zero.

For $q \leq p$, repeating the process in Step 2, we may write

$$(5) \quad \Phi = \sum a_{t_0 \dots \underline{t_{n-q-1}} \underline{(-1) \dots (-1)}}^{j_1 \dots j_{p-q} (n-q+1) \dots n} \omega_{t_0 \dots \underline{t_{n-q-1}} \underline{(-1) \dots (-1)}}^{j_1 \dots j_{p-q} (n-q+1) \dots n}$$

Step 4. We may assume $q = p$.

Assume $p > q$. If $t_0 > 0$ and $j_1 > 0$, then

$$\omega_{t_0 \dots \underline{t_{n-q-1}} \underline{(-1) \dots (-1)}}^{j_1 \dots j_{p-q} (n-q+1) (n-q+2) \dots n} - d^{(q-1)} \left((-1)^{p-q} \omega_{(t_0-1)t_1 \dots \underline{t_{n-q-1}} 0 \underline{(-1) \dots (-1)}}^{0j_1 \dots j_{p-q} (n-q+2) (n-q+3) \dots n} \right)$$

is contained in the κ -vector space generated by those $\omega_{t'_0 \dots t'_{n-q-1}}^{0j_2' \dots j_{p-q}'(n-q+1)(n-q+2) \dots n}$. Therefore, in the case $p > q$, we may assume that the representation (5) also satisfies the condition

- $t_0 = 0$ or $j_1 = 0$.

If $j_1 = 0$, the coefficient of $d^{(q)}(\Psi)$ at $\omega_{(t_0+1)t_1 \dots t_{n-q-2}(t_{n-q-1}-1)(-1) \dots (-1)}$ is

$$a_{t_0 \dots t_{n-q-1}}^{0j_2 \dots j_{p-q}(n-q+1)(n-q+2) \dots n},$$

which has to be zero. Therefore we may assume that the representation (5) satisfies the condition

- $t_0 = 0$ and $j_1 > 0$.

Repeating this process, finally we may assume that Ψ is of the form

$$\Phi = \sum a_{0 \dots 0 t_{n-p} \dots t_{n-q-1}}^{(n-p) \dots (n-q-1)(n-q+1) \dots n} \omega_{0 \dots 0 t_{n-p} \dots t_{n-q-1}}^{(n-p) \dots (n-q-1)(n-q+1) \dots n}.$$

For such a representation, the coefficient of $d^{(q)}(\Psi)$ at $\omega_{0 \dots 0 t_{n-p} \dots t_{n-q-1}}^{(n-p) \dots (n-q-2)(n-q) \dots n}$ is $a_{0 \dots 0 t_{n-p} \dots t_{n-q-1}}^{(n-p) \dots (n-q-1)(n-q+1) \dots n}$, which has to be zero. Therefore, if $p > q$, adding to Ψ a suitable element in the image of $d^{(q-1)}$, Ψ becomes zero. After these steps, we may write

$$(6) \quad \Phi = \sum a_{t_0 \dots t_{n-q-1}}^{(n-q+1) \dots n} \omega_{t_0 \dots t_{n-q-1}}^{(n-q+1) \dots n}.$$

Step 5. We may assume $m = 0$ and $t_0 = \dots = t_{n-q-1} = 0$.

If $m < 0$, it is impossible that the conditions

- $t_0, \dots, t_{n-q-1} \geq 0$,
- $t_0 + \dots + t_{n-q-1} - q - 1 = m - q - 1$

of the representation (6) are satisfied. Hence $\Psi = 0$ if $m < 0$. If $m > 0$, that is, $t_\ell > 0$ for some $0 \leq \ell \leq n - q - 1$, then

$$d^{(q-1)} \left(\omega_{t_0 \dots t_{\ell-1} (t_\ell-1) t_{\ell+1} \dots t_{n-q-1} 0}^{\ell(n-q+2)(n-q+3) \dots n} \right) = \omega_{t_0 \dots t_{n-q-1}}^{(n-q+1)(n-q+2) \dots n}.$$

Hence we may assume that the representation (6) of Ψ satisfies the conditions

- $m = 0$,
- $t_0 = \dots = t_{n-q-1} = 0$,

that is, we may assume that Ψ is generated by $\omega_{0 \dots 0}^{(n-q+1)(n-q+2) \dots n}$. This final step verifies the claim.

It is easy to check that $\omega_{0 \dots 0}^{(n-q+1)(n-q+2) \dots n}$ is in the kernel of $d^{(q)}$. To prove the theorem, it remains to show that $\omega_{0 \dots 0}^{(n-q+1)(n-q+2) \dots n}$ is not in the image of $d^{(q-1)}$. Assume the contrary; then there exist $j_1, \dots, j_q, t_0, \dots, t_n, \ell_0 \in \mathbb{Z}$ satisfying

- $t_0, \dots, t_{n-q} \geq 0$,
- $t_{n-q+2}, \dots, t_n < 0$,
- $t_0 + \dots + t_n = -q - 1$,
- $0 \leq j_1 < \dots < j_q \leq n$,
- $j_\ell \neq n - q + 1$ for all ℓ

such that

$$\frac{X_{j_{\ell_0}}}{X_{n-q}} \omega_{t_0 \dots t_{n-q} \dots t_n}^{(n-q+1)j_1 \dots \widehat{j_{\ell_0}} \dots j_q} = \omega_{0 \dots 0}^{(n-q+1)(n-q+2) \dots n}.$$

$\{n - q + 1, j_1, \dots, \widehat{j_{\ell_0}}, \dots, j_q\} = \{n - q + 1, \dots, n\}$ implies that $j_{\ell_0} \leq n - q$. This contradicts $t_{j_{\ell_0}} \geq 0$. \square

Corollary 4.6. For $0 < q < n$,

$$h^q(\mathbb{P}^n, \Omega_{\mathbb{P}^n/\kappa}^p(m)) = \begin{cases} 1, & \text{if } m = 0 \text{ and } p = q; \\ 0, & \text{otherwise.} \end{cases}$$

Definition 4.7. For $0 \leq i \leq p$, $\check{V}_{p,m}^{n-i}$ is defined to be the vector space generated by those $\omega_{\underline{(-1)} \dots \underline{(-1)} t_i \dots t_n}^{12 \dots i j_{i+1} \dots j_p}$ satisfying

- $i < j_{i+1} < \dots < j_p \leq n$,
- $t_i < -1$,
- $t_{i+1}, \dots, t_n < 0$,
- $t_i + \dots + t_n = m - p + i - 1$.

Note that the elements $\omega_{\underline{(-1)} \dots \underline{(-1)} t_i \dots t_n}^{12 \dots i j_{i+1} \dots j_p}$ satisfying the above conditions are linearly independent.

Theorem 4.8.

$$\Omega_{\mathbb{P}^n/\kappa}^p(m)^{(n)} = \check{V}_{p,m}^{n-p} \oplus \dots \oplus \check{V}_{p,m}^n \oplus \text{Image}(d^{(n-1)})$$

Proof. If $p = 0$, then

$$\text{Image}(d^{(n-1)}) = \langle d^{(n-1)}(\omega_{t_0 \underline{t_1} \dots t_n}) \mid t_0 \geq 0 \rangle = \langle \omega_{\underline{t_0} \dots t_n} \mid t_0 \geq -1 \rangle.$$

As

$$\check{V}_{p,m}^n = \langle \omega_{\underline{t_0} \dots t_n} \mid t_0 < -1 \rangle,$$

it is easy to verify the theorem in this case. If $p = n$, then

$$\begin{aligned} \text{Image}(d^{(n-1)}) &= \langle \omega_{\underline{t_0} \dots t_n}^{1 \dots n} \mid t_0 \geq 0 \rangle, \\ \check{V}_{p,m}^{n-i} &= \langle \omega_{\underline{(-1)} \dots \underline{(-1)} t_i \dots t_n}^{1 \dots n} \mid t_i < -1 \rangle \end{aligned}$$

for $0 \leq i \leq n$. It is also easy to verify the theorem in this case. So we assume that $0 < p < n$ in the rest of the proof.

In the proof, we assume that the indices j_1, \dots, j_p satisfy the condition $j_1 < \dots < j_p$. We define

$$\begin{aligned} W'_0 &:= \langle d^{(n-1)}(\omega_{\underline{t_0} \underline{t_1} \underline{t_2} \dots t_n}^{j_1 \dots j_p}) \mid j_1 = 0 \rangle, \\ W''_0 &:= \langle d^{(n-1)}(\omega_{\underline{t_0} \underline{t_1} \underline{t_2} \dots t_n}^{j_1 \dots j_p}) \mid j_1 > 0 \text{ and } t_0 > 0 \rangle, \\ W_0 &:= W'_0 + W''_0, \\ W_1 &:= \langle d^{(n-1)}(\omega_{\underline{t_0} \underline{t_1} \underline{t_2} \dots t_n}^{j_1 \dots j_p}) \mid j_1 > 0, t_0 = 0, \text{ and } t_1 < -1 \rangle. \end{aligned}$$

Note that, if $j_1 > 0$, $t_0 = 0$, and $t_1 > -1$, then

$$d^{(n-1)}(\omega_{\underline{t_0} \underline{t_1} \underline{t_2} \dots t_n}^{j_1 \dots j_p}) = \sum_{\ell=1}^p (-1)^{\ell+1} \omega_{\underline{(-1)} \underline{t_1} \dots \underline{(t_{j_\ell}+1)} \dots t_n}^{1 j_1 \dots \widehat{j_\ell} \dots j_p} = 0.$$

For $1 < i \leq p + 1$, we define

$$\begin{aligned} W_i &:= \langle d^{(n-1)}(\omega_{\underline{t_0} \underline{t_1} \underline{t_2} \dots t_n}^{j_1 \dots j_p}) \mid j_1 = 2, \dots, j_{i-1} = i, \\ &\quad t_0 = 0, t_1 = \dots = t_{i-1} = -1, \text{ and } t_i < -1 \rangle. \end{aligned}$$

Note that

$$d^{(n-1)}(\omega_{0\underline{(-1)}(-1)\dots(-1)t_{p+2}\dots t_n}^{23\dots(p+1)}) = 0.$$

Hence

$$W_{p+1} := \langle d^{(n-1)}(\omega_{t_0\underline{t_1}t_2\dots t_n}^{j_1\dots j_p}) \mid j_1 = 2, \dots, j_p = p + 1, \\ t_0 = 0, t_1 = \dots = t_p = -1 \rangle.$$

If $1 < i \leq p$ and $j_i > i + 1$, then

$$d^{(n-1)}(\omega_{0\underline{(-1)}(-1)\dots(-1)t_{i+1}\dots t_n}^{23\dots i j_i \dots j_p}) \\ = \sum_{\ell=i}^p (-1)^{i+\ell} d^{(n-1)}(\omega_{0\underline{(-1)}\dots(-1)(t_{i+1}-1)t_{i+2}\dots t_{j_\ell-1}(t_{j_\ell}+1)t_{j_\ell+1}\dots t_n}^{23\dots i(i+1)j_i\dots j_\ell\dots j_p}) \in W_{i+1}.$$

Therefore

$$\text{Image}(d^{(n-1)}) = W_0 + \dots + W_{p+1}.$$

We define

$$\Omega^{(-1)!} := \Omega_{\mathbb{P}^n/\kappa}^p(m)^{(n)}, \\ \Omega^i := \langle \omega_{\underline{(-1)}(-1)\dots(-1)t_{i+1}\dots t_n}^{12\dots i j_{i+1}\dots j_p} \rangle$$

for $0 \leq i \leq p$. Note that $\Omega^{p!} = W_{p+1}$. We will show that

$$\Omega^{(i-1)!} = \check{V}_{p,m}^{n-i} \oplus W_i \oplus \Omega^i$$

for $0 \leq i \leq p$, from which the theorem follows easily.

Case 1 ($i = 0$). If $j_1 = 0$, then

$$d^{(n-1)}(\omega_{t_0\underline{t_1}t_2\dots t_n}^{j_1\dots j_p}) = \omega_{\underline{t_0}t_1t_2\dots t_n}^{1j_2\dots j_p}.$$

If $j_1 > 0$ and $t_0 > 0$, then

$$d^{(n-1)}(\omega_{t_0\underline{t_1}t_2\dots t_n}^{j_1\dots j_p}) = -\omega_{\underline{(t_0-1)}(t_1+1)t_2\dots t_n}^{j_1\dots j_p} + \sum_{\ell=1}^p (-1)^{\ell+1} \omega_{\underline{(t_0-1)}t_1\dots(t_{j_\ell}+1)\dots t_n}^{1j_1\dots j_\ell\dots j_p}.$$

The summation in the above equation is contained in W'_0 . Hence

$$W_0 = \langle \omega_{\underline{t_0}t_1\dots t_n}^{j_1\dots j_p} \mid t_0 \geq 0 \rangle.$$

Recall that

$$\check{V}_{p,m}^n = \langle \omega_{\underline{t_0}t_1\dots t_n}^{j_1\dots j_p} \mid t_0 < -1 \rangle, \\ \Omega^{0!} = \langle \omega_{\underline{t_0}t_1\dots t_n}^{j_1\dots j_p} \mid t_0 = -1 \rangle.$$

Therefore

$$\Omega^{(-1)!} = \check{V}_{p,m}^n \oplus W_0 \oplus \Omega^{0!}.$$

Case 2 ($i = 1$). If $j_1 > 0$, $t_0 = 0$, and $t_1 < -1$, then

$$d^{(n-1)}(\omega_{t_0\underline{t_1}t_2\dots t_n}^{j_1\dots j_p}) = -\omega_{\underline{(-1)}(t_1+1)t_2\dots t_n}^{j_1\dots j_p} + \sum_{\ell=1}^p (-1)^{\ell+1} \omega_{\underline{(-1)}t_1\dots(t_{j_\ell}+1)\dots t_n}^{1j_1\dots j_\ell\dots j_p}.$$

The summation in the above equation is contained in $\check{V}_{p,m}^{n-1}$. Therefore

$$\Omega^{0!} = \check{V}_{p,m}^{n-1} \oplus W_1 \oplus \Omega^{1!}.$$

Case 3 ($1 < i \leq p$). If $t_i < -1$, then

$$d^{(n-1)} \left(\omega_{\underline{0(-1)\dots(-1)t_i\dots t_n}}^{2\dots ij_i\dots j_p} \right) = (-1)^i \omega_{\underline{(-1)\dots(-1)(t_i+1)t_{i+1}\dots t_n}}^{12\dots(i-1)j_i\dots j_p} + \sum_{\ell=i}^p (-1)^{\ell+1} \omega_{\underline{(-1)\dots(-1)t_i\dots(t_j_\ell+1)\dots t_n}}^{12\dots ij_i\dots j_\ell\dots j_p}.$$

The summation in the above equation is contained in $\check{V}_{p,m}^{n-i}$; therefore

$$\Omega^{(i-1)!} = \check{V}_{p,m}^{n-i} \oplus W_i \oplus \Omega^i!$$

□

Corollary 4.9.

$$h^n(\mathbb{P}^n, \Omega_{\mathbb{P}^n/\kappa}^p(m)) = \binom{-m-1}{n-p} \binom{-m+p}{-m}.$$

Proof. Indeed,

$$\begin{aligned} h^n(\mathbb{P}^n, \Omega_{\mathbb{P}^n/\kappa}^p(m)) &= \sum_{i=0}^p \dim \check{V}_{p,m}^{n-i} \\ &= \sum_{i=0}^p \binom{n-i}{p-i} \binom{-m+p-i-1}{n-i} \\ &= \binom{-m-1}{n-p} \sum_{i=0}^p \binom{-m+p-i-1}{-m-1} \\ &= \binom{-m-1}{n-p} \binom{-m+p}{-m}. \end{aligned}$$

□

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