ORTHOGONAL POLYNOMIAL EIGENFUNCTIONS OF SECOND-ORDER PARTIAL DIFFERERENTIAL EQUATIONS

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ABSTRACT. In this paper, we show that for several second-order partial differential equations

$$L[u] = A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} + D(x,y)u_x + E(x,y)u_y$$
$$= \lambda_n u$$

which have orthogonal polynomial eigenfunctions, these polynomials can be expressed as a product of two classical orthogonal polynomials in one variable. This is important since, otherwise, it is very difficult to *explicitly* find formulas for these polynomial solutions. From this observation and characterization, we are able to produce additional examples of such orthogonal polynomials together with their orthogonality that widens the class found by H. L. Krall and Sheffer in their seminal work in 1967. Moreover, from our approach, we can answer some open questions raised by Krall and Sheffer.

1. INTRODUCTION

We are concerned with the problem raised by Krall and Sheffer [7] (see also [3], [4], [11]): classify all second-order partial differential equations of the type

$$(1.1) \quad L[u] := A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} + D(x,y)u_x + E(x,y)u_y$$
$$= \lambda_n u, \quad (n = 0, 1, 2, \dots),$$

which have orthogonal polynomials as eigenfunctions. Up to a complex linear change of variable, they found ten such differential equations, among which include

$$(1.2)$$
$$(x^2 + y + 1)u_{xx} + 2x(y+1)u_{xy} + (y+1)^2 u_{yy} + g(xu_x + yu_y) = n(n-1+g)u,$$

and

$$(1.3)$$
$$x^2 u_{xx} + 2xy u_{xy} + (y^2 - y)u_{yy} + g\{(x-1)u_x + (y-\gamma)u_y\} = n(n-1+g)u.$$

Krall and Sheffer [7] showed that the partial differential equations (1.2) and (1.3) have at least weak orthogonal polynomial solutions (see Definition 2.1) but they were unable to determine if these polynomials are orthogonal. We remark that,

even when the differential equation (1.1) is known to have orthogonal polynomial solutions, it is not easy to explicitly determine these polynomials.

Our main results (Theorems 3.4 and Corollary 3.6) show that for most of the differential equations (1.1), including (1.2) and (1.3), orthogonal polynomial solutions of (1.1) can be expressed as products of two classical orthogonal polynomial solutions $\{p_n(k;x)\}_{n=0}^{\infty}$ and $\{q_n(y)\}_{n=0}^{\infty}$ of certain second-order ordinary differential equations.

In this way, we can explicitly explain the orthogonality of many of the orthogonal polynomial solutions of (1.1). Moreover, from our results, we are able to obtain more examples of orthogonal polynomials satisfying differential equations of the type (1.1), which do not appear in the classification by Krall and Sheffer [7]. This is due to the fact that the orthogonality, but not the positive-definiteness (see Definition 2.2), is preserved under a complex linear change of variables which Krall and Sheffer used in their classification.

2. Preliminaries

For any integer $n \geq 0$, let $\mathcal{P}_n$ be the space of real polynomials in two variables of (total) degree $\leq n$ and $\mathcal{P} = \bigcup_{n \geq 0} \mathcal{P}_n$. By a polynomial system (PS), we mean a sequence $\{\phi_{mn}\}_{m,n=0}^{\infty}$ of polynomials such that $\deg \phi_{mn} = m + n$ for $m, n \geq 0$ and $\{\phi_{n-j,j}\}_{j=0}^{n}$ are linearly independent modulo $\mathcal{P}_{n-1}$ for $n \geq 0$ ($\mathcal{P}_{-1} = \{0\}$).

A PS $\{P_{mn}\}_{m,n=0}^{\infty}$ is said to be monic if

$$P_{mn}(x, y) = x^m y^n + R_{mn}(x, y), \qquad (m, n \geq 0),$$

where $R_{mn}(x, y)$ is a polynomial of degree $\leq m + n - 1$.

A linear functional on $\mathcal{P}$ is called a moment functional. For any moment functional σ and $\psi \in \mathcal{P}$, we define the moment functionals ∂_x, ∂_y, and $\psi\sigma$ by

$$\langle \partial_x \sigma, \phi \rangle = -\langle \sigma, \partial_x \phi \rangle, \quad \langle \partial_y \sigma, \phi \rangle = -\langle \sigma, \partial_y \phi \rangle, \quad \langle \psi\sigma, \phi \rangle = \langle \sigma, \psi\phi \rangle \quad (\phi \in \mathcal{P}).$$

Definition 2.1. (see [7]) A PS $\{\phi_{mn}\}_{m,n=0}^{\infty}$ is a weak orthogonal polynomial system (WOPS) if there is a non-zero moment functional σ such that

$$\langle \sigma, \phi_{mn}\phi_{kl} \rangle = 0 \qquad \text{if } m + n \neq k + l.$$

Furthermore, if

$$\langle \sigma, \phi_{mn}\phi_{kl} \rangle = K_{mn}\delta_{mk}\delta_{nl}$$

where K_{mn} are non-zero (respectively, positive) constants, we call $\{\phi_{mn}\}_{m,n=0}^{\infty}$ an orthogonal polynomial system (OPS) (respectively, a positive-definite OPS). In this case, we say that $\{\phi_{mn}\}_{m,n=0}^{\infty}$ is a WOPS or an OPS relative to σ.

A PS $\{\phi_{mn}\}_{m,n=0}^{\infty}$ is a WOPS relative to σ if and only if $\langle \sigma, \phi_{mn}R \rangle = 0$ for any polynomial $R \in \mathcal{P}_{m+n-1}$.

For any PS $\{\phi_{mn}\}_{m,n=0}^{\infty}$, there is a unique moment functional σ, called the *canonical moment functional* of $\{\phi_{mn}\}_{m,n=0}^{\infty}$, defined by the conditions

$$\langle \sigma, 1 \rangle = 1 \text{ and } \langle \sigma, \phi_{mn} \rangle = 0 \quad (m + n \geq 1).$$

Note that if $\{\phi_{mn}\}_{m,n=0}^{\infty}$ is a WOPS relative to σ, then σ must be a non-zero constant multiple of the canonical moment functional of $\{\phi_{mn}\}_{m,n=0}^{\infty}$.

Definition 2.2. A moment functional σ is quasi-definite (respectively, positive-definite) if there is an OPS (respectively, a positive-definite OPS) relative to σ.

The following was proved in [7]; see also [3].

Proposition 2.1. *For a moment functional $\sigma \neq 0$, the following statements are equivalent.*

(i) *σ is quasi-definite (respectively, positive-definite).*

(ii) *There is a unique monic WOPS $\{P_{mn}\}_{m,n=0}^{\infty}$ relative to σ.*

(iii) *There is a monic WOPS $\{P_{mn}\}_{m,n=0}^{\infty}$ relative to σ, such that for each $n \geq 0$, $H_n := \langle \sigma, \mathbb{P}_n \mathbb{P}_n^T \rangle$ is a nonsingular (respectively, positive-definite) symmetric matrix, where $\mathbb{P}_n := (P_{n0}, P_{n-1,1}, \cdots, P_{0n})^T$.*

(iv) *$\Delta_n \neq 0$ (respectively, D_n is positive-definite) , where $\Delta_n = |D_n|$, and*

$$D_n := \begin{pmatrix} \sigma_{00} & \sigma_{10} & \sigma_{01} & \cdots & \sigma_{n0} & \cdots & \sigma_{0n} \\ \sigma_{10} & \sigma_{20} & \sigma_{11} & \cdots & \sigma_{n+1,0} & \cdots & \sigma_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \sigma_{0n} & \sigma_{1n} & \sigma_{0,n+1} & \cdots & \sigma_{nn} & \cdots & \sigma_{0,2n} \end{pmatrix} \quad (n \geq 0),$$

and $\sigma_{mn} = \langle \sigma, x^m y^n \rangle$ $(m, n \geq 0)$ are the moments of σ.

From hereon, we write a PS $\{\phi_{mn}\}_{m,n=0}^{\infty}$ as $\{\Phi_n\}_{n=0}^{\infty}$, where

$$\Phi_n = (\Phi_{n0}, \Phi_{n-1,1}, \cdots, \Phi_{0n})^T.$$

It is easy to see that if the partial differential equation (1.1) has a PS $\{\Phi_n\}_{n=0}^{\infty}$ of solutions, then it must be of the form

$$(2.1) \quad \begin{aligned} L[u] &= Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y \\ &= (ax^2 + d_1 x + e_1 y + f_1)u_{xx} + (2axy + d_2 x + e_2 y + f_2)u_{xy} \\ &\quad + (ay^2 + d_3 x + e_3 y + f_3)u_{yy} + (gx + h_1)u_x + (gy + h_2)u_y \\ &= \lambda_n u \end{aligned}$$

where $\lambda_n := an(n-1) + gn$ (see [7]).

We will always assume that $|A| + |B| + |C| \not\equiv 0$ since otherwise equation (2.1) cannot have any OPS as solutions (see [3]). Following Krall and Sheffer [7], we also assume that equation (2.1) is admissible, that is, $\lambda_m \neq \lambda_n$ for $m \neq n$ (or equivalently $an + g \neq 0$ for each $n \geq 0$) so that equation (2.1) has a unique monic PS of solutions.

Proposition 2.2. *For any OPS $\{\Phi_n\}_{n=0}^{\infty}$ relative to σ, the following two statements are equivalent.*

(i) *$\{\Phi_n\}_{n=0}^{\infty}$ satisfies equation (2.1).*

(ii) *σ satisfies*

$$M_1[\sigma] := (A\sigma)_x + (B\sigma)_y - D\sigma = 0,$$
$$M_2[\sigma] := (B\sigma)_x + (C\sigma)_y - E\sigma = 0.$$

We call $M_1[\sigma] = 0$ and $M_2[\sigma] = 0$ the *moment equations* for the partial differential equation (2.1). If we let $L^*[\cdot]$ be the formal Lagrange adjoint of $L[\cdot]$ defined by

$$L^*[u] := (Au)_{xx} + 2(Bu)_{xy} + (Cu)_{yy} - (Du)_x - (Eu)_y,$$

then $L^*[\sigma] = (M_1[\sigma])_x + (M_2[\sigma])_y$. Hence if an OPS $\{\Phi_n\}_{n=0}^{\infty}$ relative to σ satisfies equation (2.1), then

$$M_1[\sigma] = M_2[\sigma] = L^*[\sigma] = 0.$$

Using the moments σ_{mn} of σ, we may express $L^*[\sigma] = 0$, $M_1[\sigma] = 0$, and $M_2[\sigma] = 0$ as (see [3] and [7])

$$
\begin{aligned}
A_{mn}: \quad &= \langle L^*[\sigma], x^m y^n \rangle \\
&= \lambda_{m+n}\sigma_{mn} + m[d_1(m-1) + e_2 n + h_1]\sigma_{m-1,n} \\
&\quad + n[d_2 m + e_3(n-1) + h_2]\sigma_{m,n-1} \\
&\quad + e_1 m(m-1)\sigma_{m-2,n+1} + d_3 n(n-1)\sigma_{m+1,n-2} \\
&\quad + f_1 m(m-1)\sigma_{m-2,n} + f_2 mn\sigma_{m-1,n-1} + f_3 n(n-1)\sigma_{m,n-2} = 0; \\
B_{mn}: \quad &= -2\langle M_2[\sigma], x^m y^n \rangle \\
&= 2\{a(m+n) + g\}\sigma_{m,n+1} + e_2 m\sigma_{m-1,n+1} + (d_2 m + 2e_3 n + 2h_2)\sigma_{mn} \\
&\quad + 2d_3 n\sigma_{m+1,n-1} + f_2 m\sigma_{m-1,n} + 2f_3 n\sigma_{m,n-1} = 0; \\
C_{mn}: \quad &= -2\langle M_1[\sigma], x^m y^n \rangle \\
&= 2\{a(m+n) + g\}\sigma_{m+1,n} + (2d_1 m + e_2 n + 2h_1)\sigma_{mn} + d_2 n\sigma_{m+1,n-1} \\
&\quad + 2e_1 m\sigma_{m-1,n+1} + 2f_1 m\sigma_{m-1,n} + f_2 n\sigma_{m,n-1} = 0
\end{aligned}
$$

for $m, n \geq 0$, where $\sigma_{mn} = 0$ if $m < 0$ or $n < 0$.

Using the above three recurrence relations for the moments $\{\sigma_{mn}\}_{m,n=0}^{\infty}$, Krall and Sheffer [7] classified second-order partial differential equations having OPS's as solutions.

We now recall the classification of classical orthogonal polynomials in one variable, which we will need later in this paper. We refer to [10] for definitions and basic facts about OPS's in one variable.

An OPS $\{p_n\}_{n=0}^{\infty}$ (in one variable) is called a classical OPS if $\{p_n\}_{n=0}^{\infty}$ satisfies a second-order ordinary differential equation

$$(2.2) \qquad \alpha(x)y''(x) + \beta(x)y'(x) = \lambda_n y(x),$$

where $\alpha(x) = ax^2 + bx + c \, (\not\equiv 0)$, $\beta(x) = dx + e$, and $\lambda_n = an(n-1) + dn$.

Proposition 2.3. (see [10], [12]) *The second-order differential equation (2.2) has an OPS $\{p_n(x)\}_{n=0}^{\infty}$ (respectively, a positive-definite OPS) as solutions if and only if*

$$s_n := an + d \neq 0 \ \text{and} \ \alpha\left(-\frac{t_n}{s_{2n}}\right) \neq 0 \quad (n \geq 0),$$

(respectively,

$$s_n \neq 0 \ \text{and} \ \frac{s_{n-1}}{s_{2n-1}s_{2n+1}}\alpha\left(-\frac{t_n}{s_{2n}}\right) < 0 \quad (n \geq 0))$$

where $t_n := bn + e$ and $s_{-1} = 1$. Moreover, $\{p_n\}_{n=0}^{\infty}$ is orthogonal relative to a moment functional u which is any nontrivial solution of the moment equation

$$(2.3) \qquad (\alpha u)' = \beta u.$$

Later in this paper, we will also make use of the following simple fact (see [9]) that when the differential equation (2.2) is admissible, the corresponding moment equation (2.3) has a unique moment functional solution u with $\langle u, 1 \rangle = 1$ (and all other solutions are constant multiples of u).

Using Proposition 2.3, Kwon and Littlejohn [10] classified all classical OPS's in one variable up to a *real* linear change of variable. We list this classification which we will refer to later in this paper when we produce further second-order partial differential equations having orthogonal polynomial solutions.

(1) **Jacobi polynomials**: the differential equation

$$(1 - x^2)y''(x) + [(\beta - \alpha) - (\alpha + \beta + 2)x]y'(x) = -n(n + \alpha + \beta + 1)y(x)$$

has the OPS (respectively, a positive-definite OPS)

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^n \binom{n+\alpha}{k}\binom{n+\beta}{n-k}(x-1)^{n-k}(x+1)^k \quad (n \geq 0)$$

as solutions if and only if

$$\alpha + n + 1 \neq 0, \ \beta + n + 1 \neq 0 \text{ and } \alpha + \beta + 2 + n \neq 0 \quad (n \geq 0)$$

(respectively, $\alpha, \beta > -1$).

(2) **Twisted Jacobi polynomials**: the ordinary differential equation

$$(1 + x^2)y''(x) + [dx + e]y'(x) = n(n + d - 1)y(x)$$

has the OPS, which is not positive-definite,

$$\check{P}_n^{(\alpha,\beta)}(x) = \sum_{k=0}^n \binom{n+\alpha}{k}\binom{n+\beta}{n-k}(x-i)^{n-k}(x+i)^k \quad (n \geq 0)$$

as solutions if and only if $d+n \neq 0$ for $n \geq 0$. Here $i = \sqrt{-1}$ and $d = \alpha+\beta+2$, $e = i(\alpha - \beta)$.

(3) **Bessel polynomials**: the ordinary differential equation

$$x^2 y''(x) + [dx + e]y'(x) = n(n + d - 1)y(x)$$

has the OPS, which is not positive-definite,

$$B_n^{(d,e)}(x) = \sum_{k=0}^n \frac{n!\Gamma(n + k - 1 + d)}{(n-k)!k!}\left(\frac{x}{e}\right)^k \quad (n \geq 0)$$

as solutions if and only if $e \neq 0$ and $d + n \neq 0$ for $n \geq 0$. For real and complex orthogonalities of Bessel polynomials, we refer to [6], [8], and [13].

(4) **Laguerre polynomials**: the ordinary differential equation

$$xy''(x) + [\alpha + 1 - x]y'(x) = -ny(x)$$

has an OPS (respectively, a positive-definite OPS)

$$L_n^{(\alpha)}(x) = \sum_{k=0}^n \binom{n+\alpha}{n-k}\frac{(-x)^k}{k!} \quad (n \geq 0)$$

as solutions if and only if $\alpha + n + 1 \neq 0$ for $n \geq 0$ (respectively, $\alpha > -1$).

(5) **Hermite polynomials**: the ordinary differential equation

$$y''(x) - 2xy'(x) = -2ny(x)$$

has a positive-definite OPS

$$H_n(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k}{k!(n-2k)!}\frac{x^{n-2k}}{4^k} \quad (n \geq 0)$$

as solutions.

(6) **Twisted Hermite polynomials**: the ordinary differential equation

$$y''(x) + 2xy'(x) = -2ny(x)$$

has an OPS, which is not positive-definite,

$$\check{H}_n(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{1}{k!(n-2k)!} \frac{x^{n-2k}}{4^k} \quad (n \geq 0)$$

as solutions.

Finally, we need the following result characterizing the partial differential equation (2.1) which has a product of two PS's in one variable as solutions.

Proposition 2.4. (see Proposition 4.1 and Theorem 4.5 in [3]) *Let $\{\mathbb{P}_n\}_{n=0}^{\infty}$ be the monic PS of solutions to equation (1.1). If $A_y = 0$ (respectively, $C_x = 0$), then $P_{n0}(x, y) = P_{n0}(x)$ (respectively, $P_{0n}(x, y) = P_{0n}(y)$) for each $n \geq 0$ and $\{P_{n0}(x)\}_{n=0}^{\infty}$ (respectively, $\{P_{0n}(y)\}_{n=0}^{\infty}$) is a WOPS in one variable satisfying the second-order ordinary differential equation*

$$Au_{xx} + Du_x = \lambda_n u \quad (\text{respectively, } Cu_{yy} + Eu_y = \lambda_n u).$$

Moreover, if the canonical moment functional σ of the PS $\{\mathbb{P}_n\}_{n=0}^{\infty}$ is positive-definite, then $\{P_{n0}(x)\}_{n=0}^{\infty}$ (respectively, $\{P_{0n}(y)\}_{n=0}^{\infty}$) is a positive-definite classical OPS in one variable. If $A_y = C_x = B = 0$, then

$$P_{mn}(x, y) = P_{m0}(x)P_{0n}(y)$$

for $m, n \geq 0$.

3. Generating orthogonal polynomial solutions

Concerning the partial differential equation (2.1), which we assume to be admissible, we can find its unique monic PS $\{\mathbb{P}_n\}_{n=0}^{\infty}$ of solutions at least recursively. However, as mentioned earlier, it is not always easy to find an OPS solution of the equation (2.1) even when we know that it has an OPS as solutions (see also the discussion in [11]).

Moreover, as Krall and Sheffer pointed out in [7], it is sometimes very difficult to see if the partial differential equation (2.1) has an OPS of solutions even when it has a monic WOPS of solutions. Indeed, Krall and Sheffer show in [7] that the monic polynomial solutions to the equations (1.2) and (1.3) are WOPS's but were unable to show whether they have an OPS of solutions.

Let $\{p_n(k; x)\}_{n=0}^{\infty} (k = 0, 1, 2, \cdots)$ and $\{q_n(y)\}_{n=0}^{\infty}$ be PS's in one variable, both of which will be specified later. For any nontrivial function $\varrho(x)$, let

$$(3.1) \qquad \phi_{n-k,k}(x, y) = p_{n-k}(k; x)\varrho^k(x)q_k\left(\frac{y}{\varrho(x)}\right) \ (0 \leq k \leq n),$$

and $\{\Phi_n\}_{n=0}^{\infty} = \{\{\phi_{n-k,k}(x, y)\}_{k=0}^n\}_{n=0}^{\infty}$. In general, $\{\Phi_n\}_{n=0}^{\infty}$ need not be a PS but we can establish the following.

Lemma 3.1. *(see [15]) Let $\{p_n(k; x)\}_{n=0}^{\infty} (k = 0, 1, 2, \cdots)$ and $\{q_n(y)\}_{n=0}^{\infty}$ be PS's as described above and $\{\Phi_n\}_{n=0}^{\infty}$ be defined as in (3.1). In addition, suppose that $q_1^2(0) + q_2^2(0) \neq 0$ (which is automatically true when $\{q_n(y)\}_{n=0}^{\infty}$ is an OPS). Then $\{\Phi_n\}_{n=0}^{\infty}$ is a PS if and only if either*

(i) $\varrho(x)$ *is a polynomial of degree ≤ 1*

   or

(ii) $\varrho(x)$ *is not a polynomial but $\varrho^2(x)$ is a polynomial of degree ≤ 2 and $\{q_n(y)\}_{n=0}^{\infty}$ is symmetric (that is, $q_n(-y) = (-1)^n q_n(y)$ for each $n \geq 0$).*

From now on, we will always assume that one of the conditions in Lemma 3.1 holds so that $\{\Phi_n\}_{n=0}^{\infty}$ is a PS.

To initiate our study of equation (2.1), we claim that we may assume, via a suitable linear change of variables, that either $A_y = 0$ or $C_x = 0$; that is, $e_1 d_3 = 0$. If $e_1 d_3 \neq 0$, then set

$$x = a_1 s + b_1 t, \ y = a_2 s + b_2 t \quad (\Delta := a_1 b_2 - a_2 b_1 \neq 0)$$

and

$$u(x, y) = w(s, t).$$

Then in the new coordinates (s, t), the partial differential equation (2.1) becomes

$$L[w] := A^* w_{ss} + 2B^* w_{st} + C^* w_{tt} + D^* w_s + E^* w_t = \lambda_n w$$

where $A^* = as^2 + d_1^* s + e_1^* t + f_1^*$ and $e_1^* = \Delta^{-1}\{e_1 a_1^3 + b_1(e_2 - d_1)a_1^2 + b_1^2(e_3 - d_2)a_1 - b_1^3 d_3\}$, etc. Take $b_1 = 1$ and a_1 to be any real root of

$$e_1 a_1^3 + (e_2 - d_1)a_1^2 + (e_3 - d_2)a_1 - d_3 = 0.$$

Then $e_1^* = 0$ so that $A^*(s, t) = A^*(s)$. Therefore, we can consider the admissible partial differential equation

(3.2) $\quad L[u] := \alpha(x)u_{xx} + 2B(x, y)u_{xy} + C(x, y)u_{yy} + \beta(x)u_x + E(y)u_y = \lambda_n u$

where $\alpha(x) = ax^2 + bx + c$, $\beta(x) = dx + e$, and $E(y) = dy + h_2$.

We will show that for *most* of the partial differential equations (3.2), orthogonal polynomial solutions (if they exist) to differential equation (3.2) can be obtained in the form (3.1), where $\{p_n(k; x)\}_{n=0}^{\infty}$ and $\{q_n(y)\}_{n=0}^{\infty}$ are certain classical OPS's in one variable.

We now assume that $\alpha(x) (\neq 0)$ is monic and $\varrho(x)$ satisfies

(3.3) $$\alpha \varrho' = B_y \varrho$$

and let $\{p_n(k; x)\}_{n=0}^{\infty}$ be the unique monic PS of solutions to the admissible ordinary differential equation:

(3.4) $$\alpha(x)Z''(x) + \beta_{2k}(x)Z'(x) = \mu_n(k)Z(x),$$

where $\mu_n(k) = n[a(n + 2k - 1) + d]$ and

(3.5) $\quad \beta_k(x) = \beta(x) + kB_y(x, y) = (d + ka)x + e + \frac{1}{2}ke_2 \quad (k \geq 0).$

Naturally, there arises the question: when does the PS $\{\Phi_n\}_{n=0}^{\infty}$, defined in (3.1), satisfy the partial differential equation (3.2)?

For simplicity, we set

$$p(x) = p_{n-k}(k; x), \ q(\eta) = q_k(\eta), \ \eta = \frac{y}{\varrho(x)}, \ \text{and} \ u = \phi_{n-k,k}(x, y) = p(x)\varrho^k(x)q(\eta).$$

Then

$$L[u] - \lambda_n u = p\varrho^{k-2}X(x, \eta) + p'\varrho^{k-1}q'Y(x, \eta),$$

where

$$\begin{aligned}
X(x, \eta) = {} & (B_y \varrho \varrho' \eta^2 - 2B\varrho'\eta + C)q''(\eta) \\
& + [\{\alpha' - (2k-1)B_y - \beta\}\varrho\varrho'\eta - a\varrho^2\eta + 2(k-1)B\varrho' + \varrho E]q'(\eta) \\
& + [(2ak - ak^2 - dk)\varrho^2 + k(kB_y - \alpha' + \beta)\varrho\varrho']q(\eta)
\end{aligned}$$

and

$$Y(x,\eta) = (2B - 2\varrho B_y \eta)q'(\eta) = (d_2 x + f_2)q'(\eta)$$

since $\alpha \varrho'' = B_y \varrho' - \alpha' \varrho' + a\varrho$, $\alpha p'' + \beta_{2k} p' = \mu_{n-k}(k)p$, and $\mu_{n-k}(k) + ak - \lambda_n = 2ak - ak^2 - dk$. Assume that $L[u] = \lambda_n u$, that is,

(3.6) $L[u] - \lambda_n u = p\varrho^{k-2}X(x,\eta) + p'\varrho^{k-1}q'Y(x,\eta) = 0.$

Take $n = k$ in (3.6) so that $p(x) = p_0(k;x) = 1$. Then $X(x,\eta) = 0$ and so $Y(x,\eta) = 0$. Hence $d_2 = f_2 = 0$; that is,

(3.7) $2B(x,y) = 2axy + e_2 y$

and

$$
\begin{aligned}
(3.8) \quad X(x,\eta) &= \left[\{a\varrho^2 - (ax + \tfrac{1}{2}e_2)\varrho\varrho'\}\eta^2 + e_3\varrho\eta + d_3 x + f_3\right]q''(\eta) \\
&\quad + \left[\{(a-d)x + b - \tfrac{1}{2}e_2 - e\}\varrho\varrho'\eta + (d-a)\varrho^2\eta + h_2\varrho\right]q'(\eta) \\
&\quad + k\varrho\left[(2a - ak - d)(\varrho - x\varrho') + \left(\tfrac{1}{2}ke_2 - b + e\right)\varrho'\right]q(\eta) \\
&= 0.
\end{aligned}
$$

First assume that $b = e_2$. Then $\varrho(x) = \sqrt{\alpha(x)}$ is a solution to equation (3.3); consequently, equation (3.8) becomes

$$
\begin{aligned}
(3.9) \quad X(x,\eta) &= \left[\left(ac - \tfrac{1}{4}b^2\right)\eta^2 + e_3\varrho\eta + d_3 x + f_3\right]q''(\eta) \\
&\quad + \left[\left\{\left(\tfrac{1}{2}bd - ae\right)x + \tfrac{1}{4}b^2 - ac + cd - \tfrac{1}{2}be\right\}\eta + h_2\varrho\right]q'(\eta) \\
&\quad + k\left[\left(ae - \tfrac{1}{2}bd\right)x + (k-2)\left(\tfrac{1}{4}b^2 - ac\right) + \tfrac{1}{2}be - cd\right]q(\eta) \\
&= 0.
\end{aligned}
$$

Hence by differentiating (3.9) twice with respect to x, we obtain

(3.10) $[e_3\varrho'\eta + d_3]q''(\eta) + [(\tfrac{1}{2}bd - ae)\eta + h_2\varrho']q'(\eta) + k(ae - \tfrac{1}{2}bd)q(\eta) = 0$

and

(3.11) $\varrho''(x)[e_3\eta q''(\eta) + h_2 q'(\eta)] = 0$

so that either $\varrho''(x) = 0$ or $e_3 = h_2 = 0$.

Case 1. $b = e_2, a = 1$, and $b^2 - 4c = 0$: Then $\alpha(x) = (x - \gamma)^2$, $b = -2\gamma$. If we take $\varrho(x) = x - \gamma$, then equation (3.9) becomes

$$
\begin{aligned}
&[e_3(x-\gamma)\eta + d_3 x + f_3]q''(\eta) + (x-\gamma)[h_2 - (d\gamma + e)\eta]q'(\eta) \\
&\quad + k(x-\gamma)(d\gamma + e)q(\eta) = 0.
\end{aligned}
$$

When $x = \gamma$, we have $(d_3\gamma + f_3)q''(\eta) = 0$ so that $f_3 = -d_3\gamma$ and so

$$(e_3\eta + d_3)q''(\eta) + \{h_2 - (d\gamma + e)\eta\}q'(\eta) + k(d\gamma + e)q(\eta) = 0.$$

Case 2. $a = b = e_2 = 0$ and $c = 1$: Then $\alpha(x) = 1$. If we take $\varrho(x) = 1$, then $d_3 = 0$ from (3.10) and so the equation (3.9) becomes

$$(e_3\eta + f_3)q''(\eta) + (d\eta + h_2)q'(\eta) - kdq(\eta) = 0.$$

Case 3. $b = e_2$ and $b^2 - 4ac \neq 0$: Then $\varrho(x) = \sqrt{\alpha(x)}$ is not a polynomial so that $e_3 = h_2 = 0$ from (3.11). Hence, equations (3.9) and (3.10) become, respectively,

$$[(ac - \frac{1}{4}b^2)\eta^2 + d_3 x + f_3]q''(\eta) + [(\frac{1}{2}bd - ae)x + \frac{1}{4}b^2 - ac + cd - \frac{1}{2}be]\eta q'(\eta)$$

$$+ k[(ae - \frac{1}{2}bd)x + (\frac{1}{4}b^2 - ac)(k-2) + \frac{1}{2}be - cd]q(\eta) = 0,$$

and

$$(3.12) \qquad d_3 q''(\eta) + (\frac{1}{2}bd - ae)\eta q'(\eta) + (ae - \frac{1}{2}bd)kq(\eta) = 0,$$

from which we also have

$$(3.13) \qquad \left[\left(ac - \frac{1}{4}b^2\right)\eta^2 + f_3\right]q''(\eta) + \left(\frac{1}{4}b^2 - ac + cd - \frac{1}{2}be\right)\eta q'(\eta)$$

$$+ k\left[\left(\frac{1}{4}b^2 - ac\right)(k-2) + \frac{1}{2}be - cd\right]q(\eta) = 0.$$

In this case, there are several possibilities : If $\frac{1}{2}bd - ae = 0$, then $d_3 = 0$ from (3.12). If $d_3 = 0$ but $\frac{1}{2}bd - ae \neq 0$, then $q_k(\eta) = \eta^k$ from (3.12) so that $f_3 = 0$ from (3.13). If $\{q_k(\eta)\}_{k=0}^{\infty}$ is an OPS, then $\{q_k(\eta)\}_{k=0}^{\infty}$ must be a classical OPS satisfying the ordinary differential equation (3.13); moreover, $d_3 = \frac{1}{2}bd - ae = 0$ since $ac - \frac{1}{4}b^2 \neq 0$ and a second order ordinary differential equation having a given OPS as solutions is unique up to a constant multiple. In this last case, we also have $f_3 \neq 0$ by Proposition 2.3.

We now assume that $b \neq e_2$. In this case, it is easy to see that equation (3.3) has a solution $\varrho(x)(\not\equiv 0)$ such that $\varrho^2(x)$ is a polynomial only when either $a = 1$ and $b^2 - 4c > 0$ or $a = 0$ and $b = 1$.

Case 4. $a = 1$, $b^2 - 4c > 0$, and $b \neq e_2$: Then $\alpha(x) = (x - \gamma_1)(x - \gamma_2)$, $\gamma_1 \neq \gamma_2$, and we may take $\varrho(x)$ as

$$\varrho(x) = (x - \gamma_1)^{\delta}(x - \gamma_2)^{1-\delta}, \text{ where } \delta = \frac{\frac{1}{2}e_2 + \gamma_1}{\gamma_1 - \gamma_2}.$$

Hence, $\varrho^2(x)$ is a polynomial of degree ≤ 2 only for $\delta = 0, \frac{1}{2}$ or 1; that is, $e_2 = -2\gamma_1, -(\gamma_1 + \gamma_2)$, or $-2\gamma_2$. Since $b \neq e_2$, we have $e_2 = -2\gamma_1$ or $e_2 = -2\gamma_2$. For example, if $e_2 = -2\gamma_1$ so that $\varrho(x) = x - \gamma_2$, then equation (3.8) becomes

$$[(\gamma_1 - \gamma_2)(x - \gamma_2)\eta^2 + e_3(x - \gamma_2)\eta + d_3 x + f_3]q''(\eta)$$

$$+ (x - \gamma_2)[h_2 - (e + d\gamma_2)\eta]q'(\eta)$$

$$+ k(x - \gamma_2)[(\gamma_2 - \gamma_1)(k-1) + d\gamma_2 + e]q(\eta) = 0.$$

When $x = \gamma_2$, we have $(d_3\gamma_2 + f_3)q''(\eta) = 0$ so that $f_3 = -d_3\gamma_2$ and

$$[(\gamma_1 - \gamma_2)\eta^2 + e_3\eta + d_3]q''(\eta) + [h_2 - (e + d\gamma_2)\eta]q'(\eta)$$

$$+ k[(\gamma_2 - \gamma_1)(k-1) + d\gamma_2 + e]q(\eta) = 0.$$

If $e_2 = -2\gamma_2$, we have the same conclusion with γ_1 and γ_2 interchanged.

If $a = 0$, $b = 1$, and $e_2 \neq 1$, then $\alpha(x) = x + c$ and $\varrho(x) = (x + c)^{\frac{1}{2}e_2}$ so that $e_2 = 0$ or $e_2 = 2$.

Case 5. $a = 0$, $b = 1$, and $e_2 = 0$: With $\varrho(x) = 1$, equation (3.8) becomes

$$(e_3\eta + d_3 x + f_3)q''(\eta) + (d\eta + h_2)q'(\eta) - kdq(\eta) = 0$$

so that $d_3 = 0$ and

$$(e_3\eta + f_3)q''(\eta) + (d\eta + h_2)q'(\eta) - kdq(\eta) = 0.$$

Case 6. $a = 0$, $b = 1$, and $e_2 = 2$: With $\varrho(x) = x + c$, equation (3.8) becomes

$$[-(x+c)\eta^2 + e_3(x+c)\eta + d_3x + f_3]q''(\eta)$$
$$+ [(-dx - e)(x+c)\eta + d(x+c)^2\eta + h_2(x+c)]q'(\eta)$$
$$+ k(k - 1 + e - cd)(x+c)q(\eta) = 0.$$

When $x = -c$, we have $(-d_3c + f_3)q''(\eta) = 0$ so that $f_3 = d_3c$ and

$$(-\eta^2 + e_3\eta + d_3)q''(\eta) + [h_2 - (e - cd)\eta]q'(\eta) + k(k - 1 + e - cd)q(\eta) = 0.$$

In summary, we have

Theorem 3.2. *Let* $\{\Phi_n\}_{n=0}^\infty$ *be a PS as in (3.1), where* $\{p_n(k;x)\}_{n=0}^\infty$ *is a monic PS satisfying the ordinary differential equation (3.4) for each* $k \geq 0$. *If* $\{\Phi_n\}_{n=0}^\infty$ *satisfies equation (3.2), then* $d_2 = f_2 = 0$ *and we must have one of the following six cases :*

(i) *if* $\alpha(x) = (x - \gamma)^2$ (γ *real*), $e_2 = -2\gamma$, *and* $\varrho(x) = x - \gamma$, *then* $f_3 = -d_3\gamma$ *and*

$$(3.14) \qquad (e_3y + d_3)q_n''(y) + [h_2 - (d\gamma + e)y]q_n'(y) = -(d\gamma + e)nq_n(y);$$

(ii) *if* $\alpha(x) = 1$, $e_2 = 0$, *and* $\varrho(x) = 1$, *then* $d_3 = 0$ *and*

$$(3.15) \qquad (e_3y + f_3)q_n''(y) + (dy + h_2)q_n'(y) = dnq_n(y);$$

(iii) *if* $b^2 - 4ac \neq 0$, $e_2 = b$, *and* $\varrho(x) = \sqrt{\alpha(x)}$, *then* $e_3 = h_2 = 0$ *and*

$$(3.16) \qquad d_3q_n''(y) + \left(\tfrac{1}{2}bd - ae\right)yq_n'(y) = \left(\tfrac{1}{2}bd - ae\right)nq_n(y)$$

and

$$(3.17) \qquad [(ac - \tfrac{1}{4}b^2)y^2 + f_3]q_n''(y) + (\tfrac{1}{4}b^2 - ac + cd - \tfrac{1}{2}be)yq_n'(y)$$
$$= n[(ac - \tfrac{1}{4}b^2)(n - 2) + cd - \tfrac{1}{2}be]q_n(y);$$

(iv) *if* $\alpha(x) = (x - \gamma_1)(x - \gamma_2)$ ($\gamma_1 \neq \gamma_2$ *real*), $e_2 = -2\gamma_1$, *and* $\varrho(x) = x - \gamma_2$, *then* $f_3 = -d_3\gamma_2$ *and*

$$(3.18) \qquad [(\gamma_1 - \gamma_2)y^2 + e_3y + d_3]q_n''(y) + [h_2 - (e + d\gamma_2)y]q_n'(y)$$
$$= n[(\gamma_1 - \gamma_2)(n - 1) - d\gamma_2 - e]q_n(y);$$

(v) *if* $\alpha(x) = x + c$, $e_2 = 0$, *and* $\varrho(x) = 1$, *then* $d_3 = 0$ *and*

$$(3.19) \qquad (e_3y + f_3)q_n''(y) + (dy + h_2)q_n'(y) = dnq_n(y);$$

(vi) *if* $\alpha(x) = x + c$, $e_2 = 2$, *and* $\varrho(x) = x + c$, *then* $f_3 = d_3c$ *and*

$$(3.20) \quad (-y^2 + e_3y + d_3)q_n''(y) + [h_2 - (e - cd)y]q_n'(y) = n(1 - n - e + cd)q_n(y).$$

Moreover, if $\{q_n(y)\}_{n=0}^\infty$ *is an OPS in case (iii), then* $d_3 = e_3 = h_2 = \tfrac{1}{2}bd - ae = 0$.

Remark 3.3. The differential equation (3.3) defining $\varrho(x)$ has many solutions and different choices of $\varrho(x)$ may yield different differential equations for $\{q_n(y)\}_{n=0}^\infty$ (consequently, a different PS $\{q_n(y)\}_{n=0}^\infty$). For example, if we choose $\varrho(x) =$

$\sqrt{-\alpha(x)}$ in case (iii), then $\{q_n(y)\}_{n=0}^{\infty}$ satisfy ordinary differential equations (see (3.16) and (3.17))

$$d_3 q_n''(y) + (ae - \frac{1}{2}bd)yq_n'(y) = (ae - \frac{1}{2}bd)nq_n(y)$$

and

$$[(\frac{1}{4}b^2 - ac)y^2 + f_3]q_n''(y) + (ac - \frac{1}{4}b^2 + \frac{1}{2}be - cd)yq_n'(y)$$
$$= n[(\frac{1}{4}b^2 - ac)(n-2) + \frac{1}{2}be - cd]q_n(y).$$

In the first three cases above, that is, when $b = e_2$, we have from (3.7) and (3.5)

$$2B(x,y) = 2axy + by = \alpha'(x)y$$

and

$$\beta_{2k}(x) = \beta(x) + k\alpha'(x).$$

Hence if we let $\{p_n(x)\}_{n=0}^{\infty} = \{p_n(0;x)\}_{n=0}^{\infty}$, then for each $k \geq 0$, $\{p_n(k;x)\}_{n=0}^{\infty}$ is given recursively by

$$p_n(k;x) = \frac{1}{n+1}p_{n+1}'(k-1;x), \qquad k \geq 1.$$

On the other hand, in Case 2 and Case 5, where $\varrho(x) = 1$, we have

$$B(x,y) = 0, \ C(x,y) = C(y), \ \text{and} \ \beta_k(x) = \beta(x)$$

so that

$$p_n(k;x) = p_n(0;x) := p_n(x), \ k \geq 0, \ \text{and} \ \phi_{n-k,k}(x,y) = p_{n-k}(x)q_k(y);$$

see Proposition 2.4 for the details.

Conversely, we also have

Theorem 3.4. *Suppose that the sequence $\{q_n(y)\}_{n=0}^{\infty}$ satisfies one of the ordinary differential equations listed in (3.14) $\sim$ (3.20) which appear in Theorem 3.2. Then $\{\Phi_n\}_{n=0}^{\infty}$ satisfies the partial differential equation (3.2) where $d_2 = f_2 = 0$ and the corresponding additional condition is satisfied:*

(i) *$f_3 = -d_3\gamma$ if $\alpha(x) = (x-\gamma)^2$, $e_2 = -2\gamma$, and $\varrho(x) = x - \gamma$;*
(ii) *$d_3 = 0$ if $\alpha(x) = 1$, $e_2 = 0$, and $\varrho(x) = 1$;*
(iii) *$d_3 = e_3 = h_2 = 0$ if $b = e_2$, $b^2 - 4ac \neq 0$, $\frac{1}{2}bd - ae = 0$, and $\varrho(x) = \sqrt{\alpha(x)}$;*
(iv) *$f_3 = -d_3\gamma_2$ if $\alpha(x) = (x-\gamma_1)(x-\gamma_2)$ $(\gamma_1 \neq \gamma_2)$, $b \neq e_2$, $e_2 = -2\gamma_1$, and $\varrho(x) = x - \gamma_2$;*
(v) *$d_3 = 0$ if $\alpha(x) = x + c$, $e_2 = 0$, and $\varrho(x) = 1$;*
(vi) *$f_3 = d_3c$ if $\alpha(x) = x + c$, $e_2 = 2$, and $\varrho(x) = x + c$.*

Proof. We prove only case (iii) since the proofs for the other cases are essentially the same. Assume that $b = e_2$, $b^2 - 4ac \neq 0$, $\frac{1}{2}bd - ae = 0$, and $\{q_n(y)\}_{n=0}^{\infty}$ satisfies equation (3.17). Then we have

$$L[u] - \lambda_n u = p\varrho^{k-2}X(x,\eta) + p'\varrho^{k-1}q'Y(x,\eta),$$

where $u = \phi_{n-k,k}(x,y)$ and $p, q, X,$ and Y are the same as before. Now $Y = 0$ since $d_2 = f_2 = 0$ and $X = 0$ (see (3.9)) since $d_3 = e_3 = h_2 = \frac{1}{2}bd - ae = 0$ and $q_n(y)$ satisfies equation (3.17). Therefore, $L[u] = \lambda_n u$. $\qquad \square$

We now discuss the orthogonality of the PS $\{\Phi_n\}_{n=0}^\infty$ given by (3.1). It is well known (see [5], [14], [15]) that $\{\Phi_n\}_{n=0}^\infty$ is an OPS with respect to the weight function

$$w(x,y) = w_1(x)\frac{1}{\varrho(x)}w_2\left(\frac{y}{\varrho(x)}\right)$$

on the domain

$$\{(x,y)|a_1 \le x \le b_1, a_2\varrho(x) \le y \le b_2\varrho(x)\}$$

if $\{p_n(k;x)\}_{n=0}^\infty$ ($k \ge 0$) and $\{q_n(y)\}_{n=0}^\infty$ are OPS's with respect to weights $w_1(x)\varrho^{2k}(x)$ on $[a_1, b_1]$ and $w_2(y)$ on $[a_2, b_2]$, respectively, and if $\varrho(x) \ge 0$ on $[a_1, b_1]$.

For more general cases, in which orthogonalizing weights (and so orthogonalizing intervals) of $\{p_n(k;x)\}_{n=0}^\infty$ and $\{q_n(y)\}_{n=0}^\infty$ are not known explicitly, we need to discuss the orthogonality of $\{\Phi_n\}_{n=0}^\infty$ formally using moment functionals.

Theorem 3.5. *Assume that $\{p_n(k;x)\}_{n=0}^\infty$ ($k \ge 0$) and $\{q_n(y)\}_{n=0}^\infty$ are OPS's relative to $\varrho^{2k}(x)v$ and w respectively, where v and w are quasi-definite moment functionals in one variable. Then $\{\Phi_n\}_{n=0}^\infty$ is an OPS relative to the moment functional σ defined by either the formula*

$$\langle \sigma, x^m y^n \rangle = \langle \varrho^n(x)v, x^m \rangle \langle w, y^n \rangle$$

when $\varrho(x)$ is a polynomial or the formula

$$\langle \sigma, x^m y^n \rangle = \begin{cases} \langle \varrho^{2p}(x)v, x^m \rangle \langle w, y^{2p} \rangle & \text{if } n = 2p, \\ 0 & \text{if } n = 2p+1 \end{cases}$$

when $\varrho(x)$ is not a polynomial. In both cases, $m, n \in \mathbb{N}_0$. In this case, $\{\Phi_n\}_{n=0}^\infty$ is positive-definite if and only if both $\{p_n(k;x)\}_{n=0}^\infty$ and $\{q_n(y)\}_{n=0}^\infty$ are positive-definite.

Proof. Let

$$q_n(y) = \sum_{j=0}^n c_{nj}y^j \quad (c_{nn} \ne 0) \quad (n \ge 0).$$

First assume that $\varrho(x)$ is a polynomial. Then

$$\langle \sigma, \phi_{m-j,j}\phi_{n-k,k} \rangle$$
$$= \sum_{s=0}^j \sum_{t=0}^k c_{js}c_{kt}\langle \sigma, p_{m-j}(j;x)p_{n-k}(k;x)\varrho(x)^{j+k-s-t}y^{s+t} \rangle$$
$$= \sum_{s=0}^j \sum_{t=0}^k c_{js}c_{kt}\langle \varrho^{s+t}v, p_{m-j}(j;x)p_{n-k}(k;x)\varrho(x)^{j+k-s-t} \rangle \langle w, y^{s+t} \rangle$$
$$= \sum_{s=0}^j \sum_{t=0}^k c_{js}c_{kt}\langle \varrho^{j+k}v, p_{m-j}(j;x)p_{n-k}(k;x) \rangle \langle w, y^{s+t} \rangle$$
$$= \langle \varrho^{j+k}v, p_{m-j}(j;x)p_{n-k}(k;x) \rangle \langle w, q_j(y)q_k(y) \rangle$$
$$= \langle \varrho^{2k}v, p_{n-k}^2(k;x) \rangle \langle w, q_k^2(y) \rangle \delta_{mn}\delta_{jk}.$$

Secondly assume that $\varrho(x)$ is not a polynomial. Then $\{q_n(y)\}_{n=0}^{\infty}$ must be symmetric by Lemma 3.1 and, similarly as above, we have

$$\langle \sigma, \phi_{m-2j,2j}\phi_{n-2k,2k}\rangle$$

$$= \sum_{s=0}^{j}\sum_{t=0}^{k} c_{2j,2s}c_{2k,2t}\langle \sigma, p_{m-2j}(2j;x)p_{n-2k}(2k;x)\varrho(x)^{2(j+k-s-t)}y^{2(s+t)}\rangle$$

$$= \langle \varrho^{4k}v, p_{n-2k}^{2}(2k;x)\rangle\langle w, q_{2k}^{2}(y)\rangle\delta_{mn}\delta_{jk},$$

as well as

$$\langle \sigma, \phi_{m-2j,2j}\phi_{n-2k-1,2k+1}\rangle = 0,$$

and

$$\langle \sigma, \phi_{m-2j-1,2j+1}\phi_{n-2k-1,2k+1}\rangle = \langle \varrho^{4k+2}v, p_{n-2k-1}^{2}(2k+1;x)\rangle\langle w, q_{2k+1}^{2}(y)\rangle\delta_{mn}\delta_{jk}.$$

Hence $\{\Phi_n\}_{n=0}^{\infty}$ is an OPS relative to σ, which is positive-definite if and only if $\{p_n(k;x)\}_{n=0}^{\infty}$ and $\{q_n(y)\}_{n=0}^{\infty}$ are positive-definite. $\qquad\square$

From Theorem 3.2, Theorem 3.4, and Theorem 3.5, we now have

Corollary 3.6. *Assume, for each $k \geq 0$, $\{p_n(k;x)\}_{n=0}^{\infty}$ is a classical OPS satisfying the ordinary differential equation (3.4) and that $\{q_n(y)\}_{n=0}^{\infty}$ is an OPS. Then*

(i) *$\{\Phi_n\}_{n=0}^{\infty}$ is an OPS;*
(ii) *$\{q_n(y)\}_{n=0}^{\infty}$ is also a classical OPS if $\{\Phi_n\}_{n=0}^{\infty}$ satisfies the partial differential equation (3.2);*
(iii) *$\{\Phi_n\}_{n=0}^{\infty}$ satisfies the partial differential equation (3.2) (with suitable restrictions on the coefficients as in Theorem 3.4) if $\{q_n(y)\}_{n=0}^{\infty}$ is a classical OPS satisfying one of the ordinary differential equations (3.14) $\sim$ (3.20) which appear in Theorem 3.2.*

Proof. Assume that $\{p_n(0;x)\}_{n=0}^{\infty}$ is an OPS relative to a quasi-definite moment functional v. Then v must satisfy

$$(\alpha v)' = \beta v.$$

It is then easy to see that $\varrho^{2k}v$ satisfies

$$(\alpha\varrho^{2k}v)' = \beta_{2k}v.$$

Since $\varrho^{2k}v \neq 0$, $\{p_n(k;x)\}_{n=0}^{\infty}$ must be an OPS relative to $\varrho^{2k}v$ (see Proposition 2.3). Hence, $\{\Phi_n\}_{n=0}^{\infty}$ is an OPS by Theorem 3.5, which proves (i). Finally, (ii) and (iii) are immediate consequences of Theorem 3.2 and Theorem 3.4, respectively. $\qquad\square$

In closing this section, we note that the roles of x and y can be exchanged in the above discussions (see Example 4.1 below).

4. EXAMPLES

We now examine each of the six cases in Section 3. By Proposition 2.3 and Theorem 3.4 and Corollary 3.6, we have:

Case 1. We may assume that $\gamma = 0$ so that $\alpha(x) = x^2$. Let $\{p_n(k;x)\}_{n=0}^{\infty}$ and $\{q_n(y)\}_{n=0}^{\infty}$ be PS's satisfying the differential equations

$$x^2p_n''(k;x) + [(d+2k)x + e]\,p_n'(k;x) = n(n+d+2k-1)p_n(k;x);$$
$$(e_3y + d_3)q_n''(y) + (h_2 - ey)q_n'(y) = -enq_n(y).$$

Then, the PS $\{\Phi_n\}_{n=0}^\infty$, where

$$\phi_{n-k,k}(x,y) = p_{n-k}(k;x)x^k q_k\left(\frac{y}{x}\right) \quad (0 \le k \le n),$$

satisfies the partial differential equation

$$x^2 u_{xx} + 2xy u_{xy} + (y^2 + d_3 x + e_3 y)u_{yy}$$
$$+ (dx+e)u_x + (dy+h_2)u_y = n(n+d-1)u.$$

Moreover, $\{\Phi_n\}_{n=0}^\infty$ is an OPS, which cannot be positive-definite, if $e \ne 0$, $d+n \ne 0$, and $d_3 e + e_3(h_2 + e_3 n) \ne 0$ for $n \ge 0$. In this case,

$$p_n(k;x) = \frac{1}{n+1}p'_{n+1}(k-1;x) = B_n^{(d+2k,e)}(x)$$

and

$$q_n(y) = \begin{cases} L_n^{(\alpha)}\left(\frac{e}{e_3}\left(y + \frac{d_3}{e_3}\right)\right) & \text{if } e_3 \ne 0, \\ H_n\left(\sqrt{\frac{e}{2d_3}}\left(y - \frac{h_2}{e}\right)\right) & \text{if } e_3 = 0 \text{ and } ed_3 > 0, \\ \check{H}_n\left(\sqrt{-\frac{e}{2d_3}}\left(y - \frac{h_2}{e}\right)\right) & \text{if } e_3 = 0 \text{ and } ed_3 < 0, \end{cases}$$

where $\alpha = \frac{h_2}{e_3} + \frac{ed_3}{e_3^2} - 1$.

Example 4.1. Consider the differential equation (1.2) in which we replace $y+1$ by y:

(4.1) $\qquad (x^2 + y)u_{xx} + 2xy u_{xy} + y^2 u_{yy} + gx u_x + (gy-g)u_y = n(n+g-1)u.$

This differential equation (4.1) has an OPS $\{\Phi_n\}_{n=0}^\infty$, which cannot be positive-definite, of solutions if $g+n \ne 0$ $(n \ge 0)$, where

$$\phi_{n-k,k}(x,y) = \begin{cases} B_{n-k}^{(g+2k,-g)}(y)y^k H_k\left(\sqrt{\frac{-g}{2}}\frac{x}{y}\right) & \text{if } g < 0, \\ B_{n-k}^{(g+2k,-g)}(y)y^k \check{H}\left(\sqrt{\frac{g}{2}}\frac{x}{y}\right) & \text{if } g > 0. \end{cases}$$

Example 4.2. Consider the partial differential equation (1.3)

(4.2)
$$x^2 u_{xx} + 2xy u_{xy} + (y^2 - y)u_{yy} + (gx-g)u_x + (gy - g\gamma)u_y = n(n+g-1)u,$$

which is obtained from Case 1 by taking

$$d = g, e = -g, d_3 = 0, e_3 = -1, h_2 = -g\gamma.$$

Differential equation (4.2) has an OPS $\{\Phi_n\}_{n=0}^\infty$, which cannot be positive-definite, as solutions if $g+n \ne 0$ and $g\gamma + n \ne 0$ for $n \ge 0$, where

$$\phi_{n-k,k}(x,y) = B_{n-k}^{(g+2k,-g)}(x)x^k L_k^{(g\gamma-1)}\left(\frac{gy}{x}\right).$$

We remark that Example 4.1 and Example 4.2 answer open questions raised by Krall and Sheffer [7]. Krall and Sheffer only showed that the partial differential equations (4.1) and (4.2) have at least weak OPS's.

Case 2. Let $\{p_n(x)\}_{n=0}^\infty$ and $\{q_n(y)\}_{n=0}^\infty$ be PS's satisfying the respective differential equations

$$p''_n(x) + (dx+e)p'_n(x) = dn p_n(x);$$
$$(e_3 y + f_3)q''_n(y) + (dy + h_2)q'_n(y) = dn q_n(y).$$

Then the PS $\{\Phi_n\}_{n=0}^{\infty}$, where

$$\phi_{n-k,k}(x,y) = p_{n-k}(x)q_k(y) \quad (0 \leq k \leq n),$$

satisfies the partial differential equation

$$u_{xx} + (e_3 y + f_3)u_{yy} + (dx + e)u_x + (dy + h_2)u_y = dnu.$$

Moreover, $\{\Phi_n\}_{n=0}^{\infty}$ is an OPS (respectively, a positive-definite OPS) if $d \neq 0$ and $df_3 - e_3(e_3 n + h_2) \neq 0$ for $n \geq 0$ (respectively, $d < 0$ and $e_3 h_2 - df_3 > 0$). In this case, we have

$$p_n(x) = \begin{cases} H_n\left(\sqrt{-\frac{d}{2}}\left(x + \frac{e}{d}\right)\right) & \text{if } d < 0, \\ \check{H}_n\left(\sqrt{\frac{d}{2}}\left(x + \frac{e}{d}\right)\right) & \text{if } d > 0, \end{cases}$$

and

$$q_n(y) = \begin{cases} L_n^{(\alpha)}\left(-\frac{d}{e_3}\left(y + \frac{f_3}{e_3}\right)\right) & \text{if } e_3 \neq 0, \\ H_n\left(\sqrt{-\frac{d}{2f_3}}\left(y + \frac{h_2}{d}\right)\right) & \text{if } e_3 = 0 \text{ and } df_3 < 0, \\ \check{H}_n\left(\sqrt{\frac{d}{2f_3}}\left(y + \frac{h_2}{d}\right)\right) & \text{if } e_3 = 0 \text{ and } df_3 > 0, \end{cases}$$

where $\alpha = \frac{h_2}{e_3} - \frac{df_3}{e_3^2} - 1$.

Case 3. Consider the partial differential equation

$$\begin{aligned}
(4.3) \qquad & (ax^2 + bx + c)u_{xx} + (2ax + b)yu_{xy} + (ay^2 + f_3)u_{yy} \\
& + (dx + e)u_x + dyu_y = n[a(n-1) + d]u,
\end{aligned}$$

where $b^2 - 4ac \neq 0$ and $\frac{1}{2}bd - ae = 0$. Then $a \neq 0$ so that we may assume that $a = 1$, $b = 0$ and $c = \pm 1$. Then $e = 0$ and the differential equation (4.3) becomes

$$(4.4) \qquad (x^2 + c)u_{xx} + 2xyu_{xy} + (y^2 + f_3)u_{yy} + dxu_x + dyu_y = n(n + d - 1)u,$$

where $c = \pm 1$. Here, we take $\varrho(x) = \sqrt{1 + cx^2}$, that is, $\varrho(x) = \sqrt{\alpha(x)}$ if $c = 1$ and $\varrho(x) = \sqrt{-\alpha(x)}$ if $c = -1$ (see Remark 3.3).

Let $\{p_n(k;x)\}_{n=0}^{\infty}$ and $\{q_n(y)\}_{n=0}^{\infty}$ be PS's satisfying the equations

$$\begin{aligned}
(x^2 + c)p_n''(k;x) + (d + 2k)xp_n'(k;x) &= n(n + d + 2k - 1)p_n(k;x), \\
(y^2 + f_3)q_n''(y) + (d-1)yq_n'(y) &= n(n + d - 2)q_n(y).
\end{aligned}$$

Then the PS $\{\Phi_n\}_{n=0}^{\infty}$, defined by

$$\phi_{n-k,k}(x,y) = p_{n-k}(k;x)(1 + cx^2)^{\frac{k}{2}}q_k\left(\frac{y}{\sqrt{1 + cx^2}}\right) \quad (0 \leq k \leq n),$$

satisfies the differential equation (4.4). Moreover, $\{\Phi_n\}_{n=0}^{\infty}$ is an OPS (respectively, a positive-definite OPS) if $f_3 \neq 0$ and $d + n - 1 \neq 0$ for $n \geq 0$ (respectively, $c = -1$, $d > 1$ and $f_3 < 0$).

In this case, we may assume $f_3 = \pm 1$ so that

$$p_n(k;x) = \frac{1}{n+1}p_{n+1}'(k-1,x) = \begin{cases} P_n^{(\frac{d+2k-2}{2}, \frac{d+2k-2}{2})}(x) & \text{if } c = -1, \\ \check{P}_n^{(\frac{d+2k-2}{2}, \frac{d+2k-2}{2})}(x) & \text{if } c = 1, \end{cases}$$

and

$$q_n(y) = \begin{cases} P_n^{(\frac{d-3}{2},\frac{d-3}{2})}(y) & \text{if } f_3 = -1, \\ \check{P}_n^{(\frac{d-3}{2},\frac{d-3}{2})}(y) & \text{if } f_3 = 1. \end{cases}$$

When $c = f_3 = -1$, $\{\Phi_n\}_{n=0}^{\infty}$ is the circle polynomials (see [1] and [7]) and all other $\{\Phi_n\}_{n=0}^{\infty}$ are new quasi-definite OPS's, which cannot be positive-definite.

Case 4. We may assume that $\gamma_1 = 0$ and $\gamma_2 = 1$ so that $\alpha(x) = x^2 - x$. Then $e_2 = 0$ or $e_2 = -2$.

Case 4-1. Suppose $e_2 = 0$. Let $\{p_n(k;x)\}_{n=0}^{\infty}$ and $\{q_n(y)\}_{n=0}^{\infty}$ be PS's satisfying the respective differential equations

$$(x^2 - x)p_n''(k;x) + [(d+2k)x + e]\,p_n'(k;x) = n(n+d+2k-1)p_n(k;x),$$
$$(y^2 - e_3 y - d_3)q_n''(y) + [(d+e)y - h_2]\,q_n'(y) = n(n+d+e-1)q_n(y).$$

Then the PS $\{\Phi_n\}_{n=0}^{\infty}$, where

$$\phi_{n-k,k}(x,y) = p_{n-k}(k;x)(x-1)^k q_k\left(\frac{y}{x-1}\right) \quad (0 \le k \le n),$$

satisfies the partial differential equation

$$(x^2 - x)u_{xx} + 2xy u_{xy} + (y^2 + d_3 x + e_3 y - d_3)u_{yy}$$
$$+ (dx + e)u_x + (dy + h_2)u_y = n(n+d-1)u.$$

Moreover, $\{\Phi_n\}_{n=0}^{\infty}$ is an OPS (respectively, a positive-definite OPS) if $d+n \ne 0$, $e-n \ne 0$, $d+e+n \ne 0$ and $\varepsilon^2 - \xi(2n+d+e)^2 \ne 0$ for $n \ge 0$ (respectively, $e < 0$, $d+e > 0$, $\xi > 0$ and $\sqrt{\xi}(d+e) \pm \varepsilon > 0$), where $\xi = d_3 + \frac{1}{4}e_3^2$ and $\varepsilon = \frac{1}{2}(d+e)e_3 - h_2$.

In this case, we have

$$p_n(k;x) = P_n^{(d+e+2k-1,-e-1)}(2x-1)$$

and

$$q_n(y) = \begin{cases} P_n^{(\alpha,\beta)}\left(\frac{2y-e_3}{2\sqrt{\xi}}\right) & \text{if } \xi > 0, \\ \check{P}_n^{(\alpha,\beta)}\left(\frac{2y-e_3}{2\sqrt{-\xi}}\right) & \text{if } \xi < 0, \\ B_n^{(d+e,\varepsilon)}\left(y - \frac{1}{2}e_3\right) & \text{if } \xi = 0. \end{cases}$$

where $\alpha = \frac{1}{2}(d+e+\frac{\varepsilon}{\sqrt{\xi}}-2)$, $\beta = \frac{1}{2}(d+e-\frac{\varepsilon}{\sqrt{\xi}}-2)$ when $\xi > 0$ and $\alpha = \frac{1}{2}(d+e-i\frac{\varepsilon}{\sqrt{-\xi}}-2)$, $\beta = \frac{1}{2}(d+e+i\frac{\varepsilon}{\sqrt{-\xi}}-2)$ when $\xi < 0$.

Case 4-2. Suppose $e_2 = -2$. Let $\{p_n(k;x)\}_{n=0}^{\infty}$ and $\{q_n(y)\}_{n=0}^{\infty}$ be PS's satisfying the respective differential equations

$$(x^2 - x)p_n''(k;x) + [(d+2k)x + e - 2k]\,p_n'(k;x) = n(n+d+2k-1)p_n(k;x),$$
$$(y^2 + e_3 y + d_3)q_n''(y) + [h_2 - ey]\,q_n'(y) = n(n-e-1)q_n(y).$$

Then the PS $\{\Phi_n\}_{n=0}^{\infty}$, where

$$\phi_{n-k,k}(x,y) = p_{n-k}(k;x)x^k q_k\left(\frac{y}{x}\right) \quad (0 \le k \le n),$$

satisfies the partial differential equation

$$(x^2 - x)u_{xx} + 2(x-1)y u_{xy} + (y^2 + d_3 x + e_3 y)u_{yy}$$
$$+ (dx + e)u_x + (dy + h_2)u_y = n(n+d-1)u.$$

Moreover, $\{\Phi_n\}_{n=0}^\infty$ is an OPS (respectively, a positive-definite OPS) if $d + n \neq 0$, $e - n \neq 0$, $d + e + n \neq 0$ and $\varepsilon^2 - \xi(2n - e)^2 \neq 0$ for $n \geq 0$ (respectively, $e < 0$, $d + e > 0$, $\xi > 0$ and $e\sqrt{\xi} \pm \varepsilon < 0$), where $\xi = \frac{1}{4}e_3^2 - d_3$ and $\varepsilon = \frac{1}{2}e_3 e + h_2$.

In this case, we have

$$p_n(k; x) = P_n^{(d+e-1, 2k-e-1)}(2x - 1)$$

and

$$q_n(y) = \begin{cases} P_n^{(\alpha,\beta)}\left(\frac{2y+e_3}{2\sqrt{\xi}}\right) & \text{if } \xi > 0, \\ \breve{P}_n^{(\alpha,\beta)}\left(\frac{2y+e_3}{2\sqrt{-\xi}}\right) & \text{if } \xi < 0, \\ B_n^{(-e,\varepsilon)}\left(y + \frac{1}{2}e_3\right) & \text{if } \xi = 0, \end{cases}$$

where

$$\alpha = \frac{1}{2}\left(-e + \frac{\varepsilon}{\sqrt{\xi}} - 2\right), \quad \beta = \frac{1}{2}\left(-e - \frac{\varepsilon}{\sqrt{\xi}} - 2\right)$$

when $\xi > 0$ and

$$\alpha = \frac{1}{2}\left(-e - i\frac{\varepsilon}{\sqrt{-\xi}} - 2\right), \quad \beta = \frac{1}{2}\left(-e + i\frac{\varepsilon}{\sqrt{-\xi}} - 2\right)$$

when $\xi < 0$.

Example 4.3. Consider the partial differential equation

$$(4.5) \quad \begin{aligned} (x^2 - x)u_{xx} + 2xyu_{xy} + (y^2 - y)u_{yy} + \left[(\alpha + \beta + \gamma + 3)x - (\alpha + 1)\right]u_x \\ + \left[(\alpha + \beta + \gamma + 3)y - (\beta + 1)\right]u_y = n(n + \alpha + \beta + \gamma + 2)u. \end{aligned}$$

If $\alpha + 1 + n \neq 0$, $\beta + 1 + n \neq 0$, $\gamma + 1 + n \neq 0$, $\beta + \gamma + 2 + n \neq 0$ and $\alpha + \beta + \gamma + 3 + n \neq 0$ for $n \geq 0$ (respectively, $\alpha, \beta, \gamma > -1$), then the differential equation (4.5) has an OPS (called the triangle polynomials; see [1] and [7]) (respectively, a positive-definite OPS) $\{\Phi_n\}_{n=0}^\infty$ as solutions, where

$$\phi_{n-k,k}(x, y) = P_{n-k}^{(\beta+\gamma+2k+1,\alpha)}(2x - 1)(x - 1)^k P_k^{(\gamma,\beta)}\left(\frac{2y}{x - 1} - 1\right) \quad (0 \leq k \leq n).$$

The results in this example show a marked improvement in the original results stated by Krall and Sheffer in [7, p. 370].

Case 5. We may assume that $c = 0$. Let $\{p_n(k; x)\}_{n=0}^\infty$ and $\{q_n(y)\}_{n=0}^\infty$ be PS's satisfying the respective differential equations

$$xp_n''(x) + (dx + e)p_n'(x) = dnp_n(x),$$
$$(e_3 y + f_3)q_n''(y) + (dy + h_2)q_n'(y) = dnq_n(y).$$

Then the PS $\{\Phi_n\}_{n=0}^\infty$, where

$$\phi_{n-k,k}(x, y) = p_{n-k}(x)q_k(y) \quad (0 \leq k \leq n),$$

satisfies the partial differential equation

$$xu_{xx} + (e_3 y + f_3)u_{yy} + (dx + e)u_x + (dy + h_2)u_y = dnu.$$

Moreover, $\{\Phi_n\}_{n=0}^\infty$ is an OPS (respectively, a positive-definite OPS) if $d \neq 0$, $e + n \neq 0$, $df_3 - e_3(e_3 n + h_2) \neq 0$ for $n \geq 0$ (respectively, $d \neq 0$, $e \neq 0$, and $h_2 e_3 - df_3 > 0$). In this case,

$$p_n(x) = L_n^{(e-1)}(-dx)$$

and

$$
q_n(y) = \begin{cases} L_n^{(\alpha)}\left(-\frac{d}{e_3}\left(y + \frac{f_3}{e_3}\right)\right) & \text{if } e_3 \neq 0, \\ H_n\left(\sqrt{-\frac{d}{2f_3}}\left(y + \frac{h_2}{d}\right)\right) & \text{if } e_3 = 0 \text{ and } df_3 < 0, \\ \check{H}_n\left(\sqrt{\frac{d}{2f_3}}\left(y + \frac{h_2}{d}\right)\right) & \text{if } e_3 = 0 \text{ and } df_3 > 0, \end{cases}
$$

where $\alpha = e_3^{-2}(e_3 h_2 - df_3) - 1$.

Case 6. We may assume that $c = 0$. Let $\{p_n(k;x)\}_{n=0}^{\infty}$ and $\{q_n(y)\}_{n=0}^{\infty}$ be PS's satisfying the respective differential equations

$$
xp_n''(k;x) + (dx + e + 2k)p_n'(k;x) = dn p_n(k;x);
$$
$$
(y^2 - e_3 y - d_3)q_n''(y) + (ey - h_2)q_n'(y) = n(n + e - 1)q_n(y).
$$

Then the PS $\{\Phi_n\}_{n=0}^{\infty}$, where

$$
\phi_{n-k,k}(x,y) = p_{n-k}(k;x)x^k q_k\left(\frac{y}{x}\right) \quad (0 \leq k \leq n),
$$

satisfies the partial differential equation

(4.6) $xu_{xx} + 2yu_{xy} + (d_3 x + e_3 y)u_{yy} + (dx + e)u_x + (dy + h_2)u_y = dnu.$

Moreover, $\{\Phi_n\}_{n=0}^{\infty}$ is an OPS (respectively, a positive-definite OPS) if $d \neq 0$, $e + n \neq 0$, $\varepsilon^2 - \xi(e + 2n)^2 \neq 0$ for $n \geq 0$ (respectively, $d \neq 0$, $e > 0$, $\xi > 0$ and $e\sqrt{\xi} \pm \varepsilon > 0$), where $\xi = d_3 + \frac{1}{4}e_3^2$ and $\varepsilon = \frac{1}{2}e_3 e - h_2$. In this case,

$$
p_n(k;x) = L_n^{(e+2k-1)}(-dx)
$$

and

$$
q_n(y) = \begin{cases} P_n^{(\alpha,\beta)}\left(\frac{2y - e_3}{2\sqrt{\xi}}\right) & \text{if } \xi > 0, \\ \check{P}_n^{(\alpha,\beta)}\left(\frac{2y - e_3}{2\sqrt{-\xi}}\right) & \text{if } \xi < 0, \\ B_n^{(e,\varepsilon)}\left(y - \frac{1}{2}e_3\right) & \text{if } \xi = 0, \end{cases}
$$

where

$$
\alpha = \frac{1}{2}\left(e + \frac{\varepsilon}{\sqrt{\xi}} - 2\right), \quad \beta = \frac{1}{2}\left(e - \frac{\varepsilon}{\sqrt{\xi}} - 2\right)
$$

when $\xi > 0$ and

$$
\alpha = \frac{1}{2}\left(e - i\frac{\varepsilon}{\sqrt{-\xi}} - 2\right), \quad \beta = \frac{1}{2}\left(e + i\frac{\varepsilon}{\sqrt{-\xi}} - 2\right)
$$

when $\xi < 0$.

We remark that Theorem 3.4 and Corollary 3.6 are applicable to all the differential equations found by Krall and Sheffer [7] except the partial differential equation

(4.7) $3yu_{xx} + 2u_{xy} - xu_x - yu_y + nu = 0.$

Krall and Sheffer [7] showed that differential equation (4.7) has an OPS (which cannot be positive-definite by Proposition 2.4). Trivially, Theorem 3.4 is not applicable to differential equation (4.7). Anthony du Rapau (see [2]) found the monic PS of solutions to (4.7) in a closed form, which is a WOPS but not an OPS.

References

[1] P. Appell and J. K. de Fériet, *Fonctions Hypergéometriques et Hypersphériques; Polynomes d'Hermite,* Paris, Gauthier-Villars et Cie (1926).

[2] V. K. Jain, *On the orthogonal polynomials of two variables,* Indian J. Pure Appl. Math. **11**(4) (1980), 492-501. MR **81c:**42030

[3] Y. J. Kim, K. H. Kwon, and J. K. Lee, *Orthogonal polynomials in two variables and second order partial differential equations,* J. Comp. Appl. Math. **82** (1997), 239-260. MR **98m:**33031

[4] Y. J. Kim, K. H. Kwon, and J. K. Lee, *Partial differential equations having orthogonal polynomial solutions,* J. Comp. Appl. Math. **99** (1998) no. 1-2, 239-253. MR **99j:**35007

[5] T. H. Koornwinder, *Two-variable analogues of the classical orthogonal polynomials,* Theory and Applications of Special Functions, R. Askey Ed., Academic Press (1975), 435-495. MR **53:**5967

[6] H. L. Krall and O. Frink, *A new class of orthogonal polynomials: The Bessel polynomials,* Trans. Amer. Math. Soc. **65** (1949), 100-115. MR **10:**453a

[7] H. L. Krall and I. M. Sheffer, *Orthogonal polynomials in two variables,* Ann. Mat. Pura Appl. (4) **76** (1967), 325- 376. MR **37:**4499

[8] K. H. Kwon, S. S. Kim, and S. S. Han, *Orthogonality of Tchebychev sets of polynomials,* Bull. London Math. Soc. **24** (1992), 361-367. MR **93g:**33007

[9] K. H. Kwon, J. K. Lee, and B. H. Yoo, *Characterizations of classical orthogonal polynomials,* Results in Math. **24** (1993), 119-128. MR **98i:**33010

[10] K. H. Kwon and L. L. Littlejohn, *Classification of classical orthogonal polynomials,* J. Korean Math. Soc., **34** (1997), 973-1008. MR **99k:**35007

[11] L. L. Littlejohn, *Orthogonal polynomial solutions to ordinary and partial differential equations,* Proc. 2nd Intern. Symp. Orthogonal Polynomial and their Applications, M. Alfaro et al. Eds, Segovia (Spain), 1986, Lect. Notes Math. Vol. **1329**, Springer-Verlag, Berlin, 1988, 98-124. MR **89j:**33013

[12] F. Marcellán, A. Branquinho, and J. Petronilho, *Classical orthogonal polynomials: A functional approach,* Acta Appl. Math. **34** (1994), 283-303. MR **95b:**33024

[13] P. Maroni, *An integral representation for the Bessel form,* J. Comp. Appl. Math. **57** (1995), 251-260. MR **96a:**42038

[14] P. K. Suetin, *Orthogonal polynomials in two variables*, Nauka, Moscow, 1988 (in Russian). MR **90g:**42055

[15] Y. Xu, *A class of bivariate orthogonal polynomials and cubature formula,* Numer. Math. **69** (1994), 233-241. MR **96b:**65027

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