TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 353, Number 9, Pages 3789–3802
S 0002-9947(01)02842-2
Article electronically published on May 3, 2001

CRYSTAL BASES FOR $U_q(\Gamma(\sigma_1, \sigma_2, \sigma_3))$

YI MING ZOU

Abstract. We construct crystal bases for certain infinite dimensional representations of the q-deformation of the Lie superalgebra $\Gamma(\sigma_1, \sigma_2, \sigma_3)$.

1. Introduction

The notion of crystal bases has been generalized to the q-deformations of certain Lie superalgebras by [1], [11] and [14]. Since the lack of reducibility property in the superalgebra case, in order to generalize crystal base theory to the q-deformations of Lie superalgebras, one needs to modify the original definition of crystal base. In [1], the notion of pseudo-base was introduced. Roughly speaking, a crystal base for a module M in the Lie algebra case is a pair (L, B), where $L \subset M$ is a lattice over the subring $A \subset \mathbb{Q}(q)$ consisting of functions regular at $q = 0$ and B is a basis of L/qL over $\mathbb{Q}$. In the super case, one needs to allow B to contain both a $\mathbb{Q}$-basis and its negative, therefore B is not a basis, and the associated crystal is $B/\{\pm 1\}$.

In this paper, we use an approach similar to that of [1] to construct crystal bases for certain modules of the q-deformation of the universal enveloping algebra of the Lie superalgebra $\Gamma(\sigma_1, \sigma_2, \sigma_3)$. The notation $\Gamma(\sigma_1, \sigma_2, \sigma_3)$ actually stands for a one-parameter family of Lie superalgebras defined over the complex number field $\mathbb{C}$, though the corresponding q-deformations can be defined over the field $\mathbb{C}(q)$, for our purpose, we shall consider the deformations over the field $\mathbb{Q}(q)$. The construction of crystal bases in [1] depends a crucial fact about the Lie superalgebra $gl(m, n)$: this Lie superalgebra possesses a natural vector representation V with a crystal base. This fact enables [1] to construct a crystal base theory for the simple objects in a certain category of $gl(m, n)$-modules obtained by taking tensor products of V, although modules in this category are not necessarily completely reducible. Unlike the algebra $gl(m, n)$, the algebra $\Gamma(\sigma_1, \sigma_2, \sigma_3)$ does not have a natural vector representation. To construct a crystal base theory for $\Gamma(\sigma_1, \sigma_2, \sigma_3)$, we first construct a simple module $\mathbf{V}$ with a crystal base, then we show that certain simple modules possess crystal bases by studying tensor products of $\mathbf{V}$. It turns out that finite dimensional nontrivial $\Gamma(\sigma_1, \sigma_2, \sigma_3)$-modules do not have bases that behave well with respect to tensor products. The modules that we show to have crystal bases are infinite dimensional $\Gamma(\sigma_1, \sigma_2, \sigma_3)$-modules.

2. Preliminary

We use the notation adopted in [12], [13]. Recall that $G = \Gamma(\sigma_1, \sigma_2, \sigma_3)$ is defined as a contragredient Lie superalgebra with three nonzero complex numbers $\sigma_1, \sigma_2, \sigma_3$

Received by the editors February 26, 1999 and, in revised form, January 30, 2001.
2000 *Mathematics Subject Classification.* Primary 16W30, 16S80, 22E70.

satisfying $\sigma_1 + \sigma_2 + \sigma_3 = 0$, with generators e_i, f_i, h_i $(i = 1, 2, 3)$ and the defining matrix $(a_{ij})_{3\times 3}$ given by

$$\begin{pmatrix} 0 & 2\sigma_2 & 2\sigma_3 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

Let G_0 be the even part of G and let G_1 be the odd part of G, then $G_0 \cong sl(2) \otimes sl(2) \otimes sl(2)$. We denote the standard generators of G_0 by X_i, Y_i, H_i, i.e.

$$X_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y_i = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad i = 1, 2, 3.$$

We use linear functions $\epsilon_1, \epsilon_2, \epsilon_3$ to express the roots of G, where $\epsilon_i(H_j) = \delta_{ij}$. The set of even roots and the set of odd roots are

$$R_0 = \{\pm 2\epsilon_i : i = 1, 2, 3\}, \qquad R_1 = \{\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3\}.$$

We choose $\alpha_1 = \epsilon_1 - \epsilon_2 - \epsilon_3, \alpha_2 = 2\epsilon_2, \alpha_3 = 2\epsilon_3$ to be a simple root system and denote the corresponding generators of G by e_i, f_i, h_i $(i = 1, 2, 3)$. Note that $e_i = X_i, f_i = Y_i$, $h_i = H_i$ $(i = 2, 3)$, and $h_1 = -\sigma_1 H_1 + \sigma_2 H_2 + \sigma_3 H_3$. Note also that

$$\rho = (\sum_{\alpha\in\Delta_0^+} \alpha - \sum_{\alpha\in\Delta_1^+} \alpha)/2 = -\epsilon_1 + \epsilon_2 + \epsilon_3 = -\alpha_1.$$

Let $H = \langle h_1, h_2, h_3 \rangle$. We denote an element $\lambda = m_1\epsilon_1 + m_2\epsilon_2 + m_3\epsilon_3 \in H^*$ by (m_1, m_2, m_3). We also use the numerical marks to denote the elements in H^*. If $\lambda(h_i) = a_i$ $(i = 1, 2, 3)$, then we write $\lambda = [a_1, a_2, a_3]$. Let P be the set of integral weights in H^*, i.e. P is the set of $\lambda = [a_1, a_2, a_3]$ such that $a_i \in \mathbb{Z}$.

The conditions on the σ_i's imply that G depends on only one parameter (compare with [4, 2.5.2]). In order to construct our basic simple highest weight module with a crystal base, we need to assume (see Section 3) that the highest weight $\lambda = (m_1, m_2, m_3)$ of the module satisfies $(\lambda+\rho)(h_1) = 0$, i.e. $-m_1\sigma_1 + m_2\sigma_2 + m_3\sigma_3 = 0$. Thus the parameter and the highest weight are determined by each other, and if a result holds for one set of σ_i, we can adjust the corresponding highest weight for another set of σ_i to obtain a similar result (see the end of Section 3 for an example). In order to simplify our notation, we fix

$$\sigma_1 = \sigma_2 = \frac{1}{2}, \quad \sigma_3 = -1.$$

Under our assumption, the defining matrix is now

$$\begin{pmatrix} 0 & 1 & -2 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

We use $\ell_1 = 1, \ell_2 = -1, \ell_3 = 2$ to symmetrize the matrix.

Let q be an indeterminate over $\mathbb{Q}$, let $q_i = q_i^{\ell_i}$ $(i = 1, 2, 3)$, and let A be the subring of the quotient field $\mathbb{Q}(q)$ consisting of those f/g with $g(0) \neq 0$. We define the algebra $\mathcal{U}'$ to be the $\mathbb{Z}_2$-graded unital associative algebra over $\mathbb{Q}(q)$ generated by the elements E_i, F_i, $K_i^{\pm 1}$ $(i = 1, 2, 3)$, with the parities given by

$$p(E_i) = p(F_i) = 0, \quad i = 2, 3; \quad p(K_i^{\pm 1}) = 0, \quad i = 1, 2, 3; \quad p(E_1) = p(F_1) = 1,$$

and the following generating relations:

(2.1) $$K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1, \quad 1 \le i,j \le 3;$$

(2.2) $$K_iE_jK_i^{-1} = q_i^{a_{ij}}E_j, \quad K_iF_jK_i^{-1} = q_i^{-a_{ij}}F_j, \quad 1 \le i,j \le 3;$$

(2.3) $$E_iF_j - (-1)^{ab}F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, a = p(E_i), b = p(F_j), 1 \le i,j \le 3;$$

(2.4) $$E_2E_3 = E_3E_2, \quad F_2F_3 = F_3F_2;$$

(2.5) $$E_i^2E_1 - (q_i + q_i^{-1})E_iE_1E_i + E_1E_i^2 = 0, \quad i = 2,3;$$
$$F_i^2F_1 - (q_i + q_i^{-1})F_iF_1F_i + F_1F_i^2 = 0, \quad i = 2,3;$$

(2.6) $$E_1^2 = F_1^2 = 0.$$

As in [1], to define the Hopf algebra structure, we use the parity operator σ on $\mathcal{U}'$, which is defined by $\sigma(x) = (-1)^{p(x)}x$ for the generators x of $\mathcal{U}'$. Let $\mathcal{U} = \mathcal{U}' \oplus \mathcal{U}'\sigma$ with the algebra structure given by $\sigma^2 = 1$ and $\sigma u \sigma = \sigma(u)$ for $u \in \mathcal{U}'$. Let $p(i) = p(E_i)$, $(i = 1,2,3)$. The Hopf algebra structure on $\mathcal{U}$ has the comultiplication Δ, the antipode S and the counit ε defined by

(2.7) $$\begin{aligned} &\Delta(\sigma) = \sigma \otimes \sigma, \qquad \Delta(K_i) = K_i \otimes K_i, \\ &\Delta(E_i) = E_i \otimes K_i^{-1} + \sigma^{p(i)} \otimes E_i, \Delta(F_i) = F_i \otimes 1 + \sigma^{p(i)}K_i \otimes F_i; \end{aligned}$$

(2.8) $$S(\sigma) = \sigma, S(E_i) = -\sigma^{p(i)}E_iK_i, S(F_i) = -\sigma^{p(i)}K_i^{-1}F_i, S(K_i) = K_i^{-1};$$

(2.9) $$\varepsilon(E_i) = 0, \quad \varepsilon(F_i) = 0, \quad \varepsilon(\sigma) = \varepsilon(K_i) = 1.$$

There exists an algebra anti-automorphism η of $\mathcal{U}$ defined by

(2.10) $$\eta(\sigma) = \sigma, \eta(K_i) = K_i, \eta(E_i) = q_iF_iK_i^{-1}, \eta(F_i) = q_i^{-1}K_iE_i,$$

and $\eta(uv) = \eta(v)\eta(u)$, for all $u, v \in \mathcal{U}$. Note that $\eta^2 = id$ and

(2.11) $$\Delta \circ \eta = (\eta \otimes \eta) \circ \Delta.$$

The adjoint action of $\mathcal{U}'$ on itself is given by

(2.12) $$ad_qx(y) = \sum (-1)^{p(b_i)p(y)}a_iyS(b_i),$$

where $\Delta x = \sum a_i \otimes b_i$.

Define a $\mathbb{Q}$-algebra anti-automorphism θ of $\mathcal{U}'$ by

$$\theta(E_i) = F_i, \quad \theta(F_i) = E_i, \quad \theta(K_i) = K_i^{-1}, \quad \theta(q) = q^{-1},$$

and $\theta(uv) = \theta(v)\theta(u)$, for all $u, v \in \mathcal{U}'$.

Introduce the following elements of $\mathcal{U}'$:

(2.13) $$\begin{aligned} E_{121} &= ad_qE_3(E_1) = E_3E_1 - q_3^{-1}E_1E_3, \\ E_{112} &= ad_qE_2(E_1) = E_2E_1 - q_2^{-1}E_1E_2, \\ E_{111} &= ad_qE_3ad_qE_2(E_1) = ad_q(E_3E_2)(E_1), \\ E_0 &= (q_2 + q_2^{-1})E_1E_{111} + (q_3 + q_3^{-1})E_{111}E_1 \\ &\quad + (q_3q_2^{-1} - q_3^{-1}q_2)E_{121}E_{112}, \end{aligned}$$

and let

$$F_{212} = \theta E_{121}, \quad F_{221} = \theta E_{112}, \quad F_{222} = \theta E_{111}, \quad F_0 = \theta E_0. \tag{2.14}$$

Let $\mathcal{U}^+$, $\mathcal{U}^-$, $\mathcal{U}^0$ be the subalgebras of $\mathcal{U}'$ generated by the E_i, the F_i, and the $K_i^{\pm 1}$ $(i = 1, 2, 3)$ respectively. Then $\mathcal{U}' = \mathcal{U}^-\mathcal{U}^0\mathcal{U}^+$ and $\mathcal{U}' \cong \mathcal{U}^- \otimes \mathcal{U}^0 \otimes \mathcal{U}^+$ as $\mathbb{Q}(q)$-vector spaces.

For $\delta = (\delta_1, \delta_2, \delta_3, \delta_4)$, where $\delta_i = 0$ or 1, and $m = (m_1, m_2, m_3), m_i \in \mathbb{Z}_{\geq 0}$, let

$$\begin{aligned} E^{(\delta,m)} &= E_{111}^{\delta_1} E_{121}^{\delta_2} E_{112}^{\delta_3} E_1^{\delta_4} E_0^{m_1} E_2^{m_2} E_3^{m_3}, \\ F^{(\delta,m)} &= F_{222}^{\delta_1} F_{212}^{\delta_2} F_{221}^{\delta_3} F_1^{\delta_4} F_0^{m_1} F_2^{m_2} F_3^{m_3}. \end{aligned} \tag{2.15}$$

For $t = (t_1, t_2, t_3)$, $t_i \in \mathbb{Z}$, let $K^t = K_1^{t_1} K_2^{t_2} K_3^{t_3}$. Then by [13], a PBW type theorem holds, i.e., the K^t form a basis of $\mathcal{U}^0$, the elements of the form $E^{(\delta,m)}$ (resp. $F^{(\delta,m)}$) form a basis of $\mathcal{U}^+$ (resp. $\mathcal{U}^-$), and the elements of the form $F^{(\delta,m)} K^t E^{(\delta',m')}$ form a basis of $\mathcal{U}'$.

Following [1], we define the category $\mathcal{O}_{int}$ of $\mathcal{U}$-modules and the modified Kashiwara operators. The category $\mathcal{O}_{int}$ consists of $\mathcal{U}$-modules M such that the following conditions hold:

(i) M is a weight module $M = \sum_{\lambda \in P} M_\lambda$, where

$$M_\lambda = \{u \in M : K_i u = q_i^{\lambda(h_i)} u, i = 1, 2, 3\}.$$

(ii) $\dim M_\lambda < \infty$ for any $\lambda \in P$.

(iii) For $i = 2, 3$, M is locally $\mathcal{U}_i$-finite, where $\mathcal{U}_i$ is the subalgebra of $\mathcal{U}$ generated by $E_i, F_i, K_i^{\pm 1}$.

(iv) For any λ such that $M_\lambda \neq 0$, $\lambda(h_1) \geq 0$.

(v) For $\lambda \in P$ such that $\lambda(h_1) = 0$, $E_1 M_\lambda = F_1 M_\lambda = 0$.

The modified Kashiwara operators for the modules M in $\mathcal{O}_{int}$ are defined as follows. For $i = 2, 3$, $n \geq 0$, let $F_i^{(n)} = F_i^n / [n]_i!$ be as usual. For $u \in M$ of weight λ, let

$$u = \sum_{k \geq 0, -\lambda(h_i)} F_i^{(k)} u_k$$

be the unique expression such that $E_i u_k = 0$ for each k.

For $i = 3$, we define

$$\tilde{E}_3 u = \sum_k F_3^{(k-1)} u_k, \qquad \tilde{F}_3 u = \sum_k F_3^{(k+1)} u_k. \tag{2.16}$$

For $i = 2$, we let $\ell_k = (\lambda + k\alpha_2)(h_2)$ and define

$$\tilde{E}_2 u = \sum_k q_2^{\ell_k - 2k + 1} F_2^{(k-1)} u_k, \qquad \tilde{F}_2 u = \sum_k q_2^{-\ell_k + 2k + 1} F_2^{(k+1)} u_k. \tag{2.17}$$

For $i = 1$, we define $\tilde{E}_1 u = q^{-1} K_1 E_1 u$, $\tilde{F}_1 = F_1 u$.

Definition 2.1 ([1, Def. 2.3 and Def. 2.4])**.** Let $M \in \mathcal{O}_{int}$. A crystal base of M is a pair (L, B) such that:

(A1) L generates M as a vector space over $\mathbb{Q}(q)$.

(A2) $\sigma L = L$ and L has a weight decomposition $L = \bigoplus_{\lambda \in P} L_\lambda$ with $L_\lambda = L \cap M_\lambda$.

(A3) $\tilde{E}_i L \subset L$ and $\tilde{F}_i L \subset L$ for $i = 1, 2, 3$.

(C1) B is a subset of L/qL such that $\sigma b = \pm b$ for any $b \in B$, and B has a weight decomposition $B = \bigsqcup_{\lambda \in P} B_\lambda$ with $B_\lambda = B \cap (L_\lambda / qL_\lambda)$.

(C2) B is a pseudo-base of L/qL, that is, $B = B' \cup (-B')$ for a $\mathbb{Q}$ base B' of L/qL.

(C3) $\tilde{E}_i B \subset B \sqcup \{0\}$ and $\tilde{F}_i B \subset B \sqcup \{0\}$, for $i = 1, 2, 3$.

(C4) For any $b, b' \in B$ and $i = 1, 2, 3$, $b = \tilde{F}_i b' \iff b = \tilde{E}_i b$.

The crystal associated to the crystal base (L, B) is $B/\{\pm 1\}$.

Definition 2.2. We call a symmetric bilinear form $(,)$ on a $\mathcal{U}$-module M η-invariant if it satisfies

$$(um, m') = (m, \eta(u)m') \tag{2.18}$$

for all $m, m' \in M$ and $u \in \mathcal{U}$. We say that a crystal base (L, B) for a $\mathcal{U}$-module M is polarizable if there exists an η-invariant form $(,)$ on M such that $(L, L) \subset A$, and with respect to the induced $\mathbb{Q}$-valued symmetric bilinear form $(,)_0$ on L/qL,

$$(b, b')_0 = \begin{cases} \pm 1, & \text{if } b' = \pm b, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{for all } b, b' \in B. \tag{2.19}$$

By [1, Thm. 2.12], a $\mathcal{U}$-module M with a polarizable crystal base is completely reducible.

We now consider the tensor products of $\mathcal{U}$-modules. Let (L, B) be a crystal base of a $\mathcal{U}$-module M. For $b \in B$ and $i = 2, 3$, we define

$$\varepsilon_i(b) = max\{n \in \mathbb{Z}_{\geq 0} : \tilde{E}_i^n b \neq 0\},$$
$$\varphi_i(b) = max\{n \in \mathbb{Z}_{\geq 0} : \tilde{F}_i^n b \neq 0\}.$$

Note that $\mathrm{wt}(b)(h_i) = \varphi_i(b) - \varepsilon_i(b), i = 2, 3$.

Let $M_1, M_2 \in \mathcal{O}_{int}$ and suppose that they have crystal bases (L_1, B_1) and (L_2, B_2) respectively. Set $L = L_1 \otimes_A L_2$ and $B = B_1 \otimes B_2$. Then (L, B) is a crystal base for $M_1 \otimes M_2$ and by [1, Prop. 2.8], the actions of $\tilde{E}_i, \tilde{F}_i$ on $b_1 \otimes b_2$ ($b_1 \in B_1$ and $b_2 \in B_2$) are given by

$$\begin{aligned} \tilde{E}_1(b_1 \otimes b_2) &= \begin{cases} \tilde{E}_1(b_1) \otimes b_2, & \text{if } \mathrm{wt}(b_1)(h_1) > 0, \\ \sigma b_1 \otimes \tilde{E}_1(b_2), & \text{if } \mathrm{wt}(b_1)(h_1) = 0, \end{cases} \\ \tilde{F}_1(b_1 \otimes b_2) &= \begin{cases} \tilde{F}_1(b_1) \otimes b_2, & \text{if } \mathrm{wt}(b_1)(h_1) > 0, \\ \sigma b_1 \otimes \tilde{F}_1(b_2), & \text{if } \mathrm{wt}(b_1)(h_1) = 0; \end{cases} \end{aligned} \tag{2.20}$$

$$\begin{aligned} \tilde{E}_2(b_1 \otimes b_2) &= \begin{cases} b_1 \otimes \tilde{E}_2(b_2), & \text{if } \varphi_2(b_2) \geq \varepsilon_2(b_1), \\ \tilde{E}_2(b_1) \otimes b_2, & \text{if } \varphi_2(b_2) < \varepsilon_2(b_1), \end{cases} \\ \tilde{F}_2(b_1 \otimes b_2) &= \begin{cases} b_1 \otimes \tilde{F}_2(b_2), & \text{if } \varphi_2(b_2) > \varepsilon_2(b_1), \\ \tilde{F}_2(b_1) \otimes b_2, & \text{if } \varphi_2(b_2) \leq \varepsilon_2(b_1); \end{cases} \end{aligned} \tag{2.21}$$

$$\begin{aligned} \tilde{E}_3(b_1 \otimes b_2) &= \begin{cases} \tilde{E}_3(b_1) \otimes b_2, & \text{if } \varphi_3(b_1) \geq \varepsilon_3(b_2), \\ b_1 \otimes \tilde{E}_3(b_2), & \text{if } \varphi_3(b_1) < \varepsilon_3(b_2), \end{cases} \\ \tilde{F}_3(b_1 \otimes b_2) &= \begin{cases} \tilde{F}_3(b_1) \otimes b_2, & \text{if } \varphi_3(b_1) > \varepsilon_3(b_2), \\ b_1 \otimes \tilde{F}_3(b_2), & \text{if } \varphi_3(b_1) \leq \varepsilon_3(b_2). \end{cases} \end{aligned} \tag{2.22}$$

3. The basic module and the main result

Since a PBW type theorem holds for $\mathcal{U}'$, we can define highest weight modules for $\mathcal{U}$ as usual. We call a $\mathcal{U}$-module M a highest weight module if there is a weight vector v of M such that $M = \mathcal{U}^-(v)$, if this is the case, we call v a highest weight vector. We call a weight vector v maximal if $\mathcal{U}^+(v) = 0$. For a highest weight module M with a highest weight vector v, the action of σ on M can be specified by the action of σ on v.

By [4, Thm. 8] (compare with [8, Prop. 2.1]), the simple $\mathcal{U}$-module $L(\lambda)$ with highest weight $\lambda = m_1\epsilon_1 + m_2\epsilon_2 + m_3\epsilon_3$ is finite dimensional only if $m_i \in \mathbb{Z}_{\geq 0}$ $(i = 1, 2, 3)$. Consider condition (iv) in the definition of the category $\mathcal{O}_{int}$. If $L(\lambda)$ is finite dimensional with

$$m_1 > 0 \quad \text{and} \quad \lambda(h_1) = -\frac{1}{2}m_1 + \frac{1}{2}m_2 - m_3 \geq 0,$$

then $m_2 > 0$. By using the action of the Weyl group $W \cong \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$ of G, we see that there is a weight μ of the module $L(\lambda)$ such that $\mu(h_1) < 0$. If $m_1 = 0$, then $L(\lambda)$ is finite dimensional if and only if $m_2 = m_3 = 0$, i.e., $L(\lambda)$ is the trivial module. Since condition (iv) (alternatively $\mu(h_1) \leq 0$ for all weights μ of a $\mathcal{U}$-module in the category) is needed in working with the tensor products of crystal bases, we should consider simple highest weight $\mathcal{U}$-modules with $m_1 < 0$.

Let $\lambda = (-2, 0, 1) = [0, 0, 1]$ (recall that the numbers inside $(,)$ stand for the coefficients of the ϵ_i's and the numbers in $[,]$ stand for the numerical marks), denote the $\mathcal{U}$-module $L(\lambda)$ by $\mathbf{V}$. Let v_0 be a highest weight vector of $\mathbf{V}$ and let the action σ on $\mathbf{V}$ be given by $\sigma v_0 = v_0$. Note that $\mathbf{V}$ is infinite dimensional since $\lambda(H_1) = -2$.

Proposition 3.1. *The $\mathcal{U}$-module $\mathbf{V}$ has a polarizable crystal base $(\mathbf{L}, \mathbf{B})$ with the associated crystal graph (the repeating block is $3, 1, 2, 1$):*

$$\boxed{0} \xrightarrow{3} \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \xrightarrow{1} \boxed{4} \xrightarrow{3} \boxed{5} \xrightarrow{1} \boxed{6} \xrightarrow{2} \boxed{7} \xrightarrow{1} \boxed{8} \cdots \tag{3.1}$$

Proof. Define a set of vectors $\{v_i\}_{i\geq 0}$ of $\mathbf{V}$ according to the action of the F_i's described in the given crystal graph, i.e.

$$v_1 = F_3v_0, \quad v_2 = F_1v_1, \quad v_3 = F_2v_2, \quad v_4 = F_1v_3, \ldots .$$

We claim that $\{v_i\}_{i\geq 0}$ is a basis of $\mathbf{V}$ with the action of E_i, F_i $(i = 1, 2, 3)$ given by

$$\begin{aligned} F_1v_i &= \begin{cases} 0, & i \text{ even}, \\ v_{i+1}, & i \text{ odd}, \end{cases} \\ F_2v_i &= \begin{cases} 0, & i \neq 4n+2, \\ v_{i+1}, & i = 4n+2, \end{cases} \\ F_3v_i &= \begin{cases} 0, & i \neq 4n, \\ v_{i+1}, & i = 4n; \end{cases} \end{aligned} \tag{3.2}$$

$$E_1v_i = \begin{cases} 0, & i = 0 \text{ or odd}, \\ [n]v_{i-1}, & i = 4n, n \geq 1, \\ [n+2]v_{i-1}, & i = 4n+2, n \geq 0, \end{cases} \tag{3.3}$$

where $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$ (note that $\ell_1 = 1$ and $q_1 = q$).

$$E_2 v_i = \begin{cases} 0, & i \neq 4n+3, \\ v_{i-1}, & i = 4n+3, \end{cases} \quad E_3 v_i = \begin{cases} 0, & i \neq 4n+1, \\ v_{i-1}, & i = 4n+1. \end{cases} \tag{3.4}$$

These formulas will imply that the subspace of $\mathbf{V}$ spanned by $\{v_i\}_{i\geq 0}$ is a submodule of $\mathbf{V}$, thus must be the whole $\mathbf{V}$ since $\mathbf{V}$ is simple. We verify these formulas at the first block:

$$v_0 \xrightarrow{3} v_1 \xrightarrow{1} v_2 \xrightarrow{2} v_3 \xrightarrow{1} .$$

The verification at any other block is similar.

Consider v_0. If $F_1 v_0 \neq 0$, then since $E_i F_1 v_0 = 0$ for $i = 1, 2, 3$, $F_1 v_0$ generates a proper submodule of $\mathbf{V}$, contradicting the simplicity of $\mathbf{V}$. Similarly, $F_2 v_0 = 0$. Also by $U_q(sl(2))$-theory, $v_1 = F_3 v_0 \neq 0$ and $F_3 v_1 = 0$.

Consider v_1. By (2.4) $F_2 v_1 = F_2 F_3 v_0 = F_3 F_2 v_0 = 0$. Since $\mathbf{V}$ is infinite dimensional, $v_2 = F_1 v_1 \neq 0$.

Consider v_2. By (2.5)

$$F_3 v_2 = F_3 F_1 F_3 v_0 = \frac{1}{q_3 + q_3^{-1}} (F_3^2 F_1 + F_1 F_3^2) v_0 = 0.$$

Since $F_1^2 = 0$, $F_1 v_2 = 0$. The weight of v_2 is $wt(v_2) = [2, 1, 0]$. By (2.3), $E_2 v_2 = 0$, so $U_q(sl(2))$-theory implies $v_3 = F_2 v_2 \neq 0$ and $F_2 v_3 = 0$.

Consider v_3. By (2.4), $F_3 v_3 = 0$, then $F_1 v_3 \neq 0$.

Observe that the weights of the vectors in $\{v_i\}_{i\geq 0}$ are given by

$$\begin{aligned} wt(v_{4n}) &= [n, 0, 1], wt(v_{4n+1}) = [n+2, 0, -1], \\ wt(v_{4n+2}) &= [n+2, 1, 0], wt(v_{4n+3}) = [n+1, -1, 0]; \end{aligned} \tag{3.5}$$

the actions of E_i $(i = 1, 2, 3)$ are then clear.

We denote the images of $\pm v_i$ in $\mathbf{L}/q\mathbf{L}$ also by $\pm v_i$ and let

$$\mathbf{L} = \bigoplus_{i\geq 0} A v_i, \qquad \mathbf{B} = \{\pm v_i : i \geq 0\}.$$

To see that $(\mathbf{L}, \mathbf{B})$ is a crystal base for $\mathbf{V}$, we only need to verify conditions (A3), (C3) and (C4) in Definition 2.1. For $i = 2, 3$, by formulas (2.2)–(2.7), we see that $\mathbf{V}$ is a direct sum of one- or two-dimensional modules for the subalgebra $\mathcal{U}_i = \langle E_i, F_i, K_i^{\pm 1} \rangle$, so conditions (A3), (C3) and (C4) clearly hold. For $i = 1$, $\tilde{F}_1 = F_1$ and

$$\begin{aligned} \tilde{E}_1 v_i = q^{-1} K_1 E_1 v_i &= \begin{cases} 0, & i = 0 \text{ or odd}, \\ \frac{q^{2n}-1}{q^2-1} v_{i-1}, & i = 4n, n \geq 1, \\ \frac{q^{2n+4}-1}{q^2-1} v_{i-1}, & i = 4n+2, n \geq 0, \end{cases} \\ &= \begin{cases} 0, & i = 0 \text{ or odd}, \\ v_{i-1}, & i = 4n, n \geq 1, \\ v_{i-1}, & i = 4n+2, n \geq 0, \end{cases} \quad (\operatorname{mod} q\mathbf{L}). \end{aligned} \tag{3.6}$$

Thus (A3), (C3) and (C4) also hold for $\tilde{E}_1$ and $\tilde{F}_1$.

To show that $(\mathbf{L}, \mathbf{B})$ is polarizable, we define a symmetric bilinear form on $\mathbf{V}$ by letting

$$(v_0, v_0) = 1 \qquad \text{and} \qquad (uv_0, u'v_0) = (v_0, \eta(u)u'v_0),$$

for all $u, u' \in \mathcal{U}^- \oplus \mathcal{U}^-\sigma$. Then $(,)$ is η-invariant and $(v_i, v_j) = 0$ if $i \neq j$. To prove that $(\mathbf{L}, \mathbf{L}) \subset A$, we use induction on n to prove that

$$(v_{4n+i}, v_{4n+i}) \in A, \qquad \text{for } n \geq 0 \text{ and } 0 \leq i \leq 3. \tag{3.7}$$

For $n = 0$, since $K_3 v_0 = q_3^{\lambda(h_3)} v_0 = q_3 v_0$, we have

$$\begin{aligned}(v_1, v_1) &= (F_3 v_0, F_3 v_0) = (v_0, \eta(F_3) F_3 v_0) \\ &= (v_0, q_3^{-1} K_3 E_3 F_3 v_0) = (v_0, v_0) = 1.\end{aligned}$$

Then

$$(v_2, v_2) = (F_1 v_1, F_1 v_1) = (v_1, q^{-1} K_1 E_1 F_1 v_1) = (v_1, (q^2+1) v_1) = q^2 + 1.$$

Similarly to the case $n = 0$, we have $(v_3, v_3) = (v_2, v_2)$.

For $n > 0$, by formulas (3.3) and (3.4) we have

$$\begin{aligned}(v_{4n}, v_{4n}) &= \frac{q^{2n}-1}{q^2-1}(v_{4n-1}, v_{4n-1}), (v_{4n+1}, v_{4n+1}) = (v_{4n}, v_{4n}), \\ (v_{4n+2}, v_{4n+2}) &= \frac{q^{2n+4}-1}{q^2-1}(v_{4n+1}, v_{4n+1}), (v_{4n+3}, v_{4n+3}) = (v_{4n+2}, v_{4n+2}).\end{aligned} \tag{3.8}$$

Thus by induction, we see that $(v_i, v_i) \in A$ $(i \geq 0)$, hence $(\mathbf{L}, \mathbf{L}) \subset A$. From these computations we also see that the induced $\mathbb{Q}$-valued symmetric bilinear form $(,)_0$ on $\mathbf{L}/q\mathbf{L}$ satisfies (2.19). □

By the results in Section 2.4 of [1], we have the following corollary.

Corollary 3.2. *For all integers $n > 0$, the $\mathcal{U}$-module $\mathbf{V}^{\otimes n}$ is completely reducible.*

Now we can state our main result.

Theorem 3.3. *For any $\mu = [m, 0, n]$ $(m, n \in \mathbb{Z}_{\geq 0})$, the simple $\mathcal{U}$-module $L(\mu)$ has a polarizable crystal base.*

We will give the proof of Theorem 3.3 in the next section. Although we will see that there are simple $\mathcal{U}$-modules $L(\lambda)$ with $\lambda(h_2) > 0$ which possess polarizable crystal bases, we cannot expect that all simple $\mathcal{U}$-modules $L(\lambda)$ with $\lambda(h_i) \geq 0$ $(i = 1, 2, 3)$ possess crystal bases for the same reason as discussed at the beginning of this section. It is clear that if we choose $\sigma_1 = \sigma_3 = \frac{1}{2}$ and $\sigma_2 = -1$ instead, then similar results hold, in particular, the statement in Theorem 3.3 can be changed to

Theorem 3.3′. *For any $\mu = [m, n, 0]$ $(m, n \in \mathbb{Z}_{\geq 0})$, the simple $\mathcal{U}$-module $L(\mu)$ has a polarizable crystal base.*

4. Proof of Theorem 3.3

Consider the decomposition of $\mathbf{V}\otimes\mathbf{V}$. The actions of $\tilde{E}_i$ and $\tilde{F}_i$ on $\mathbf{B}\otimes\mathbf{B}$ can be obtained from (2.20)–(2.22) and (3.1). We have

$$\tilde{E}_1(v_i\otimes v_j)=\begin{cases}\tilde{E}_1(v_i)\otimes v_j, & i\neq 0,\\ \sigma v_0\otimes\tilde{E}_1(v_j), & i=0,\end{cases}$$
$$\tilde{F}_1(v_i\otimes v_j)=\begin{cases}\tilde{F}_1(v_i)\otimes v_j, & i\neq 0,\\ \sigma v_0\otimes\tilde{F}_1(v_j), & i=0.\end{cases}\tag{4.1}$$

$$\tilde{E}_2(v_i\otimes v_j)=\begin{cases}v_i\otimes\tilde{E}_2(v_j), & \text{otherwise},\\ \tilde{E}_2(v_i)\otimes v_j, & \text{if } i=4m+3, j\neq 4n+2,\end{cases}$$
$$\tilde{F}_2(v_i\otimes v_j)=\begin{cases}v_i\otimes\tilde{F}_2(v_j), & \text{if } i\neq 4m+3, j=4n+2,\\ \tilde{F}_2(v_i)\otimes v_j, & \text{otherwise};\end{cases}\tag{4.2}$$

$$\tilde{E}_3(v_i\otimes v_j)=\begin{cases}\tilde{E}_3(v_i)\otimes v_j, & \text{otherwise},\\ v_i\otimes\tilde{E}_3(v_j), & \text{if } i\neq 4m, j=4n+1,\end{cases}$$
$$\tilde{F}_3(v_i\otimes v_j)=\begin{cases}\tilde{F}_3(v_i)\otimes v_j, & \text{if } i=4m, j\neq 4n+1,\\ v_i\otimes\tilde{F}_3(v_j), & \text{otherwise}.\end{cases}\tag{4.3}$$

Let

$$HW(\mathbf{B}\otimes\mathbf{B})=\{x\in\mathbf{B}\otimes\mathbf{B}:\tilde{E}_i x=0, i=1,2,3\}.$$

The elements of $HW(\mathbf{B}\otimes\mathbf{B})$ provide a set of linear independent maximal vectors in $\mathbf{V}\otimes\mathbf{V}$.

Proposition 4.1. *We have*

$$HW(\mathbf{B}\otimes\mathbf{B})=\{v_0\otimes v_j: j=4n+1, n\geq 0\}\cup\{v_{4m+3}\otimes v_{4n+2}: m,n\geq 0\}.$$

Proof. We first consider elements of the form $v_0\otimes v_j$. By (4.1), $\tilde{E}_1(v_0\otimes v_j)=0$ if and only if $\tilde{E}_1 v_j=0$, that is, $j=0$ or j is odd (see (3.3)). The case $j=0$ is clear, consider the case j is odd. By (4.2), (4.3), (3.4) we have $\tilde{E}_2(v_0\otimes v_j)=v_0\otimes\tilde{E}_2(v_j)=0\Leftrightarrow j=4n+1$ and $\tilde{E}_3(v_0\otimes v_j)=\tilde{E}_3(v_0)\otimes v_j=0$. Therefore $v_0\otimes v_{4n+1}\in HW(\mathbf{B}\otimes\mathbf{B})$.

Then we consider the case $i\neq 0$. We have $\tilde{E}_1(v_i\otimes v_j)=0\Leftrightarrow\tilde{E}_1(v_i)=0\Leftrightarrow i$ is odd. For $i=4n+1$, formulas (4.3) and (3.4) imply that

$$\tilde{E}_3(v_i\otimes v_j)=\begin{cases}v_i\otimes\tilde{E}_3(v_j), & j=4m+1,\\ \tilde{E}_3(v_i)\otimes v_j, & j\neq 4m+1,\end{cases}\neq 0.$$

For $i=4n+3$, we have

$$\tilde{E}_2(v_i\otimes v_j)=\begin{cases}\tilde{E}_2(v_i)\otimes v_j, & j\neq 4m+2,\\ v_i\otimes\tilde{E}_2(v_j), & j=4m+2,\end{cases}=0\Leftrightarrow j=4m+2.$$

and $\tilde{E}_3(v_{4n+3}\otimes v_{4m+2})=\tilde{E}_3(v_{4n+3})\otimes v_{4m+2}=0$. Therefore the desired result follows. □

Note that

$$\begin{aligned} wt(v_0 \otimes v_0) &= (-4, 0, 2) = [0, 0, 2], \\ wt(v_0 \otimes v_{4n+1}) &= (-2n-4, 0, 0) = [n+2, 0, 0], \\ wt(v_{4m+3} \otimes v_{4n+2}) &= (-2m-2n-6, 0, 0) = [m+n+3, 0, 0]. \end{aligned} \tag{4.4}$$

Lemma 4.2. *Let $\lambda_1 = (-2, 0, 0) = [1, 0, 0]$, then the $\mathcal{U}$-module $L(\lambda_1)$ has a polarizable crystal base.*

Remark. Since the highest weight of $\mathbf{V}$ is $\lambda = [0, 0, 1]$, we see that Theorem 3.3 follows immediately from Lemma 4.2.

Proof. Let $\mu = wt(v_0 \otimes v_1) = (-4, 0, 0) = [2, 0, 0]$. Then by Proposition 4.1 and the results in Section 2.4 of [1], the $\mathcal{U}$-module $L(\mu)$ has a polarizable crystal base. Let $M(\mu)$ be the Verma module (see [2, p. 72]) with the highest weight μ and let $M(\lambda_1)$ be the Verma module with the highest weight λ_1. Let $v_\mu \in M(\mu)$ (resp. $v_{\lambda_1} \in M(\lambda_1)$) be a highest weight vector. Then

$$L(\mu) \cong M(\mu)/\langle F_2 v_\mu, F_3 v_\mu \rangle, \qquad L(\lambda_1) \cong M(\lambda_1)/\langle F_2 v_{\lambda_1}, F_3 v_{\lambda_1} \rangle.$$

Let v be either v_μ or v_{λ_1}. Then

$$F^{(\delta,m)} v = F_{222}^{\delta_1} F_{212}^{\delta_2} F_{221}^{\delta_3} F_1^{\delta_4} F_0^m v, \quad \delta_i = 0, 1, \quad m \in \mathbb{Z}_{\geq 0},$$

form a basis of $L(\mu)$ or $L(\lambda_1)$. The weights of these elements are

$$\begin{aligned} wt(F^{(\delta,m)} v) = wt(v) &- (\delta_1 + \delta_2 + \delta_3 + \delta_4 + 2m)\epsilon_1 \\ &+ (-\delta_1 + \delta_2 - \delta_3 + \delta_4)\epsilon_2 + (-\delta_1 - \delta_2 + \delta_3 + \delta_4)\epsilon_3. \end{aligned} \tag{4.5}$$

Thus $wt(F^{(\delta,m)} v_{\lambda_1})(h_1) = m + 1 + \delta_1 + 2\delta_2 - \delta_3 \geq 0$ and $L(\lambda_1) \in \mathcal{O}_{int}$.

Formula (4.5) implies that the decomposition of $L(\mu)$ and $L(\lambda_1)$ into simple $\mathcal{U}_i$-modules ($i = 2, 3$) are the same, and $L(\mu) \cong L(\lambda_1)$ as $\mathcal{U}_i$-modules ($i = 2, 3$).

The corresponding weights of $L(\mu)$ and $L(\lambda_1)$ have different numerical marks on h_1, however, the difference is just 1. Therefore, by applying an argument similar to the proof of Proposition 3.1 (in particular, the computations (3.3) and (3.6)), we see that if (L_μ, B_μ) is a polarizable crystal base for $L(\mu)$ with

$$L_\mu = \{F_{i_1} \cdots F_{i_k} v_\mu : (i_1 \cdots i_k) \in J\},$$

where J is a certain set of sequences formed by the numbers in $\{1, 2, 3\}$, then

$$L_{\lambda_1} = \{F_{i_1} \cdots F_{i_k} v_{\lambda_1} : (i_1 \cdots i_k) \in J\},$$

with the corresponding B_{λ_1} will define a polarizable crystal base for $L(\lambda_1)$. □

5. Crystal graphs

From the proof of Lemma 4.2 we see that we can identify the crystal graph of $L([m, 0, 0])$ $(m > 0)$ with the crystal graph of $L([2, 0, 0])$. We also identify the crystal graph of $L([2, 0, 0])$ with the connected component generated by $\boxed{0} \otimes \boxed{1}$ in the crystal graph of $\mathbf{V} \otimes \mathbf{V}$. Write

$$\boxed{i} \otimes \boxed{j} = \boxed{\begin{array}{c} i \\ \hline j \end{array}}, \qquad i, j \geq 0.$$

The crystal graph of $L([2, 0, 0])$ is given by the diagram

$$\begin{array}{ccccccccccc}
\boxed{\frac{0}{1}} & \xrightarrow{1} & \boxed{\frac{0}{2}} & \xrightarrow{2} & \boxed{\frac{0}{3}} & \xrightarrow{1} & \boxed{B_0} & \xrightarrow{1} & \boxed{B_1} & \xrightarrow{1} & \cdots \\
 & & {\scriptstyle 3}\downarrow & & \downarrow{\scriptstyle 3} & & & & & & \\
 & & \boxed{\frac{1}{2}} & \xrightarrow{2} & \boxed{\frac{1}{3}} & & & & & & \\
 & & {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} & & & & & & \\
 & & \boxed{\frac{2}{2}} & \xrightarrow{2} & \boxed{\frac{2}{3}} & \xrightarrow{2} & \boxed{\frac{3}{3}} & \xrightarrow{1} & \boxed{B'_1} & \xrightarrow{1} & \cdots
\end{array} \tag{5.1}$$

where the blocks $\boxed{B_i}$ $(i = 0, 1, 2, ...)$ are given by

$$\begin{array}{ccccccccc}
\xrightarrow{1} & \boxed{\frac{4i}{4}} & \xrightarrow{3} & \boxed{\frac{4i+1}{4}} & \xrightarrow{1} & \boxed{\frac{4i+2}{4}} & \xrightarrow{2} & \boxed{\frac{4i+3}{4}} & \xrightarrow{1} \\
 & & & \downarrow{\scriptstyle 3} & & \downarrow{\scriptstyle 3} & & \downarrow{\scriptstyle 3} & \\
 & & & \boxed{\frac{4i+1}{5}} & \xrightarrow{1} & \boxed{\frac{4i+2}{5}} & \xrightarrow{2} & \boxed{\frac{4i+3}{5}} & \\
 & & & & & & & \downarrow{\scriptstyle 1} & \\
 & & & & & & & \boxed{\frac{4i+4}{5}} &
\end{array} \tag{5.2}$$

and the blocks $\boxed{B'_j}$ $(j = 1, 2, ...)$ are given by

$$\xrightarrow{1} \boxed{\frac{4j}{3}} \xrightarrow{3} \boxed{\frac{4j+1}{3}} \xrightarrow{1} \boxed{\frac{4j+2}{3}} \xrightarrow{2} \boxed{\frac{4j+3}{3}} \xrightarrow{1} . \tag{5.3}$$

These graphs can be verified by using (4.1)-(4.3) and (3.1).

The crystal graph of $L([0, 0, n])$ can be identified with the connected component generated by $\boxed{0} \otimes \boxed{0} \otimes \cdots \otimes \boxed{0}$ (n copies) in the crystal graph of $\mathbf{V}^{\otimes n}$, and is given by

$$\underbrace{\boxed{0} \otimes \boxed{0} \otimes \cdots \otimes \boxed{0}}_{n \text{ copies}} \xrightarrow{3} \boxed{B_1} \xrightarrow{3} \boxed{B_2} \xrightarrow{3} \cdots , \tag{5.4}$$

where the blocks $\boxed{B_i}$ $(i = 0, 1, 2, ...)$ are given by

$$
\begin{array}{ccccccccc}
\xrightarrow{3} & \boxed{1,1} & \xrightarrow{1} & \boxed{1,2} & \xrightarrow{2} & \boxed{1,3} & \xrightarrow{1} & \boxed{1,4} & \xrightarrow{3} \\
 & \downarrow{\scriptstyle 3} & & \downarrow{\scriptstyle 3} & & \downarrow{\scriptstyle 3} & & & \\
 & \boxed{2,1} & \xrightarrow{1} & \boxed{2,2} & \xrightarrow{2} & \boxed{2,3} & \xrightarrow{1} & \boxed{2,4} & \\
 & \downarrow{\scriptstyle 3} & & \downarrow{\scriptstyle 3} & & \downarrow{\scriptstyle 3} & & \downarrow{\scriptstyle 3} & \\
 & \boxed{3,1} & \xrightarrow{1} & \boxed{3,2} & \xrightarrow{2} & \boxed{3,3} & \xrightarrow{1} & \boxed{3,4} & \\
 & \downarrow{\scriptstyle 3} & & \downarrow{\scriptstyle 3} & & \downarrow{\scriptstyle 3} & & \downarrow{\scriptstyle 3} & \\
 & \vdots & & \vdots & & \vdots & & \vdots & \\
 & \downarrow{\scriptstyle 3} & & \downarrow{\scriptstyle 3} & & \downarrow{\scriptstyle 3} & & \downarrow{\scriptstyle 3} & \\
 & \boxed{n,1} & \xrightarrow{1} & \boxed{n,2} & \xrightarrow{2} & \boxed{n,3} & \xrightarrow{1} & \boxed{n,4} &
\end{array}
$$

in which

$$\boxed{s,t} = \boxed{4i+t} \otimes \underbrace{\boxed{1} \otimes \cdots \otimes \boxed{1}}_{s-1 \text{ copies}} \otimes \underbrace{\boxed{0} \otimes \cdots \otimes \boxed{0}}_{n-s \text{ copies}}, \qquad 1 \le s \le n, \quad 1 \le t \le 4.$$

The crystal graph of $L([m, 0, n])$ $(m, n > 0)$ can be identified with the connected component generated by

$$\underbrace{\boxed{0} \otimes \boxed{0} \otimes \cdots \otimes \boxed{0}}_{n \text{ copies}} \otimes \boxed{\begin{array}{c} 0 \\ \hline 1 \end{array}}$$

in the crystal graph of $L([0, 0, n]) \otimes L([m, 0, 0])$ (or $L([0, 0, n]) \otimes L([2, 0, 0])$). We should just give the crystal graph in the case $n = 1$ as an example.

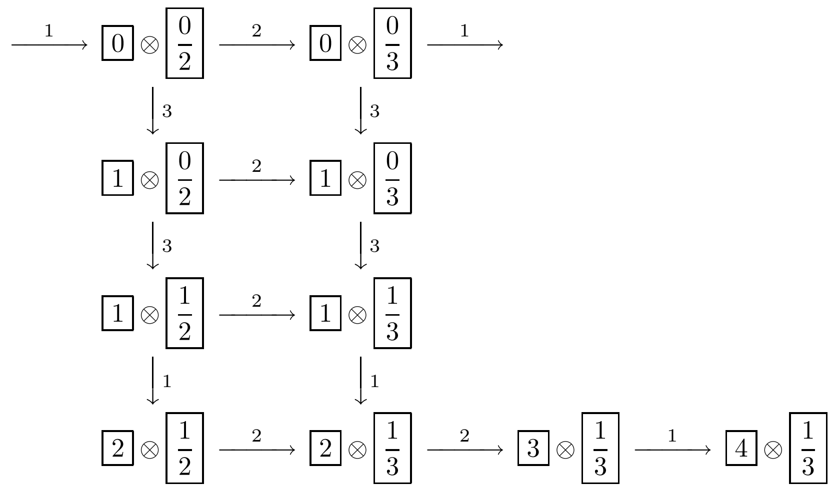
Example. The crystal graph of $L([m, 0, 1])$ is given by

$$
\text{(5.4)} \qquad
\begin{array}{ccccccccc}
\boxed{0} \otimes \boxed{\begin{array}{c} 0 \\ \hline 1 \end{array}} & \xrightarrow{1} & \boxed{D} & \xrightarrow{1} & \boxed{D_0} & \xrightarrow{1} & \boxed{D_1} & \xrightarrow{1} & \cdots \\
\downarrow{\scriptstyle 3} & & & & & & & & \\
\boxed{C_0} & \xrightarrow{3} & \boxed{C_1} & \xrightarrow{3} & \cdots & & & &
\end{array}
$$

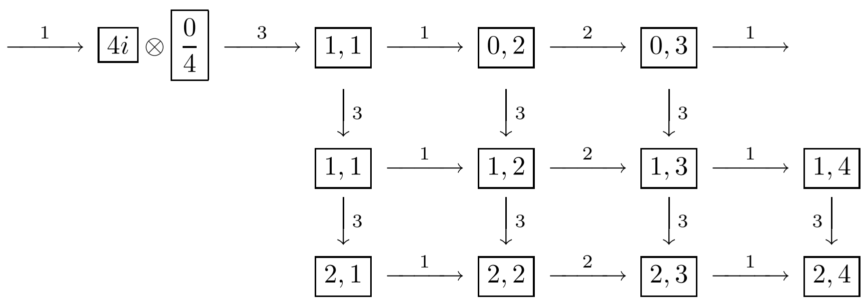
where the blocks $\boxed{C_i}$ $(i = 0, 1, 2, ...)$ are given by

$$\xrightarrow{3} \boxed{4i+1} \otimes \boxed{\begin{array}{c} 0 \\ \hline 1 \end{array}} \xrightarrow{1} \boxed{4i+2} \otimes \boxed{\begin{array}{c} 0 \\ \hline 1 \end{array}} \xrightarrow{2} \boxed{4i+3} \otimes \boxed{\begin{array}{c} 0 \\ \hline 1 \end{array}} \xrightarrow{1} \boxed{4i+4} \otimes \boxed{\begin{array}{c} 0 \\ \hline 1 \end{array}} \xrightarrow{3},$$

the block $\boxed{D}$ is given by



and the blocks $\boxed{D_i}$ $(i = 0, 1, 2, ...)$ are given by



in which

$$\boxed{0,t} = \boxed{4i+t} \otimes \boxed{\frac{0}{4}}, \qquad 1 \le t \le 3,$$

$$\boxed{1,t} = \boxed{4i+t} \otimes \boxed{\frac{1}{4}}, \qquad 1 \le t \le 4,$$

$$\boxed{2,t} = \boxed{4i+t} \otimes \boxed{\frac{1}{5}}, \qquad 1 \le t \le 4.$$

Remark. From graph (5.1) we see that

$$\tilde{E}_i\left(\boxed{\frac{0}{1}} \otimes \boxed{\frac{0}{2}}\right) = 0, \qquad i = 1, 2, 3.$$

So $\boxed{\frac{0}{1}} \otimes \boxed{\frac{0}{2}}$ generates a connected component in the crystal graph of $L([2,0,0]) \otimes L([2,0,0])$. Since

$$wt\left(\boxed{\frac{0}{1}} \otimes \boxed{\frac{0}{2}}\right) = [4, 1, 1],$$

we see that there are $L(\nu)$ with $\nu(h_2) > 0$ which possess polarizable crystal bases.

References

1. G. Benkart, S.-J. Kang and M. Kashiwara, *Crystal Bases for the quantum superalgebra $U_q(\mathfrak{gl}(m,n))$*, J. of Amer. Math. Soc. **13(2)** (2000), 295-331. MR **2000m:**17015
2. J. C. Jantzen, *Lectures on Quantum Groups*, Graduate Studies in Mathematics, vol. 6, AMS, 1995. MR **96m:**17029
3. A. Joseph, *Quantum Groups and Their Primitive Ideals*, Ergebnisse der Mathematik (3) **29**, Springer-Verlag, Berlin, 1995. MR **96d:**17015
4. V. Kac, *Lie superalgebras*, Adv. Math. **26** (1977), 8-96. MR **58:**5803
5. ———, *Representations of classical Lie superalgebras*, Lecture Notes in Math., vol. 676, Springer, Berlin, 1978, pp. 597-626. MR **80f:**17006
6. M. Kashiwara, *Crystalizing the q-analogue of universal enveloping algebras*, Commun. Math. Phys. **133** (1990), 249-260. MR **92b:**17018
7. ———, *On crystal bases of the q-analogue of universal enveloping algebras*, Duke Math. J. **63(2)** (1991), 465-516. MR **93b:**17045
8. ———, *Crystal base and Littelmann's refined Demazure character formula*, Duke Math. J. **71** (1993), 839-858. MR **95b:**17019
9. ———, *On crystal bases*, Representations of Groups, CMS conference Proc., vol. 16, AMS, 1995, pp. 155-197. MR **97a:**17016
10. M. Kashiwara and T. Nakashima, *Crystal graphs for representations of the q-analogue of classical Lie algebras*, J. of Algebra **165** (1994), 295-345. MR **95c:**17025
11. I. M. Musson and Y. M. Zou, *Crystal bases for $U_q(osp(1,2r))$*, J. of Algebra **210** (1998), 514-534. MR **99j:**17024
12. Y. M. Zou, *Finite-dimensional representations of $\Gamma(\sigma_1,\sigma_2,\sigma_3)$*, J. Algebra **169(3)** (1994), 827-846. MR **95j:**17005
13. ———, *Deformation of the universal enveloping algebra of $\Gamma(\sigma_1,\sigma_2,\sigma_3)$*, Canad. Math. Bull. **39(4)** (1996), 499-506. MR **97m:**17021
14. ———, *Crystal bases for $U_q(sl(2,1))$*, Proc. of Amer. Math. Soc. **127(8)** (1999), 2213-2223. MR **99j:**17028

Department of Mathematical Sciences, University of Wisconsin-Milwaukee, Milwaukee, Wisconsin 53201

E-mail address: `ymzou@uwm.edu`