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CONVERGENCE OF ASYMPTOTIC DIRECTIONS

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ABSTRACT. We study convergence properties of asymptotic directions of unbounded sets in normed spaces. The links between the continuity of a set-valued map and the convergence of asymptotic directions are examined. The results are applied to investigate continuity properties of marginal functions and asymptotic directions of level sets.

1. INTRODUCTION

The concept of asymptotic cone (also called recession cone or horizon cone) has been introduced to study unbounded sets. As far as we know the first work devoted to asymptotic cones of convex sets is [45] by Steinitz. Further contributions are credited to Choquet [9], Fenchel [19], Klee [20], Rockafellar [41] and some others. The case of nonconvex sets was first investigated by Debreu [14], but most progress has been made quite recently by Dedieu [15], Luc [26, 27, 28], Luc and Théra [32], Penot [36, 37, 38], Rockafellar and Wets [42, 43], Zalinescu [46, 47]. In these works the authors develop calculus rules for asymptotic cones and asymptotic functions in general settings and apply them to several topics of applied mathematics, especially to nonconvex optimization. In Agadi and Penot [3] an analogy between asymptotic cones and usual tangent cones is displayed and in Luc and Théra [32] a link is established between the asymptotic function of a function and its derivative with support, thereby showing that the roles the asymptotic cones and asymptotic functions play in the study of sets and functions at remote points is similar to the roles the tangent cones and conventional derivatives play at finite points.

The purpose of the present paper is to investigate convergence properties of asymptotic cones in relation with continuity of set-valued maps. The paper is organized as follows. In the next section we study continuity properties of cone-valued maps. Section 3 is focused on the convergence of asymptotic directions of a family of sets. In Section 4 further properties are established for asymptotic directions in relation with several operations on set-valued maps. Applications to the study of marginal functions are presented in Section 5 and applications to the convergence of asymptotic directions of level sets are given in Section 6. A result on the extreme desirability condition that guarantees the existence of equilibria in unbounded exchange economies is also obtained in Section 6. The final section contains a short discussion on a possible study of asymptotic directions with respect to the weak topology following a suggestion of a referee.

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2. Continuity of cone-valued maps

Throughout this paper, unless otherwise mentioned, Ω is a metrizable topological space, X is a real normed vector space, B_X is the closed unit ball and S_X is the unit sphere in X. For a set $A \subseteq X$, $intA$ stands for the interior of A while $c\ell A$ is its closure. If Y is another normed space, then the product space $X \times Y$ is equipped with the max norm. In particular, one has $B_{X\times Y} = B_X \times B_Y$. Let R be a set-valued map from Ω to X. We recall some definitions concerning continuity properties of R; most of them are standard, sometimes under a different terminology (see [6], [21], [43]).

a) R is said to be *lower (*resp. *upper) continuous* at $w_0 \in \Omega$ if for every open set $U \subseteq X$ with $U \cap R(w_0) \neq \emptyset$ (resp. $R(w_0) \subseteq U$), one can find a neighborhood W of w_0 in Ω such that $U \cap R(w) \neq \emptyset$ (resp. $R(w) \subseteq U$) for every $w \in W$.

b) R is said to be *closed* at w_0 if whenever a sequence $\{(w_n, x_n)\}_{n=1}^{\infty}$ from the graph of R converges to some limit $(w_0, x_0) \in \Omega \times X$, this limit belongs to the graph of R.

c) R is said to be *upper hemicontinuous* or *upper Hausdorff continuous* at $w_0 \in \Omega$ if for each $\varepsilon > 0$ there exists a neighborhood W_ε of w_0 such that $R(W_\varepsilon) \subseteq R(w_0) + \varepsilon B_X$.

d) R is said to be *boundedly compact* at $w_0 \in \Omega$ if for every sequence $\{(w_n, x_n)\}_{n=1}^{\infty}$ in the graph of R with $\{w_n\}_{n=1}^{\infty}$ converging to w_0 and $\{x_n\}_{n=1}^{\infty}$ bounded, the sequence $\{x_n\}_{n=1}^{\infty}$ admits a convergent subsequence.

e) Now let us suppose that R is a cone-valued map. We shall say that R is *cosmically upper continuous (resp. cosmically lower continuous, resp. cosmically closed)* at $w_0 \in \Omega$ if the map $w \longmapsto R(w) \cap B_X$ is upper continuous (resp. lower continuous, resp. closed) at w_0.

We also recall [4], [17], [18] that given a cone $C \subseteq X$ and a positive ε, the conic ε-neighborhood of C is the set $C_\varepsilon(C) := \{x \in X : d(x, C) < \varepsilon||x||\} \cup \{0\}$, where $d(x, C)$ stands for the distance from x to C. Since for $\varepsilon > 1$ the set $C_\varepsilon(C)$ is the whole space, we shall assume $\varepsilon \in (0, 1)$ when speaking of conic neighborhoods; in particular for $C = \{0\}$ we have $C_\varepsilon(C) = \{0\}$. Moreover, for a vector $v \in X$, by writing $C_\varepsilon(v)$ we mean the conic ε-neighborhood of the ray $\{tv : t \geq 0\}$.

f) A cone-valued map R is said to be *conically upper continuous* at $w_0 \in \Omega$ if for each $\varepsilon > 0$ there exists a neighborhood W of w_0 such that $R(W) \subseteq C_\varepsilon(R(w_0))$.

It is easy to show that a cone-valued map is closed at w_0 if it is conically upper continuous at w_0 (and, by Proposition 2.1 below, a fortiori when it is cosmically upper continuous at this point). The relationships between some of the preceding concepts for general set-valued maps are well known. Let us compare them in the case of cone-valued maps.

Proposition 2.1. *For any cone-valued map R, the following assertions hold:*

i) If R is upper continuous at $w_0 \in \Omega$, then there is a neighborhood W of w_0 such that $R(W) \subseteq c\ell R(w_0)$;

ii) R is conically upper continuous at $w_0 \in \Omega$ if it is cosmically upper continuous at this point;

iii) R is cosmically upper continuous at $w_0 \in \Omega$ if and only if for every bounded closed set $B \subseteq X$, the map $w \longmapsto R(w) \cap B$ is upper continuous at w_0;

iv) R is cosmically upper continuous at $w_0 \in \Omega$ if it is conically upper continuous at this point and if $R(w_0)$ is locally compact;

v) R is cosmically lower continuous (resp. cosmically closed) at $w_0 \in \Omega$ if and only if it is lower continuous (resp. closed) at w_0.

vi) R is cosmically closed at $w_0 \in \Omega$ if and only if it is closed at w_0.

Proof. Let us prove the first assertion. By upper continuity, for the open set $U := \{x \in X : d(x, R(w_0)) < 1\}$ there exists a neighborhood W of w_0 in Ω such that $R(W) \subseteq U$. Since $R(w_0)$ and $R(W)$ are cones, the latter inclusion shows that $d(v, R(w_0)) = 0$ for every $v \in R(W)$ which means that $R(W) \subseteq c\ell R(w_0)$.

To prove ii) assume that R is cosmically upper continuous at w_0. Since the case $R(w_0) = \{0\}$ is trivial, we may assume that $R(w_0) \neq \{0\}$. Let $C_\varepsilon(R(w_0))$ be a conic ε-neighborhood of $R(w_0)$, with $\varepsilon \in (0, 1)$. Then there is a neighborhood W of w_0 in Ω such that $R(W) \cap B_X$ is contained in $C_\varepsilon(R(w_0)) \cup B(0, \varepsilon)$. Hence $R(W) \cap S_X$ is contained in $C_\varepsilon(R(w_0))$. By homogeneity $R(W) \subseteq C_\varepsilon(R(w_0))$ and R is conically upper continuous at w_0.

Now, it is obvious that the condition given in iii) implies cosmic upper continuity of R at w_0. For the converse, let B be a closed bounded set in X. There is a positive λ such that $\lambda B \subseteq B_X$. The map $w \longmapsto R(w) \cap (\lambda B)$ is upper continuous at w_0 because it is the intersection of the upper continuous map : $w \longmapsto R(w) \cap B_X$ with the closed subset λB. The upper continuity of the map $w \longmapsto R(w) \cap B$ follows by observing that $R(w) \cap B = \frac{1}{\lambda}\,[R(w) \cap \lambda B]$.

As to assertion iv), for every open set U containing $R(w_0) \cap B_X$ that is compact, there is a positive ε such that U contains the set $\{x \in X : d(x, R(w_0) \cap B_X) \leq \varepsilon\}$. Hence, if $R(W) \subseteq \{x \in X : d(x, R(w_0)) \leq \varepsilon ||x||\}$ for some neighborhood W of w_0, then $R(W) \cap B_X$ is contained in U. Thus R is cosmically upper continuous at w_0.

Let us turn to assertion v). For the necessary condition, let U be any open set in X with $U \cap R(w_0) \neq \emptyset$. We can find a smaller bounded open set $U_0 \subseteq U$ such that $U_0 \cap R(w_0) \neq \emptyset$. There exists a positive λ such that $\lambda U_0 \subseteq B_X$. Hence $\lambda U_0 \cap R(w_0) \neq \emptyset$ and $\lambda U_0 \cap (R(w_0) \cap B_X) \neq \emptyset$. By cosmic lower continuity, one can find a neighborhood W of w_0 in Ω such that $\lambda U_0 \cap (R(w) \cap B_X) \neq \emptyset$ for $w \in W$. Consequently $\lambda U_0 \cap R(w) \neq \emptyset$ and $U_0 \cap R(w) \neq \emptyset$ for all $w \in W$.

For the sufficient condition, suppose that $U \cap (R(w_0) \cap B_X) \neq \emptyset$ where U is an open set in X. There exists a smaller open set $U_0 \subseteq U \cap int\ B_X$ such that $U_0 \cap (R(w_0) \cap B_X) \neq \emptyset$. Hence $U_0 \cap R(w_0) \neq \emptyset$ and by lower continuity of R, $U_0 \cap R(w) \neq \emptyset$ for w close to w_0. This implies the cosmic lower continuity of R at w_0 because $U_0 \cap R(w) = U_0 \cap (R(w) \cap B_X)$.

For the last assertion, the equivalence of the cosmic closedness and the closedness of R at w_0 is straighforward. □

It should be noticed that the sufficiency part of assertion v) can also be derived from a more general fact given in [31], [38] saying that if a set-valued map R_1 is lower continuous at w_0 while for a set-valued map R_2 there is a subset $A \subseteq R_2(w_0)$ with the property that for every $x \in A$, there is a neighborhood W of w_0 and a neighborhood U of x such that $U \subseteq R_2(w)$ for each $w \in W$, then the map $w \longmapsto R_1(w) \cap R_2(w)$ is lower continuous at w_0 provided

$$c\ell(R_1(w_0) \cap R_2(w_0)) = c\ell(R_1(w_0) \cap A).$$

In our case R_2 is the map $w \longmapsto B_X$ and $A = \ int\ B_X$. Furthermore, Proposition 2.1 (i) shows that when a map is closed-valued, upper continuity is a very stringent property, especially for cone-valued maps: when $R(w_0)$ is closed, the cone-valued map R is upper continuous at w_0 if and only if there is a neighborhood W of w_0 such that $R(W) \subseteq R(w_0)$.

In order to study the effect of some operations upon conic continuity and cosmic continuity for cone-valued maps we need some preliminary results of independent interest. In particular, in order to study the preservation of convergence under a linear map, let us say that a positively homogeneous (or a linear) map L from X to a real normed space Y is *expanding at* 0 on a cone $P \subseteq X$ if there is a positive α such that $\|L(x)\| \geq \alpha\|x\|$ for all $x \in P$. This property implies $rB_Y \cap L(P) \subset L(\alpha^{-1}rB_X \cap P)$ for each $r > 0$, an openness property at 0 on P. We will give concrete criteria for such a property after the following lemma.

Lemma 2.2. *Given two cones P, Q in X, the following assertions are equivalent:*
a) there exist $\alpha,\ \beta > 0$ such that $C_\alpha(P) \cap C_\beta(Q) = \{0\}$;
b) there exists $\gamma > 0$ such that $P \cap C_\gamma(Q) = \{0\}$;
c) the map $(x, y) \mapsto x - y$ is expanding at 0 on the cone $P \times Q$.
These assertions are satisfied whenever P, Q are closed, P is locally compact and $P \cap Q = \{0\}$.

Proof. Since assertion a) is clearly stronger than assertion b), let us prove that it follows from b) by showing that for any $\alpha, \beta \in (0, 1)$, and for any unit vector $x \in C_\alpha(P) \cap C_\beta(Q)$ there exist $y \in P \cap C_\gamma(Q)$ for $\gamma = (1-\alpha)^{-1}(\alpha+\beta)$ such that $y \neq 0$. By definition, we can find $y \in P$, $z \in Q$ such that $\|x - y\| < \alpha$, $\|x - z\| < \beta$. Then $\|y\| > 1 - \alpha > 0$ and $\|y - z\| < \alpha + \beta$. Thus $\|y - z\| < (1-\alpha)^{-1}(\alpha+\beta)\|y\|$ and $y \in P \cap C_\gamma(Q)$. It follows that when assertion b) holds, for $\alpha = \beta \leq \gamma(\gamma+2)^{-1}$ one has $C_\alpha(P) \cap C_\beta(Q) = \{0\}$.

Let us show that a) implies c). This can be done by a standard contradiction argument. However, here we provide a proof that expresses the dependence of the expanding coefficient and the magnitude of the conic neighborhoods. More precisely we show that if $C_\varepsilon(P) \cap C_\varepsilon(Q) = \{0\}$, then $\alpha \|(x, y)\| \leq \|x - y\|$ for any $(x, y) \in P \times Q$ whenever $\alpha \leq 2\varepsilon(1+\varepsilon)^{-1}$. We may suppose $\|(x, y)\| = 1$; we have to show that the inequality $\|x - y\| < \alpha$ leads to a contradiction. In fact, setting $z := (1/2)(x + y)$, this inequality yields $\|z - x\| = (1/2)\|x - y\| = \|z - y\|$ hence

$$\|z\| \geq \max(\|x\| - \|z - x\|, \|y\| - \|z - y\|) \geq \max(\|x\|, \|y\|) - \alpha/2 = 1 - \alpha/2$$

and

$$\|z - x\| < (1/2)\alpha \leq (1/2)\alpha(1 - \alpha/2)^{-1}\|z\| = \alpha(2-\alpha)^{-1}\|z\| \leq \varepsilon\|z\|$$

and similarly $\|z - y\| < \varepsilon\|z\|$, a contradiction with $C_\varepsilon(P) \cap C_\varepsilon(Q) = \{0\}$.

Conversely, let us show that $C_\varepsilon(P) \cap C_\varepsilon(Q) = \{0\}$ whenever $\varepsilon \leq \alpha(2+\alpha)^{-1}$ and $\alpha\|(x, y)\| \leq \|x - y\|$ for any $(x, y) \in P \times Q$. In fact, if there exists $z \in C_\varepsilon(P) \cap C_\varepsilon(Q)$ with $\|z\| = 1$, for some $(x, y) \in P \times Q$, we have

$$\|x - y\| = \|(x - z) - (y - z)\| < 2\varepsilon;$$

hence $\|(x, y)\| \geq 1 - \varepsilon > (1-\varepsilon)(1/2\varepsilon)\|x - y\| \geq (1-\varepsilon)(1/2\varepsilon)\alpha\|(x, y)\|$, a contradiction with $\varepsilon \leq \alpha(2+\alpha)^{-1}$.

Now let us suppose P is closed and locally compact. If assertion b) does not hold we can find sequences $\{\gamma_n\}_{n=1}^\infty \to 0$, $\{x_n\}_{n=1}^\infty$ in P with norm one such that $d(x_n, Q) < \gamma_n$ for each n. Since P is locally compact and closed, taking a subsequence if necessary, we may assume that $\{x_n\}_{n=1}^\infty$ has a limit x in P with $\|x\| = 1$. Since Q is closed, we get $x \in Q$, a contradiction. ☐

In the following criteria, we say that L is open on its image if the image $L(B_X)$ is a neighborhood of the origin in $L(X)$.

Lemma 2.3. *Let P be a cone in X and let L be a continuous linear map from X to Y. Each of the following conditions is sufficient for L to be expanding at 0 on P:*

a) L is an isomorphism;

b) P is closed, locally compact and $P \cap KerL = \{0\}$;

c) L is open on its image and $C_\alpha(N) \cap P = \{0\}$ for some $\alpha > 0$, where $N := KerL$ is the kernel of L and N has a topological supplement;

d) P is closed, L is open on its image and the kernel $KerL$ of L is finite dimensional with $P \cap KerL = \{0\}$.

Proof. It is obvious that conditions a) and b) imply that L is expanding at 0 on P. We now show that condition c) also does it. Indeed, if not, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of elements in P with $\|x_n\| = 1$ and $\{L(x_n)\}_{n=1}^{\infty} \to 0$. By decomposing X into a topological direct sum of $N := \mathrm{Ker}L$ and a subspace $X_0 \subseteq X$ one can express x_n above as $a_n + b_n$ with $a_n \in N$ and $b_n \in X_0$. It follows that the sequence $\{b_n\}_{n=1}^{\infty}$ converges to 0 because $\lim_{n\mapsto\infty} L(b_n) = \lim_{n\mapsto\infty} L(a_n + b_n) = 0$ and the restriction of L to X_0 is an isomorphism onto $L(X)$. Consequently, $\lim_{n\mapsto\infty} \|x_n - a_n\| = 0$ and for n large enough we get $x_n \in C_\alpha(N) \cap P = \{0\}$, a contradiction. Assertion d) is a consequence of the preceding one and of the previous lemma. □

Lemma 2.4. *Let P be a cone in X and let L be a continuous linear map from X to Y. If L is expanding at 0 on P, then for some $\gamma > 0$ it is expanding on $C_\gamma(P)$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that $L(C_\delta(P)) \subseteq C_\varepsilon(L(P))$.*

Proof. Let $\alpha > 0$ be such that $\|L(x)\| \geq \alpha\|x\|$ for each $x \in P$. Let us show that for any $\beta \in (0, \alpha)$ we can find $\gamma \in (0, 1)$ such that $\|L(w)\| \geq \beta\|w\|$ for each $w \in C_\gamma(P)$. Given $w \in C_\gamma(P)$ with $\|w\| = 1$ we can find $x \in P$ such that $\|x - w\| < \gamma$, so that $\|x\| \geq 1 - \gamma$ and

$$\|L(w)\| \geq \|L(x)\| - \|L\|\|w - x\| \geq \alpha(1 - \gamma) - \|L\|\gamma \geq \beta$$

provided $\gamma \leq (\alpha - \beta)(\alpha + \|L\|)^{-1}$.

Suppose to the contrary that there is some $\varepsilon > 0$ such that for each $n \geq 1$ the inclusion $L(C_{1/n}(P)) \subseteq C_\varepsilon(L(P))$ does not hold, i.e. one can find $x_n \in P$ such that $\|x_n\| = 1$, $d(x_n, P) < 1/n$ and

$$d(L(x_n), L(P)) \geq \varepsilon\|L(x_n)\|$$

for every $n \geq 1$. Let $y_n \in P$ be such that $\|x_n - y_n\| < 1/n$. Then, as L is expanding at 0 on P, we have for some $\alpha > 0$

$$\alpha \leq \|L(x_n)\| \leq \varepsilon^{-1} d(L(x_n), L(P)) \leq \varepsilon^{-1}\|L(x_n) - L(y_n)\| \leq \varepsilon^{-1}\|L\|/n,$$

a contradiction for n large. □

It is worthwhile noticing that the expanding condition on L in the preceding lemma can not be dropped. This can be seen by the following example. Let $X = \mathbb{R}^3, Y = \mathbb{R}^2, L(x_1, x_2, x_3) = (x_1, x_2), P = \{0\} \times \mathbb{R}^2$. Then for any positive number δ one has $L(C_\delta(P)) = Y$; nevertheless $C_\varepsilon(L(P)) = C_\varepsilon(\{0\} \times \mathbb{R}) \neq Y$. In this example $\mathrm{Ker}L \subseteq P$.

The following result is an immediate consequence of the first assertion of the preceding lemma and of the fact that when a continuous linear map is expanding at 0 on a cone, it is expanding at 0 on its closure.

Lemma 2.5. *Assume that R is a cone-valued map from Ω to X that is conically upper continuous at ω_0 and L is a continuous linear map from X to Y that is expanding at 0 on $R(\omega_0)$. Then there is a neighborhood W of ω_0 such that L is expanding at 0 on the cone cl($R(W)$).*

Theorem 2.6. *Let R, R_1, R_2 (resp. S) be cone-valued maps from Ω to X (resp. Y) and let L be a linear continuous map from X to Y. Assume that the above cone-valued maps are conically upper continuous at w_0. Then the following assertions hold.*

i) $R_1 \cup R_2$ is conically upper continuous at w_0;

ii) $R_1 \cap R_2$ is conically upper continuous at w_0 provided one of the following conditions holds:

a) there exists $c > 0$ such that $d(x, R_1(w_0) \cap R_2(w_0)) \leq cd(x, R_1(w_0)) + cd(x, R_2(w_0))$ for all $x \in X$;

b) either $R_1(w_0)$ or $R_2(w_0)$ is locally compact and both are closed;

c) for any $\gamma > 0$ there exist $\alpha, \beta > 0$ such that $C_\alpha(R_1(w_0)) \cap C_\beta(R_2(w_0)) \subset C_\gamma(R_1(w_0) \cap R_2(w_0))$.

iii) $R \times S$ is conically upper continuous at w_0;

iv) $L \circ R$ is conically upper continuous at w_0 if L is expanding at 0 on $R(\omega_0)$;

v) $R_1 + R_2$ is conically upper continuous at w_0 if $R_1(w_0)$ and $R_2(w_0)$ are closed, one of them is locally compact and $R_1(w_0) \cap -R_2(w_0) = \{0\}$.

Proof. Assertion i) is obvious, taking into account the following inequality

$$d(x, R_1(w_0) \cup R_2(w_0)) \leq \min\{d(x, R_1(w_0)), d(x, R_2(w_0))\}$$

for all $x \in X$.

Clearly condition c) of ii) entails that $R_1 \cap R_2$ is conically upper continuous. Taking $\alpha = \beta = \gamma/2c$ we see that condition a) of ii) implies condition c) of ii). Let us show that condition b) also implies condition c). Let us set $P := R_1(w_0)$, $Q := R_2(w_0)$. If the condition c) does not hold, we can find a positive ε, a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in C_{1/n}(P) \cap C_{1/n}(Q)$ with $\|x_n\| = 1$ such that $d(x_n, P \cap Q) > \varepsilon$ for each $n = 1, 2, ...$ Since P or Q is locally compact, we may assume that $\{x_n\}_{n=1}^{\infty}$ converges to some x with $\|x\| = 1$. It follows that $x \in P \cap Q$ because P and Q are closed. Thus $\{d(x_n, P \cap Q)\}_{n=1}^{\infty}$ tends to 0, a contradiction.

For assertion iii) it suffices to observe that for the supremum norm on $X \times Y$ one has $C_\varepsilon(R(w_0)) \times C_\varepsilon(S(w_0)) \subseteq C_\varepsilon(R(w_0) \times S(w_0))$,

Assertion iv) is obtained by Lemma 2.4 with $P = R(w_0)$.

The last assertion is derived from assertion iv) by considering the map $R = R_1 \times R_2$ from Ω to the product space $X \times X$ and the linear map L from $X \times X$ to X defined by $L(x, y) = x + y$, and by observing that the map R is conically upper continuous at w_0 according to assertion iii) and L is expanding at 0 on $R(\omega_0)$ by Lemma 2.2. ◻

It is worth noticing that in the preceding theorem condition b) does not imply condition a) even in a finite dimensional space. To see this, let us consider the cones (denoted by C_1 and C_2) generated by the sets $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = 1,\ x_2 \geq 0,\ 0 \leq x_3 \leq \sqrt{x_2}\}$ and $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = 1,\ x_2 \leq 0,\ 0 \leq x_3 \leq \sqrt{-x_2}\}$, respectively. Their intersection is the cone $\{t(1, 0, 0) \in \mathbb{R}^3 : t \geq 0\}$. It is evident that these cones are locally compact. Nevertheless condition a) does not hold. In fact, for each $n \geq 1$ one has $d((1, 0, 1/n), C_1 \cap C_2) = 1/n$, and $d((1, 0, 1/n), C_1)$

$= d((1,0,1/n), C_2) \leq 1/n^2$, which shows that there is no constant $c > 0$ as in condition a).

Theorem 2.7. *Let R, R_1, R_2 (resp. S) be cone-valued maps from Ω to X (resp. Y) and let L be a linear continuous map from X to Y. Assume that the above cone-valued maps are cosmically upper continuous at w_0. Then the following assertions are true.*

i) $R_1 \cup R_2$ *is cosmically upper continuous at* w_0*;*

ii) $R_1 \cap R_2$ *is cosmically upper continuous at* w_0*;*

iii) $R \times S$ *is cosmically upper continuous at* w_0 *provided* $R(w_0)$ *and* $S(w_0)$ *are locally compact;*

iv) $L \circ R$ *is cosmically upper continuous at* w_0 *if* $R(\omega_0)$ *is closed and* L *is expanding at* 0 *on* $R(w_0)$*;*

v) $R_1 + R_2$ *is cosmically upper continuous at* w_0 *if* $R_1(w_0)$ *and* $R_2(w_0)$ *are closed, locally compact and* $R_1(w_0) \cap -R_2(w_0) = \{0\}$.

Proof. The first three assertions can be reduced to classical upper continuity results (see Theorems 7.3.8, 7.3.10 and 7.3.14 of [21].)

Let us prove iv). For this purpose, let V be an open set containing $L(R(\omega_0)) \cap B_Y$. Set $U = L^{-1}(V)$ and $A = L^{-1}(B_Y)$. Since L is continuous, U is open and A is closed. Moreover, U contains the closed set $R(\omega_0) \cap A$. It follows from Lemma 2.5 that there is a neighborhood W of ω_0 such that $B := cl(R(W)) \cap A$ is bounded. Since the inclusion $R(\omega_0) \cap B \subseteq R(\omega_0) \cap A \subseteq U$ holds, in view of Proposition 2.1 (iii) there exists a neighborhood $W_0 \subseteq W$ of ω_0 such that $R(W_0) \cap B \subseteq U$. It follows that

$$L(R(W_0)) \cap B_Y \subseteq L(R(W_0) \cap A) \subseteq L(R(W_0) \cap B) \subseteq L(U) \subseteq V.$$

This establishes iv).

The last assertion is deduced from the preceding one by a technique similar to the one in the proof of Theorem 2.6. ☐

3. Convergence of asymptotic directions

Let A be a nonempty subset of the normed vector space X. We recall that the *asymptotic cone* (or *recession cone*) of A is the cone, denoted by $\mathrm{Rec}(A)$, consisting of all limits of sequences $t_n x_n$ where $x_n \in A$, and t_n are positive numbers converging to 0. In [26] the recession cone of A (in an arbitrary topological vector space) is defined as the cone $\bigcap_{V \in \mathcal{B}} \mathrm{c}\ell\, \mathrm{cone}\, (A \cap V)$ where $\mathcal{B}$ is the filter generated by open sets in X whose complements are bounded, and $c\ell\, \mathrm{cone}\, (A \cap V)$ stands for the closure of the cone generated by $A \cap V$. In the above definition we adopt the convention that $\mathrm{c}\ell\, \mathrm{cone}\, (A \cap V) = \{0\}$ if $A \cap V = \emptyset, A \neq \emptyset$. As noted in [26], [27] these definitions are equivalent in normed spaces, as we suppose it throughout. The relations between these cones in general spaces are analyzed in [47]. Although our feeling is that they coincide (by using nets instead of sequences in the definition of asymptotic cones), no precise argument has been known to prove or disprove it. The reader is referred to [10], [14, 15, 16], [26, 27, 28, 29], [37], [38], [42], [43], [46], [47] for more details on asymptotic cones of arbitrary sets.

In [42] the convergence of a family of sets in the cosmic topology has been investigated in a finite dimensional setting. In this section we shall further study the link between the convergence of a family of sets and the convergence of asymptotic directions.

Let M be a set-valued map from Ω to X. We define the map of asymptotic directions of M, denoted by R_M, as

$$R_M(w) = \text{Rec } (M(w)) \quad \text{for } w \in \Omega \ .$$

It is clear that $R_M(w)$ is a closed cone and is nonempty whenever $M(w) \neq \emptyset$. If $M(w)$ is bounded, $R_M(w) = \{0\}$. The converse is also true in finite dimensional spaces and some particular cases of infinite dimension (see [26]). Note that the map R_M is different from the asymptotic map studied in [15], [16], [27], [46], which is obtained by taking the asymptotic cone of the graph of M when Ω is a topological vector space.

Before studying the link between the continuity of R_M and that of M, let us say (see also [29], [37], [38]) that a subset $A \subseteq X$ is *asymptotically compact* if every sequence $\{x_n/||x_n||\}_{n=1}^{\infty}$ with $x_n \in A$ and $\{||x_n||\}_{n=1}^{\infty} \to \infty$, admits a convergent subsequence. It is said to be *boundedly compact* if $A \cap B$ is compact for every bounded closed set B in X.

Observe that some similar notions (asymptotic compactness in [15], condition (CB) in [26], condition [AB] in [47]) have been introduced in the literature, however all of them imply the above one. The following notions concerning set-valued maps are new.

Definition 3.1. The set-valued map M is said to be *recessively upper continuous* at w_0 if for every open set U containing $R_M(w_0) \cap B_X$, there is a neighborhood W of w_0 such that $\text{Rec}(M(W)) \cap B_X \subseteq U$.

The set-valued map M is said to be *recessively lower continuous* at w_0 if for every $v \in R_M(w_0)$ with $v \neq 0$, for every $\varepsilon > 0$, there exists a neighborhood W of w_0 such that $M(w) \cap C_\varepsilon(v)$ is unbounded for all $w \in W$.

The set-valued map M is said to be *asymptotically compact* at w_0 if for every sequence $\{(w_n, x_n)\}_{n=1}^{\infty}$ from the graph of M with $\{w_n\}_{n=1}^{\infty}$ converging to w_0 and $\{\|x_n\|\}_{n=1}^{\infty}$ unbounded, one can extract from the sequence $\{x_n/\|x_n\|\}_{n=1}^{\infty}$ a convergent subsequence.

It is evident that if M is asymptotically compact at w_0, then its value at this point is asymptotically compact. The converse is not true in general. On the other hand, if there is a neighborhood W of w_0 such that the set $M(W)$ is asymptotically compact, then M is asymptotically compact at this point. In particular a constant map is asymptotically compact if and only if its value is asymptotically compact.

Let us first give some immediate sufficient conditions for recessive continuities.

Proposition 3.2. *Each of the first four properties below entails the recessive upper continuity of M at $w_0 \in \Omega$, while each of the rest entails the recessive lower continuity of M at $w_0 \in \Omega$:*

1) M is locally bounded at w_0;

2) M is upper hemicontinuous (in particular, M is upper continuous) at w_0;

3) $Rec(M(w_0))$ is locally compact and for each positive ε, there is a bounded set $B_\varepsilon \subseteq X$ and a neighborhood W of w_0 such that $M(w) \subseteq C_\varepsilon(Rec\ (M(w_0))) \cup B_\varepsilon$ for all $w \in W$;

4) M is cone-valued and M is cosmically upper continuous at w_0;

5) M is bounded-valued at w_0;

6) M is cone-valued and lower continuous at w_0;

7) $M(w_0) \subseteq M(w)$ for every w sufficiently close to w_0.

Proof. The cases 1), 2), 5), 7) are obvious. In 3), let U be an open set containing $R_M(w_0) \cap B_X$. There is a positive ε such that $C_\varepsilon(R_M(w_0)) \cap B_X \subseteq U$. Let W be a neighborhood of w such that $M(W) \subseteq C_\varepsilon(R_M(w_0)) \cup B_\varepsilon$ with B_ε bounded. One has

$$\text{Rec}\ (M(W)) \subseteq C_\varepsilon(R_M(w_0))$$

and by this, Rec $(M(W)) \cap B_X \subseteq U$. To see 4) and 6) it suffices to note that if $M(w)$ is a cone, then $\text{Rec}\, M(w) = c\ell\, M(w)$. □

Let us recall [42] (see also [10] Lemma 3.6 and [38]) that the outer horizon limit of M at ω_0, denoted by $\limsup^\infty_{w\to w_0}\ M(w)$ is the set consisting of all limits of sequences $\{t_n x_n\}$ with $\{t_n\}_{n=1}^\infty \downarrow 0$, $x_n \in M(w_n)$ and $\{w_n\}_{n=1}^\infty \to w_0$.

Proposition 3.3. *Assume that X is finite dimensional. Then M is recessively upper continuous at ω_0 if and only if*

$$\limsup_{w\to w_0}{}^{\infty}\ M(w) \subseteq R_M(w_0)\ . \tag{1}$$

Proof. Let us denote by $\mathcal{W}$ the filter of neighborhoods of w_0. In view of the compactness of the unit ball, it is clear that M is recessively upper continuous at w_0 if and only if

$$\bigcap_{W\in\mathcal{W}} \text{Rec}\ (M(W)) \subseteq R_M(w_0)\ . \tag{2}$$

Moreover, on the one hand, it is evident that

$$\limsup_{w\to w_0}{}^{\infty}\ M(w) \subseteq \bigcap_{W\in\mathcal{W}} \text{Rec}\ (M(W))\ . \tag{3}$$

On the other hand, if $v \in \bigcap_{W\in\mathcal{W}} \text{Rec}(M(W))$, for each $W \in \mathcal{W}$ there are a sequence of positive numbers $\{t_n^W\}_{n=1}^\infty \downarrow 0$, $x_n^W \in M(w_n^W), w_n^W \in W$ such that $\lim_{n\to\infty}\ t_n^W x_n^W = v$. Using a diagonal process we choose $\{t_n^{W_n}\}_{n=1}^\infty \downarrow 0$, $\{w_n^{W_n}\}_{n=1}^\infty \to w_0$, $x_n^{W_n} \in M(w_n^{W_n})$ such that

$$\lim_{n\to\infty}\ t_n^{W_n}\ x_n^{W_n} = v\ .$$

This means that v belongs to the outer horizon limit of $M(w)$. Thus, equality in (3) holds. The equivalence between (1) and the recessive upper continuity is established. □

We notice that the condition that $\dim X < \infty$ is important for the equivalence of (1) and the recessive upper continuity. To see this, consider the following map from $[0,1]$ to ℓ^2. Let us denote by e_n the vector of ℓ^2 whose components are all zero except for the nth component equal to 1. Define

$$M(w) = \begin{cases} \{0\} & \text{if } w = 0, \\ \text{cone}\{e_n, e_{n+1}, \cdots\} & \text{if } w \in [\frac{1}{n}\ ,\ \frac{1}{n+1}), \quad n = 1, 2, \cdots\ . \end{cases}$$

It is easy to see that

$$\limsup_{w\to 0}{}^{\infty}\ M(w) = \{0\} \subseteq R_M(0)\ .$$

However the map is not recessively upper continuous at 0. Neither is the map R_M (which is identical to M) cosmically upper continuous there.

As a matter of fact, the equivalence of (1) and the recessive upper continuity can be guaranteed under a milder condition. Namely, suppose that M is asymptotically compact at w_0 (this is always true in a finite dimensional space). Then M is recessively upper continuous at w_0 if and only if $\limsup^{\infty}_{w\to w_0} \; M(w) \subseteq R_M(w_0)$. To see this, it suffices to observe that (2) and (1) are unconditionally equivalent, while (2) is a consequence of recessive upper continuity. Under the asymptotic compactness condition, (2) implies recessive upper continuity (using the same technique as in the finite dimensional case), hence the equivalence.

Theorem 3.4. *Assume that $M(w_0) \neq \emptyset$. Each of the following conditions is sufficient for R_M to be closed at w_0:*

i) M is upper hemicontinuous at w_0;

ii) M is closed, lower continuous at w_0 and convex-valued in a neighborhood of w_0;

iii) M is closed at w_0, convex-valued in a neighborhood W of w_0 and there is a compact set $K \subseteq X$ such that $M(w) \cap K \neq \emptyset$ for every $w \in W$.

Proof. Under the first condition, the conclusion is a direct consequence of Proposition 3.2. For ii) and iii), let $v_n \in R_M(w_n)$, $n = 1, 2, \cdots$, with $\lim_{n\to\infty} (w_n, v_n) = (w_0, v_0)$. Let x_0 be an arbitrary point in $M(w_0)$. To show $v_0 \in R_M(w_0)$, it suffices to prove that $x_0 + tv_0 \in M(w_0)$ for every positive t. Assuming the assumptions of (ii) hold, we can find $x_n \in M(w_n)$ converging to x_0 as n tends to ∞. Since $M(w_n)$ is convex, without loss of generality we may assume that $x_n + tv_n \in M(w_n)$. By the closedness of M at w_0, the limit $\lim_{n\to\infty} (x_n + tv_n) = x_0 + tv_0$ must lie in $M(w_0)$. Thus $v_0 \in R_M(w_0)$.

Under condition iii), take any point $x_n \in M(w_n) \cap K$. Since K is compact, one may assume that $\{x_n\}_{n=1}^{\infty}$ converges to some $x_0 \in K$. Actually $x_0 \in M(w_0)$ because M is closed at w_0. As before, it can be supposed that $x_n + tv_n \in M(w_n)$ for all $t \geq 0$. One has again $x_0 + tv_0 \in M(w_0)$, which shows that $v_0 \in R_M(w_0)$. □

The above proof also reveals that the lower continuity hypothesis in condition ii) and the existence of K in condition iii) can be replaced by the following weaker condition (called lower semicontinuity of M at (w_0, X) in [37]): for any sequence $\{w_n\}_{n=1}^{\infty}$ converging to w_0, there exists a convergent sequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $x_{n_k} \in M(w_{n_k})$, where $\{w_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{w_n\}_{n=1}^{\infty}$. This observation is also valid for Theorem 3.7 to come.

Theorem 3.5. *Assume $M(w_0) \neq \emptyset$. Then the following properties hold:*

i) R_M is upper continuous at w_0 if M is upper hemicontinuous at this point;

ii) R_M is lower continuous at w_0 if M is recessively lower continuous at this point, and asymptotically compact-valued on a neighborhood of w_0.

Proof. For the first assertion it suffices to observe that there exists a neighborhood W of w_0 such that $M(W) \subseteq M(w_0) + B_X$ if M is upper hemicontinuous at w_0. We have then $R_M(w) \subseteq \text{Rec}\,(M(W)) \subseteq R_M(w_0)$ for all $w \in W$.

To prove ii), let $v \in R_M(w_0)$. The case $v = 0$ is trivial, so we may assume $||v|| = 1$. Let U be any open set containing v. There is a positive $\varepsilon < \frac{1}{2}$ such that $B(v, \varepsilon)$ (the ball centered at v of radius ε) is contained in U. We show that there can be found a positive $\delta < \varepsilon$ such that the conic δ-neighborhood $C_\delta(v)$ of the ray $[v] := \{tv : t \geq 0\}$ is contained in $\text{cone}(B(v, \varepsilon))$. Indeed, if not, there is a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that

$$d(x_n, [v]) < \frac{1}{n}\, ||x_n|| \,, \quad \text{but} \quad x_n \notin \text{cone}(B(v, \varepsilon)) \,.$$

It is obvious that $||x_n|| \neq 0$. Moreover $d(x_n/||x_n||\ ,\ [v]) < 1/n$. Let $t_n \geq 0$ be such that $||x_n/||x_n|| - t_n v|| < 1/n$. Then $\lim_{n\to\infty} ||x_n/||x_n|| - t_n v|| = 0$. This implies $\lim_{n\to\infty} t_n = 1$ and $\lim_{n\to\infty} x_n/||x_n|| = v$. Hence $x_n/||x_n|| \in B(v,\varepsilon)$ whenever n is sufficiently large. For these n, $x_n \in \text{cone}(B(v,\varepsilon))$, a contradiction. By the same technique one can prove, by taking a smaller δ if necessary, that $u \in C_\delta(v)$ and $\|u\| = 1$ imply $u \in B(v,\epsilon)$. With such δ in hand, using recessive lower continuity we are able to find a neighborhood W of w_0 such that $M(w) \cap C_\delta(v)$ is unbounded for $w \in W$. Since $M(w)$ is recessively compact, the set $M(w) \cap C_\delta(v)$ admits an asymptotic direction $v_w, ||v_w|| = 1$ and $v_w \in C_\delta(v)$. Hence $v_w \in R_M(w) \cap B(v,\epsilon) \subseteq R_M(w) \cap U$, showing the lower continuity of R_M. □

We recall further [42] that the *inner horizon limit* of M at w_0 is the set, denoted by $\liminf^\infty_{w\to w_0} M(w)$, consisting of all limits $\lim_{w\to w_0} t_w x_w$, where $t_w > 0$, $\lim_{w\to w_0} t_w = 0$ and $x_w \in M(w)$.

Corollary 3.6. *Assume that X is finite dimensional and M is recessively lower continuous at w_0. Then*

$$\liminf_{w\to w_0}{}^{\infty}\ M(w) \supseteq R_M(w_0)\ .$$

Proof. Observe that in finite dimensional spaces every set-valued map is asymptotically compact-valued. Hence by Theorem 3.5, R_M is lower continuous at w_0 and one has

$$R_M(w_0) \subseteq \liminf_{w\to w_0}\ R_M(w)\ ,$$

where "lim inf" is understood in the Kuratowski-Painlevé sense. By Proposition 3.9 of [42] (or by a direct verification),

$$\liminf_{w\to w_0}\ R_M(w) \subseteq \liminf_{w\to w_0}{}^{\infty}\ M(w).$$

Hence the required inclusion is obtained. □

It should be noticed that the inclusion of Corollary 3.6 does not imply the lower continuity of R_M at w_0. A counterexample can be given as follows. Let M be the set-valued map from $[0,1]$ to $\mathbb{R}$, defined by

$$M(w) = \begin{cases} \{\frac{1}{w}\} & \text{if } 0 < w \leq 1, \\ \{t : t \geq 0\} & \text{if } w = 0. \end{cases}$$

Then $R_M(0) = \{t : t \geq 0\} = \liminf^\infty_{w\to w_0} M(w)$. However, $R_M(w) = \{0\}$ for $w \neq 0$ and R_M is not lower continuous at w_0.

Theorem 3.7. *Assume that $M(w_0) \neq \emptyset$. Each of the following conditions is sufficient for the map R_M to be cosmically upper continuous, hence conically upper continuous at w_0:*

i) M is recessively upper continuous at w_0;

ii) M is asymptotically compact at w_0 and R_M is closed at w_0; (in particular if either of conditions ii) and iii) of Theorem 3.4 holds).

Proof. Since $R_M(w) \subseteq$ Rec $(M(W))$ for every $w \in W$, the first condition implies at once cosmic upper continuity.

Assume that condition ii) holds. If R_M is not cosmically upper continuous at w_0, there can be found a sequence $\{w_n\}_{n=1}^\infty$ converging to w_0, an open set U containing $R_M(w_0) \cap B_X$ and $v_n \in R_M(w_n) \cap B_X$ such that $v_n \notin U$, $n = 1, 2, \cdots$. In particular $v_n \neq 0$, and one may assume that $\{||v_n||\}_{n=1}^\infty$ converges to some $r \in (0,1]$.

For each n, there is a sequence $\{x_n^k\}_{k=1}^\infty \subseteq M(w_n)$ with $\lim_{k\to\infty} ||x_n^k|| = \infty$ and $\lim_{k\to\infty} x_n^k/||x_n^k|| = v_n/||v_n||$. Choose $x_n^{k(n)}$ with the property that $\|x_n^{k(n)}\| \geq n$ and $\|x_n^{k(n)}/\|x_n^{k(n)}\| - v_n/\|v_n\|\| \leq 1/n$. By asymptotic compactness, one can extract a convergent subsequence $\{x_{n_i}^{k(n_i)}/\|x_{n_i}^{k(n_i)}\|\}_{i=1}^\infty$, say with limit $v_0 \neq 0$. Then $\{v_{n_i}/\|v_{n_i}\|\}_{i=1}^\infty$ also converges to v_0. Hence $\lim_{i\to\infty} v_{n_i} = rv_0$. This limit belongs to $R_M(w_0)$ because R_M is closed at w_0. We arrive at a contradiction with $v_n \in B_X \setminus U$ for all $n \geq 1$. □

Notice also that none of the conditions appearing in Theorem 3.7 is necessary for cosmic upper continuity. A counterexample is given as follows. Let $M : [0,1] \to \mathbb{R}$ be defined by

$$M(w) := \begin{cases} \{1/w\} & \text{if } 0 < w \leq 1, \\ \{0\} & \text{if } w = 0. \end{cases}$$

Then $R_M(w) = \{0\}$ for all $w \in [0,1]$ so that R_M is closed at ω_0. It is also upper continuous (hence cosmically upper continuous). Nevertheless,

$$\limsup_{w\to w_0}{}^{\infty} M(w) = \{t : t \geq 0\}$$

and the map M is not recessively upper continuous at 0.

The results of Proposition 3.3 and Theorem 3.7 reveal that actually relation (1) presents a condition stronger than the cosmic upper continuity of R_M, but weaker than the upper continuity of R_M.

We end this section introducing another type of recessive continuity. Let us say that M is *recessively conically upper continuous at* w_0 if for every $\epsilon > 0$ there exists a neighborhood W of w_0 such that $\mathrm{Rec}(M(W)) \subseteq C_\epsilon(R_M(w_0))$. Observe that recessive upper continuity implies recessive conic upper continuity. The converse implication is also valid provided $R_M(w_0)$ is locally compact. In particular in finite dimensional spaces there is no distinction between recessive upper continuity and recessive conic upper continuity. Observe also that if M is recessively conically upper continuous at w_0, then the map R_M is conically upper continuous at that point. The example given after Theorem 3.7 shows that the converse does not hold, that is a map M with R_M conically upper continuous is not necessarily recessively conically upper continuous.

4. Operations with set-valued maps

As we have seen in the preceding section, the concepts of cosmic upper continuity, conic upper continuity and recessive upper continuity are useful in the study of asymptotic directions of set-valued maps. We shall further investigate them in relation with the most important operations on set-valued maps, such as union, intersection, sum and composition with linear maps.

Let us say that a subset A of X is *asymptotable* if for each $v \in \mathrm{Rec}(A)$ and each sequence of positive numbers $\{t_n\}_{n=1}^\infty$ converging to ∞ there exists a sequence $\{v_n\}_{n=1}^\infty$ converging to v such that $t_n v_n \in A$ for all n. It is clear that a set $A \subseteq X$ satisfying the condition (CD) of [26] (namely for each $v \in \mathrm{Rec}(A)$ there exists a bounded set B such that $(tv + B) \cap A \neq \emptyset$ for every $t \geq 0$) is asymptotable. Since A is asymptotable if and only if

$$\liminf_{t\downarrow 0} tA = \limsup_{t\downarrow 0} tA$$

the above condition can be considered as a tangentability condition at infinity. This interpretation can be performed in a precise way using the stereographic mapping method of [39] (see also [42]).

Lemma 4.1. *Let A, A_1 and A_2 be nonempty subsets of X and let B be a nonempty subset of another normed space Y. Then the following properties hold:*

i) $\mathrm{Rec}(A_1 \cup A_2) = \mathrm{Rec}(A_1) \cup \mathrm{Rec}(A_2)$*;*

ii) $\mathrm{Rec}(A_1 \cap A_2) \subseteq \mathrm{Rec}(A_1) \cap \mathrm{Rec}(A_2)$. *Equality holds provided A_1 and A_2 are closed convex with $A_1 \cap A_2 \neq \emptyset$;*

iii) $\mathrm{Rec}(A_1 + A_2) \subseteq \mathrm{Rec}(A_1) + \mathrm{Rec}(A_2)$ *provided one of the sets A_1 and A_2 is asymptotically compact, and* $\mathrm{Rec}(A_1) \cap -\mathrm{Rec}(A_2) = \{0\}$*;*

iv) $\mathrm{Rec}(A_1) + \mathrm{Rec}(A_2) \subseteq \mathrm{Rec}(A_1 + A_2)$ *provided at least one of the sets A_1 and A_2 is asymptotable;*

v) $\mathrm{Rec}(A) \times \mathrm{Rec}(B) \supseteq \mathrm{Rec}(A \times B)$. *Equality holds provided at least one of the sets A and B is asymptotable.*

Proof. This is essentially the result of Theorem 2.5 and Theorem 2.12 of [26] (Chapter 1) (see also [43]), except for the last statement which is evident. A stronger condition was used for iii) in the above mentioned theorems, namely it was required that there is a bounded set $D \subseteq X$ such that the cone generated by $A_1 \backslash D$ can be generated by a compact set (not containing zero). However, as the proof given there reveals, the asymptotic compactness is sufficient to ensure the result. □

The following result is an improvement of Theorem 2.5 of [42] in infinite dimensions.

Lemma 4.2. *Let A be a nonempty subset of X and let L be a continuous linear map from X to Y. Then one has*

$$L(\mathrm{Rec}(A)) \subseteq \mathrm{Rec}(L(A)) .$$

Equality holds under each of the following conditions:

a) L is open on its image and $L^{-1}(L(A)) = A$;

b) L is open on its image with a finite dimensional kernel and $KerL \cap \mathrm{Rec}(A) = \{0\}$*;*

c) A is asymptotically compact and $KerL \cap \mathrm{Rec}(A) = \{0\}$*.*

Proof. Let $v \in L(\mathrm{Rec}(A))$. There exist $u \in \mathrm{Rec}(A)$ with $L(u) = v$, a sequence $\{x_n\}_{n=1}^{\infty} \subseteq A$ and a sequence of positive numbers $\{t_n\}_{n=1}^{\infty}$ converging to 0 such that $\lim_{n\to\infty} t_n x_n = u$. By the continuity of L, one has $v = \lim_{n\to\infty} L(t_n x_n) \in \mathrm{Rec}(L(A))$.

Under condition a), let $v \in \mathrm{Rec}(L(A))$, that is $v = \lim_{n\to\infty} t_n y_n$ for $y_n \in L(A)$ and $t_n > 0$ with $\lim_{n\to\infty} t_n = 0$. Since L is open, given $u \in L^{-1}(v)$ we can find a sequence $\{u_n\}_{n=1}^{\infty}$ in X with $\lim_{n\to\infty} u_n = u$ and $L(u_n) = t_n y_n$ for all $n = 1, 2, \ldots$. Setting $x_n = t_n^{-1} u_n$, we have $x_n \in L^{-1}(L(A)) = A$ so that $u \in \mathrm{Rec}(A)$. Consequently $v \in L(\mathrm{Rec}(A))$.

Suppose now that condition b) holds. Let us decompose $X = X_0 \oplus \mathrm{Ker}L$ as in the proof of Lemma 2.3 so that the restriction L_0 of L to X_0 is an isomorphism. Let $v \in \mathrm{Rec}(L(A))$, that is $v = \lim_{n\to\infty} t_n y_n$ for $y_n \in L(A)$ and $t_n > 0$ with $\lim_{n\to\infty} t_n = 0$. Let $x_n = a_n + b_n$ with $a_n \in X_0, b_n \in \mathrm{Ker}L$ be such that $y_n = L(x_n)$. Consider the sequence $\{t_n b_n\}_{n=1}^{\infty}$. We claim that it is bounded. In fact, if not, by taking a subsequence if necessary, we may assume that $\lim_{n\to\infty} t_n \|b_n\| = \infty$ and that $\{b_n/\|b_n\|\}_{n=1}^{\infty}$ converges to some $u_1 \in \mathrm{Ker}L$. Then $\{(a_n + b_n)/\|b_n\|\}_{n=1}^{\infty}$ converges to u_1 because L_0 is an isomorphism onto its image and the sequence $\{t_n a_n\}_{n=1}^{\infty}$ converges to $u_2 := L_0^{-1}(v)$. We arrive at the contradiction that $u_1 \in \mathrm{Rec}(A) \cap \mathrm{Ker}L$. We have shown that $\{t_n b_n\}_{n=1}^{\infty}$ is bounded. Since Ker$L$ is finite dimensional, one may assume that it has a limit $u_1 \in \mathrm{Ker}L$. Moreover, as before, the sequence $\{t_n a_n\}_{n=1}^{\infty}$ converges to $u_2 := L_0^{-1}(v)$. Thus $u_1 + u_2 = \lim_{n\to\infty} t_n x_n \in \mathrm{Rec}(A)$ and $v = L(u_1 + u_2) \in L(\mathrm{Rec}(A))$.

Finally, assume that c) holds. As in the preceding case, let $v \in \mathrm{Rec}(L(A))$, that is $v = \lim_{n\to\infty} t_n y_n$ for $y_n \in L(A)$ and $t_n > 0$ with $\lim_{n\to\infty} t_n = 0$. Let $x_n \in A$ be such that $y_n = L(x_n)$. If $\{||x_n||\}_{n=1}^{\infty}$ is bounded, $\lim_{n\to\infty} t_n x_n = 0$. Consequently $v = \lim_{n\to\infty} t_n L(x_n) = \lim_{n\to\infty} L(t_n x_n) = 0 \in L(\mathrm{Rec}(A))$. If $\{||x_n||\}_{n=1}^{\infty}$ is unbounded, by asymptotic compactness, one may assume that $\{x_n/||x_n||\}_{n=1}^{\infty}$ converges to some $u \in \mathrm{Rec}(A)$. The sequence $\{t_n||x_n||\}_{n=1}^{\infty}$ is bounded, otherwise one should have

$$L(u) = \lim_{n\to\infty} \frac{v}{t_n||x_n||} = 0 \quad \text{with} \quad ||u|| = 1 ,$$

contradicting the condition $\mathrm{Ker}L \cap \mathrm{Rec}(A) = \{0\}$. Therefore we may assume that $\{t_n||x_n||\}_{n=1}^{\infty}$ converges to some $\alpha \geq 0$. By this,

$$v = \lim_{n\to\infty} L(t_n||x_n||\frac{x_n}{||x_n||}) = \alpha L(u) \in L(\mathrm{Rec}(A))$$

and the inclusion becomes an equality. □

We note that the inclusion stated in Lemma 4.2 may be strict. For instance, let L be the projection $(x, y) \longmapsto (x, 0)$ in $\mathbb{R}^2$ and $A = \{(x, x^2) : x \geq 0\}$. Then we have $\mathrm{Rec}(A) = \{(0, x) : x \geq 0\}$ and $L(\mathrm{Rec}(A)) = \{0\}$, while $\mathrm{Rec}(L(A)) = \{(x, 0) : x \in \mathbb{R}\}$. Furthermore, in condition c), the asymptotic compactness assumption cannot be dropped. A counterexample can be given as follows. Let $A = \bigcup_{n=1}^{\infty} \{n^2 e_n\} \subseteq \ell^2$, where $e_1, e_2, ...$ represents the canonical Hilbert basis in ℓ^2 as in Section 3. Then A is not asymptotically compact and $\mathrm{Rec}(A) = \{0\}$. Consider the continuous linear map L from ℓ^2 to ℓ^2 defined by

$$L(e_n) = \frac{1}{n} e_1 \quad \text{for all} \quad n \geq 1 .$$

It is clear that L is continuous and $\mathrm{Ker}L \cap \mathrm{Rec}(A) = \{0\}$. Nevertheless $\mathrm{Rec}(L(A)) = \mathrm{Rec}(\bigcup_{n=1}^{\infty} \{ne_1\}) \neq \{0\}$ and the inclusion is strict.

For later use, let us derive two more general facts about cones in normed spaces.

Lemma 4.3. *Let K_1 and K_2 be two closed cones in X. Then for every open set U containing $K_1 \cap K_2 \cap B_X$ there can be found two open sets U_1, U_2 in X such that $K_1 \cap B_X \subseteq U_1, K_2 \cap B_X \subseteq U_2$ and $U_1 \cap U_2 \subseteq U$.*

Proof. Consider the two closed sets $K_1 \cap B_X \setminus U$ and $K_2 \cap B_X \setminus U$. They do not intersect. Therefore there exist two disjoint open sets V_1 and V_2 that contain $K_1 \cap B_X \setminus U$ and $K_2 \cap B_X \setminus U$ respectively. Take $U_1 := V_1 \cup U$ and $U_2 := V_2 \cup U$ to obtain the required sets. □

Lemma 4.4. *Let K_X and K_Y be locally compact cones in X and Y respectively. Then for every open set U containing $(K_X \times K_Y) \cap B_{X\times Y}$ there can be found $\varepsilon > 0$ such that $(K_X \cap B_X + \varepsilon B_X) \times (K_Y \cap B_Y + \varepsilon B_Y) \subseteq U$.*

Proof. Remember first that the product space $X \times Y$ is endowed with the max norm, hence $B_{X\times Y} = B_X \times B_Y$. Furthermore for every $(x, y) \in (K_X \times K_Y) \cap$

$B_{X\times Y}$ there exists $\varepsilon_{(x,y)} > 0$ such that $(x,y) + \varepsilon_{(x,y)} B_X \times B_Y \subseteq U$. Using the compactness of the set $(K_X \times K_Y) \cap B_{X\times Y}$ we get an $\varepsilon > 0$ common for every $(x,y) \in (K_X \times K_Y) \cap B_{X\times Y}$. This ε satisfies our requirements. □

In the remaining part of this section, M_1, M_2 and M are set-valued maps from Ω to X; N is a set-valued map from Ω to Y; and L is a continuous linear map from X to Y. The two following theorems deal with the cosmic continuity and the conic continuity of asymptotic directions of set-valued maps.

Theorem 4.5. *Assume that R_{M_i}, $i = 1, 2$, R_M and R_N are cosmically upper continuous at w_0. Then the following assertions are true:*

i) $R_{M_1\cup M_2}$ is cosmically upper continuous at w_0;

ii) $R_{M_1\cap M_2}$ is cosmically upper continuous at w_0 provided $R_{M_1\cap M_2}(w_0) = R_{M_1}(w_0) \cap R_{M_2}(w_0)$;

iii) $R_{M\times N}$ is cosmically upper continuous at w_0 provided the following conditions hold:

a) $R_{M\times N}(w_0) = R_M(w_0) \times R_N(w_0)$;

b) $R_M(w_0)$ and $R_N(w_0)$ are locally compact;

iv) $R_{L\circ M}$ is cosmically upper continuous at w_0 provided one of the following conditions holds:

c) L is open with a finite dimensional kernel and Ker $L \cap R_M(w_0) = \{0\}$;

d) $M(w)$ is asymptotically compact for all w sufficiently close to w_0 and Ker $L \cap R_M(w_0) = \{0\}$;

v) $R_{M_1+M_2}$ is cosmically upper continuous at w_0 provided the following conditions hold:

e) $R_{M_1}(w_0) \cap -R_{M_2}(w_0) = \{0\}$;

f) $M_1(w)$ and $M_2(w)$ are asymptotically compact for all w sufficiently close to w_0;

g) $R_{M_1}(w_0) \times R_{M_2}(w_0) \subseteq R_{M_1\times M_2}(w_0)$.

Proof. We apply Theorem 2.7 and Lemmas 4.1, 4.2 to achieve our proof. The first three assertions are immediate.

To prove assertion iv) let us notice that for w sufficiently close to w_0 one has $\text{Ker}L \cap R_M(w) = \{0\}$. This follows from Proposition 2.1 and from the fact that being cosmically upper continuous, R_M is also conically upper continuous. By conditions c) and d) and Lemma 4.2 one has $L \circ R_M(w) = R_{L\circ M}(w)$ for $w \in W$. Therefore it suffices to apply Theorem 2.7 to obtain that $R_{L\circ M}$ is cosmically upper continuous at w_0.

The last assertion is obtained from the preceding one by using the map $M := M_1 \times M_2$, the linear map L from $X \times X$ to X defined by $L(x,y) = x + y$, and by observing that condition f) implies condition d) for the product map, while condition g) ensures that $R_{M_1\times M_2}$ is cosmically upper continuous in view of assertion iii). □

Theorem 4.6. *Assume that R_{M_i}, $i = 1, 2$, R_M and R_N are conically upper continuous at w_0. Then the following assertions are true:*

i) $R_{M_1\cup M_2}$ is conically upper continuous at w_0;

ii) $R_{M_1\cap M_2}$ is conically upper continuous at w_0 provided the following conditions hold:

a) $R_{M_1\cap M_2}(w_0) = R_{M_1}(w_0) \cap R_{M_2}(w_0)$;

b) *Either there is* $c > 0$ *such that*

$$d(x, R_{M_1}(w_0) \cap R_{M_2}(w_0)) \leq c(d(x, R_{M_1}(w_0)) + d(x, R_{M_2}(w_0)))$$

for every $x \in X$, *or at least one of the sets* $R_{M_1}(w_0)$ *and* $R_{M_2}(w_0)$ *is locally compact;*

iii) $R_{M\times N}$ *is conically upper continuous at* w_0 *provided* $R_{M\times N}(w_0) = R_M(w_0) \times R_N(w_0)$*;*

iv) $R_{L\circ M}$ *is conically upper continuous at* w_0 *provided one of the following conditions holds:*

c) *L is open with a finite dimensional kernel and* $KerL \cap R_M(w_0) = \{0\}$*;*

d) $M(w)$ *is asymptotically compact for all* w *sufficiently close to* w_0 *and* $KerL \cap R_M(w_0) = \{0\}$*;*

v) $R_{M_1+M_2}$ *is conically upper continuous at* w_0 *provided the following conditions hold:*

e) $R_{M_1}(w_0) \cap -R_{M_2}(w_0) = \{0\}$*;*

f) $M_1(w)$ *and* $M_2(w)$ *are asymptotically compact for all* w *sufficiently close to* w_0;

g) $R_{M_1}(w_0) \times R_{M_2}(w_0) \subseteq R_{M_1\times M_2}(w_0)$.

Proof. The technique of the preceding theorem goes through by using Theorem 2.6 instead of Theorem 2.7. ☐

Now we turn our study to the recessive upper continuity of set-valued maps using asymptotic directions.

Theorem 4.7. *Assume that* M_i $(i = 1, 2)$, M *and* N *are recessively upper continuous at* w_0. *Then the following assertions hold.*

i) $M_1 \cup M_2$ *is recessively upper continuous at* w_0*;*

ii) $M_1 \cap M_2$ *is recessively upper continuous at* w_0 *provided* $R_{M_1\cap M_2}(w_0) = R_{M_1}(w_0) \cap R_{M_2}(w_0)$*;*

iii) $M \times N$ *is recessively upper continuous at* w_0 *provided the following conditions hold:*

a) $R_{M\times N}(w_0) = R_M(w_0) \times R_N(w_0)$*;*

b) $R_M(w_0)$ *and* $R_N(w_0)$ *are locally compact;*

iv) $L \circ M$ *is recessively upper continuous at* w_0 *provided one of the following conditions holds:*

c) *L is open with a finite dimensional kernel and* $KerL \cap R_M(w_0) = \{0\}$*;*

d) M *is asymptotically compact at* w_0 *and* $KerL \cap R_M(w_0) = \{0\}$*;*

v) $M_1 + M_2$ *is recessively upper continuous at* w_0 *provided the following conditions hold:*

e) $R_{M_1}(w_0) \cap -R_{M_2}(w_0) = \{0\}$*;*

f) M_1 *and* M_2 *are asymptotically compact at* w_0;

g) $R_{M_1}(w_0) \times R_{M_2}(w_0) \subseteq R_{M_1\times M_2}(w_0)$.

Proof. Assertion i) is a direct consequence of the definitions and of Lemma 4.1.

For assertion ii) let U be an open set containing the set $R_{M_1\cap M_2}(w_0) \cap B_X$, which is identical to $R_{M_1}(w_0) \cap R_{M_2}(w_0) \cap B_X$. By Lemma 4.3, one can find two open sets U_1, U_2 in X such that

$$R_{M_i}(w_0) \cap B_X \subseteq U_i, \qquad i = 1, 2,$$
$$U_1 \cap U_2 \subseteq U.$$

It follows from recessive upper continuity that there exists a neighborhood W of w_0 such that $\mathrm{Rec}(M_i(W)) \cap B_X \subseteq U_i, i = 1, 2$. These inclusions and Lemma 4.1 imply

$$\begin{aligned}\mathrm{Rec}\,((M_1 \cap M_2)(W)) \cap B_X &\subseteq \mathrm{Rec}\,(M_1(W) \cap M_2(W)) \cap B_X \\ &\subseteq \mathrm{Rec}\,(M_1(W)) \cap \mathrm{Rec}\,(M_2(W)) \cap B_X \\ &\subseteq U_1 \cap U_2 \subseteq U,\end{aligned}$$

which establishes assertion ii).

As to assertion iii), let U be an open set containing the set $R_{M\times N}(w_0) \cap B_{X\times Y}$. In view of conditions a), b) and of Lemma 4.4 there exist two open sets $U_1 \subseteq X$, $U_2 \subseteq Y$ containing $R_M(w_0) \cap B_X$ and $R_N(w_0) \cap B_Y$ respectively such that $U_1 \times U_2 \subseteq U$. By recessive upper continuity there exists a neighborhood W of w_0 such that $\mathrm{Rec}(M(W)) \cap B_X \subseteq U_1$ and $\mathrm{Rec}(N(W)) \cap B_Y \subseteq U_2$. These inclusions and Lemma 4.1 imply

$$\begin{aligned}\mathrm{Rec}\,((M \times N)(W)) \cap B_{X\times Y} &\subseteq \mathrm{Rec}\,(M(W) \times N(W)) \cap B_{X\times Y} \\ &\subseteq \mathrm{Rec}\,(M(W)) \times \mathrm{Rec}\,(N(W)) \cap B_{X\times Y} \\ &\subseteq U_1 \times U_2 \subseteq U,\end{aligned}$$

which proves assertion iii).

Now we proceed to assertion iv). Suppose to the contrary that the composition is not recessively upper continuous at w_0, i.e. there are an open set $V \subseteq Y$ containing $R_{L\circ M}(w_0) \cap B_Y$, a decreasing basis of neighborhoods $\{W_n\}_{n=1}^{\infty}$ of w_0, vectors $v_n \in B_Y \cap \mathrm{Rec}\,(L \circ M(W_n)) \backslash V$, $n = 1, 2, \cdots$.

By the definition of asymptotic directions, for every $n \geq 1$, there exist $w_n^k \in W_n$, $x_n^k \in M(w_n^k)$, $y_n^k = L(x_n^k)$ and $t_n^k > 0$ such that $\lim_{k\to\infty} t_n^k = 0$, and $v_n = \lim_{k\to\infty} t_n^k y_n^k$ with $\lim_{k\to\infty} \|y_n^k\| = \infty$. Since L is continuous, one also has $\lim_{k\to\infty} \|x_n^k\| = \infty$.

Assume that condition d) holds. For each $n \geq 1$ choose an integer $k(n) \geq n$ with the property that $||v_n - t_n^{k(n)} y_n^{k(n)}|| \leq 1/n$ and $||x_n^{k(n)}|| \geq n$. Observe that $\lim_{k\to\infty} w_n^{k(n)} = w_0$. By the asymptotic compactness of M at w_0 one may assume that $\{x_n^{k(n)}/\|x_n^{k(n)}\|\}_{k=1}^{\infty}$ converges to some $u \neq 0$ which belongs to $R_M(w_0)$ because M is recessively upper continuous at w_0. Consider the sequence $\{t_n^{k(n)}\|x_n^{k(n)}\|\}_{k=1}^{\infty}$. We may assume that it converges to some $r \in [0, \infty]$. Note that $r \neq 0$, otherwise one would have $\lim_{n\to\infty} L(t_n^{k(n)} x_n^{k(n)}) = 0$, hence $\lim_{n\to\infty} v_n = 0$ which contradicts the fact that $v_n \notin V$. Note further that $r \neq \infty$ because in the contrary case one would have $0 = \lim_{n\to\infty} v_n/(t_n^{k(n)}\|x_n^{k(n)}\|) = \lim_{n\to\infty} L(x_n^{k(n)}/\|x_n^{k(n)}\|) = L(u)$ with $u \neq 0$, a contradiction with $\mathrm{Ker}L \cap R_M(w_0) = \{0\}$. Thus

$$\begin{aligned}\lim_{n\to\infty} v_n &= \lim_{n\to\infty} L(t_n^{k(n)} x_n^{k(n)}) = L(ru) \\ &\in L(R_M(w_0)) \cap B_Y \\ &\subseteq R_{L\circ M}(w_0) \cap B_Y \\ &\subseteq V,\end{aligned}$$

a contradiction with $v_n \notin V$.

Now assume condition c) hold. Decomposing $X = X_0 \oplus \mathrm{Ker}\, L$ as in the proof of Lemma 4.2 we can express $x_n^k = a_n^k + b_n^k$ with $a_n^k \in X_0, b_n^k \in \mathrm{Ker}L$. Since the restriction L_0 of L to X_0 is an isomorphism, one has $\lim_{k\to\infty} t_n^k a_n^k = \lim_{k\to\infty} L_0^{-1}(t_n^k y_n^k)$ $= L_0^{-1}(v_n)$. Furthermore, since $\mathrm{Ker}L$ is finite dimensional, as in the proof of Lemma

4.2 one may assume that $\lim_{k\to\infty} t_n^k b_n^k$ exists. Let $u_n = \lim t_n^k x_n^k \in \mathrm{Rec}(M(W_n))$ with $L(u_n) = v_n$. A similar technique shows that $\{\|u_n\|\}_{n=1}^{\infty}$ is bounded, say majorized by a positive $\beta > 1/\|L\|$. Let V_0 be an open subset in V which contains $R_{L\circ M}(w_0)\cap B_Y$ and has the property that $y \in V$ whenever $y \in B_Y$ and $y/\beta\|L\| \in V_0$ (take for instance $V_0 = t^{-1}V \cup (B_Y \setminus t^{-1}B_Y)$ with $t = \beta\|L\|$). We note that $R_{L\circ M}(w_0) = L(R_M(w_0))$ by Lemma 4.2 and by the asymptotic compactness of $M(w_0)$. Then in view of the recessive upper continuity of M, there is $n_0 > 0$ such that

$$\mathrm{Rec}(M(W_n)) \cap \frac{1}{\|L\|} B_X \subseteq L^{-1}(V_0)$$

for all $n \geq n_0$. In particular for such n one has $u_n/\beta\|L\| \in L^{-1}(V_0)$. Hence $v_n/\beta\|L\| \in V_0$ and $v_n \in V$ for all $n \geq n_0$, a contradiction. Thus assertion iv) is established.

The last assertion is deduced from the previous ones by a standard argument. □

Observe that in condition d) of this theorem it is not sufficient to require $M(w_0)$ to be asymptotically compact. To see this, let us consider the set-valued map M from $[0,1]$ to ℓ^2 defined by $M(0) = \{0\}$ and $M(w) = \bigcup_{n=1}^{\infty} \{n^2 e_n\} \subseteq \ell^2$, if $w \neq 0$. Let L be the linear map defined by $L(e_n) = e_1/n$ as in the counterexample given after Lemma 4.2. Then M is recessively upper continuous at 0. Despite this, the composition $L \circ M$ is not recessively upper continuous at 0. Note that M is not asymptotically compact at this point.

Up to this moment we have extensively exploited the standard approach of deriving a continuity criterion for a sum of two set-valued maps from a continuity criterion for a composition of a set-valued map with a linear map. Below we shall use a direct approach to establish a sufficient condition for recessive (resp. cosmic, resp. conic) upper continuity of a sum, which is not a consequence of the counterpart for a composition. The next two lemmas have some links with Lemma 2.2.

Lemma 4.8. *Assume that M_1 and M_2 are recessively upper continuous at w_0 and that $R_{M_1}(w_0) \cap -R_{M_2}(w_0) = \{0\}$. Then there exists a neighborhood W_0 of w_0 such that $R_{M_1}(W_0) \cap -R_{M_2}(W_0) = \{0\}$.*

Proof. Let $S_i := R_{M_i}(w_0)\cap S_X$, $i = 1, 2$. These sets are closed disjoint subsets of S_X. Since S_X is normal, there are disjoint open subsets U_1, U_2 of S_X which contain S_1 and S_2 respectively. Then the sets $(0,\infty)U_1$ and $(0,\infty)U_2$ are disjoint open subsets in X and their unions with $intB_X$ contain $R_{M_1}(w_0) \cap B_X$ and $R_{M_2}(w_0) \cap B_X$ respectively. By recessive upper continuity there is a neighborhood W_0 of w_0 such that $\mathrm{Rec}\, M_1(W_0) \cap B_X \subseteq intB_X \cup (0,\infty)U_1$ and $\mathrm{Rec}\, M_2(W_0) \cap B_X \subseteq intB_X \cup (0,\infty)U_2$. Hence $\mathrm{Rec}\, M_1(W_0) \subseteq \{0\} \cup (0,\infty)U_1$ and $\mathrm{Rec}\, M_2(W_0) \subseteq \{0\} \cup (0,\infty)U_2$ by homogeneity. □

Lemma 4.9. *Assume that $R_{M_1}(w_0) \cap -R_{M_2}(w_0) = \{0\}$ and one of the following conditions holds:*

a) *R_{M_1} and R_{M_2} are cosmically upper continuous at w_0;*

b) *R_{M_1} and R_{M_2} are conically upper continuous at w_0 with $R_{M_1}(w_0)$ or $R_{M_2}(w_0)$ locally compact.*

Then there exists a neighborhood W_0 of w_0 such that $R_{M_1}(w) \cap -R_{M_2}(w) = \{0\}$ for all $w \in W_0$.

Proof. The technique of the preceding lemma applies. ☐

Theorem 4.10. *Assume that M_1 and M_2 are recessively upper continuous at w_0. Then the following conditions are sufficient for the sum $M_1 + M_2$ to be recessively upper continuous at w_0 :*

a) $R_{M_1}(w_0) \cap -R_{M_2}(w_0) = \{0\}$;

b) $R_{M_1}(w_0) + R_{M_2}(w_0) \subseteq R_{M_1+M_2}(w_0)$;

c) There is a neighborhood W of w_0 such that $M_1(W)$ and $M_2(w_0)$ are asymptotically compact.

Proof. Suppose to the contrary that the sum is not recessively upper continuous at w_0, i.e. there are an open set U containing $R_{M_1+M_2}(w_0) \cap B_X$, a basis of neighborhoods $\{W_n\}_{n=1}^\infty \subseteq W \cap W_0$ of w_0, (where W_0 is determined by Lemma 4.8), and vectors $v_n \in B_X \cap \mathrm{Rec}([M_1 + M_2](W_n))\backslash U$, $n = 1, 2, \cdots$.

By the definition of asymptotic directions, for each $n \geq 1$, there exist $w_n^k \in W_n$, $y_n^k \in M_1(w_n^k)$, $z_n^k \in M_2(w_n^k)$, $x_n^k = y_n^k + z_n^k$ and $t_n^k > 0$ such that $\lim_{k\to\infty} t_n^k = 0$, $\lim_{k\to\infty} \|x_n^k\| = \infty$ and $v_n = \lim_{k\to\infty} t_n^k x_n^k$.

We claim that for n sufficiently large one has

$$\lim_{k\to\infty} \|y_n^k\| = \infty \quad \text{and} \quad \lim_{k\to\infty} \|z_n^k\| = \infty.$$

Indeed, if not, say for $i = 1, 2, ..., \{\|z_{n_i}^k\|\}_{k=1}^\infty$ are bounded, then one should have $v_{n_i} \in \mathrm{Rec}(M_1(W_{n_i})) \subseteq \mathrm{Rec}([M_1 + M_2](W_{n_i}))$ and $v_{n_i} \notin U$ for all $i \geq 1$, which contradicts the recessive upper continuity of M_1. We claim further that for such n the sequence $\{t_n^k y_n^k\}_{k=1}^\infty$ possesses a subsequence (say with the same notation) converging to some $a_n \in \mathrm{Rec}(M_1(W_n))$. Indeed, by the asymptotic compactness of $M_1(W_n)$ (because $W_n \subseteq W$) one may assume that the sequence $\{y_n^k/\|y_n^k\|\}_{k=1}^\infty$ converges to some $u_n \in \mathrm{Rec}(M_1(W_n))$. Consider the sequence $\{t_n^k\|y_n^k\|\}_{k=1}^\infty$ which is assumed to converge to some $r \in [0, \infty]$. Observe that $r \neq \infty$, otherwise one should obtain $0 = \lim_{k\to\infty} v_n/(t_n^k\|y_n^k\|) = \lim_{k\to\infty} (y_n^k/\|y_n^k\| + z_n^k/\|y_n^k\|)$, which leads to a contradiction $u_n \in R_{M_1}(W_0) \cap -R_{M_2}(W_0), u_n \neq 0$ (Lemma 4.8). By this $a_n = \lim_{k\to\infty} t_n^k y_n^k = r u_n$.

Consequently $\lim_{k\to\infty} t_n^k z_n^k = v_n - a_n$, denoted by b_n. A similar argument shows that the sequences $\{\|a_n\|\}_{n=1}^\infty$ and $\{\|b_n\|\}_{n=1}^\infty$ are bounded, say majorized by $\beta > 0$. Using a diagonal process and the asymptotic compactness of $M_1(W)$ one may assume that $\{a_n\}_{n=1}^\infty$ converges to some $a \in R_{M_1}(w_0)$. Since $M_2(w_0)$ is asymptotically compact, the set $R_{M_2}(w_0) \cap \beta B_X$ is compact. This and the recessive upper continuity of M_2 imply the convergence of $\{b_n\}_{n=1}^\infty$ (or eventually a subsequence of $\{b_n\}_{n=1}^\infty$), say $\lim_{n\to\infty} b_n = b \in R_{M_2}(w_0)$. Thus, $\lim_{n\to\infty} v_n = a + b$. In view of condition c), this limit lies in $R_{M_1+M_2}(w_0) \cap B_X$, hence in U. We arrive at a contradiction with $v_n \notin U$ and establish the recessive upper continuity of the sum. ☐

Theorem 4.11. *Assume that M_1 and M_2 are cosmically (resp. conically) upper continuous at w_0. Then the following conditions are sufficient for the sum $M_1 + M_2$ to be cosmically (resp. conically) upper continuous at w_0 :*

a) $R_{M_1}(w_0) \cap -R_{M_2}(w_0) = \{0\}$;

b) $R_{M_1}(w_0) + R_{M_2}(w_0) \subseteq R_{M_1+M_2}(w_0)$;

c) There is a neighborhood W of w_0 such that $M_1(w), w \in W$, and $M_2(w_0)$ are asymptotically compact.

Proof. Apply the technique of Theorem 4.10, using Lemma 4.9 instead of Lemma 4.8. □

Note that in condition c) of Theorem 4.10 the set $M_2(W)$ is not required to be asymptotically compact nor is M_2 assumed to be asymptotically compact at w_0. If both of the sets $M_1(W)$ and $M_2(W)$ are asymptotically compact, then the maps M_1 and M_2 are asymptotically compact at w_0. In this event Theorem 4.7, v) is applicable (with condition g) instead of condition b) of Theorem 4.10). Similar observations can be made concerning Theorem 4.11.

Furthermore, one can provide conditions for the intersection, sum, product, composition of recessively lower continuous set-valued maps to be recessively lower continuous. However, except for the union and for the case of cone-valued maps already studied in Section 2, such sufficient conditions are of extremely particular character, so we do not mention them in this study. For instance, the product $M \times N$ is not recessively lower continuous even if M is a constant cone-valued map, while N is a nonzero constant point-valued map; or $M_1 \cap M_2$ may be not recessively lower continuous even when their values are closed convex in a finite dimensional space and $M_1(w) \cap M_2(w) \neq 0$ for every w.

Finally, note also that a result similar to Theorems 4.7 and 4.10 can be obtained for recessive conic upper continuity by a similar argument. We leave it to the interested readers.

5. Marginal functions

Let f be a function from $\Omega \times X$ to the extended real line $\mathbb{R} \cup \{\infty\}$ and let M be a set-valued map from Ω to X. The marginal function (also called value or performance function) we are going to study is defined by

$$\varphi(w) := \inf_{x \in M(w)} f(w,x)\ , \quad \text{for} \quad w \in \Omega\ .$$

The set $S(w) := \{x \in M(w) : \varphi(w) = f(w,x)\}$ is the solution set corresponding to w.

An important issue of optimization theory is to find conditions which guarantee continuity properties of φ. It is known that φ is lower semicontinuous provided f is lower semicontinuous and M is upper continuous, compact-valued ; and φ is upper semicontinuous provided f is upper semicontinuous and M is lower continuous (see [6], [8]). In the case of non-compact values, especially unbounded values, the upper continuity condition, as we have seen in Section 2, is a very restrictive one. Many efforts have been made to replace this requirement by a milder one (see for instance [6], [7], [22], [24], [25], [31], [36], [44] to name but a few). Below we present some sufficient conditions for the continuity of φ without upper continuity of M. From now on f is suppposed to be continuous.

Let us denote by $f^M_\infty(w_0\ ;\ \cdot)$ the recession function of f with respect to M at w_0, defined by

$$f^M_\infty(w_0, v) = \inf\{\lim_{n\to\infty} t_n f(w_n, x_n) \\ : \{t_n, w_n\}_{n=1}^\infty \to (0_+, w_0), \{t_n x_n\}_{n=1}^\infty \to v, x_n \in M(w_n)\}.$$

Note that this function differs from the recession function of the function $f(w_0, \cdot)$ with w_0 fixed.

Theorem 5.1. *Assume that $M(w_0) \neq \emptyset$ and the following conditions hold:*

i) There is a neighborhood W of w_0 such that $M(W)$ is boundedly relatively compact;

ii) M is closed, lower continuous, recessively upper continuous and asymptotically compact at w_0;

iii) $f_\infty^M(w_0, v) > 0$ for all $v \in R_M(w_0) \backslash \{0\}$.

Then φ is continuous at w_0 and $\varphi(w_0)$ is attained.

Proof. In view of the conditions recalled above for upper semicontinuity of φ it suffices to show that φ is lower semicontinuous at w_0. This can be done if we are able to find a positive α such that

$$\varphi(w) = \inf_{x \in M(w) \cap B(0,\alpha)} f(w,x) \tag{1}$$

for w sufficiently close to w_0. Moreover $\varphi(w_0)$ is attained because $M(w) \cap B(0, \alpha)$ is compact (remember that the map $w \to M(w) \cap B(0, \alpha)$ is closed and takes its values in the relatively compact set $M(W) \cap B(0, \alpha)$). Suppose to the contrary that (1) is not true, i.e. there is a sequence $\{w_n\}_{n=1}^\infty$ converging to w_0, $x_n \in M(w_n)$ with $||x_n|| \geq n$, such that for $n \geq 1$,

$$f(w_n, x_n) < \inf_{x \in M(w_n) \cap B(0,n)} f(w_n, x) \; .$$

By asymptotic compactness we may assume that $\{x_n/||x_n||\}_{n=1}^\infty$ converges to some $v \in X$, $v \neq 0$. In view of recessive upper continuity one has $v \in R_M(w_0)$. Now, pick any $a_0 \in M(w_0)$. By the lower continuity of M there is $a_n \in M(w_n)$ such that $\lim_{n\to\infty} a_n = a_0$. Without loss of generality we may also assume that $||a_n|| \leq n$, for all $n \geq 1$, so that $f(w_n, x_n) < f(w_n, a_n)$.

Hence $f_\infty^M(w_0, v) \leq \liminf_{n\to\infty} 1/||x_n|| \, [f(w_n, x_n) - f(w_n, a_n)] \leq 0$ (here we use the fact that $\lim f(w_n, a_n) = f(w_0, a_0)$ is finite). This contradicts iii) and the proof is complete. □

Note that the recessive upper continuity requirement in condition ii) of the preceding proposition can be dispensed with if the inequality in condition iii) is satisfied for all $v \in (\limsup_{w\to w_0}^\infty M(w)) \setminus \{0\}$.

Theorem 5.2. *Assume that $M(w_0) \neq \emptyset$ and the following conditions hold.*

i) M is closed, asymptotically compact, lower continuous at w_0 and convex-valued on some neighborhood W of w_0 such that $M(W)$ is boundedly relatively compact;

ii) f is quasiconvex in x for every fixed $w \in W$;

iii) $S(w_0)$ is bounded, nonempty.

Then φ is continuous at w_0.

Proof. As in the proof of Theorem 5.1, it suffices to show relation (1) with $\alpha > \|x_0\|$ for some $x_0 \in S(w_0)$. Supposing that (1) is not true, we can find a sequence $\{w_n\}_{n=1}^\infty$ converging to w_0, a sequence $\{x_n\}_{n=1}^\infty$ with $\lim_{n\to\infty} ||x_n|| = \infty$ and $x_n \in M(w_n)$ such that

$$f(w_n, x_n) < \inf_{x \in M(w_n) \cap B(0,\alpha)} f(w_n, x) \; .$$

By asymptotic compactness, we may assume that $\{x_n/||x_n||\}_{n=1}^\infty$ converges to some $v_0 \in X$, $v_0 \neq 0$. Moreover, since M is lower continuous at w_0, we can find $\bar{x}_n \in$

$M(w_n)$ with $\lim_{n\to\infty} \bar{x}_n = x_0$. For n sufficiently large, one has $\bar{x}_n \in B(0,\alpha)$, consequently $f(w_n, x_n) < f(w_n, \overline{x}_n)$. For every fixed positive t, one has

$$\overline{x}_n + \frac{t}{||x_n||}\,(x_n - \overline{x}_n) \in M(w_n)$$

if n is large enough. It follows from the closedness of M at w_0 that $x_0 + tv_0 \in M(w_0)$. Moreover, since f is quasiconvex in x, one also has

$$\begin{aligned} f(w_n, \overline{x}_n + \frac{t}{||x_n||}\,(x_n - \overline{x}_n)) &\leq \max\{f(w_n, \overline{x}_n)\ ,\ f(w_n, x_n)\} \\ &\leq f(w_n, \overline{x}_n). \end{aligned}$$

Consequently, $f(w_0, x_0 + tv_0) \leq f(w_0, x_0)$. Actually, we have equality because $x_0 \in S(w_0)$. Since t is arbitrary, we arrive at the contradiction that $S(w_0) \ni x_0 + tv_0$ for all $t > 0$. □

Observe that if X is finite dimensional, then the recessive compactness and the boundedly compact valuedness property of M are automatically satisfied. This case was studied in [13].

It is worthwhile noticing that without quasiconvexity of f, Theorem 5.3 may fail. Indeed, let us define M from $\mathbb{R}$ to $\mathbb{R}^2$ by

$$M(w) = \{(w,t) \in \mathbb{R}^2 : t \geq w^2\}$$

for $w \in \mathbb{R}$, and a function f on $\mathbb{R}^2$ by

$$f(x,y) = y - 2x^2$$

for $(x,y) \in \mathbb{R}^2$. It can be verified that for $w_0 = 0$, all the conditions of Theorem 5.2 are satisfied except for the second one. The performance function is not continuous at $w_0 = 0$, for its value at this point is 0, while at other points its value is $-\infty$.

In the remaining part of this section we consider the case where M is a convex polyhedral set in a finite dimensional space. More precisely, M is given by a system of linear equalities and inequalities

$$\begin{aligned} \langle a_i(\omega), x\rangle \ + \ \alpha_i(x) \leq 0, &\qquad i = 1, \cdots, p, \\ \langle a_j(\omega), x\rangle \ + \ \alpha_j(\omega) = 0, &\qquad j = p+1, \cdots, q+p\ , \end{aligned}$$

where $x \in X$, a finite dimensional space; $\alpha_1, \cdots, \alpha_{p+q}$ are real-valued continuous functions on Ω, $a_1, \cdots, a_{q+p}$ are continuous vector functions from Ω to X', the dual of X.

Lemma 5.3. *Assume that $M(w_0) \neq \emptyset$. Then R_M is cosmically upper continuous at w_0.*

Proof. We know that $R_M(w)$ consists of the solutions to the homogeneous system

$$\begin{aligned} \langle a_i(\omega), x\rangle \ \leq 0, &\qquad i = 1, \cdots, p, \\ \langle a_j(\omega), x\rangle \ = 0, &\qquad j = p+1, \cdots, q+p\ . \end{aligned}$$

Moreover, with B_X being compact, it is clear that the maps

$$\begin{aligned} w &\longmapsto \{x \in X : \langle a_i(w), x\rangle \ \leq 0\}, \qquad i = 1, ..., p, \\ w &\longmapsto \{x \in X : \langle a_i(w), x\rangle \ = 0\}, \qquad j = p+1, ..., p+q, \end{aligned}$$

are cosmically upper continuous. By Theorem 2.7, their intersection (i.e. R_M) is cosmically upper continuous. □

Corollary 5.4. *Assume that for the preceding data the following conditions hold:*
i) M *is lower continuous at* w_0 *and* $M(w_0) \neq \emptyset$;
ii) f *is quasiconvex in* x *for every fixed* w;
iii) $f^M_\infty(w_0, v) > 0$ *for every* $v \in R_M(w_0)\backslash\{0\}$.
Then φ *is continuous at* w_0.

Proof. Observe that the last two conditions of the corollary imply that $S(w_0)$ is nonempty and compact. Moreover, since X is finite dimensional, condition i) of Theorem 5.2 holds. By this, φ is continuous at w_0. One can also see that in view of Lemma 5.3 the conditions of Theorem 5.1 are satisfied; therefore the result of that theorem can also be applied to our case. □

Corollary 5.5. *Assume that* $M(w) \neq \emptyset$ *for* w *close to* w_0 *and conditions ii), iii) of Corollary 5.4 hold. Then for every neighborhood* W *of* w_0*, there exists an open set* $W_0 \subseteq W$ *such that* φ *is continuous on* W_0.

Proof. By Lemma 5.3, it follows from condition iii) of Corollary 5.4 that there exists a neighborhood $W_1 \subseteq W$ of w_0 such that $f^M_\infty(w, v) > 0$ for every $w \in W_1$, $v \in R_M(w)\backslash\{0\}$. In view of Proposition 2.1 of [20] (which says that the map M is closed and lower continuous on an open dense subset of W), there exists an open set $W_0 \subseteq W_1$ such that M is lower continuous on W_0. Now it remains to apply Corollary 5.4 to every point $w \in W_0$. □

In the case where f is linear with respect to the variable x, a result stronger than Corollary 5.5 has been obtained in [31]. Namely, it was shown that there can be found a solution $x(w) \in M(w)$ which depends smoothly on w such that $\varphi(w) = f(w, x(w))$ for all $w \in W_0$ if the functions a_i, α_i are smooth.

6. Asymptotic directions of level sets

Let f be a function defined on a Banach space X with values in the extended real line $\mathbb{R} \cup \{\infty\}$. Let us define the level set map associated with f as a set-valued map from $\mathbb{R}$ to X by

$$L_f(\alpha) := \{x \in X : f(x) \leq \alpha\}$$

for all $\alpha \in \mathbb{R} \cup \{\infty\}$. Level sets (called also sublevel sets) play an important role in the study of functions (see [6], [33], [41] and the references given there). They characterize, for instance, continuity properties and other properties relative to the structure of functions (convexity, connectedness, etc.). In optimization the feasible solution set of a problem is often expressed as a level set of a constraint function. In this section we shall establish some convergence properties of asymptotic directions of level sets with the help of the results of the previous sections. An application is made to derive the extreme desirability condition in an unbounded exchange economy. We shall need the following result of [33].

Lemma 6.1. *The level set map* L_f *has the following properties:*
i) L_f *is closed on* $\mathbb{R}$ *if and only if* f *is lower semicontinuous;*
ii) L_f *is lower continuous on its domain if and only if every local minimizer of* f *with finite value is a global minimizer.*

Theorem 6.2. *Assume that* f *is a lower semicontinuous, quasiconvex function from* X *to* $\mathbb{R} \cup \{\infty\}$*. Then the following assertions hold:*
i) R_{L_f} *is closed on the domain of* L_f *if every local minimizer of* f *with finite value is a global minimizer;*

ii) R_{L_f} *is cosmically upper continuous on the domain of* L_f *if every local minimizer of* f *with finite value is a global minimizer and if the domain of* f *is asymptotically compact.*

Proof. Let α be an element of the domain of the level set map L_f. In view of the preceding lemma, under the condition of assertion i), L_f is lower continuous, convex-valued and closed at α. By Theorem 3.4, R_{L_f} is closed at this point. If in addition the domain of f is asymptotically compact, then according to Theorem 3.7, R_{L_f} is cosmically upper continuous at this point. The proof is complete. □

Corollary 6.3. *Assume that* f *is a lower semicontinuous quasiconvex function from a finite dimensional space* X *to* $\mathbb{R} \cup \{\infty\}$ *and every local minimizer of* f *with finite value is a global minimizer. Then* R_{L_f} *is cosmically upper continuous on its domain.*

Proof. This follows from Theorem 6.2 and the fact that every subset of a finite dimensional space is asymptotically compact. □

Note that the conclusion of Theorem 6.2 (ii) is not always true without the asymptotic compactness assumption. This can be seen by the following example. Let X be the product space $\mathbb{R} \times Y$, where Y is an infinite dimensional space. Define a function f on X by

$$f(t,y) = \begin{cases} 0 & \text{if } t = 0 \text{ and } y = 0, \\ \|y\|/t & \text{if } t > 0, \\ \infty & \text{if } t < 0 \text{ or } t = 0 \text{ and } y \neq 0. \end{cases}$$

It is evident that the requirements of Theorem 6.2 (ii) are satisfied, except for the asymptotic compactness of the domain of f. For all $\alpha \geq 0$, one sees that $R_{L_f}(\alpha) = L_f(\alpha)$ coincides with the cone generated by the set $\{1\} \times \alpha B_Y$. Hence R_{L_f} is not cosmically upper continuous at $\alpha \geq 0$.

Let us now discuss the so-called "extreme desirability condition" that guarantees the existence of equilibria in unbounded exchange economies. Let f be a continuous function from a finite dimensional space X to $\mathbb{R}$. We say that *the extreme desirability condition* is satisfied at a level $\alpha \in \mathbb{R}$ if every nonzero asymptotic direction v of $L_f(\alpha)$ is decreasing in the sense that for every $x \in L_f(\alpha)$ there exists $t > 0$ such that $f(x + tv) < f(x)$. The interested reader is referred to [34], [35] for more about this condition and its role in economics. We shall show that under suitable assumptions, if the extreme desirability condition is satisfied at some level, it is so at any level sufficiently close to that level.

Proposition 6.4. *Suppose the function* f *is continuous and quasiconvex on a finite dimensional space* X *with values in* $\mathbb{R} \cup \{\infty\}$, *and that:*

i) *Every local minimizer of* f *with finite value is a global minimizer;*

ii) *The extreme desirability condition is satisfied at a level* α *with* $L_f(\alpha) \neq \emptyset$*;*

iii) *For each asymptotic direction* v *of this level with* $\|v\| = 1$, *one has*

$$\lim_{t\to\infty} f(x + tv) \geq \liminf_{n\to\infty} f(x_n)$$

whenever $\lim_{n\to\infty} \|x_n\| = \infty$ *and* $\lim_{n\to\infty} x_n/\|x_n\| = v$.

Then there exists a positive ε *such that the extreme desirability condition is satisfied at any level in* $(-\infty, \alpha + \varepsilon)$.

Proof. Suppose to the contrary that there are a sequence $\{\alpha_n\}_{n=1}^{\infty}$ decreasing to α, vectors $v_n \in R_{L_f}(\alpha_n)$ with $\|v_n\| = 1$ and $y_n \in L_f(\alpha_n)$ such that $f(y_n) = \min\{f(y_n + tv_n) : t \geq 0\}$. Without loss of generality one may assume that $\{v_n\}_{n=1}^{\infty}$ converges to some vector v_0. In view of Theorem 6.2 (i), $v_0 \in R_{L_f}(\alpha)$. Now we show that actually for n sufficiently large, $v_n \in R_{L_f}(\alpha)$. Indeed, if not, for an arbitrary fixed $x \in L_f(\alpha)$ one can find $t_n \geq 0$ such that $f(x + tv_n) > \alpha$ for all $t \geq t_n$. Let $s_n > t_n$ with $\lim_{n\to\infty} s_n = \infty$ and let $x_n = x + s_n v_n$. Then on one hand one has that $\lim_{n\to\infty} \|x_n\| = \infty$, $\lim_{n\to\infty} x_n/\|x_n\| = v_0$ and $\liminf_{n\to\infty} f(x_n) \geq \alpha$. On the other hand, by the extreme desirability condition at the level α and by the quasiconvexity of f, one can find $\lambda > 0$ such that $f(x+tv_0) \leq \alpha - \lambda$ for t sufficiently large. This implies that $\lim_{n\to\infty} f(x + tv_0) < \alpha$ and leads to a contradiction with condition iii). By this, for n sufficiently large, one has $v_n \in R_{L_f}(\alpha)$. For these n, one also has $f(y_n) > \alpha$, because otherwise y_n could not minimize the function $t \mapsto f(y_n + tv_n)$ over $\mathbb{R}_+$ according to the extreme desirability condition. Consequently $f(y_n + tv_n) \geq \alpha$ for all $t \geq 0$. Let $r_n \geq \|y_n\|^2$ with $\lim_{n\to\infty} r_n = \infty$. Putting $x_n = y_n + r_n v_n$, we see that $\lim_{n\to\infty} \|x_n\| = \infty$, $\lim_{n\to\infty} x_n/\|x_n\| = v_0$ and arrive at the same contradiction as above. The proof is complete. □

Finally, let us make a comment on Theorem 1 (page 487) of [34] stating that for a continuous quasiconvex function f the extreme desirability condition is satisfied at all levels $\alpha \in \{f(x) : x \in X\}$ if and only if the following condition (called the no-half-lines condition) holds: there do not exist x and $v \neq 0$ such that $f(x) = f(x+tv)$ for all $t \geq 0$. This result, however, can not be extended to all levels $\alpha \in \mathbb{R}$. For instance, for the function $f(x) := x/(1 + |x|)$ on $\mathbb{R}$ the no-half-lines condition is satisfied, but the extreme desirability condition does not hold at the level 1. This example also shows that the extreme desirability condition is not continuous in the sense that if it is satisfied at all the levels strictly lower than $\alpha \in \mathbb{R}$, it may fail at the level α.

7. Conclusion

Our investigation has been centered around the properties linking asymptotic analysis and perturbations of the data. Up to now several applications of asymptotic analysis have already been obtained and many are in progress. For a broad view of applications to optimization we refer to [5]–[7], [43], [46] for scalar problems and to [28], [47] for vector problems. Applications to mechanics, variational inequalities and other topics of functional analysis are found in [1], [7], [29], [30]. A study of continuity properties with respect to weaker topologies in the image space X as suggested by a referee, is possible and worth further attention because it should offer a larger range of applications. In fact the research along this direction has already been initiated in [5], [6], [7]. In these works the authors define asymptotic cones in the weak topology. However, in order to obtain their results they impose additional conditions that make these cones identical with the asymptotic cones defined by the norm topology. In order to really exploit the advantages of weak topologies, further investigations are needed and we shall address this topic in a future work.

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