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COLORING $\mathbb{R}^n$

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ABSTRACT. If $1 \leq m \leq n$ and $A \subseteq \mathbb{R}$, then define the graph $G(A, m, n)$ to be the graph whose vertex set is $\mathbb{R}^n$ with two vertices $x, y \in \mathbb{R}^n$ being adjacent iff there are distinct $u, v \in A^m$ such that $\|x - y\| = \|u - v\|$. For various m and n and various A, typically $A = \mathbb{Q}$ or $A = \mathbb{Z}$, the graph $G(A, m, n)$ can be properly colored with ω colors. It is shown that in some cases such a coloring $\varphi : \mathbb{R}^n \longrightarrow \omega$ can also have the additional property that if $\alpha : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is an isometric embedding, then the restriction of φ to $\alpha(A^m)$ is a bijection onto ω.

Erdős [1] proved that there is a function $\varphi : \mathbb{R}^2 \longrightarrow \omega$ such that whenever $x, y \in \mathbb{R}^2$ are distinct and the distance between them is rational (that is, $\|x - y\| \in \mathbb{Q}$), then $\varphi(x) \neq \varphi(y)$. There have been various generalizations of this result, including extensions to higher dimensions – to $\mathbb{R}^3$ by Erdős & Komjáth [2] and then to arbitrary $\mathbb{R}^n$ by Komjáth [5]. Another proof of Komjáth's theorem, as well as proofs of some other similar theorems, can be found in [7]. In another direction, there is the recent improvement by Komjáth [6] who showed that the function $\varphi : \mathbb{R}^2 \longrightarrow \omega$ could, in addition, be required to satisfy the following interesting condition: if $\ell \subseteq \mathbb{R}^2$ is a line and $a \in \ell$, then φ maps $\{x \in \ell : \|x - a\| \in \mathbb{Q}\}$ onto ω. In this paper, Komjáth's improvement is extended to arbitrary $\mathbb{R}^n$.

Theorem 1. *There is a function $\varphi : \mathbb{R}^n \longrightarrow \omega$ such that for any line $\ell \subseteq \mathbb{R}^n$ and $a \in \ell$, the restriction of φ to $\{x \in \ell : \|x - a\| \in \mathbb{Q}\}$ is a bijection onto ω.*

Komjáth [4] proved some similar types of theorems related to sets having the Steinhaus property. A subset $B \subseteq \mathbb{R}^2$ is said to have the **Steinhaus property** if, for any isometry $\alpha : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, there is exactly one lattice point in $\alpha(B)$ or, in other words, $|\alpha(B) \cap \mathbb{Z}^2| = 1$. In a very recent preprint, Jackson & Mauldin [3] settle a long-standing open problem by proving the existence of a set having the Steinhaus property. Earlier, Komjáth [4] had proved that there is a subset $B \subseteq \mathbb{R}^2$ such that for any isometry $\alpha : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, there is exactly one point in $\alpha(B) \cap \mathbb{Q}^2$. We improve the Komjáth result by showing that $\mathbb{R}^2$ can be partitioned into countably many sets each having this property. Moreover, we will prove the following n-dimensional extension of the Komjáth result.

Theorem 2. *There is a function $\varphi : \mathbb{R}^n \longrightarrow \omega$ such that for any isometry $\alpha : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, the restriction of φ to $\alpha(\mathbb{Q}^n)$ is a bijection onto ω.*

Notice that Theorem 1 can be rephrased in a manner similar to the way that Theorem 2 is phrased. We will often consider isometric embeddings $\alpha : \mathbb{R}^m \longrightarrow \mathbb{R}^n$,

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but we will refer to them as isometries, even when $m < n$. Thus, the image of an isometry $\alpha : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is just an m-dimensional hyperplane. Theorem 1 asserts that there is a function $\varphi : \mathbb{R}^n \longrightarrow \omega$ such that for any isometry $\alpha : \mathbb{R} \longrightarrow \mathbb{R}^n$, the restriction of φ to $\alpha(\mathbb{Q})$ is a bijection onto ω. Both Theorems 1 and 2 are consequences of the more general Theorem 3.

Suppose $1 \leq m \leq n$ and $A \subseteq \mathbb{R}$. Then we define $G(A, m, n)$ to be the graph having vertex set $\mathbb{R}^n$ in which two distinct vertices x, y are **adjacent** iff $\{x, y\} \subseteq \alpha(A^m)$ for some isometry $\alpha : \mathbb{R}^m \longrightarrow \mathbb{R}^n$. We will sometimes refer to the elements of ω as **colors**. A function $\varphi : D \longrightarrow \omega$, where $D \subseteq \mathbb{R}^n$, will be referred to as a **coloring**, and it is **proper** if $\varphi(x) \neq \varphi(y)$ whenever $x, y \in D$ are adjacent. The graph associated with Theorem 1 is $G(\mathbb{Q}, 1, n)$. Komjáth's theorem in [5] asserts that this graph has chromatic number $\aleph_0$.

Whenever we have $1 \leq m \leq n$ and $A \subseteq \mathbb{R}$, it will be understood that any reference to a graph is to the graph $G(A, m, n)$.

Theorem 3. *Let $1 \leq m \leq n$ and $A \subseteq \mathbb{R}$ be such that the following two conditions hold:*

(1) *A is a countable subring of $\mathbb{R}$ and $1 \in A$;*
(2) *for any finite $F \subseteq \mathbb{R}^n$ and isometry $\alpha : \mathbb{R}^m \longrightarrow \mathbb{R}^n$, there is $z \in \alpha(A^m) \setminus F$ which is not adjacent to any $y \in F \setminus \alpha(A^m)$.*

Then there is a coloring $\varphi : \mathbb{R}^n \longrightarrow \omega$ such that for any isometry $\alpha : \mathbb{R}^m \longrightarrow \mathbb{R}^n$, the restriction of φ to $\alpha(A^m)$ is a bijection onto ω.

The proof of Theorem 3 will be presented in §1. In §2 we show how Theorem 3 implies Theorems 1 and 2. Another consequence of Theorem 3 is also given in that section. Finally, we make a connection with sets having the Steinhaus property, which concerns the graph $G(\mathbb{Z}, 2, 2)$.

1. The Proof of Theorem 3

In this section we give a proof of Theorem 3. The proof will rely heavily on the proof of Komjáth's theorem that the chromatic number of $\mathbb{R}^n$ is $\aleph_0$ as given in [7]. We present a summary of that proof in a form suitable for our needs here.

We will think of $\mathbb{R}$ as an ordered field. Since A is countable, we can find a countable real-closed field $\mathbb{F} \subseteq \mathbb{R}$ such that $A \subseteq \mathbb{F}$. We will take $\mathbb{F}$ to be fixed for the remainder of this proof. Notice that if x, y are adjacent, then $\|x - y\| \in \mathbb{F}$. If $X \subseteq \mathbb{R}$ and $R \subseteq \mathbb{R}^k$ for some $k < \omega$, then we say that R is X-**definable** if it is definable in the ordered field $\mathbb{R}$ by a formula in which parameters from $X \cup \mathbb{F}$ are allowed. We say that $a \in \mathbb{R}^k$ is X-definable if $\{a\}$ is X-definable.

Let T be a transcendence basis for $\mathbb{R}$ over $\mathbb{F}$ which is to be fixed for the remainder of this proof. (Note that the existence of T cannot be proved without some use of the Axiom of Choice.) Then each $a \in \mathbb{R}^n$ is T-definable. In fact there is a unique smallest finite subset $S \subseteq T$ such that a is S-definable; we will refer to this set as the **support** of a, and denote it by $supp(a)$. When it is convenient, we will consider $supp(a)$ to be an ordered set: thus, if $supp(a) = \{t_0, t_1, t_2, \ldots, t_{s-1}\}$, where $t_0 < t_1 < \cdots < t_{s-1}$, then we will sometimes let $supp(a) = \langle t_0, t_1, t_2, \ldots, t_{s-1}\rangle$. For any subset $X \subseteq \mathbb{R}^n$, let $supp(X) = \bigcup\{supp(a) : a \in X\}$. Let $b_0 = (0, 0, \ldots, 0) \in \mathbb{R}^m$, and for $1 \leq j \leq m$ let $b_j = (0, 0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^m$, which has its unique 1 preceded by $j - 1$ 0's. It follows from (1) that whenever $\alpha, \beta : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ are isometries $\{\alpha(b_0), \alpha(b_1), \ldots, \alpha(b_m)\} \subseteq \beta(A^m)$, then $\alpha(A^m) = \beta(A^m)$. Thus, for each isometry $\alpha : \mathbb{R}^m \longrightarrow \mathbb{R}^n$, each element of $\alpha(A^m)$ is $\{\alpha(b_0), \alpha(b_1), \dots, \alpha(b_m)\}$-definable and therefore, $supp(\alpha(A))$ is finite. In fact, $supp(\alpha(A^m)) = supp(\alpha(b_0)) \cup supp(\alpha(b_1)) \cup \cdots \cup supp(\alpha(b_m))$.

For $s < \omega$, a subset $B \subseteq \mathbb{R}^s$ is a **special** s**-box** if there are rationals $p_0 < q_0 < p_1 < q_1 < \cdots < p_{s-1} < q_{s-1}$ such that $B = (p_0, q_0) \times (p_1, q_1) \times \cdots \times (p_{s-1}, q_{s-1})$. Each of the intervals (p_i, q_i) is a **factor** of B. Let $\langle f_r : r < \omega \rangle$ be a list of all $\emptyset$-definable analytic functions $f : B \longrightarrow \mathbb{R}^n$, where B is a special s-box for some $s < \omega$, and let B_r be the domain of f_r.

The following lemma, which is Lemma 7 of [7], is a key fact which is used repeatedly.

Lemma 1.1. *Suppose that B is a special s-box and $g : B \longrightarrow \mathbb{R}$ is a $\emptyset$-definable analytic function such that $g(\overline{t}) = 0$ for some $\overline{t} \in B \cap T^s$. Then $g(\overline{x}) = 0$ for every $\overline{x} \in B$.* □

Associate with each $x \in \mathbb{R}^n$ the set $\Psi(x)$ of colors, where $r \in \Psi(x)$ iff $supp(x) = \langle t_0, t_1, \dots, t_{s-1}\rangle$ and $x = f_r(t_0, t_1, \dots, t_{s-1})$. The crucial facts about the sets $\Psi(x)$ are contained in the next two lemmas. The first follows from the Implicit Function Theorem and the Tarski-Seidenberg Theorem on the elimination of quantifiers in $\mathbb{R}$. The second can be deduced from Lemma 1.1.

Lemma 1.2. *If $x \in \mathbb{R}^n$, then $\Psi(x) \neq \emptyset$.* □

Lemma 1.3. *If $x, y \in \mathbb{R}^n$ are adjacent in $G(A, m, n)$ (or even if $0 < \|x - y\| \in \mathbb{F}$), then $\Psi(x) \cap \Psi(y) = \emptyset$.* □

By Lemma 1.2, there is a coloring $\psi : \mathbb{R}^n \longrightarrow \omega$ such that $\psi(x) \in \Psi(x)$ for each $x \in \mathbb{R}^n$, and from Lemma 1.3 we get that any such ψ is proper.

The coloring φ will be constructed inductively; that is, we will construct an increasing sequence $\langle \varphi_k : k < \omega \rangle$ of functions, and then let φ be its union. This sequence of functions will be defined from two sequences $d_0, d_1, d_2, \dots$ and $e_0, e_1, e_2, \dots$ of colors. For each $k < \omega$, we let

$$D_k = \{x \in \mathbb{R}^n : \Psi(x) \cap \{d_0, d_1, \dots, d_{k-1}\} \neq \emptyset\},$$

and then let $\varphi_k : D_k \longrightarrow \omega$ be such that if $x \in D_k$ then $\varphi_k(x) = e_m$, where $m < k$ is the least for which $d_m \in \Psi(x)$. Whenever we have $d_0, d_1, \dots, d_{k-1}$, we will assume that D_k has been defined in this way, and if, in addition, we have $e_0, e_1, \dots, e_{k-1}$, then we also assume that φ_k has been defined. Of course, for each k we must have that φ_k is a proper coloring of D_k; we will say that the finite sequence $d_0, d_1, \dots, d_{k-1}, e_0, e_1, \dots, e_{k-1}$ is **acceptable** if φ_k is a proper coloring.

At the beginning of stage k, we have $d_0, d_1, \dots, d_{k-1}$ and $e_0, e_1, \dots, e_{k-1}$, and thus also D_k and φ_k. Then, at stage k, we will obtain d_k, e_k, D_{k+1} and φ_{k+1}. There are two requirements which must be taken care of in this construction: the domain of φ should be $\mathbb{R}^n$; and for each isometry $\alpha : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ and color r, there should be some $z \in \alpha(A^m)$ such that $\varphi(z) = r$. The first of these requirements is easily handled by the following lemma.

Lemma 1.4. *If $d_0, d_1, \dots, d_{k-1}, e_0, e_1, \dots, e_{k-1}$ is acceptable and if d_k is any color, then there is a color e_k such that $d_0, d_1, \dots, d_{k-1}, d_k, e_0, e_1, \dots, e_{k-1}, e_k$ is acceptable.*

Proof. By Lemma 1.3, we can choose any $e_k \notin \{e_0, e_1, \dots, e_{k-1}\}$. □

We now turn to taking care of the second requirement.

Lemma 1.5. *Suppose that $d_0, d_1, \dots, d_{k-1}$ are colors and that $\alpha : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is an isometry. Then there is $z \in \alpha(A^m) \setminus D_k$ such that z is not adjacent to any $y \in D_k \setminus \alpha(A^m)$ and $supp(z) = supp(\alpha(A^m))$.*

Proof. We begin this proof by showing that condition (2) of Theorem 3 can be improved to the following:

(2′) *for any finite $F \subseteq \mathbb{R}^n$ and isometry $\alpha : \mathbb{R}^m \longrightarrow \mathbb{R}^n$, there is $z \in \alpha(A^m) \setminus F$ which is not adjacent to any $y \in F \setminus \alpha(A^m)$ and is such that $supp(z) = supp(\alpha(A^m))$.*

For each $\bar{t}$, which is properly contained in $supp(\alpha(A^m))$, the set of elements in $\alpha(A^m)$ having support contained in $\bar{t}$ lie in some $(m-1)$-dimensional hyperplane of $\mathbb{R}^n$. (Otherwise, we would have that $supp(\alpha(A^m)) \subseteq \bar{t}$.) Therefore, $\{a \in A^m : supp(\alpha(a)) \subseteq \bar{t}\}$ is contained in an $(m-1)$-dimensional hyperplane of $\mathbb{R}^m$. Thus, the set S of elements in $a \in A^m$ for which $supp(\alpha(a))$ is different from $supp(A^m)$ is contained in the union of finitely many $(m-1)$-dimensional hyperplanes, Clearly, there are finitely many $v_0, v_1, \dots, v_p \in A^m$ such that $A^m \subseteq (v_0 + (A^m \setminus S)) \cup (v_1 + (A^m \setminus S)) \cup \cdots \cup (v_p + (A^m \setminus S))$. Let $\beta_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the isometry defined by $\beta_i(x) = x + (\alpha(v_i) - \alpha(0))$. Then $\beta_i(\alpha(A^m)) = \alpha(A^m)$ for each $i \leq p$, and $\alpha(A^m) \subseteq \beta_0(\alpha(A^m \setminus S)) \cup \beta_1(\alpha(A^m \setminus S)) \cup \cdots \cup \beta_p(\alpha(A^m \setminus S))$. By (2) we let $x \in \alpha(A^m) \setminus (\beta_0(F) \cup \beta_1(F) \cup \cdots \cup \beta_p(F))$ be such that it is not adjacent to any $y \in (\beta_0(F) \cup \beta_1(F) \cup \cdots \cup \beta_p(F)) \setminus \alpha(A^m)$. There is $i \leq p$ such that $x \in \beta_i(\alpha(A^m \setminus S))$. Then $x \notin \beta_i(F)$ and x is not adjacent to any point in $\beta_i(F) \setminus \alpha(A^m)$. Therefore, $z = \beta_i^{-1}(x)$ is as required.

We now return to the proof of the lemma. Consider an equivalence relation on $D_k \setminus \alpha(A^m)$ obtained in the following way. The points $y, y' \in D_k \setminus \alpha(A^m)$ are equivalent if $\Psi(y) \cap \{d_0, d_1, \dots, d_{k-1}\} = \Psi(y') \cap \{d_0, d_1, \dots, d_{k-1}\}$ and their supports are equivalent over $supp(\alpha(A^m))$ in the following sense: if $supp(y) = \langle t_0, t_1, \dots, t_{s-1}\rangle$, $supp(y') = \langle t'_0, t'_1, \dots, t'_{s-1}\rangle$, $u \in supp(\alpha(A^m))$ and $j < s$, then $t_j < u$ iff $t'_j < u$ and $u < t_j$ iff $u < t'_j$. Clearly, there are only finitely many equivalence classes.

We show that if y and y' are equivalent and $x \in \alpha(A^m)$, then y is adjacent to x iff y' is adjacent to x; in fact, we will show that if y is adjacent to x, then $\|y - x\| = \|y' - x\|$. So, suppose that y and y' are equivalent and y is adjacent to $x \in \alpha(A^m)$. Then let $\bar{t}, \bar{t}'$ be their supports, so that $y = f_{d_i}(\bar{t})$ and $y' = f_{d_i}(\bar{t}')$. Let B be a special box for which $supp(y) \cup supp(\alpha(A^m)), supp(y') \cup supp(\alpha(A^m)) \in B$ and on which there is a $\emptyset$-definable analytic function g for which $g(supp(y), supp(\alpha(A^m))) = \|y - x\|^2$ and $g(supp(y'), supp(\alpha(A^m))) = \|y' - x\|^2$. Since this $\|y - x\|^2 \in \mathbb{F}$, it follows from Lemma 1.1 that g is constant on B, so that $\|y - x\| = \|y' - x\|$.

Now let $Y \subseteq D_k \setminus \alpha(A^m)$ be a finite set which meets every equivalence class. Then, by (2′), we can choose $z \in \alpha(A^m) \setminus D_k$ such that $supp(z) = supp(\alpha(A^m))$ and z is not adjacent to any $y \in Y$. Then z is not adjacent to any $y \in D_k \setminus \alpha(A^m)$, thereby proving the lemma. □

We say that an isometry $\alpha : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ has **type** $\tau = \langle i_0, i_1, \dots, i_m\rangle$ if the following hold for each $j \leq m$:

- $i_j \in \Psi(\alpha(b_j))$;
- $B_{i_j} = B_{i_0}$ for each $j \leq m$;
- $f_{i_j}(supp(\alpha(A^m))) = \alpha(b_j)$.

We will call the box B_{i_0} the **domain** of τ. It is possible for an isometry not to have a type, and it is also possible that an isometry have more than one type.

Lemma 1.6. *For every isometry $\alpha : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ there is an isometry $\gamma : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ such that $\gamma(A^m) = \alpha(A^m)$ and γ has a type.*

Proof. Using an argument like the one at the beginning of the proof of Lemma 1.5, we see that there is a point $v \in A^m$ such that $supp(\alpha(v + b_0)) = supp(\alpha(v + b_1)) = \cdots = supp(\alpha(v + b_m))$. Let $\gamma : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be such that $\gamma(x) = \alpha(v + x)$. ☐

If $C_0, C_1, \dots, C_{k-1}, B$ are special boxes, then we say that B is **refining** over $C_0, C_1, \dots, C_{k-1}$ if, whenever J is a factor of some C_j and I is a factor of B, then either $J \cap I = \emptyset$ or $J \supseteq I$.

Lemma 1.7. *Let τ be the type of an isometry, and let $C_0, C_1, \dots, C_{k-1}$ be special boxes. Then there are types $\tau_0, \tau_1, \dots, \tau_p$ with domains $C_k, C_{k+1}, \dots, C_{k+p}$ respectively such that the following hold:*

- *for $j \leq p$, C_{k+j} is refining over $C_0, C_1, \dots, C_{k+j-1}$;*
- *for any isometry α of type τ, there is some $j \leq p$ such that α has type τ_j.*

Proof. Let $\tau = \langle i_0, i_1, \dots, i_m \rangle$ and let B be the domain of τ. Let Q be the finite set of rationals which are the endpoints of the factors of the special boxes $C_0, C_1, \dots, C_{k-1}$ and B. Let $\mathcal{B}$ be the finite set of all special boxes whose factors have endpoints in Q. Then let $C_k, C_{k+1}, \dots, C_{k+p}$ be those special boxes which are minimal (with respect to inclusion) in $\mathcal{B}$ and which are included in B. For each $j \leq p$, let $\tau_j = \langle i_{0j}, i_{1j}, \dots, i_{mj} \rangle$, where each i_{rj} is such that $f_{i_{rj}} = f_{i_r} | C_{k+j}$. It is clear that the conditions in the lemma are met. ☐

Lemma 1.8. *Suppose that $d_0, d_1, \dots, d_{k-1}, e_0, e_1, \dots, e_{k-1}$ is acceptable and e_k is a color. Suppose that α is an isometry of type $\tau = \langle i_0, i_1, \dots, i_m \rangle$ such that $\varphi_k(z) \neq e_k$ for all $z \in \alpha(A^m)$. Suppose that B, the domain of τ, is refining over $B_{d_0}, B_{d_1}, \dots, B_{d_{k-1}}$. Then there is a color d_k such that $B_{d_k} = B$, $d_0, d_1, \dots, d_{k-1}, d_k, e_0, e_1, \dots, e_{k-1}, e_k$ is acceptable, and for any isometry $\beta : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ of type τ, there is $w \in \beta(A^m)$ such that $\varphi_{k+1}(w) = e_k$.*

Proof. By Lemma 1.5, let $z \in \alpha(A^m) \setminus D_k$ be such that $supp(z) = supp(\alpha(A^m))$ and z is not adjacent to any $y \in D_k \setminus \alpha(A^m)$. Let $supp(\alpha(A^m)) = \bar{t}$ and let $a = \alpha^{-1}(z)$. Then $a = (a_1, a_2, \dots, a_m) \in \mathbb{R}^m$. Let $a_0 = 1 - (a_1 + a_2 + \cdots + a_m)$. We now let d_k be such that $f_{d_k} : B \longrightarrow \mathbb{R}^n$ is the analytic function defined by

$$f_{d_k}(\overline{x}) = a_0 f_{i_0}(\overline{x}) + a_1 f_{i_1}(\overline{x}) + \cdots + a_m f_{i_m}(\overline{x}).$$

Then $B_{d_k} = B$. Note that $f_{d_k}(\bar{t}) = z$ since

$$\begin{aligned} f_{d_k}(\bar{t}) &= a_0 f_{i_0}(\bar{t}) + a_1 f_{i_1}(\bar{t}) + \cdots + a_m f_m(\bar{t}) \\ &= a_0 \alpha(b_0) + a_1 \alpha(b_1) + \cdots + a_m \alpha(b_m) \\ &= \alpha(a_0 b_0 + a_1 b_1 + \cdots + a_m b_m) \\ &= \alpha(a) = z. \end{aligned}$$

It is clear that $\varphi_{k+1}(z) = e_k$. We will show that for every isometry β having type τ there is $w \in \beta(A^m)$ for which $\varphi_{k+1}(w) = e_k$. Consider β having type τ, and let $\overline{s} = supp(\beta(A^m))$. Then let $w = \beta(a) = f_{d_k}(\overline{s})$.

Clearly, $w \in \beta(A^m)$. To show that $\varphi_{k+1}(w) = e_k$, it suffices to show that $w \in D_{k+1} \setminus D_k$.

We show that $w \in D_{k+1}$ by showing that $d_k \in \Psi(w)$. Since $w = f_{d_k}(\overline{s})$, we need only show that $supp(w) = \overline{s}$. If not, then there is color p such that $f_p(\overline{s}') = w$ and $\overline{s}'$ is properly contained in $\overline{s}$. Without loss of generality, we can assume that s_0 is the unique real in $\overline{s}$ but not in $\overline{s}'$. Thus, we can let $w = f_p(\overline{s}') = f_{d_k}(\overline{s}', s_0)$. Since s_0 is not $\overline{s}'$-definable, it follows that for some open neighborhood U of s_0, if $s \in U$, then $f_p(\overline{s}') = f_{d_k}(\overline{s}', s)$. Let $r \in U$ be a rational, and then $f_{d_k}(\overline{s}', s_0) = f_{d_k}(\overline{s}', r)$. It then easily follows from Lemma 1.3 that $supp(z) \neq \overline{t}$, which is a contradiction.

Next, we must show that $w \notin D_k$. For a contradiction, suppose that $m < k$ and $d_m \in \Psi(w)$. Thus $w = f_{d_m}(\overline{s})$. Since B_{d_k} is refining, $B_{d_k} \subseteq B_{d_m}$, so it follows from Lemma 1.1 that f_{d_m} and f_{d_k} agree on B_{d_k}. Therefore, $z = f_{d_m}(\overline{t})$, contradicting that $z \notin D_k$.

It remains to prove that φ_{k+1} is a proper coloring. Clearly, there is no $w \in D_{k+1}$ adjacent to z such that $\varphi_{k+1}(w) = \varphi_{k+1}(z)$. Consider arbitrary $z' \in D_{k+1} \setminus D_k$ and some $w' \in D_{k+1}$ adjacent to it, with the intent of showing that $\varphi_{k+1}(w') \neq \varphi_{k+1}(z')$. Then $z' = f_{d_k}(\overline{t}')$ for some $\overline{t}'$. Let $m' \leq k$ be minimal such that $w' = f_{d_{m'}}(\overline{t''})$. Since B_{d_k} is refining, we can find a special box C such that $\langle \overline{t}', \overline{t}'' \rangle, \langle \overline{t}, \overline{t}'' \rangle \in C$ and then let $g : C \longrightarrow \mathbb{R}$ be the $\emptyset$-definable analytic function such that $g(\overline{x}', \overline{x}'') = \|f_{d_k}(\overline{x}') - f_{m'}(\overline{x}'')\|^2$. Then $g(\overline{t}', \overline{t}'') = \|z' - w'\|^2 \in \mathbb{F}$, so it follows from Lemma 1.1 that g is constant. We can find $\overline{s}$ such that $\langle \overline{s}, \overline{t} \rangle \in C$. Then $g(\overline{s}, \overline{t}) = \|z' - w'\|^2 \in \mathbb{F}$. Let $v = f_{m'}(\overline{s})$. Then $v \in D_{k+1}$, and v and z are adjacent. Therefore, $\varphi_{k+1}(v) \neq \varphi_{k+1}(z)$, so to complete the proof it suffices to show that $\varphi_{k+1}(w') = \varphi_{k+1}(v)$.

Suppose $\varphi_{k+1}(w') \neq \varphi_{k+1}(v)$. Then there is $m < m'$ such that $d_m \in \Psi(v)$, so that $f_m(\overline{s}) = f_{m'}(\overline{s})$. It follows from Lemma 1.1, that $f_m(\overline{t''}) = f_{m'}(\overline{t''}) = w''$, which contradicts the minimality of m'. □

We finish off the proof of Theorem 3. We are constructing the two sequences $d_0, d_1, d_2, \ldots$ and $e_0, e_1, e_2, \ldots$. At each stage k we have the first k terms of each sequence, and $d_0, d_1, \ldots, d_{k-1}, e_0, e_1, \ldots, e_{k-1}$ is acceptable. There are the two requirements mentioned just before Lemma 1.4.

For the first of these, by Lemma 1.2, it suffices that $\omega = \{d_0, d_1, d_2, \ldots\}$. So at some stage k we are concerned that d gets into this sequence. By Lemma 1.4, we can let $d_k = d$ and then get e_k such that $d_0, d_1, \ldots, d_{k-1}, d_k, e_0, e_1, \ldots, e_{k-1}, e_k$ is acceptable.

To meet the second requirement, it suffices by Lemma 1.6 to show that for every type τ and color r, if α has type τ, then there is $z \in \alpha(A^m)$ such that $\varphi(z) = r$. So at some stage k we will consider τ and r. Let $C_0, C_1, \ldots, C_{k-1}$ be the special boxes $B_{d_0}, B_{d_1}, \ldots, B_{d_{k-1}}$. Apply Lemma 1.7 to get types $\tau_0, \tau_1, \ldots, \tau_p$ with domains $C_k, C_{k+1}, \ldots, C_{k+p}$. Now apply Lemma 1.8 $p+1$ times, at the jth time using τ_j, to get acceptable $d_0, d_1, \ldots, d_{k+p}, e_0, e_1, \ldots, e_{k+p}$. Clearly, the second requirement will be met, completing the proof of Theorem 3.

2. The Consequences

To derive Theorem 1 from Theorem 3, it suffices to show that when $A = \mathbb{Q}$ conditions (1) and (2) of Theorem 3 hold. Condition (1) is obvious. Condition (2) follows from the following lemma which is from Komjáth [6]. The proof presented here is a little different from the one in [6].

Lemma 2.1. *Let $F \subseteq \mathbb{R}^n$ be a finite set of points and $\alpha : \mathbb{R} \longrightarrow \mathbb{R}^n$ be an isometry. Then there is $x \in \alpha(\mathbb{Q}) \setminus F$ such that $\|x - y\| \notin \mathbb{Q}$ for all $y \in F \setminus \alpha(\mathbb{Q})$.*

Proof. Without loss of generality we can assume that $n = 2$ and α is such that $\alpha(x) = (x, 0)$ for all $x \in \mathbb{R}$. If $(a, b) \in F$ and for two distinct rationals q and r, both $\|(a, b)-(q, 0)\|$ and $\|(a, b)-(r, 0)\|$ are rational, then $a, b^2 \in \mathbb{Q}$. Thus, by appropriate scaling and translating, we can assume that if $(a, b) \in F$ and $\|(a, b) - (q, 0)\|$ is rational, where $0 < q \in \mathbb{Q}$, then a and b^2 are integers. Let c be a positive integer such that $c > a + b^2$ whenever $(a, b) \in F$ and let $x = (c, 0)$. To see that x is as required, let $y = (a, b) \in F \setminus \alpha(\mathbb{Q})$. Then $b \neq 0$ and $d^2 = (c-a)^2 + b^2 = \|x-y\|^2$ is an integer, so if d is rational, it also must be an integer. But $c-a < d < c-a+1$, so d is not an integer. □

To derive Theorem 2 from Theorem 3, it suffices to show that (1) and (2) hold when $A = \mathbb{Q}$. Again, (1) is trivial. The following lemma shows that (2) holds.

Lemma 2.2. *Let $F \subseteq \mathbb{R}^n$ be a finite set of points. Then there is $x \in \mathbb{Q}^n \setminus F$ such that $\|x - y\|^2 \notin \mathbb{Q}$ for all $y \in F \setminus \mathbb{Q}^n$.*

Proof. Let $m = |F|$. Let $B \subseteq \mathbb{Q}$ be such that $|B| = m + 1$ and $B^n \cap F = \emptyset$. Consider some $y = \langle y_0, y_1, \dots, y_{n-1}\rangle \in F \setminus \mathbb{Q}^n$. Then there is $j < n$ such that $y_j \notin \mathbb{Q}$. Hence, if $u, v \in \mathbb{Q}^n$ agree except at the j-th coordinate, then not both $\|u-y\|^2 \in \mathbb{Q}$ and $\|v-y\|^2 \in \mathbb{Q}$. Therefore, for every $y \in F \setminus \mathbb{Q}^n$, there are at most $(m+1)^{n-1}$ points $u \in B^n$ such that $\|u-y\|^2 \in \mathbb{Q}$. It follows that there are at most $m(m+1)^{n-1} < |B^n|$ points $u \in B^n$ such that $\|u-y\|^2 \in \mathbb{Q}$ for some $y \in F \setminus \mathbb{Q}^n$. Therefore, there is $x \in B$ such that $\|x-y\|^2 \notin \mathbb{Q}$ for every $y \in F \setminus \mathbb{Q}^n$. □

The preceding lemma remains true if $\mathbb{Q}$ is replaced by any countable subfield $\mathbb{F} \subseteq \mathbb{R}$. Thus, we get the following corollary extending Theorem 2.

Corollary 2.3. *Let $\mathbb{F} \subseteq \mathbb{R}$ be any countable subfield. Then there is a coloring $\varphi : \mathbb{R}^n \longrightarrow \omega$ such that for any isometry $\alpha : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, the restriction of φ to $\alpha(\mathbb{F}^n)$ is a bijection onto ω.*

Komjáth [4] proved that there is a subset $B \subseteq \mathbb{R}^2$ such that whenever $\alpha : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is an isometry, then $|\alpha(B) \cap \mathbb{Z}| = 1$. In fact, we can partition $\mathbb{R}^2$ into countably many such sets since Theorem 3 applies when $A = \mathbb{Z}$, $m = 1$ and $n = 2$. This can be extended to all n using the following lemma.

Lemma 2.4. *Let $F \subseteq \mathbb{R}^n$ be a finite set of points and $\alpha : \mathbb{R} \longrightarrow \mathbb{R}^n$ an isometry. Then there is $x \in \alpha(\mathbb{Z}) \setminus F$ such that $\|x - y\| \notin \mathbb{Z}$ for all $y \in F \setminus \alpha(\mathbb{Z})$.*

Proof. If there is $x \in \alpha(\mathbb{R}) \cap (F \setminus \alpha(\mathbb{Z}))$ then choose that point. Otherwise, let x be as in Lemma 2.1. □

Corollary 2.5. *There is a coloring $\varphi : \mathbb{R}^n \longrightarrow \omega$ such that for any isometry $\alpha : \mathbb{R} \longrightarrow \mathbb{R}^n$, the restriction of φ to $\alpha(\mathbb{Z})$ is a bijection onto ω.*

The question of whether there is such a result for the graph $G(\mathbb{Z}, 2, 2)$ appears to be open. A positive answer would result in a partition of $\mathbb{R}^2$ into countably many sets each having the Steinhaus property. Theorem 3 cannot be used to get such a partition since the set $F = \{(\frac{3}{5}, k + \frac{4}{5}) : k = 0, 1, \dots, 4\}$ is a counterexample to (2) (for $\alpha : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ being the identity isometry).

There is also the question concerning the graphs $G(\mathbb{Q}, m, n)$ when $2 \leq m < n$. For $m = 2, 3$, this question also appears to be open. However, if $4 \leq m < n$, then there is no such result since for any isometry $\alpha : \mathbb{R}^m \longrightarrow \mathbb{R}^n$, $\alpha(\mathbb{Q}^m)$ is not a maximal clique of $G(\mathbb{Q}, m, n)$ by Lagrange's Theorem on sums of 4 squares.

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