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COMPOSITE BANK-LAINE FUNCTIONS AND A QUESTION OF RUBEL

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Dedicated to the memory of Steve Bank and Lee Rubel

Abstract. A Bank-Laine function is an entire function E satisfying $E'(z) = \pm 1$ at every zero of E. We determine all Bank-Laine functions of form $E = f \circ g$, with f, g entire. Further, we prove that if h is a transcendental entire function of finite order, then there exists a path tending to infinity on which h and all its derivatives tend to infinity, thus establishing for finite order a conjecture of Rubel.

1. Introduction

A Bank-Laine function is an entire function E such that $E'(z) = \pm 1$ at every zero z of E, these arising from differential equations in the following way [1, 12]. Let A be an entire function, and let f_1, f_2 be linearly independent solutions of

$$w'' + A(z)w = 0, \tag{1}$$

normalized so that the Wronskian $W = W(f_1, f_2) = f_1 f_2' - f_1' f_2$ satisfies $W = 1$. Then $E = f_1 f_2$ is a Bank-Laine function and

$$4A = (E'/E)^2 - 2E''/E - 1/E^2, \quad E''' + 4AE' + 2A'E = 0. \tag{2}$$

Conversely, if E is any Bank-Laine function, then [3] the function A defined by (2) is entire, and E is the product of linearly independent normalized solutions of (1). There has been extensive work in recent years concerning the exponent of convergence $\lambda(f_j)$ of the zeros of solutions f_j, in connection with the order of growth $\rho(A)$ of the coefficient A, these are defined by

$$\lambda(f_j) = \limsup_{r \to \infty} \frac{\log^+ N(r, 1/f_j)}{\log r}, \quad \rho(A) = \limsup_{r \to \infty} \frac{\log^+ T(r, A)}{\log r}. \tag{3}$$

In particular it has been conjectured that

$$A \quad \text{transcendental}\,, \quad \rho(A) < \infty, \quad \max\{\lambda(f_1), \lambda(f_2)\} < \infty \tag{4}$$

implies that $\rho(A)$ is a positive integer: see [1, 20, 23, 24] for partial results in this direction.

There are a number of methods of constructing Bank-Laine functions, although it seems to be relatively difficult to make examples having finite order and associated via (2) with transcendental coefficients. A method observed by Shen [25] uses the Mittag-Leffler theorem: if (a_n) is a complex sequence tending to infinity without

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repetition, then there exists a Bank-Laine function F with zero-sequence (a_n). However, if the sequence (a_n) has finite exponent of convergence, the Bank-Laine function F so constructed may nevertheless have infinite order, and this will always be the case if all the a_n are real and $\sum |a_n|^{-1} < \infty$ [15, Theorem 1]. Further methods of construction, which do give rise to Bank-Laine functions E of finite order with A transcendental, were given in [14], using quasiconformal modifications, and in [15], using an elementary variational method.

The investigations of the present paper were prompted by the following question: to what extent is it possible to construct Bank-Laine functions as compositions $E = f \circ g$ of transcendental entire functions f, g? Examples of composite Bank-Laine functions include the following:

(a) if F is a Bank-Laine function with $F(0) = 0$ and $f(w) = F(w)/w$, then $E(z) = f(e^z)$ is a Bank-Laine function;

(b) if f is an entire function such that $f(a) = 0$ implies that $f'(a)^2(1 - a^2) = 1$, then $E(z) = f(\sin z)$ is a Bank-Laine function.

Functions f as in (b) may easily be constructed using the Mittag-Leffler theorem. We will prove the following theorem.

Theorem 1.1. *Suppose that f and g are non-constant, with f meromorphic in the plane and g entire, and that the composition $E = f \circ g$ is such that $E(z) = 0$ implies that $E'(z) = \pm 1$. Then one of the following holds.*

(i) f has no zeros.

(ii) f has one zero, at w, and either g omits the value w, or $(g(z) - w)f'(w)$ is a Bank-Laine function.

(iii) f has at least two zeros, and g has the form

$$g(z) = B_1 z^2 + B_2 z + B_3,$$

with B_1, B_2, B_3 constants. Further, if $f(w) = 0$, then $f'(w)^2(B_2^2 - 4B_1(B_3 - w)) = 1$.

(iv) f has at least two zeros, and g has the form

$$g(z) = B_1 e^{2bz} + B_2 + B_3 e^{-2bz}, \tag{5}$$

with b, B_1, B_2, B_3 constants. Further, if $f(w) = 0$, then either g omits the value w, or $4b^2 f'(w)^2((B_2 - w)^2 - 4B_1 B_3) = 1$.

Obviously, if E is entire then f can have at most one pole in Theorem 1.1. The next result shows that Bank-Laine functions of finite order cannot arise as compositions of transcendental functions.

Theorem 1.2. *Let E be a Bank-Laine function of finite order. Then E is pseudoprime, that is, E has no factorization of form $E = f \circ g$, with f, g transcendental and g entire, f meromorphic in the plane.*

The term *pseudoprime* comes from the language of factorization theory [7], and there is a substantial literature involving classes of pseudoprime functions. For example, entire functions F with finitely many fixpoints are pseudoprime [5], as are entire functions of finite order with sum of Nevanlinna deficiencies 2 [6].

To prove Theorem 1.1, it suffices to establish the following result.

Theorem 1.3. *Let g be an entire function, and let a_1, a_2, b_1, b_2 be complex numbers with $a_1 \neq a_2$ and $b_1 b_2 \neq 0$ and such that $g(z) = a_j$ implies that $g'(z) = \pm b_j$, for $j = 1, 2$. Then g is a polynomial of degree at most 2, or has the form (5).*

The proof of Theorem 1.3 requires the following auxiliary result, which has some interest in its own right.

Theorem 1.4. *Let E be a Bank-Laine function of finite order, associated via (2) with a transcendental coefficient A. Then 0 is an asymptotic value of E.*

That Theorem 1.4 should hold for E and A as in the hypotheses is perhaps not surprising, given that E is generally small where A is large, by (2), but the difficulties arise from the exceptional set on which the logarithmic derivatives in (2) are large. On the other hand, it would be interesting to know whether it is possible to delete the hypothesis in Theorem 1.4 that E has finite order. We remark that for a non-constant polynomial A, it was proved in [13] that there exists a path γ tending to infinity on which every solution of (1) tends to 0, and it would also be interesting to know whether this stronger assertion holds when A is transcendental.

Theorem 1.4 depends on the next result, in which we resolve, for functions of finite order, the following question of Rubel [18, pp. 595-596]: if f is a transcendental entire function, must there exist a path tending to infinity on which $f(z)$ and its derivative $f'(z)$ both tend to infinity?

Theorem 1.5. *Let f be transcendental and meromorphic in the plane, of order less than $\rho < \infty$, and with finitely many poles. Let (v_j) be a complex sequence such that $v_j \to \infty$ without repetition, and let $n_0 > 0$ be such that*

$$\sum |v_j|^{-n_0} < \infty. \tag{6}$$

Let

$$n_2 = 3n_0 + 5\rho + 4. \tag{7}$$

Then there exists a path γ tending to infinity and not meeting the discs $B(v_j, |v_j|^{-n_2})$, such that for each non-negative integer m and each positive real number c we have

$$\lim_{z \to \infty, z \in \gamma} \frac{\log |f^{(m)}(z)|}{\log |z|} = +\infty \tag{8}$$

and

$$\int_\gamma |f^{(m)}(z)|^{-c} |dz| < \infty. \tag{9}$$

Here we use the standard notation $B(a, r)$ for the open disc of centre a and radius r. If f is any transcendental entire function, the existence of a path γ on which f tends to infinity is established by the classical theorem of Iversen, a result substantially strengthened in [10, 17]. Moreover, Rossi proved in [22] that γ can be chosen so that $z^{-n} f^{(m)}(z)$ is unbounded on γ, for every pair of positive integers m, n.

2. Lemmas needed for the theorems

Lemma 2.1 ([16]). *Suppose that G is transcendental and meromorphic in the plane, of order less than $\rho < \infty$. Then there exists an unbounded uncountable set of positive real numbers R such that the length $L(r, R, G)$ of the level curves $|G(z)| = R$ lying in $|z| < r$ satisfies $L(r, R, G) \leq r^{(4+\rho)/2}$ for $r \geq \log R$.*

The next lemma is Tsuji's well-known estimate for harmonic measure [27, p.116]. Let D be a domain in the complex plane, regular for the Dirichlet problem. For $0 < t < \infty$, define $\theta_D^*(t)$ to be the angular measure of the intersection of D with the circle $S(0,t)$ of centre 0 and radius t, except that $\theta_D^*(t) = \infty$ if the whole circle $S(0,t)$ lies in D.

Lemma 2.2 ([27]). *Let z_0 be in D, with $2|z_0| \leq r/2 < \infty$, and let D_r be the component of $D \cap B(0,r)$ containing z_0. Then*

$$\omega(z_0, S(0,r), D_r) \leq 13 \exp\left(-\pi \int_{2|z_0|}^{r/2} \frac{dt}{t\theta_D^*(t)} \right).$$

Lemma 2.3. *Let $0 < 4\rho < \sigma < \infty$. Let z_0, z_1 be complex numbers with $|z_1 - z_0| > \sigma$. Let U be a domain in $\rho < |z - z_0| < \infty$ such that $S(z_0,t)$ meets $\mathbf{C}\backslash U$ for all t with $\rho \leq t \leq \sigma$. Then*

$$\omega(z_1, S(z_0,\rho), U) \leq 26(\rho/\sigma)^{1/2}. \tag{10}$$

Proof. Let

$$w = \phi(z) = \frac{1}{z - z_0}, \quad w_1 = \phi(z_1), \quad V = \phi(U).$$

Then $S(0,s)$ meets $\mathbf{C}\backslash V$ for $1/\sigma \leq s \leq 1/\rho$. Hence Lemma 2.2 gives

$$\begin{aligned}
\omega(z_1, S(z_0,\rho), U) &= \omega(w_1, S(0,1/\rho), V) \\
&\leq 13 \exp\left(-\pi \int_{2/\sigma}^{1/2\rho} \frac{dt}{t\theta_V^*(t)} \right) \\
&\leq 13 \exp\left(-\frac{1}{2} \log(\sigma/4\rho) \right) \\
&= 26(\rho/\sigma)^{1/2}.
\end{aligned}$$

$\square$

Lemma 2.4. *Let (z_j) be a complex sequence such that $z_j \to \infty$ without repetition, and with $|z_j| > 2$, and let $N_1 > 0$ be such that*

$$\sum |z_j|^{-N_1} < \infty. \tag{11}$$

Let h be transcendental and meromorphic in the plane, of order less than $\rho < \infty$, and with finitely many poles. Let

$$N_2 > 3N_1 + 2\rho. \tag{12}$$

For $m = 1, 2$, let H_m be the union of the closures of the $B(z_j, |z_j|^{-N_m})$.

Next, let $R_1 > 4$ be so large that

$$h^{-1}(\{\infty\}) \subseteq B(0, \frac{1}{2}R_1), \quad M(R_1, h) = \max\{|h(z)| : |z| = R_1\} > e^4, \tag{13}$$

and

$$\log|h(z)| \leq |z/2|^\rho \quad \text{for} \quad |z| \geq R_1, \tag{14}$$

and

$$(\frac{1}{2}R_1)^{N_2 - N_1} > 4, \qquad \sum_{|z_j| > \frac{1}{2}R_1} 26|z_j|^{\rho + \frac{1}{2}(N_1 - N_2)} \leq \sum_{|z_j| > \frac{1}{2}R_1} 26|z_j|^{-N_1} < 1. \tag{15}$$

Let w_0 lie outside H_1, with

$$|w_0| > R_1, \quad |h(w_0)| > M_1^2, \quad M_1 > M(R_1, h)^2, \tag{16}$$

and let C_0 be the component of the set $\{z \in \mathbf{C} \backslash H_2 : |h(z)| > M_1\}$ in which w_0 lies.

Let $M > 0$. Then there exist arbitrarily large R satisfying

$$S(0,R) \cap H_1 = \emptyset, \tag{17}$$

and such that

$$\frac{\log^+(\sup\{|h(z)| : z \in C_0, \quad |z| = R\})}{\log R} > M. \tag{18}$$

In particular, C_0 is unbounded.

Note that in (15) we use (12).

Proof. If $S(0,R) \subseteq C_0$ for arbitrarily large R satisfying (17) then the result is obvious. We assume henceforth that $R_2 > R_1$ is such that $S(0,R)$ meets the complement of C_0 for all $R \geq R_2$ satisfying (17).

Let $M_2 > 0$, and assume the existence of an unbounded subset E_0 of (R_2, ∞) such that R satisfies (17) for all R in E_0 and such that

$$\log|h(z)| \leq M_2 \log|z| \quad \text{for} \quad z \in C_0, \quad |z| = R \in E_0, \tag{19}$$

such a set E_0 obviously existing if C_0 is bounded. Let R be large, in E_0, and let D be the component of $C_0 \cap B(0,R)$ in which w_0 lies.

Let $D = D_0$, and form domains as follows. If D_m has been formed, choose (if possible) a z_{j_m} such that $S(z_{j_m}, |z_{j_m}|^{-N_2})$ meets the boundary ∂D_m and $S(z_{j_m}, \lambda_m) \subseteq D_m$ for some λ_m with $|z_{j_m}|^{-N_2} \leq \lambda_m \leq |z_{j_m}|^{-N_1}$. If such a z_{j_m} exists, we set $D_{m+1} = D_m \cup B(z_{j_m}, \lambda_m)$, while if no such z_{j_m} exists, we halt the process. Since z_j tends to infinity with j, the process must terminate after the formation of some D_n, and we set $G = D_n$.

It follows easily from the construction that $D_m \subseteq B(0,R)$ and $\partial D_{m+1} \subseteq \partial D_m$ for each m. Thus ∂G is contained in a union of arcs of $S(0,R)$ on which (19) holds, and arcs of level curves $|h(z)| = M_1$, as well as arcs of circles $S(z_j, |z_j|^{-N_2})$.

Suppose that $S_j = S(z_j, |z_j|^{-N_2})$ meets ∂G. Then $|z_j| > \frac{1}{2}R_1$ and $S(z_j, t)$ meets the complement of G for all t with $|z_j|^{-N_2} \leq t \leq |z_j|^{-N_1}$. Thus Lemma 2.3 and (15) give

$$\omega(w_0, S_j, G) \leq 26|z_j|^{\frac{1}{2}(N_1 - N_2)}. \tag{20}$$

To estimate $\omega(w_0, S(0,R), G)$, we note first that $G \subseteq C_0 \cup H_1$. Hence, using (15), we have $\theta_G^*(t) \leq 2\pi$ for $R_2 \leq t \leq R$ and $t \notin E_1$, in which E_1 has measure less than 2. Thus, with d_j positive constants independent of R_2 and R, Lemma 2.2 gives

$$\omega(w_0, S(0,R), G) \leq d_1 \exp\left(-\pi \int_{[2R_2, R/2] \backslash E_1} \frac{dt}{t 2\pi}\right) \leq d_2 (R/R_2)^{-1/2}. \tag{21}$$

Since (14) gives

$$\log|h(z)| \leq |z_j|^\rho \quad \text{for} \quad |z - z_j| \leq 1, \quad |z| \geq R_1,$$

the two-constants theorem and (16), (19), (20) and (21) lead to

$$\begin{aligned} 2\log M_1 &\leq \log|h(w_0)| \\ &\leq \log M_1 + \sum_{|z_j|>\frac{1}{2}R_1} 26|z_j|^{\rho+\frac{1}{2}(N_1-N_2)} + d_3(R/R_2)^{-1/2}M_2\log R, \end{aligned}$$

which is plainly impossible if R is large enough, using (13), (15) and (16) again. □

3. Proof of Theorem 1.5

Take a sequence $S_k, k = 0, 1, \ldots$, such that $f^{-1}(\{\infty\}) \subseteq B(0, \frac{1}{4}S_0)$ and

$$4 < 2S_k < S_{k+1} \tag{22}$$

for each k, and such that

$$\sum_{j=0}^{k} \overline{n}(r, 1/f^{(j)}) < r^{\rho} \tag{23}$$

for $r \geq S_k$. For each k, let $u_{1,k}, \ldots, u_{\lambda_k,k}$ be the distinct points in $S_k \leq |z| < S_{k+1}$ at which at least one of $f, \ldots, f^{(k)}$ vanishes, arranged so as to have non-decreasing modulus. Then (23) gives

$$\sum_{p=1}^{\lambda_k} |u_{p,k}|^{-\rho-1} \leq \sum_{q=1}^{\infty} (2^q S_k)^{\rho}(2^{q-1}S_k)^{-\rho-1} = S_k^{-1}2^{\rho+1}. \tag{24}$$

Arranging the v_μ and $u_{p,k}, 1 \leq p \leq \lambda_k, k = 0, 1, \ldots$, into a sequence (z_j), with $|z_j|$ non-decreasing, it follows from (6), (7), (22) and (24) that

$$\sum_{j=1}^{\infty} |z_j|^{-n_1} < \infty, \quad n_1 = n_0 + \rho + 1, \quad n_2 > 3n_1 + 2\rho. \tag{25}$$

For $m = 1, 2$, define

$$B_m = \bigcup_{j=1}^{\infty} B(z_j, |z_j|^{-n_m}), \tag{26}$$

and let H_m be the union of the corresponding closed discs. We can assume that $|z_j| > 2$ and that all $f^{(m)}, m \geq 0$, are non-zero on all of the circles $S(z_j, |z_j|^{-n_2})$, decreasing n_2 slightly, if necessary, to ensure this. Choose a strictly increasing sequence of positive integers M_k such that, for each non-negative integer k, we have

$$|f^{(k)}(z)/f^{(m)}(z)| \leq |z|^{M_k} \quad \text{for} \quad 0 \leq m \leq k, \quad |z| \geq S_k, \quad z \notin B_2. \tag{27}$$

Such M_k exist, by a standard application of the differentiated Poisson-Jensen formula [8, p.22]. Define

$$g_k(z) = f^{(k)}(z)z^{-2M_k}, \tag{28}$$

and choose large R_k, such that (17) holds for $R = R_k$ and

$$R_k > 4S_k, \quad M(R_k, g_k) > e^4, \quad \log|g_k(z)| \leq |z/2|^{\rho} \quad \text{for} \quad |z| \geq R_k, \tag{29}$$

and, using (25),

$$\sum_{|z_j|>\frac{1}{2}R_k} 26|z_j|^{-n_1} < (k+1)^{-2}, \qquad \sum_{|z_j|>\frac{1}{2}R_k} 26|z_j|^{\rho+\frac{1}{2}(n_1-n_2)} < 1. \tag{30}$$

Finally, choose T_k using Lemma 2.1, such that all $f^{(m)}, m \geq 0$, are non-zero on the level curves $|g_k(z)| = T_k$, and such that

$$T_k > 2^k, \quad T_k > M(R_k, g_k)^2, \quad L(r, T_k, g_k) \leq r^{(4+\rho)/2} \quad \text{for} \quad r \geq \log T_k. \tag{31}$$

Suppose now that k is a non-negative integer and that w_k has been chosen, satisfying

$$|w_k| > R_k, \quad |g_k(w_k)| > T_k^2, \quad w_k \notin H_1, \tag{32}$$

such a w_k plainly existing for $k = 0$. Let C_k be the component of the set $\{z \in \mathbf{C} \backslash H_2 : |g_k(z)| > T_k\}$ in which w_k lies. Then by (29), (30), (31) and Lemma 2.4, there exist an unbounded set E_k such that for all R in E_k we have (17) and

$$\sup\{|g_k(z)| : z \in C_k, |z| = R\} > R^{2M_{k+1}}. \tag{33}$$

In particular C_k is unbounded.

Lemma 3.1. *There exist arbitrarily large R in E_k with*

$$\sup\{|g_{k+1}(z)| : z \in C_k, |z| = R\} > T_{k+1}^2.$$

Proof. This is obviously the case if $S(0, R) \subseteq C_k$ for arbitrarily large R in E_k, and so we assume that $S(0, R)$ meets the complement of C_k for all sufficiently large R in E_k. Let R be large, in E_k, and using (28) and (33) choose ζ_1 in C_k with $|\zeta_1| = R$ and

$$|g_k(\zeta_1)| > R^{2M_{k+1}}, \quad |f^{(k)}(\zeta_1)| > R^{2M_k + 2M_{k+1}}. \tag{34}$$

Follow the circle $|z| = R$ counterclockwise until the first point ζ_2 of intersection with ∂C_k. By (17) and (28) we must have

$$|g_k(\zeta_2)| \leq T_k, \quad |f^{(k)}(\zeta_2)| \leq T_k R^{2M_k}. \tag{35}$$

If $|g_{k+1}(z)| \leq T_{k+1}^2$ for all z on the open arc σ of the circle $S(0, R)$ from ζ_1 to ζ_2, then integration of $f^{(k+1)}$ and (34) and (35) give

$$R^{2M_k + 2M_{k+1}} \leq T_k R^{2M_k} + 2\pi T_{k+1}^2 R^{2M_{k+1}+1},$$

an obvious contradiction if R is large enough. □

We now construct the path γ. Using Lemma 3.1, choose w_{k+1} in $C_k \backslash H_1$ such that $|w_{k+1}| > R_{k+1}$ and $|g_{k+1}(w_{k+1})| > T_{k+1}^2$. For each non-negative integer k, let γ_k be a path from w_k to w_{k+1} in the closure of C_k, consisting of part of the ray $\arg z = \arg w_k$, part of the circle $S(0, |w_{k+1}|)$ and part of the boundary ∂C_k. We can assume that all the $f^{(m)}, m \geq 0$, are non-zero on the ray $\arg z = \arg w_k$ and the circle $S(0, |w_{k+1}|)$, adjusting w_k and w_{k+1} slightly, if necessary, to achieve this. Since $|w_k| > R_k$ and $|g_k(z)| \geq T_k > M(R_k, g_k)^2$ on γ_k, we see that the union of the γ_k forms a path γ which tends to infinity. Also, if $0 \leq m \leq k$, then (27) gives

$$|f^{(m)}(z)| \geq |z|^{-M_k} |f^{(k)}(z)| = |z|^{M_k} |g_k(z)| \geq T_k |z|^{M_k}, \tag{36}$$

for z in the closure of C_k and, since $T_k \to \infty$ and $M_k \to \infty$, this proves (8).

To prove (9) we assume that c is a positive constant, m is a non-negative integer and that k is a positive integer, large compared to m. If a circle $S_j = S(z_j, |z_j|^{-n_2})$ meets γ_k, then (36) gives

$$\int_{S_j \cap \gamma_k} |f^{(m)}(z)|^{-c} |dz| \leq 2\pi |z_j|^{-n_2}. \tag{37}$$

Next, the contributions of the ray $\arg z = \arg w_k$ and the circle $S(0, |w_{k+1}|)$ to the integral $\int_{\gamma_k} |f^{(m)}(z)|^{-c}|dz|$ have sum at most

$$T_k^{-c} \int_{R_k}^{\infty} t^{-cM_k} dt + 2\pi T_k^{-c} |w_{k+1}|^{1-cM_k} < T_k^{-c} \tag{38}$$

if k is large enough. Also, if $T = \{z \in \gamma_k : |g_k(z)| = T_k\}$, then (31) and (36) give

$$\begin{aligned} &\int_T |f^{(m)}(z)|^{-c}|dz| \\ &\quad \le T_k^{-c} \left((\log T_k)^{(4+\rho)/2} + \sum_{q=0}^{\infty} (2^q \log T_k)^{-cM_k} (2^{q+1} \log T_k)^{(4+\rho)/2} \right) \\ &\quad \le 2T_k^{-c} (\log T_k)^{(4+\rho)/2} \end{aligned}$$

if k is large enough. Combining the last estimate with (37) and (38), and using (30), (31) and the fact that $f^{(m)} \neq 0$ on γ, we obtain (9).

4. Proof of Theorem 1.4

Let E and A be as in the hypotheses. We apply Theorem 1.5, with $f = A$ and the v_j the points in $|z| > 2$ at which $E(z) = 0$. Choose $n_0 > 0$ such that (6) holds, and define n_2 by (7). Standard estimates based on the differentiated Poisson-Jensen formula [8, p.22] give a constant $M > 0$ such that

$$|E'(z)/E(z)|^2 + |E''(z)/E(z)| \le |z|^M$$

for all z with $|z| > 4$ and lying outside the union of the discs $B_j = B(v_j, |v_j|^{-n_2})$. Theorem 1.5 gives a path γ tending to infinity and not meeting the B_j, such that

$$\lim_{z \to \infty, z \in \gamma} \frac{\log |A(z)|}{\log |z|} = +\infty,$$

and it follows at once from (2) that $E(z)$ tends to 0 as z tends to infinity on γ.

5. Polynomial coefficients

Lemma 5.1. *Suppose that n is a non-negative integer and $A(z) = a_n z^n + \ldots + a_0$ is a polynomial of degree n, with the coefficients a_j complex numbers. Suppose that E is the product $E = f_1 f_2$ of linearly independent solutions f_j of (1), normalized so that $W(f_1, f_2) = 1$. Finally, suppose that there exist constants $b \neq 0$ and $c > 0$ such that $E(z) = b$ implies that $|E'(z)| \le c$. Then $n = 0$.*

Proof. Suppose that $n > 0$. It is well known that E has order $(n+2)/2$, as may be seen by applying the Wiman-Valiron theory [9] to the first equation of (2). If the equation $E(z) = b$ has finitely many solutions, then $E(z) = P(z)e^{Q(z)} + b$, with P, Q polynomials, and Q of degree $(n+2)/2$. But then $E(z) = 0$ has infinitely many solutions, with $E'(z) = -b(P'(z)/P(z) + Q'(z)) \neq O(1)$, and this is impossible since E is a Bank-Laine function.

We suppose henceforth that the equation $E(z) = b$ has infinitely many solutions, and may assume that infinitely many of these lie in the region S_0 given by

$$S_0 = \{z : \quad |z| \ge R_0 > 0, \quad |\arg z - \theta_0| \le \pi/(n+2)\}, \tag{39}$$

in which θ_0 is real and satisfies $\arg a_n + (n+2)\theta_0 = 0 \,(\mathrm{mod}\, 2\pi)$, so that $\arg z = \theta_0$ is a critical ray for (1) [11]. By Hille's asymptotic method [11], the equation (1) has *principal* solutions u_1, u_2 in S_0 satisfying

$$u_j(z) = A(z)^{-1/4}(1+o(1))\exp((-1)^j iZ), \tag{40}$$

and

$$u_j'(z) = (-1)^j iA(z)^{1/4}(1+o(1))\exp((-1)^j iZ), \tag{41}$$

with

$$Z = \int_{R_0 e^{i\theta_0}}^{z} A(t)^{1/2}dt = \frac{2a_n^{1/2}}{n+2} z^{(n+2)/2}(1+o(1)). \tag{42}$$

Obviously, there exist complex numbers B_1, B_2, B_3, not all 0, such that

$$E = B_1 u_1^2 + B_2 u_1 u_2 + B_3 u_2^2$$

and so, in S_0, using (40), (41) and (42),

$$E(z) = A(z)^{-1/2}\left(B_1 e^{-2iZ}(1+o(1)) + B_2(1+o(1)) + B_3 e^{2iZ}(1+o(1))\right) \tag{43}$$

and

$$E'(z) = -2iB_1 e^{-2iZ}(1+o(1)) + o(1) + 2iB_3 e^{2iZ}(1+o(1)). \tag{44}$$

It follows from (43) that there exists $d > 0$ such that if z_1 is in S_0, with $|z_1|$ large and $E(z_1) = b$ then

$$\max\{|B_1 e^{-2iZ(z_1)}|, |B_3 e^{2iZ(z_1)}|\} > d|A(z_1)|^{1/2}.$$

But then (44) gives $|E'(z_1)| > 2c$ if $|z_1|$ is large enough, and this contradiction proves the lemma. $\square$

6. Proof of Theorem 1.3: preliminaries

Let b_1 and b_2 be non-zero complex numbers. Let g be a transcendental entire function such that $g(z) = 1$ implies that $g'(z) = \pm b_1$, and $g(z) = -1$ implies that $g'(z) = \pm b_2$. Set

$$H = \frac{(g')^2 + \lambda(g-1) - b_1^2}{g^2 - 1}, \quad b_2^2 - 2\lambda - b_1^2 = 0. \tag{45}$$

Then H is an entire function, and applying the Wiman-Valiron theory [9] to g shows that $H \not\equiv 0$. Also

$$T(r,H) = m(r,H) \leq m(r, 1/(g-1)) + m(r, 1/(g+1)) + s(r,g), \tag{46}$$

in which $s(r,g)$ denotes any quantity which is $o(T(r,g))$ as r tends to infinity outside a set of finite logarithmic measure. Differentiating (45) gives

$$g'(2g'' + \lambda - 2gH) = H'(g^2-1)$$

from which we see at once that if H is a non-zero constant then g has the form (5). Assuming henceforth that H is non-constant we have, since $g'(z) = 0$ implies that $g(z) \neq \pm 1$,

$$N(r, 1/g') \leq N(r, 1/H') \leq T(r,H) + s(r,H)$$

and, using (46),

$$N(r, 1/g') + N(r, 1/(g-1)) + N(r, 1/(g+1)) \leq 2T(r,g) + s(r,g). \tag{47}$$

We apply Ahlfors' theory of covering surfaces to g, using the notation of [8, pp.144-149]. Let $0<\varepsilon<1/2$ and

$$S(r)=\frac{1}{\pi}\int_0^r\int_0^{2\pi}\frac{t|g'(te^{i\theta})|^2}{(1+|g(te^{i\theta})|^2)^2}d\theta dt,\quad L(r)=\int_0^{2\pi}\frac{r|g'(re^{i\theta})|}{1+|g(re^{i\theta})|^2}d\theta \tag{48}$$

for $r>0$. Since g has no poles, [8, Theorem 5.5] implies that there is a positive constant h depending only on ε such that

$$S(r)\leq n(r)-n_1(r)+hL(r)\quad\text{for}\quad r>0. \tag{49}$$

Here $n(r)$ denotes the total multiplicity of all the islands [8, p.145] for g over $B(\pm 1,\varepsilon)$ in $B(0,r)$, and $n_1(r)$ denotes the sum of the excesses [8, p.147] of these islands. If an island has multiplicity p and connectivity q (each necessarily at least 1), then the excess of the island is $(p-1)+(q-1)$. Let $n^s(r)$ count the number of simple islands over $B(\pm 1,\varepsilon)$ in $B(0,r)$, and let $n^m(r)$ and $n_1^m(r)$ be the total multiplicity and total excess of all islands in $B(0,r)$ of multiplicity at least 2.

By (49), the Ahlfors-Shimizu characteristic $T_0(r,g)$ [8, p.12] satisfies

$$T_0(r,g)\leq N(r)-N_1(r)+h\int_0^r\frac{L(t)dt}{t},\quad N(r)=\int_0^r\frac{n(t)dt}{t}. \tag{50}$$

We define $N_1(r),N^s(r),N^m(r),N_1^m(r)$ in the same way as $N(r)$. Further, by [19] and the fact [8, p.13] that $T_0(r,g)=T(r,g)+O(1)$, we have

$$\int_0^r\frac{L(t)dt}{t}=s(r,g). \tag{51}$$

Lemma 6.1. *We have*

$$T(r,g)\leq 2N^s(r)+s(r,g). \tag{52}$$

Proof. If an island D over $B(\pm 1,\varepsilon)$ has multiplicity $p\geq 2$ and connectivity q, then the excess of D is

$$(p-1)+(q-1)\geq p-1=p(1-1/p)\geq p/2.$$

Thus $N^m(r)\leq 2N_1^m(r)$ and (50) and (51) give

$$2T(r,g)\leq 2N^s(r)+N^m(r)+s(r,g). \tag{53}$$

Next, if an island D' has multiplicity $p'\geq 1$ then one of the equations $g(z)=\pm 1$ has p' solutions in D'. Hence

$$n^s(r)+n^m(r)\leq n(r,1/(g-1))+n(r,1/(g+1))$$

and (53) gives

$$2T(r,g)\leq N^s(r)+N(r,1/(g-1))+N(r,1/(g+1))+s(r,g). \tag{54}$$

Combining (47) and (54) we get

$$2T(r,g)+N(r,1/g')\leq N^s(r)+2T(r,g)+s(r,g)$$

and so

$$N(r,1/g')\leq N^s(r)+s(r,g). \tag{55}$$

Suppose again that D is an island of multiplicity $p\geq 2$ and connectivity q. Then by the Riemann-Hurwitz formula [26, p.7], the number y of critical points of g in D, counting multiplicity, satisfies

$$q-2=p(-1)+y,\quad y=p+q-2\geq p-1=p(1-1/p)\geq p/2.$$

Thus (55) gives

$$N^m(r) \leq 2N(r, 1/g') \leq 2N^s(r) + s(r, g),$$

which, on substitution into (53), gives (52). Lemma 6.1 is proved. □

Next, set

$$E_1 = \frac{g-1}{b_1}, \quad E_2 = \frac{g+1}{b_2}. \tag{56}$$

Then E_1, E_2 are Bank-Laine functions. As in Section 1 define entire functions A_1, A_2 by

$$4A_j = (E_j'/E_j)^2 - 2E_j''/E_j - 1/E_j^2. \tag{57}$$

Lemma 6.2. *Let $d = \pm 1$. There exists a positive constant c_1, depending only on b_1 and b_2, such that if g maps the domain D univalently onto $B(d, 1/4)$, then $D^* = \{z \in D : 1/8 < |g(z) - d| < 1/4\}$ has area $A^* \geq c_1$.*

Proof. The function $G = 4(g - d)$ maps D univalently onto $B(0, 1)$, and we denote by $\psi(w)$ the inverse function $G^{-1}(w)$. The Koebe distortion theorem [21, p.9] gives $|\psi'(w)| \geq c_2$ for $1/2 \leq |w| \leq 3/4$, in which c_2 is a positive constant depending only on b_1 and b_2. Lemma 6.2 now follows at once from the standard formula for A^* as the integral of $|\psi'(w)|^2$. □

Lemma 6.3. *We have*

$$T(r, g) = O(r^2), \quad r \to \infty. \tag{58}$$

Proof. We have, using Lemma 6.2,

$$n^s(r) = O(r^2), \quad N^s(r) = O(r^2),$$

and the result follows using Lemma 6.1. □

Proposition 6.1. *At least one of A_1, A_2 is a polynomial.*

The proof of Proposition 6.1 will require the remainder of this section. Assume that A_1, A_2 are both transcendental. Since E_1, E_2 have finite order, by Lemma 6.3, each E_j has asymptotic value 0, using Theorem 1.4, and so 1 and -1 are asymptotic values of g. Further, we may choose positive constants M_1, M_2 and a sequence $v_q \to \infty$ such that

$$|E_j'(z)/E_j(z)|^2 + 2|E_j''(z)/E_j(z)| \leq |z|^{M_1}, \quad j = 1, 2, \tag{59}$$

for all z with $|z| > 2$ and lying outside the set

$$V = \bigcup_{q=1}^{\infty} B(v_q, |v_q|^{-M_2}), \quad \sum_{q=1}^{\infty} |v_q|^{-M_2} < \infty. \tag{60}$$

By (56), (57) and (59), there exists $M_3 > 4$ such that if $|z| > M_3$ and z lies outside V with $|A_j(z)| > |z|^{M_1+1}$, then $E_j(z)$ is small and $|g(z) + (-1)^j| < 1/8$.

Choose

$$M_4 > M(M_3, A_1) + M(M_3, A_2) + M(M_3, g) + 2$$

and, for $j = 1, 2$, let D_j be an unbounded component of the set $\{z : |A_j(z)z^{-M_1-1}| > M_4\}$, such a D_j existing since A_j is assumed transcendental. By the choice of M_3, M_4, the domains D_1, D_2 lie in $|z| > M_3$ and $D_1 \cap D_2 \subseteq V$.

Since g has asymptotic values $1, -1$, there exist at least two components D_3, D_4 of the set $\{z : |g(z)| > M_4\}$, both lying in $|z| > M_3$, and $D_3 \cap (D_1 \cup D_2) \subseteq V, D_4 \cap (D_1 \cup D_2) \subseteq V$, again by the choice of M_3, M_4.

For $t \geq M_3$, let $\theta_j(t)$ be the angular measure of the intersection of D_j with the circle $S(0,t)$. Define $\theta_j^*(t)$ to equal $\theta_j(t)$, except that $\theta_j^*(t) = \infty$ if the whole circle $S(0,t)$ lies in D_j. There exists $M_5 > 0$ such that $\theta_j(t) > 0$ for each j and for all $t \geq M_5$. Since the intersection of D_j and $D_{j'}$, for $j \neq j'$, lies in V, we have $\theta_j(t) = \theta_j^*(t)$ for all t outside a set F_0 of finite measure. Hence, by Lemma 6.3 and a standard application of Lemma 2.2, we have

$$\int_{M_5}^{r} \frac{\pi dt}{t\theta_j(t)} \leq \int_{M_5}^{r} \frac{\pi dt}{t\theta_j^*(t)} + O(1) \leq 2\log r + O(1) \tag{61}$$

as $r \to \infty$.

Let $\theta(t) = \pi$ for $t \in F_0$, and let

$$\theta(t) = 2\pi - \sum_{j=1}^{4} \theta_j(t) \geq 0, \quad t \notin F_0. \tag{62}$$

For $t \notin F_0$ a standard application of the Cauchy-Schwarz inequality gives

$$16 \leq (2\pi - \theta(t)) \sum_{j=1}^{4} \frac{1}{\theta_j(t)} \leq 2\pi \sum_{j=1}^{4} \frac{1}{\theta_j(t)}, \tag{63}$$

and so

$$8\log r - O(1) \leq \sum_{j=1}^{4} \int_{M_5}^{r} \frac{\pi dt}{t\theta_j(t)}. \tag{64}$$

Combining (61) and (64) we see that, for each j,

$$\int_{M_5}^{r} \frac{\pi dt}{t\theta_j(t)} = 2\log r + O(1) \tag{65}$$

as $r \to \infty$.

Lemma 6.4. *g has order 2, mean type.*

Proof. Lemma 6.4 follows at once from Lemma 6.3 and (65). □

We now have, as $r \to \infty$, using (63) again,

$$\sum_{j=1}^{4} \int_{M_5}^{r} \frac{\pi dt}{t\theta_j(t)} \geq \int_{M_5}^{r} \frac{16\pi dt}{t(2\pi - \theta(t))} - O(1) \geq 8\log r - O(1) + \int_{M_5}^{r} \frac{4\theta(t)dt}{\pi t}$$

and hence, using (65),

$$\int_{M_5}^{r} \frac{\theta(t)dt}{t} = O(1).$$

It follows that if the positive integer n is large, then

$$\int_{2^n}^{2^{n+1}} t\theta(t)dt \leq 4^{n+1} \int_{2^n}^{2^{n+1}} \frac{\theta(t)dt}{t} = o(4^n),$$

and so

$$\int_{1}^{r} t\theta(t)dt = o(r^2) \tag{66}$$

as $r \to \infty$.

Now suppose that $d = \pm 1$ and r is large, and that D is a simple island for g over $B(d, 1/4)$, lying in $M_5 < |z| < r$. Then D obviously cannot meet D_3 or D_4, and $D^* = \{z \in D : 1/8 < |g(z) - d| < 1/4\}$ has $D^* \cap D_1 \subseteq V$, $D^* \cap D_2 \subseteq V$. Since D^* has area $A^* \geq c_1$, by Lemma 6.2, and the sum of A^* over all such D is at most

$$2 \int_1^r t\theta(t) dt + O(1),$$

we obtain from (66),

$$n^s(r) = o(r^2), \quad N^s(r) = o(r^2),$$

and so $T(r, g) = o(r^2)$, using Lemma 6.1. This contradicts Lemma 6.4, and proves Proposition 6.1.

7. Proof of Theorem 1.3

Suppose that g is as in the hypotheses. Applying a linear transformation we may assume without loss of generality that $a_1 = 1, a_2 = -1$. Thus the functions E_1, E_2 defined by (56) are Bank-Laine functions, and A_1, A_2 as defined by (57) are entire.

If g is a non-constant polynomial then so is E_1, and it follows at once from (57) that $A_1 \equiv 0$. Thus E_1 has degree at most 2 and so has g. We assume henceforth that g is transcendental.

The function g thus satisfies the assumptions of Section 6, and by Proposition 6.1 at least one of A_1, A_2, say A_1, is a polynomial. But then, by Lemma 5.1, A_1 must be constant, and (5) follows at once.

8. Proof of Theorem 1.2

Suppose that f and g are transcendental, with g entire and f meromorphic in the plane, and that the composition $E = f \circ g$ is a Bank-Laine function of finite order. Define the entire function A by (2).

Then f has order 0 [8, p.53], and consequently has infinitely many zeros. It follows from Theorem 1.1 that g has the form (5), without loss of generality with $b = 1/2$, so that E and A have period $2\pi i$. We may therefore write

$$E(z) = F(e^z) = F_1(e^z) + F_2(e^{-z}), \quad A(z) = B(e^z) = B_1(e^z) + B_2(e^{-z}),$$

in which F_1, F_2, B_1, B_2 are entire functions.

Lemma 8.1. *F_1, F_2, B_1, B_2 have order 0.*

Proof. Let ρ be large and positive, and choose a real t with $|F_1(\rho e^{it})| = M(\rho, F_1)$. Next, choose $z_0 = \log \rho + is, -\pi \leq s \leq \pi$, such that $e^{z_0} = \rho e^{it}$. We then have

$$\begin{aligned} M(\rho, F_1) &= |F(\rho e^{it})| + O(1) = |E(z_0)| + O(1) \\ &\leq \exp\left(c_1 |z_0|^{M_1}\right) \leq \exp\left(c_2 (\log \rho)^{M_1}\right), \end{aligned}$$

for some positive constants c_1, c_2, M_1 independent of ρ. □

The proof of Theorem 1.2 now proceeds using ideas from the theory of periodic differential equations [2]. Since (1) has linearly independent solutions each having zeros with finite exponent of convergence, [4, Theorem 4] shows that B is a rational function. Substituting $E(z) = F(e^z), A(z) = B(e^z)$ into the second equation of (2) gives a third order linear differential equation for F, with coefficients which are rational functions. Since F has order 0, a standard application of the Wiman-Valiron theory [9] shows that F is a rational function, and so is f. This contradiction proves the theorem.

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