

EXISTENCE OF CURVES WITH PRESCRIBED TOPOLOGICAL SINGULARITIES

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ABSTRACT. Throughout this paper we study the existence of irreducible curves C on smooth projective surfaces Σ with singular points of prescribed topological types $\mathcal{S}_1, \dots, \mathcal{S}_r$. There are *necessary* conditions for the existence of the type $\sum_{i=1}^r \mu(\mathcal{S}_i) \leq \alpha C^2 + \beta C \cdot K + \gamma$ for some fixed divisor K on Σ and suitable coefficients α, β and γ , and the main *sufficient* condition that we find is of the same type, saying it is *asymptotically proper*. Ten years ago general results of this quality were not known even for the case $\Sigma = \mathbb{P}_{\mathbb{C}}^2$.

An important ingredient for the proof is a vanishing theorem for invertible sheaves on the blown up Σ of the form $\mathcal{O}_{\tilde{\Sigma}}(\pi^*D - \sum_{i=1}^r m_i E_i)$, deduced from the Kawamata-Viehweg Vanishing Theorem. Its proof covers the first part of the paper, while the middle part is devoted to the existence theorems. In the last part we investigate our conditions on ruled surfaces, products of elliptic curves, surfaces in $\mathbb{P}_{\mathbb{C}}^3$, and K3-surfaces.

1. INTRODUCTION

General Assumptions and Notations. Throughout this paper Σ will be a smooth projective surface over \mathbb{C} .

Given distinct points $z_1, \dots, z_r \in \Sigma$, we denote by $\pi : \tilde{\Sigma} = \text{Bl}_{\underline{z}}(\Sigma) \rightarrow \Sigma$ the blow up of Σ in $\underline{z} = (z_1, \dots, z_r)$, and the exceptional divisors π^*z_i will be denoted by E_i , $i = 1, \dots, r$. We shall write $\tilde{C} = \text{Bl}_{\underline{z}}(C)$ for the strict transform of a curve $C \subset \Sigma$.

For any smooth surface S we will denote by $\text{Div}(S)$ the group of divisors on S and by K_S its canonical divisor. If D is any divisor on S , $\mathcal{O}_S(D)$ shall be a corresponding invertible sheaf. $|D|_l = \mathbb{P}(H^0(S, \mathcal{O}_S(D)))$ denotes the system of curves linearly equivalent to D , while we use the notation $|D|_a$ for the system of curves algebraically equivalent to D (cf. [Har77] Ex. V.1.7), that is the reduction of the connected component of Hilb_S , the Hilbert scheme of all curves on S , containing any curve algebraically equivalent to D (cf. [Mum66] Chapter 15). We will use the notation $\text{Pic}(S)$ for the Picard group of S , that is $\text{Div}(S)$ modulo linear equivalence (denoted by \sim_l), $\text{NS}(S)$ for the Néron-Severi group, that is $\text{Div}(S)$ modulo algebraic

Received by the editors December 1, 2000 and, in revised form, March 22, 2001.

2000 *Mathematics Subject Classification*. Primary 14H10, 14H15, 14H20; Secondary 14J26, 14J27, 14J28, 14J70.

Key words and phrases. Algebraic geometry, singularity theory.

The first author was partially supported by the DFG Schwerpunkt “Globale Methoden in der komplexen Geometrie”. The second author was supported in part by the Herman Minkowsky–Minerva Center for Geometry at Tel–Aviv University, and by grant no. G0419-039.06/95 from the German-Israeli Foundation for Research and Development.

The authors would like to express their thanks to Gert-Martin Greuel, Christoph Lossen, and Eugenii Shustin for bringing the subject to their attention and for many helpful discussions.

equivalence (denoted by \sim_a), and $\text{Num}(S)$ for $\text{Div}(S)$ modulo numerical equivalence (denoted by \sim_n). Note that for all examples of surfaces Σ which we consider in Section 5 $\text{NS}(\Sigma)$ and $\text{Num}(\Sigma)$ coincide.

Given a curve $C \subset \Sigma$ we will write $p_a(C)$ for its arithmetical genus and $g(C)$ for the geometrical one.

Let Y be a Zariski topological space. We say a subset $U \subseteq Y$ is *very general* if it is an at most countable intersection of open dense subsets of Y . A statement is said to hold for points $z_1, \dots, z_r \in Y$ (or $\underline{z} \in Y^r$) *in very general position* if there is a suitable very general subset $U \subseteq Y^r$, contained in the complement of the closed subvariety $\bigcup_{i \neq j} \{z \in Y^r \mid z_i = z_j\}$ of Y^r , such that the statement holds for all $\underline{z} \in U$. The main results of this paper will only be valid for points in very general position.

Given distinct points $z_1, \dots, z_r \in \Sigma$ and non-negative integers m_1, \dots, m_r we denote by $X(\underline{m}; \underline{z}) = X(m_1, \dots, m_r; z_1, \dots, z_r)$ the zero-dimensional subscheme of Σ defined by the ideal sheaf $\mathcal{J}_{X(\underline{m}; \underline{z})/\Sigma}$ with stalks

$$\mathcal{J}_{X(\underline{m}; \underline{z})/\Sigma, z} = \begin{cases} \mathfrak{m}_{\Sigma, z_i}^{m_i}, & \text{if } z = z_i, i = 1, \dots, r, \\ \mathcal{O}_{\Sigma, z}, & \text{else.} \end{cases}$$

We call a scheme of the type $X(\underline{m}; \underline{z})$ a *generic fat point scheme*.

For a reduced curve $C \subset \Sigma$ we define the zero-dimensional subscheme $X^{es}(C)$ of Σ via the ideal sheaf $\mathcal{J}_{X^{es}(C)/\Sigma}$ with stalks

$$\mathcal{J}_{X^{es}(C)/\Sigma, z} = I^{es}(C, z) = \{g \in \mathcal{O}_{\Sigma, z} \mid f + \varepsilon g \text{ is equisingular over } \mathbb{C}[\varepsilon]/(\varepsilon^2)\},$$

where $f \in \mathcal{O}_{\Sigma, z}$ is a local equation of C at z . $I^{es}(C, z)$ is called the *equisingularity ideal* of the singularity (C, z) , and it is of course $\mathcal{O}_{\Sigma, z}$ whenever z is a smooth point. If x, y are local coordinates of Σ at z , then $I^{es}(C, z)/(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ can be identified with the tangent space of the equisingular stratum in the semiuniversal deformation of (C, z) (cf. [Wah74], [DH88], and Definition 4.1). We call $X^{es}(C)$ the *equisingularity scheme of C* .

If $X \subset \Sigma$ is any zero-dimensional scheme with ideal sheaf \mathcal{J}_X and if $L \subset \Sigma$ is any curve with ideal sheaf \mathcal{L} , we define the *residue scheme* $X : L \subset \Sigma$ by the ideal sheaf $\mathcal{J}_{X:L/\Sigma} = \mathcal{J}_X : \mathcal{L}$ with stalks

$$\mathcal{J}_{X:L/\Sigma, z} = \mathcal{J}_{X, z} : l_z,$$

where $l_z \in \mathcal{O}_{\Sigma, z}$ is a local equation for L and “:” denotes the ideal quotient. This naturally leads to the definition of the *trace scheme* $X \cap L \subset L$ via the ideal sheaf $\mathcal{J}_{X \cap L/L}$ given by the exact sequence

$$0 \longrightarrow \mathcal{J}_{X:L/\Sigma}(-L) \xrightarrow{\cdot L} \mathcal{J}_{X/\Sigma} \longrightarrow \mathcal{J}_{X \cap L/L} \longrightarrow 0.$$

Given topological singularity types¹ $\mathcal{S}_1, \dots, \mathcal{S}_r$ and a divisor $D \in \text{Div}(\Sigma)$, we denote by $V_{|D|}(\mathcal{S}_1, \dots, \mathcal{S}_r)$ the locally closed subspace of $|D|_l$ of reduced curves in the linear system $|D|_l$ having precisely r singular points of types $\mathcal{S}_1, \dots, \mathcal{S}_r$. Analogously, $V_{|D|}(m_1, \dots, m_r) = V_{|D|}(\underline{m})$ denotes the locally closed subspace of $|D|_l$ of reduced curves having precisely r ordinary singular points of multiplicities m_1, \dots, m_r (cf. [Los98] 1.3.2).

The spaces $V = V_{|D|}(\mathcal{S}_1, \dots, \mathcal{S}_r)$ respectively $V = V_{|D|}(\underline{m})$ are the main objects of interest of this paper. We say V is *T-smooth* at $C \in V$ if the germ (V, C)

¹For the definition of a singularity type and more information see [Los98] 1.2.

is smooth of the (expected) dimension $\dim |D|_l - \deg(X)$, where $X = X^{es}(C)$ respectively $X = X(\underline{m}; \underline{z})$ with $\text{Sing}(C) = \{z_1, \dots, z_r\}$. By [Los98] Proposition 2.1 T -smoothness of V at C is implied by the vanishing of $H^1(\Sigma, \mathcal{J}_{X/\Sigma}(C))$.

It is the aim of this paper to give sufficient conditions for the non-emptiness of V in terms of the linear system $|D|_l$ and invariants of the imposed singularities. The results are generalisations of known results for $\mathbb{P}_{\mathbb{C}}^2$, and for an overview on these we refer to [Los98] Chapter 4.

We basically follow the ideas described in [Los98] 4.1.2. The case of ordinary singularities (Corollary 3.4) is treated by applying a vanishing theorem for generic fat point schemes (Theorem 2.1), and the more interesting case of prescribed topological types $\mathcal{S}_1, \dots, \mathcal{S}_r$ is then dealt with by gluing local equations into a curve with ordinary singularities. Upper bounds for the minimal possible degrees of these local equations can be taken from the $\mathbb{P}_{\mathbb{C}}^2$ -case (cf. [Los98] Theorem 4.2).

Thus the main results of this paper are the following theorems and their corollaries Corollary 3.4 and Corollary 4.4.

Theorem 2.1. *Let $m_1, \dots, m_r \in \mathbb{N}_0$, and $D \in \text{Div}(\Sigma)$ be a divisor satisfying the following three conditions*

$$(2.1) \quad (D - K_{\Sigma})^2 \geq 2 \sum_{i=1}^r (m_i + 1)^2,$$

$$(2.2) \quad (D - K_{\Sigma}).B > \max\{m_i \mid i = 1, \dots, r\} \\ \text{for any irreducible curve } B \\ \text{with } B^2 = 0 \text{ and } \dim |B|_a > 0, \text{ and}$$

$$(2.3) \quad D - K_{\Sigma} \text{ is nef.}$$

Then for $z_1, \dots, z_r \in \Sigma$ in very general position and $\nu > 0$

$$H^{\nu} \left(\text{Bl}_{\underline{z}}(\Sigma), \pi^* D - \sum_{i=1}^r m_i E_i \right) = 0.$$

In particular,

$$H^{\nu}(\Sigma, \mathcal{J}_{X(\underline{m}; \underline{z})/\Sigma}(D)) = 0.$$

Theorem 3.3. *Given $m_1, \dots, m_r \in \mathbb{N}_0$, not all zero, and $z_1, \dots, z_r \in \Sigma$, $r \geq 1$, in very general position. Let $L \in \text{Div}(\Sigma)$ be very ample over \mathbb{C} , and let $D \in \text{Div}(\Sigma)$ be such that*

$$(3.3) \quad h^1(\Sigma, \mathcal{J}_{X(\underline{m}; \underline{z})/\Sigma}(D - L)) = 0, \text{ and}$$

$$(3.4) \quad D.L - 2g(L) \geq m_i + m_j \text{ for all } i, j.$$

Then there exists a curve $C \in |D|_l$ with ordinary singular points of multiplicity m_i at z_i for $i = 1, \dots, r$ and no other singular points. Furthermore,

$$h^1(\Sigma, \mathcal{J}_{X(\underline{m}; \underline{z})/\Sigma}(D)) = 0,$$

and in particular, $V_{|D|}(\underline{m})$ is T -smooth at C .

If in addition (3.5) $D^2 > \sum_{i=1}^r m_i^2$, then C can be chosen to be irreducible and reduced.

Theorem 4.3 (Existence). *Let $\mathcal{S}_1, \dots, \mathcal{S}_r$ be singularity types, and suppose there exists an irreducible curve $C \in |D|_l$ with $r+r'$ ordinary singular points $z_1, \dots, z_{r+r'}$*

of multiplicities $m_1, \dots, m_{r+r'}$ respectively as its only singularities such that $m_i = s(\mathcal{S}_i) + 1$, for $i = 1, \dots, r$, and

$$h^1(\Sigma, \mathcal{J}_{X(\underline{m}; \underline{z})/\Sigma}(D)) = 0.$$

Then there exists an irreducible curve $\tilde{C} \in |D|_i$ with r singular points of types $\mathcal{S}_1, \dots, \mathcal{S}_r$ and r' ordinary singular points of multiplicities $m_{r+1}, \dots, m_{r+r'}$ as its only singularities.²

Of course, combining the vanishing theorem Theorem 2.1 with the existence theorems Theorem 3.3 and Theorem 4.3 we get sufficient numerical conditions for the existence of curves with certain singularities (see Corollaries 3.4 and 4.4, and see Section 5 for special surfaces).

Given any scheme X and any coherent sheaf \mathcal{F} on X , we will often write $H^\nu(\mathcal{F})$ instead of $H^\nu(X, \mathcal{F})$ when no ambiguity can arise. Moreover, if $\mathcal{F} = \mathcal{O}_X(D)$ is the invertible sheaf corresponding to a divisor D , we will usually use the notation $H^\nu(X, D)$ instead of $H^\nu(X, \mathcal{O}_X(D))$.

Section 2 is devoted to the proof of the vanishing theorem Theorem 2.1, and the following sections, Section 3 and Section 4, are concerned with the existence theorems Theorem 3.3 and Theorem 4.3, while in Section 5 we calculate the conditions which we have found in the case of ruled surfaces, products of elliptic curves, surfaces in $\mathbb{P}_{\mathbb{C}}^3$, and K3-surfaces.

We finally would like to remark that one could replace topological singularity types by analytical singularity types. Making the obvious changes the result of Theorem 4.3 remains valid. For more information on the equianalytical singularity scheme, T-smoothness in the analytical case, etc. see [Los98].³

2. THE VANISHING THEOREM

Theorem 2.1. *Let $m_1, \dots, m_r \in \mathbb{N}_0$, and $D \in \text{Div}(\Sigma)$ be a divisor satisfying the following three conditions*

$$(2.1) \quad (D - K_\Sigma)^2 \geq 2 \sum_{i=1}^r (m_i + 1)^2,$$

$$(2.2) \quad (D - K_\Sigma).B > \max\{m_i \mid i = 1, \dots, r\} \\ \text{for any irreducible curve } B \\ \text{with } B^2 = 0 \text{ and } \dim |B|_a > 0, \text{ and}$$

$$(2.3) \quad D - K_\Sigma \text{ is nef.}$$

Then for $z_1, \dots, z_r \in \Sigma$ in very general position and $\nu > 0$

$$H^\nu \left(\text{Bl}_{\underline{z}}(\Sigma), \pi^* D - \sum_{i=1}^r m_i E_i \right) = 0.$$

In particular,

$$H^\nu(\Sigma, \mathcal{J}_{X(\underline{m}; \underline{z})/\Sigma}(D)) = 0.$$

²Here, of course, $\underline{m} = (m_1, \dots, m_{r+r'})$ and $\underline{z} = (z_1, \dots, z_{r+r'})$. See Definition 4.1 for the definition of $s(\mathcal{S}_i)$.

³At the time of proofreading the bounds for $s(\mathcal{S}_i)$ have been considerably improved and even corresponding bounds for analytical singularity types have been found by E. Shustin. This improves also the existence conditions in Corollary 4.4 and Section 5 without changing the proofs. For details we refer to E. Shustin, *Analytic order of singular and critical points*, Preprint 2001.

Proof. By the Kawamata-Viehweg Vanishing Theorem (cf. [Kaw82] and [Vie82]) it suffices to show that $A = (\pi^*D - \sum_{i=1}^r m_i E_i) - K_{\tilde{\Sigma}}$ is big and nef, i.e. we have to show:

- (a) $A^2 > 0$, and
- (b) $A.B' \geq 0$ for any irreducible curve B' in $\tilde{\Sigma} = \text{Bl}_{\underline{z}}(\Sigma)$.

Note that $A = \pi^*(D - K_{\Sigma}) - \sum_{i=1}^r (m_i + 1)E_i$, and thus by hypothesis (2.1) we have

$$A^2 = (D - K_{\Sigma})^2 - \sum_{i=1}^r (m_i + 1)^2 > 0,$$

which gives condition (a).

For condition (b) we observe that an irreducible curve B' on $\tilde{\Sigma}$ is either the strict transform of an irreducible curve B in Σ or is one of the exceptional curves E_i . In the latter case we have

$$A.B' = A.E_i = m_i + 1 > 0.$$

We may, therefore, assume that $B' = \tilde{B}$ is the strict transform of an irreducible curve B on Σ having multiplicity $\text{mult}_{z_i}(B) = n_i$ at z_i , $i = 1, \dots, r$. Then

$$A.B' = (D - K_{\Sigma}).B - \sum_{i=1}^r (m_i + 1)n_i,$$

and thus condition (b) is equivalent to (b'): $(D - K_{\Sigma}).B \geq \sum_{i=1}^r (m_i + 1)n_i$. Since \underline{z} is in very general position Lemma 2.2 applies. Using the Hodge Index Theorem, hypothesis (2.1), Lemma 2.2, and the Cauchy-Schwarz inequality we get the following sequence of inequalities:

$$\begin{aligned} ((D - K_{\Sigma}).B)^2 &\geq (D - K_{\Sigma})^2 \cdot B^2 \\ &\geq \left(2 \sum_{i=1}^r (m_i + 1)^2\right) \cdot B^2 \\ &\geq 2 \left(\sum_{i=1}^r (m_i + 1)^2\right) \cdot \left(\sum_{i=1}^r n_i^2 - n_{i_0}\right) \\ &= \sum_{i=1}^r (m_i + 1)^2 \cdot \sum_{i=1}^r n_i^2 + \left(\sum_{i=1}^r (m_i + 1)^2 \cdot \left(\sum_{i=1}^r n_i^2 - 2n_{i_0}\right)\right) \\ &\geq \left(\sum_{i=1}^r (m_i + 1)n_i\right)^2 + \left(\sum_{i=1}^r (m_i + 1)^2 \cdot \left(\sum_{i=1}^r n_i^2 - 2n_{i_0}\right)\right), \end{aligned}$$

where $i_0 \in \{1, \dots, r\}$ is such that $n_{i_0} = \min\{n_i | n_i \neq 0\}$. Since $D - K_{\Sigma}$ is nef, condition (b') is satisfied as soon as we have

$$(2.4) \quad \sum_{i=1}^r n_i^2 \geq 2n_{i_0},$$

which leaves us with the case

$$n_{i_0} = 1, \text{ and } n_j = 0 \text{ for } j \neq i_0.$$

In this situation (b') reads

$$(D - K_\Sigma).B \geq m_{i_0} + 1.$$

Note that since the z_i are in very general position and $z_{i_0} \in B$ we have that $B^2 \geq 0$ and $\dim |B|_a > 0$. If $B^2 > 0$ then we are done by the Hodge Index Theorem and hypothesis (2.1), since $D - K_\Sigma$ is nef:

$$(D - K_\Sigma).B \geq \sqrt{(D - K_\Sigma)^2} \geq \sqrt{\sum_{i=1}^r (m_i + 1)^2} \geq m_{i_0} + 1.$$

It remains to consider the case where $B^2 = 0$ which is covered by hypothesis (2.2).

For the “in particular” part we just note that

$$H^\nu(\Sigma, \mathcal{J}_{X(\underline{m}; \underline{z})/\Sigma}(D)) = H^\nu\left(\tilde{\Sigma}, \pi^*D - \sum_{i=1}^r m_i E_i\right).$$

□

A proof for the next lemma in the case $r = 1$ in [EL93]. For the $\mathbb{P}_{\mathbb{C}}^2$ -case we also refer to [Xu94]. The slightly more general situation can be proved analogously. See also [GS84] for a generalisation in the case when one knows more about the singular points than just their multiplicities.

Lemma 2.2. *Let $\underline{z} = (z_1, \dots, z_r) \in \Sigma^r$ be in very general position, $\underline{n} \in \mathbb{N}_0^r$, and let $B \subset \Sigma$ be an irreducible curve with $\text{mult}_{z_i}(B) \geq n_i$, then*

$$B^2 \geq \sum_{i=1}^r n_i^2 - \min\{n_i | n_i \neq 0\}.$$

Remark 2.3. Condition (2.3) is in several respects “expectable”. First, the theorem is a corollary of the Kawamata-Vieweg Vanishing Theorem, and if we take all m_i to be zero, our assumptions should basically be the same, i.e. $D - K_\Sigma$ nef and big. The latter is more or less just (2.1). Secondly, we want to apply the theorem to an existence problem. A divisor being nef means it is somehow close to being effective; or better, its linear system is close to being non-empty. If we want that some linear system $|D|_l$ contains a curve with certain properties, then it seems not to be so unreasonable to restrict to systems where $|D - K_\Sigma|_l$, or even $|D - L - K_\Sigma|_l$ with L some fixed divisor, is already of positive dimension, thus nef.

In many interesting examples condition (2.2) turns out to be obsolete or easy to handle. So finally the most restrictive obstruction seems to be (2.1). In particular the coefficient 2 is not optimal; e.g. in $\mathbb{P}_{\mathbb{C}}^2$ one may achieve a coefficient close to one (cf. [Xu95] Theorem 3). In view of equation (2.4) we could easily improve the coefficient at the expense of more complicated conditions.

Remark 2.4. Using the universal property of $|B|_a$ as a component of the Hilbert scheme of all curves on Σ , one can show that an algebraic system $|B|_a$ of dimension greater than zero with B irreducible and $B^2 = 0$ gives rise to a fibration $f : \Sigma \rightarrow H$ of Σ over a smooth projective curve H whose fibres are just the elements of $|B|_a$. In this situation, H is just the normalization of the curve $|B|_a$.

3. EXISTENCE THEOREM FOR GENERIC FAT POINT SCHEMES

Throughout the proof of the existence theorem we need the following two lemmata, the latter one as a kind of induction step more or less.

Lemma 3.1. *Let L be very ample over \mathbb{C} on the smooth projective surface Σ , and let $z, z' \in \Sigma$ be two distinct points. Then there is a smooth curve through z and z' in $|L|_l$.*

Proof. Considering the embedding into $\mathbb{P}_{\mathbb{C}}^n$ defined by L there is an $n-2$ -dimensional family of hyperplane sections going through two fixed points of Σ , which in local coordinates w. l. o. g. is given by the family of equations $\{a_1x_1 + \dots + a_{n-1}x_{n-1} = 0 \mid (a_1 : \dots : a_{n-1}) \in \mathbb{P}_{\mathbb{C}}^{n-2}\}$. Since the local analytic rings of Σ in every point are smooth, hence, in particular complete intersections, they are given as $\mathbb{C}\{x_1, \dots, x_n\}$ modulo some ideal generated by $n-2$ power series f_1, \dots, f_{n-2} forming a regular sequence, and thus, having $n-2$ free indeterminates in our family of equations, a generic equation g will lead to a regular sequence f_1, \dots, f_{n-2}, g , i.e. the hyperplane section defined by g is smooth in each of the two points, and thus everywhere. \square

Lemma 3.2. *Let $L \subset \Sigma$ be a smooth curve and $X \subset \Sigma$ a zero-dimensional scheme. If $D \in \text{Div}(\Sigma)$ such that*

$$(3.1) \quad h^1(\Sigma, \mathcal{J}_{X:L/\Sigma}(D-L)) = 0, \text{ and}$$

$$(3.2) \quad \deg(X \cap L) \leq D.L + 1 - 2g(L),$$

then

$$h^1(\Sigma, \mathcal{J}_{X/\Sigma}(D)) = 0.$$

Proof. Condition (3.2) implies

$$\begin{aligned} 2g(L) - 2 &< D.L - \deg(X \cap L) = \deg(\mathcal{O}_L(D)) + \deg(\mathcal{J}_{X \cap L/L}(D)) \\ &= \deg(\mathcal{J}_{X \cap L/L}(D)), \end{aligned}$$

and thus by Riemann-Roch (cf. [Har77] IV.1.3.4)

$$h^1(\mathcal{J}_{X \cap L/L}(D)) = 0.$$

Consider now the exact sequence

$$0 \longrightarrow \mathcal{J}_{X:L/\Sigma}(D-L) \xrightarrow{\cdot L} \mathcal{J}_{X/\Sigma}(D) \longrightarrow \mathcal{J}_{X \cap L/L}(D) \longrightarrow 0.$$

The result then follows from the corresponding long exact cohomology sequence

$$0 = H^1(\mathcal{J}_{X:L/\Sigma}(D-L)) \longrightarrow H^1(\mathcal{J}_{X/\Sigma}(D)) \longrightarrow H^1(\mathcal{J}_{X \cap L/L}(D)) = 0.$$

\square

Theorem 3.3. *Given $m_1, \dots, m_r \in \mathbb{N}_0$, not all zero, and $z_1, \dots, z_r \in \Sigma$, $r \geq 1$, in very general position. Let $L \in \text{Div}(\Sigma)$ be very ample over \mathbb{C} , and let $D \in \text{Div}(\Sigma)$ be such that*

$$(3.3) \quad h^1(\Sigma, \mathcal{J}_{X(\underline{m}; \underline{z})/\Sigma}(D-L)) = 0, \text{ and}$$

$$(3.4) \quad D.L - 2g(L) \geq m_i + m_j \text{ for all } i, j.$$

Then there exists a curve $C \in |D|_l$ with ordinary singular points of multiplicity m_i at z_i for $i = 1, \dots, r$ and no other singular points. Furthermore,

$$h^1(\Sigma, \mathcal{J}_{X(\underline{m}; \underline{z})/\Sigma}(D)) = 0,$$

and in particular, $V_{|D|}(\underline{m})$ is T -smooth at C .

If in addition (3.5) $D^2 > \sum_{i=1}^r m_i^2$, then C can be chosen to be irreducible and reduced.

Proof. W. l. o. g. $m_i \geq 1$ for all $i = 1, \dots, r$. For the convenience of notation we set $z_{r+1} := z_1$ and $m_{r+1} := m_1$. Since L is very ample, we may choose by Lemma 3.1 smooth curves $L_j \in |L|_l$ through z_j and z_{j+1} for $j = 1, \dots, r$. Writing X for $X(\underline{m}; \underline{z})$ we introduce zero-dimensional schemes X_j for $j = 1, \dots, r$ by

$$\mathcal{J}_{X_j/\Sigma, z} = \begin{cases} \mathcal{J}_{X/\Sigma, z}, & \text{if } z \neq z_j, \\ \mathfrak{m}_{\Sigma, z_j} \cdot \mathcal{J}_{X/\Sigma, z_j}, & \text{if } z = z_j. \end{cases}$$

Step 1: $h^1(\mathcal{J}_{X_j/\Sigma}(D)) = 0$.

By condition (3.4) we get

$$(3.6) \quad \deg(X_j \cap L_j) \leq m_j + m_{j+1} + 1 \leq D.L + 1 - 2g(L),$$

and the exact sequence

$$0 \longrightarrow \mathcal{J}_{X/\Sigma} \longrightarrow \mathcal{J}_{X_j: L_j/\Sigma} \longrightarrow \mathfrak{m}_{\Sigma, z_{j+1}}^{m_{j+1}-1} / \mathfrak{m}_{\Sigma, z_{j+1}}^{m_{j+1}} \longrightarrow 0$$

implies with the aid of (3.3)

$$(3.7) \quad h^1(\mathcal{J}_{X_j: L_j/\Sigma}(D - L)) = 0.$$

(3.6) and (3.7) allow us to apply Lemma 3.2 in order to obtain

$$h^1(\mathcal{J}_{X_j/\Sigma}(D)) = 0.$$

Step 2: For each $j = 1, \dots, r$ there exists a curve $C_j \in |D|_l$ with an ordinary singular point of multiplicity m_j at z_j and with $\text{mult}_{z_i}(C_j) \geq m_i$ for $i \neq j$.

Consider the exact sequence

$$0 \longrightarrow \mathcal{J}_{X_j/\Sigma} \longrightarrow \mathcal{J}_{X/\Sigma} \longrightarrow \mathfrak{m}_{\Sigma, z_j}^{m_j} / \mathfrak{m}_{\Sigma, z_j}^{m_j+1} \longrightarrow 0$$

twisted by D and the corresponding long exact cohomology sequence

$$(3.8) \quad \begin{array}{c} H^0(\mathcal{J}_{X/\Sigma}(D)) \rightarrow \mathfrak{m}_{\Sigma, z_j}^{m_j} / \mathfrak{m}_{\Sigma, z_j}^{m_j+1} \rightarrow H^1(\mathcal{J}_{X_j/\Sigma}(D)) \rightarrow H^1(\mathcal{J}_{X/\Sigma}(D)) \rightarrow 0. \\ \parallel \text{Step 1} \\ 0 \end{array}$$

Thus we may choose the C_j to be given by a section in $H^0(\mathcal{J}_{X/\Sigma}(D))$ where the m_j tangent directions at z_j are all different.

Step 3: The base locus of $\mathbb{P}(H^0(\mathcal{J}_{X/\Sigma}(D)))$ is $\{z_1, \dots, z_r\}$.

Suppose $w \in \Sigma$ was an additional base point and define the zero-dimensional scheme $X \cup \{w\}$ by

$$\mathcal{J}_{X \cup \{w\}/\Sigma, z} = \begin{cases} \mathcal{J}_{X/\Sigma, z}, & \text{if } z \neq w, \\ \mathfrak{m}_{\Sigma, w} \cdot \mathcal{J}_{X/\Sigma, w}, & \text{if } z = w. \end{cases}$$

Choosing a generic, and thus smooth, curve $L_w \in |L|_l$ through w we may deduce as in Step 1

$$h^1(\mathcal{J}_{X \cup \{w\}/\Sigma}(D)) = 0,$$

and thus as in Step 2

$$h^0(\mathcal{J}_{X/\Sigma}(D)) = h^0(\mathcal{J}_{X \cup \{w\}/\Sigma}(D)) + 1.$$

But by assumption w is a base point, and thus

$$h^0(\mathcal{J}_{X/\Sigma}(D)) = h^0(\mathcal{J}_{X \cup \{w\}/\Sigma}(D)),$$

which gives us the desired contradiction.

Step 4: $\exists C \in \mathbb{P}(H^0(\mathcal{J}_{X/\Sigma}(D))) \subseteq |D|_l$ with an ordinary singular point of multiplicity m_i at z_i for $i = 1, \dots, r$ and no other singular points.

Because of Step 2 the generic element in $\mathbb{P}(H^0(\mathcal{J}_{X/\Sigma}(D)))$ has an ordinary singular point of multiplicity m_i at z_i and is by Bertini's Theorem (cf. [Har77] III.10.9.2) smooth outside its base locus.

For two generic curves $C, C' \in \mathbb{P}(H^0(\mathcal{J}_{X/\Sigma}(D)))$ the intersection multiplicity in z_i is $i(C, C'; z_i) = m_i^2$. Thus, if condition (3.5) is fulfilled then C and C' have an additional intersection point outside the base locus of $\mathbb{P}(H^0(\mathcal{J}_{X/\Sigma}(D)))$, and Bertini's Theorem (cf. [Wae73] §47, Satz 4) implies that the generic curve in $\mathbb{P}(H^0(\mathcal{J}_{X/\Sigma}(D)))$ is irreducible.

Step 5: $h^1(\mathcal{J}_{X/\Sigma}(D)) = 0$.

This follows immediately from equation (3.8).

Step 6: $V_{|D|}(\underline{m})$ is T-smooth at C .

By [GLS98b], Lemma 2.7, we have

$$\mathcal{J}_{X/\Sigma} \subseteq \mathcal{J}_{X^{es}(C)/\Sigma},$$

and thus by Step 5

$$h^1(\mathcal{J}_{X^{es}(C)/\Sigma}(D)) = 0,$$

which proves the claim. \square

Corollary 3.4. *Let $m_1, \dots, m_r \in \mathbb{N}_0$, not all zero, $r \geq 1$, and let $L \in \text{Div}(\Sigma)$ be very ample over \mathbb{C} . Suppose $D \in \text{Div}(\Sigma)$ such that*

$$(3.9) \quad (D - L - K_\Sigma)^2 \geq 2 \sum_{i=1}^r (m_i + 1)^2,$$

$$(3.10) \quad (D - L - K_\Sigma).B > \max\{m_1, \dots, m_r\} \\ \text{for any irreducible curve } B \subset \Sigma \\ \text{with } B^2 = 0 \text{ and } \dim |B|_a \geq 1,$$

$$(3.11) \quad D - L - K_\Sigma \text{ is nef, and}$$

$$(3.12) \quad D.L - 2g(L) \geq m_i + m_j \text{ for all } i, j.$$

Then for $z_1, \dots, z_r \in \Sigma$ in very general position there exists a curve $C \in |D|_l$ with ordinary singular points of multiplicity m_i at z_i for $i = 1, \dots, r$ and no other singular points. Furthermore,

$$h^1(\Sigma, \mathcal{J}_{X(\underline{m}; \underline{z})/\Sigma}(D)) = 0,$$

and in particular, $V_{|D|}(\underline{m})$ is T-smooth in C .

If in addition (3.13) $D^2 > \sum_{i=1}^r m_i^2$, then C can be chosen to be irreducible and reduced.

Proof. Follows from Theorems 3.3 and 2.1. \square

Remark 3.5. In view of condition (3.9) condition (3.13) is satisfied if the following condition is fulfilled:

$$(3.14) \quad D^2 + (2D - L - K_\Sigma) \cdot (L + K_\Sigma) + 4 \sum_{i=1}^r m_i + 2r > 0.$$

Proof. Suppose (3.13) was not satisfied, then

$$\begin{aligned} 2 \sum_{i=1}^r m_i^2 &\geq 2D^2 = D^2 + (D - L - K_\Sigma)^2 + (2D - L - K_\Sigma) \cdot (L + K_\Sigma) \\ &\geq D^2 + 2 \sum_{i=1}^r m_i^2 + 4 \sum_{i=1}^r m_i + 2r + (2D - L - K_\Sigma) \cdot (L + K_\Sigma). \end{aligned}$$

Hence,

$$D^2 + (2D - L - K_\Sigma) \cdot (L + K_\Sigma) + 4 \sum_{i=1}^r m_i + 2r \leq 0,$$

which implies that (3.14) is sufficient. \square

4. EXISTENCE THEOREM FOR GENERAL EQUISINGULARITY SCHEMES

Notation. In the following we will denote by $\mathbb{C}[x, y]_d$, respectively by $\mathbb{C}[x, y]_{\leq d}$ the \mathbb{C} -vector spaces of polynomials of degree d , respectively of degree at most d . If $f \in \mathbb{C}[x, y]_{\leq d}$ we denote by $f_k \in \mathbb{C}[x, y]_k$ for $k = 0, \dots, d$ the homogeneous part of degree k of f , and thus $f = \sum_{k=0}^d f_k$. By $\underline{a} = (a_{k,l} | 0 \leq k+l \leq d)$ we will denote the coordinates of $\mathbb{C}[x, y]_{\leq d}$ with respect to the basis $\{x^k y^l | 0 \leq k+l \leq d\}$.

For any $f \in \mathbb{C}[x, y]_{\leq d}$ the tautological family

$$\mathbb{C}[x, y]_{\leq d} \times \mathbb{C}^2 \supset \bigcup_{g \in \mathbb{C}[x, y]_{\leq d}} \{g\} \times g^{-1}(0) \longrightarrow \mathbb{C}[x, y]_{\leq d}$$

induces a deformation of the plane curve singularity $(f^{-1}(0), 0)$ whose base space is the germ $(\mathbb{C}[x, y]_{\leq d}, f)$ of $\mathbb{C}[x, y]_{\leq d}$ at f . Given any deformation $(X, x) \hookrightarrow (\mathcal{X}, x) \rightarrow (S, s)$ of a plane curve singularity (X, x) , we will denote by $S^{es} = (S^{es}, s)$ the germ of the equisingular stratum of (S, s) . Thus, fixing $f \in \mathbb{C}[x, y]_{\leq d}$, $\mathbb{C}[x, y]_{\leq d}^{es} = (\mathbb{C}[x, y]_{\leq d}^{es}, f)$ is the (local) equisingular stratum of $\mathbb{C}[x, y]_{\leq d}$ at f .

Definition 4.1. (i) We say the family $\mathbb{C}[x, y]_{\leq d}$ is *T-smooth* at $f \in \mathbb{C}[x, y]_{\leq d}$ if for any $e \geq d$ there exists a $\Lambda \subset \{(k, l) \in \mathbb{N}_0^2 | 0 \leq k+l \leq d\}$ with $\#\Lambda = \tau^{es}$ such that $\mathbb{C}[x, y]_{\leq e}^{es}$ is given by equations

$$a_{k,l} = \phi_{k,l}(\underline{a}_{(1)}, \underline{a}_{(2)}), \quad (k, l) \in \Lambda,$$

with $\phi_{k,l} \in \mathbb{C}\{\underline{a}_{(1)}, \underline{a}_{(2)}\}$ where $\underline{a}_{(0)} = (a_{k,l} | (k, l) \in \Lambda)$, $\underline{a}_{(1)} = (a_{k,l} | 0 \leq k+l \leq d, (k, l) \notin \Lambda)$, and $\underline{a}_{(2)} = (a_{k,l} | d+1 \leq k+l \leq e)$, and where $\tau^{es} = \dim_{\mathbb{C}}(\mathbb{C}\{x, y\}/I^{es}(f^{-1}(0), 0))$ is the codimension of the equisingular stratum in the base space of the semiuniversal deformation of $(f^{-1}(0), 0)$.

- (ii) A polynomial $f \in \mathbb{C}[x, y]_{\leq d}$ is said to be a *good representative* of the singularity type \mathcal{S} in $\mathbb{C}[x, y]_{\leq d}$ if it meets the following conditions:
- (a) $\text{Sing}(f^{-1}(0)) = \left\{ p \in \mathbb{C}^2 \mid f(p) = 0, \frac{\partial f}{\partial x}(p) = 0, \frac{\partial f}{\partial y}(p) = 0 \right\} = \{0\}$,
 - (b) $(f^{-1}(0), 0) \sim_t \mathcal{S}$,
 - (c) f_d is reduced, and
 - (d) $\mathbb{C}[x, y]_{\leq d}$ is T-smooth at f .
- (iii) Given a singularity type \mathcal{S} we define $s(\mathcal{S})$ to be the minimal number d such that \mathcal{S} has a good representative of degree d .

Remark 4.2. (i) The condition for T-smoothness just means that for any $e \geq d$ the equisingular stratum $\mathbb{C}[x, y]_{\leq e}^{es}$ is smooth at the point f of the expected codimension in $(\mathbb{C}[x, y]_{\leq e}, f)$.

(ii) The condition (c) just says that the affine curve $f_d = 0$ intersects the line at infinity transversally in d different points. Since by a linear change of coordinates this can always be realised, the bounds for $s(\mathcal{S})$ given in [Los98] Theorem 4.2 and Remark 4.3 do apply here.

(iii) For refined results using the techniques of the following proof we refer to [Shu99].

Theorem 4.3 (Existence). *Let $\mathcal{S}_1, \dots, \mathcal{S}_r$ be singularity types, and suppose there exists an irreducible curve $C \in |D|_l$ with $r+r'$ ordinary singular points $z_1, \dots, z_{r+r'}$ of multiplicities $m_1, \dots, m_{r+r'}$ respectively as its only singularities such that $m_i = s(\mathcal{S}_i) + 1$, for $i = 1, \dots, r$, and*

$$h^1(\Sigma, \mathcal{I}_{X(\underline{m}; \underline{z})/\Sigma}(D)) = 0.$$

*Then there exists an irreducible curve $\tilde{C} \in |D|_l$ with r singular points of types $\mathcal{S}_1, \dots, \mathcal{S}_r$ and r' ordinary singular points of multiplicities $m_{r+1}, \dots, m_{r+r'}$ as its only singularities.*⁴

Idea of the proof. The basic idea is to glue locally at the z_i equations of good representatives for the \mathcal{S}_i into the curve C . Let us now explain in more detail what we mean by this.

If $g_i = \sum_{k+l=0}^{m_i-1} a_{k,l}^{i,fix} x_i^k y_i^l$, $i = 1, \dots, r$, are good representatives of the \mathcal{S}_i , then we are looking for a family F_t , $t \in (\mathbb{C}, 0)$, in $H^0(\Sigma, \mathcal{O}_\Sigma(D))$ which in local coordinates x_i, y_i at z_i looks like

$$F_t^i = \sum_{k+l=0}^{m_i-1} t^{m_i-1-k-l} \tilde{a}_{k,l}^i(t) x_i^k y_i^l + h.o.t.,$$

where the $\tilde{a}_{k,l}^i(t)$ should be convergent power series in t with $\tilde{a}_{k,l}^i(0) = a_{k,l}^{i,fix}$. Replacing g_i by some arbitrarily small multiple $\lambda_i g_i$ the curve defined by F_0 is an arbitrarily small deformation of C inside some suitable linear system, thus it is smooth outside $z_1, \dots, z_{r+r'}$ and has ordinary singular points in $z_1, \dots, z_{r+r'}$. For $t \neq 0$, on the other hand, F_t^i can be transformed, by $(x_i, y_i) \mapsto (tx_i, ty_i)$, into a member of some family

$$\tilde{F}_t^i = \sum_{k+l=0}^{m_i-1} \tilde{a}_{k,l}^i(t) x_i^k y_i^l + h.o.t., \quad t \in \mathbb{C},$$

⁴Here, of course, $\underline{m} = (m_1, \dots, m_{r+r'})$ and $\underline{z} = (z_1, \dots, z_{r+r'})$.

with

$$\tilde{F}_0^i = g_i.$$

Using now the T-smoothness property of g_i , $i = 1, \dots, r$, we can choose the $\tilde{a}_{k,l}^i(t)$ such that this family is equisingular. Hence, for small $t \neq 0$, the curve given by F_t will have the right singularities at the z_i . Finally, the knowledge on the singularities of the curve defined by F_0 and the conservation of Milnor numbers will ensure that the curve given by F_t has no further singularities, for $t \neq 0$ sufficiently small.

The proof will be done in several steps. First of all we are going to fix some notation by choosing a basis of $H^0(\Sigma, \mathcal{O}_\Sigma(D))$ which reflects the “independence” of the coordinates at the different z_i ensured by $h^1(\Sigma, \mathcal{J}_{X(\underline{m}; \underline{z})/\Sigma}(D)) = 0$ (Step 1.1), and by choosing good representatives for the \mathcal{S}_i (Step 1.2). In a second step we are making an “Ansatz” for the family F_t , and, for the local investigation of the singularity type, we are switching to some other families \tilde{F}_t^i , $i = 1, \dots, r$ (Step 2.1). We, then, reduce the problem of F_t , for $t \neq 0$ small having the right singularities, to a question about the equisingular strata of some families of polynomials (Step 2.2), which in Step 2.3 will be solved. The final step serves to show that the curves F_t have only the singularities which we controlled in the previous steps.

Proof. Step 1.1: Parametrise $|D|_l = \mathbb{P}(H^0(\mathcal{O}_\Sigma(D)))$.

Consider the following exact sequence:

$$0 \longrightarrow \mathcal{J}_{X(\underline{m}; \underline{z})/\Sigma}(D) \longrightarrow \mathcal{O}_\Sigma(D) \longrightarrow \bigoplus_{i=1}^{r+r'} \mathcal{O}_{\Sigma, z_i} / \mathfrak{m}_{\Sigma, z_i}^{m_i} \longrightarrow 0.$$

Since $h^1(\mathcal{J}_{X(\underline{m}; \underline{z})/\Sigma}(D)) = 0$, the long exact cohomology sequence gives

$$H^0(\mathcal{O}_\Sigma(D)) = \bigoplus_{i=1}^{r+r'} \mathbb{C}\{x_i, y_i\} / (x_i, y_i)^{m_i} \oplus H^0(\mathcal{J}_{X(\underline{m}; \underline{z})/\Sigma}(D)),$$

where x_i, y_i are local coordinates of (Σ, z_i) .

We, therefore, can find a basis $\{s_{k,l}^i, s_j \mid j = 1, \dots, e, i = 1, \dots, r+r', 0 \leq k+l \leq m_i - 1\}$ of $H^0(\mathcal{O}_\Sigma(D))$, with $e = h^0(\mathcal{J}_{X(\underline{m}; \underline{z})/\Sigma}(D))$, such that⁵

- C is the curve defined by s_1 ,
- $(s_j)_{z_i} = \sum_{|\alpha| \geq m_i} B_\alpha^{j,i} x_i^{\alpha_1} y_i^{\alpha_2}$ for $j = 1, \dots, e, i = 1, \dots, r+r'$,
- $(s_{k,l}^j)_{z_i} = \begin{cases} x_i^k y_i^l + \sum_{|\alpha| \geq m_i} A_{\alpha,k,l}^{j,i} x_i^{\alpha_1} y_i^{\alpha_2}, & \text{if } i = j, \\ \sum_{|\alpha| \geq m_i} A_{\alpha,k,l}^{j,i} x_i^{\alpha_1} y_i^{\alpha_2}, & \text{if } i \neq j. \end{cases}$

Let us now denote the coordinates of $H^0(\mathcal{O}_\Sigma(D))$ w.r.t. this basis by $(\underline{a}, \underline{b}) = (\underline{a}^1, \dots, \underline{a}^{r+r'}, \underline{b})$ with $\underline{a}^i = (a_{k,l}^i \mid 0 \leq k+l \leq m_i - 1)$ and $\underline{b} = (b_j \mid j = 1, \dots, e)$.

Thus the family

$$F_{(\underline{a}, \underline{b})} = \sum_{i=1}^{r+r'} \sum_{k+l=0}^{m_i-1} a_{k,l}^i s_{k,l}^i + \sum_{j=1}^e b_j s_j, \quad (\underline{a}, \underline{b}) \in \mathbb{C}^N \text{ with } N = e + \sum_{i=1}^{r+r'} \binom{m_i+1}{2},$$

parametrises $H^0(\mathcal{O}_\Sigma(D))$.

⁵Throughout this proof we will use the multi-index notation $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ and $|\alpha| = \alpha_1 + \alpha_2$.

Step 1.2: By the definition of $s(\mathcal{S}_i)$ and since $s(\mathcal{S}_i) = m_i - 1$, we may choose good representatives

$$g_i = \sum_{k+l=0}^{m_i-1} a_{k,l}^{i,fix} x_i^k y_i^l \in \mathbb{C}[x_i, y_i]_{\leq m_i-1}$$

for the \mathcal{S}_i , $i = 1, \dots, r$. Let $\underline{a}^{i,fix} = (a_{k,l}^{i,fix} \mid 0 \leq k+l \leq m_i-1)$ and $\underline{a}^{fix} = (\underline{a}^{1,fix}, \dots, \underline{a}^{r,fix})$. We should remark here that for any $\lambda_i \neq 0$ the polynomial $\lambda_i g_i$ is also a good representative, and thus, replacing g_i by $\lambda_i g_i$, we may assume that the $a_{k,l}^{i,fix}$ are arbitrarily close to 0.

Step 2: We are going to glue the good representatives for the \mathcal{S}_i into the curve C . More precisely, we are constructing a subfamily F_t , $t \in (\mathbb{C}, 0)$, in $H^0(\mathcal{O}_\Sigma(D))$ such that, if $C_t \in H^0(\Sigma, \mathcal{O}_\Sigma)$ denotes the curve defined by F_t ,

- (1) $z_1, \dots, z_{r+r'}$ are the only singular points of the irreducible reduced curve C_0 , and they are ordinary singularities of multiplicities $m_i - 1$, for $i = 1, \dots, r$, and m_i , for $i = r+1, \dots, r+r'$ respectively,
- (2) locally in z_i , $i = 1, \dots, r$, the F_t , for small $t \neq 0$, can be transformed into members of a fixed \mathcal{S}_i -equisingular family,
- (3) while for $i = r+1, \dots, r+r'$ and $t \neq 0$ small C_t has an ordinary singularity of multiplicity m_i in z_i .

Step 2.1: “Ansatz” and first reduction for local investigation.

Let us make the following “Ansatz”:

$$\begin{aligned} b_1 = 1, \quad b_2 = \dots = b_e = 0, \quad \underline{a}^i = 0, \quad \text{for } i = r+1, \dots, r+r', \\ a_{k,l}^i = t^{m_i-1-k-l} \cdot \tilde{a}_{k,l}^i, \quad \text{for } i = 1, \dots, r, \quad 0 \leq k+l \leq m_i-1. \end{aligned}$$

This gives rise to a family

$$F_{(t, \underline{\tilde{a}})} = s_1 + \sum_{i=1}^r \sum_{k+l=0}^{m_i-1} t^{m_i-1-k-l} \tilde{a}_{k,l}^i s_{k,l}^i \in H^0(\mathcal{O}_\Sigma(D))$$

with $t \in \mathbb{C}$ and $\underline{\tilde{a}} = (\tilde{a}^1, \dots, \tilde{a}^r)$ where $\tilde{a}^i = (\tilde{a}_{k,l}^i \mid 0 \leq k+l \leq m_i-1) \in \mathbb{C}^{N_i}$ with $N_i = \binom{m_i+1}{2}$.

Fixing $i \in \{1, \dots, r\}$, in local coordinates at z_i the family looks like

$$F_{(t, \underline{\tilde{a}})}^i := (F_{(t, \underline{\tilde{a}})})_{z_i} = \sum_{k+l=0}^{m_i-1} t^{m_i-1-k-l} \tilde{a}_{k,l}^i x_i^k y_i^l + \sum_{|\alpha| \geq m_i} \varphi_\alpha^i(t, \underline{\tilde{a}}) x_i^{\alpha_1} y_i^{\alpha_2},$$

with

$$\varphi_\alpha^i(t, \underline{\tilde{a}}) = B_\alpha^{1,i} + \sum_{j=1}^r \sum_{k+l=0}^{m_j-1} t^{m_j-1-k-l} \tilde{a}_{k,l}^j A_{\alpha,k,l}^{j,i}.$$

For $t \neq 0$ the transformation $\psi_t^i : (x_i, y_i) \mapsto (tx_i, ty_i)$ is indeed a coordinate transformation, and thus $F_{(t, \underline{a})}^i$ is contact equivalent⁶ to

$$\tilde{F}_{(t, \underline{a})}^i := t^{-m_i+1} \cdot F_{(t, \underline{a})}^i(tx_i, ty_i) = \sum_{k+l=0}^{m_i-1} \tilde{a}_{k,l}^i x_i^k y_i^l + \sum_{|\alpha| \geq m_i} t^{1+|\alpha|-m_i} \varphi_\alpha^i(t, \underline{a}) x_i^{\alpha_1} y_i^{\alpha_2}.$$

Note that for this new family in $\mathbb{C}\{x_i, y_i\}$ we have

$$\tilde{F}_{(0, \underline{a}^{fix})}^i = \sum_{k+l=0}^{m_i-1} a_{k,l}^{i, fix} x_i^k y_i^l = g_i,$$

and hence it gives rise to a deformation of $(g_i^{-1}(0), 0)$.

Step 2.2: Reduction to the investigation of the equisingular strata of certain families of polynomials.

It is basically our aim to verify the \underline{a} as convergent power series in t such that the corresponding family is equisingular. However, since the $\tilde{F}_{(t, \underline{a})}^i$ are power series in x_i and y_i , we cannot right away apply the T-smoothness property of g_i , but we rather have to reduce to polynomials. For this let e_i be the determinacy bound⁷ of \mathcal{S}_i and define

$$\begin{aligned} \hat{F}_{(t, \underline{a})}^i &:= \sum_{k+l=0}^{m_i-1} \tilde{a}_{k,l}^i x_i^k y_i^l + \sum_{|\alpha|=m_i}^{e_i} t^{1+|\alpha|-m_i} \varphi_\alpha^i(t, \underline{a}) x_i^{\alpha_1} y_i^{\alpha_2} \\ &\equiv \tilde{F}_{(t, \underline{a})}^i \pmod{(x_i, y_i)^{e_i+1}}. \end{aligned}$$

Thus $\hat{F}_{(t, \underline{a})}^i$ is a family in $\mathbb{C}[x_i, y_i]_{\leq e_i}$, and still

$$\hat{F}_{(0, \underline{a}^{fix})}^i = \tilde{F}_{(0, \underline{a}^{fix})}^i = g_i.$$

We claim that it suffices to find $\underline{a}(t) \in \mathbb{C}\{t\}$ with $\underline{a}(0) = a_{k,l}^{i, fix}$, $i = 1, \dots, r$, $0 \leq k+l \leq m_i-1$, such that the families $\hat{F}_t^i := \hat{F}_{(t, \underline{a}(t))}^i$, $t \in (\mathbb{C}, 0)$, are in the equisingular strata $\mathbb{C}[x_i, y_i]_{\leq e_i}^{es}$, for $i = 1, \dots, r$.

Since then we have, for small⁸ $t \neq 0$,

$$g_i = \hat{F}_0^i \sim_t \hat{F}_t^i \sim_t \tilde{F}_{(t, \underline{a}(t))}^i \sim_t F_{(t, \underline{a}(t))}^i = (F_{(t, \underline{a}(t))})_{z_i(t)},$$

by the e_i -determinacy and since ψ_t^i is a coordinate change for $t \neq 0$, which proves condition (2). Note that the singular points z_i will move with t .

⁶Let $f, g \in \mathcal{O}_n = \mathbb{C}\{x_1, \dots, x_n\}$ be two convergent power series in n indeterminates. We call f and g *contact equivalent*, if $\mathcal{O}_n/(f) \cong \mathcal{O}_n/(g)$, and we write in this case $f \sim_c g$. Equivalently, we could ask the germs $(V(f), 0)$ and $(V(g), 0)$ to be isomorphic, that is, ask the singularities to be *analytically equivalent*. Cf. [DP00] Definition 9.1.1 and Definition 3.4.19.

⁷A power series $f \in \mathcal{O}_n = \mathbb{C}\{x_1, \dots, x_n\}$ (respectively the singularity $(V(f), 0)$ defined by f) is said to be *finitely determined* with respect to some equivalence relation \sim if there exists some positive integer e such that $f \sim g$ whenever f and g have the same e -jet. If f is finitely determined, the smallest possible e is called the *determinacy bound*. Isolated singularities are finitely determined with respect to analytical equivalence and hence, for $n = 2$, as well with respect to topological equivalence. Cf. [DP00] Theorem 9.1.3 and Footnote 8.

⁸Here $f \sim_t g$, for two convergent power series $f, g \in \mathcal{O}_2 = \mathbb{C}\{x, y\}$, means that the singularities $(V(f), 0)$ and $(V(g), 0)$ are *topologically equivalent*, that is, there exists a homeomorphism $\Phi : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ with $\Phi(V(f), 0) = (V(g), 0)$, which of course means, that this is correct for suitably chosen representatives. Note that if f and g are contact equivalent, then there even exists an analytic coordinate change Φ , that is, $f \sim_c g$ implies $f \sim_t g$.

It remains to verify conditions (1) and (3). Setting $F_t := F_{(t, \tilde{\mathbf{a}}(t))} \in H^0(\mathcal{O}_\Sigma(D))$, for $t \in (\mathbb{C}, 0)$, we find that

$$F_0 = s_1 + \sum_{j=1}^r \sum_{k+l=m_j-1} a_{k,l}^{j,fix} s_{k,l}^j$$

is an element inside the linear system $\mathcal{D} = \{\lambda_0 s_1 + \sum_{j=1}^r \lambda_j s^j \mid (\lambda_0 : \dots : \lambda_r) \in \mathbb{P}_{\mathbb{C}}^r\}$, where $s^j = \sum_{k+l=m_j-1} a_{k,l}^{j,fix} s_{k,l}^j$. Locally at z_i , $i = 1, \dots, r+r'$, \mathcal{D} induces a deformation of (C, z_i) with equations

$$\lambda_i \cdot (g_i)_{m_i-1} + h.o.t., \quad \text{if } i = 1, \dots, r,$$

and

$$\lambda_0 \cdot \left(\sum_{|\alpha|=m_i} B_\alpha^{1,i} x_i^{\alpha_1} y_i^{\alpha_2} \right) + \sum_{j=1}^r \lambda_j \cdot \left(\sum_{k+l=m_j-1} a_{k,l}^{j,fix} \sum_{|\alpha|=m_i} A_{\alpha,k,l}^{j,i} x_i^{\alpha_1} y_i^{\alpha_2} \right) + h.o.t., \quad \text{if } i = r+1, \dots, r+r',$$

respectively. Thus any element of \mathcal{D} has ordinary singularities of multiplicity $m_i - 1$ at z_i for $i = 1, \dots, r$, and since s_1 has an ordinary singularity of multiplicity m_i at z_i for $i = r+1, \dots, r+r'$, so has a generic element of \mathcal{D} . Moreover, a generic element of \mathcal{D} has no more singular points than the special element s_1 and thus has singularities precisely in $\{z_1, \dots, z_{r+r'}\}$. Replacing the g_i by some suitable multiples, we may assume that the curve defined by F_0 is a generic element of \mathcal{D} , which proves (1). Similarly, we note that F_t in local coordinates at z_i , for $i = r+1, \dots, r+r'$, looks like

$$\begin{aligned} & \sum_{|\alpha|=m_i} B_\alpha^{1,i} x_i^{\alpha_1} y_i^{\alpha_2} + \sum_{j=1}^r \sum_{k+l=0}^{m_j-1} t^{m_j-1-k-l} \tilde{a}_{k,l}^j(t) \sum_{|\alpha|=m_i} A_{\alpha,k,l}^{j,i} x_i^{\alpha_1} y_i^{\alpha_2} + h.o.t. \\ &= \sum_{|\alpha|=m_i} \left(B_\alpha^{1,i} + \sum_{j=1}^r \sum_{k+l=0}^{m_j-1} t^{m_j-1-k-l} \tilde{a}_{k,l}^j(t) A_{\alpha,k,l}^{j,i} \right) x_i^{\alpha_1} y_i^{\alpha_2} + h.o.t., \end{aligned}$$

and thus, for $t \neq 0$ sufficiently small, the singularity of F_t at z_i will be an ordinary singularity of multiplicity m_i , which gives (3).

Step 2.3: Find $\tilde{\mathbf{a}}(t) \in \mathbb{C}\{t\}^n$ with $\tilde{\mathbf{a}}(0) = (a_{k,l}^{i,fix}, i = 1, \dots, r, 0 \leq k+l \leq m_i-1)$, $n = \sum_{i=1}^r \binom{m_i+1}{2}$, such that the families $\hat{F}_t^i = \hat{F}_{(t, \tilde{\mathbf{a}}(t))}^i$, $t \in (\mathbb{C}, 0)$, are in the equisingular strata $\mathbb{C}[x_i, y_i]_{\leq e_i}^{es}$, for $i = 1, \dots, r$.

In the sequel we adopt the notation of Definition 4.1 adding indices i in the obvious way.

Since $\mathbb{C}[x_i, y_i]_{\leq m_i-1}$ is T-smooth at g_i , for $i = 1, \dots, r$, there exist $\Lambda_i \subseteq \{(k, l) \mid 0 \leq k+l \leq m_i-1\}$ and power series $\phi_{k,l}^i \in \mathbb{C}\{\tilde{\mathbf{a}}_{(1)}^i, \tilde{\mathbf{a}}_{(2)}^i\}$, for $(k, l) \in \Lambda_i$, such that the equisingular stratum $\mathbb{C}[x_i, y_i]_{\leq e_i}^{es}$ is given by the $\tau^{es,i} = \#\Lambda_i$ equations

$$\tilde{a}_{k,l}^i = \phi_{k,l}^i(\tilde{\mathbf{a}}_{(1)}^i, \tilde{\mathbf{a}}_{(2)}^i), \quad \text{for } (k, l) \in \Lambda_i.$$

Setting $\Lambda = \bigcup_{j=1}^r \{j\} \times \Lambda_j$ we use the notation

$$\tilde{\mathbf{a}}_{(0)} = (\tilde{\mathbf{a}}_{(0)}^1, \dots, \tilde{\mathbf{a}}_{(0)}^r) = (\tilde{a}_{k,l}^i \mid (i, k, l) \in \Lambda)$$

and, similarly $\tilde{\underline{a}}_{(1)}$, $\tilde{\underline{a}}_{(2)}$, $\underline{a}_{(1)}^{i,fix}$, $\underline{a}_{(0)}^{fix}$, and $\underline{a}_{(1)}^{fix}$. Moreover, setting $\tilde{\varphi}^i(t, \tilde{\underline{a}}_{(0)}) = \left(t^{|\alpha| - m_i} \varphi_\alpha^i(t, \tilde{\underline{a}}_{(0)}, \underline{a}_{(1)}^{fix}) \mid m_i \leq |\alpha| \leq e_i \right)$, we define an analytic map germ

$$\Phi : \left(\mathbb{C} \times \mathbb{C}^{\tau^{es,1}} \times \cdots \times \mathbb{C}^{\tau^{es,r}}, (0, \underline{a}_{(0)}^{fix}) \right) \rightarrow \left(\mathbb{C}^{\tau^{es,1}} \times \cdots \times \mathbb{C}^{\tau^{es,r}}, 0 \right)$$

by

$$\Phi_{k,l}^i(t, \tilde{\underline{a}}_{(0)}) = \tilde{a}_{k,l}^i - \phi_{k,l}^i \left(\underline{a}_{(1)}^{i,fix}, t \cdot \tilde{\varphi}^i(t, \tilde{\underline{a}}_{(0)}) \right), \quad \text{for } (i, k, l) \in \Lambda,$$

and we consider the system of equations

$$\Phi_{k,l}^i(t, \tilde{\underline{a}}_{(0)}) = 0, \quad \text{for } (i, k, l) \in \Lambda.$$

One easily verifies that

$$\left(\frac{\partial \Phi_{k,l}^i}{\partial \tilde{a}_{\kappa,\lambda}^j}(0, \underline{a}_{(0)}^{fix}) \right)_{(i,k,l),(j,\kappa,\lambda) \in \Lambda} = \text{id}_{\mathbb{C}^n}.$$

Thus by the Inverse Function Theorem there exist $\tilde{a}_{k,l}^i(t) \in \mathbb{C}\{t\}$ with $\tilde{a}_{k,l}^i(0) = a_{k,l}^{i,fix}$ such that

$$\tilde{a}_{k,l}^i(t) = \phi_{k,l}^i \left(\underline{a}_{(1)}^{i,fix}, t \cdot \tilde{\varphi}^i(t, \tilde{\underline{a}}_{(0)}(t)) \right), \quad (i, k, l) \in \Lambda.$$

Now, setting $\tilde{\underline{a}}_{(1)}(t) \equiv \underline{a}_{(1)}^{fix}$, the families $\hat{F}_t^i = \hat{F}_{(t, \tilde{\underline{a}}(t))}^i$ are in the equisingular strata $\mathbb{C}[x_i, y_i]_{\leq e_i}^{es}$, for $i = 1, \dots, r$.

Step 3: It finally remains to show that F_t , for small $t \neq 0$, has no other singular points than $z_1(t), \dots, z_r(t), z_{r+1}, \dots, z_{r+r'}$.

Since for any $i = 1, \dots, r + r'$ the family F_t , $t \in (\mathbb{C}, 0)$, induces a deformation of the singularity (C_0, z_i) there are, by the conservation of Milnor numbers (cf. [DP00], Chapter 6), (Euclidean) open neighbourhoods $U(z_i) \subset \Sigma$ and $V(0) \subset \mathbb{C}$ such that for any $t \in V(0)$

$$(4.1) \quad \begin{aligned} \text{Sing}(C_t) &\subset \bigcup_{i=1}^{r+r'} U(z_i), \\ &\text{i.e. singular points of } C_t \\ &\text{come from singular points of } C_0, \end{aligned}$$

$$(4.2) \quad \mu(C_0, z_i) = \sum_{z \in \text{Sing}(F_t^i) \cap U(z_i)} \mu(F_t^i, z), \quad i = 1, \dots, r + r'.$$

For $i = r + 1, \dots, r + r'$ condition (4.2) implies

$$(m_i - 1)^2 = \mu(C_0, z_i) \geq \mu(F_t^i, z_i) = (m_i - 1)^2,$$

and thus z_i must be the only critical point of F_t^i in $U(z_i)$, in particular,

$$\text{Sing}(C_t) \cap U(z_i) = \{z_i\}.$$

Let now $i \in \{1, \dots, r\}$. For $t \neq 0$ fixed, we consider the transformation defined by the coordinate change ψ_t^i ,

$$\begin{array}{ccc} \mathbb{C}^2 & \supset & U(z_i) & \longrightarrow & U_t(z_i) & \subset & \mathbb{C}^2 \\ & & \downarrow \Psi & & \downarrow \Psi & & \\ & & (x_i, y_i) & \mapsto & \left(\frac{1}{t} x_i, \frac{1}{t} y_i \right), & & \end{array}$$

and the transformed equations

$$\tilde{F}_t^i(x_i, y_i) = t^{-m_i+1} F_t^i(tx_i, ty_i) = 0.$$

Condition (4.2) then implies,

$$(m_i - 2)^2 = \mu(C_0, z_i) = \sum_{z \in \text{Sing}(F_t^i) \cap U(z_i)} \mu(F_t^i, z) = \sum_{z \in \text{Sing}(\tilde{F}_t^i) \cap U_t(z_i)} \mu(\tilde{F}_t^i, z).$$

For $t \neq 0$ very small $U_t(z_i)$ becomes very large, so that, by shrinking $V(0)$ we may suppose that for any $0 \neq t \in V(0)$

$$\text{Sing}(g_i) \subset U_t(z_i),$$

and that for any $z \in \text{Sing}(g_i)$ there is an open neighbourhood $U(z) \subset U_t(z_i)$ such that

$$\mu(g_i, z) = \sum_{z' \in \text{Sing}(\tilde{F}_t^i) \cap U(z)} \mu(\tilde{F}_t^i, z').$$

If we now take into account that g_i has precisely one critical point, z_i , on its zero level, and that the critical points on the zero level of \tilde{F}_t^i all contribute to the Milnor number $\mu(g_i, z_i)$, then we get the following sequence of inequalities:

$$\begin{aligned} (m_i - 2)^2 - \mu(\mathcal{S}_i) &= \sum_{z \in \text{Sing}(g_i)} \mu(g_i, z) - \sum_{z \in \text{Sing}(g_i^{-1}(0))} \mu(g_i, z) \\ &\leq \sum_{z \in \text{Sing}(\tilde{F}_t^i) \cap U_t(z_i)} \mu(\tilde{F}_t^i, z) - \sum_{z \in \text{Sing}((\tilde{F}_t^i)^{-1}(0)) \cap U_t(z_i)} \mu(\tilde{F}_t^i, z) \\ &= \sum_{z \in \text{Sing}(F_t^i) \cap U(z_i)} \mu(F_t^i, z) - \sum_{z \in \text{Sing}((F_t^i)^{-1}(0)) \cap U(z_i)} \mu(F_t^i, z) \\ &\leq \mu(C_0, z_i) - \mu(F_t^i, z_i) = (m_i - 2)^2 - \mu(\mathcal{S}_i). \end{aligned}$$

Hence all inequalities must have been equalities, and, in particular,

$$\text{Sing}(C_t) \cap U(z_i) = \text{Sing}((F_t^i)^{-1}(0)) \cap U(z_i) = \{z_i\},$$

which in view of condition (4.1) finishes the proof.

Note that C_t , being a small deformation of the irreducible reduced curve C_0 , will again be irreducible and reduced. \square

Corollary 4.4. *Let $L \in \text{Div}(\Sigma)$ be very ample over \mathbb{C} . Suppose that $D \in \text{Div}(\Sigma)$ and $\mathcal{S}_1, \dots, \mathcal{S}_r$ are topological singularity types with $\mu(\mathcal{S}_1) \geq \dots \geq \mu(\mathcal{S}_r)$ such that*

$$(4.3) \quad (D - L - K_\Sigma)^2 \geq \frac{414}{5} \sum_{\mu(\mathcal{S}_i) \leq 38} \mu(\mathcal{S}_i) + 58 \sum_{\mu(\mathcal{S}_i) \geq 39} \left(\sqrt{\mu(\mathcal{S}_i)} + \frac{13}{2\sqrt{29}} \right)^2,$$

$$(4.4) \quad (D - L - K_\Sigma).B > \begin{cases} \sqrt{\frac{207}{5}} \sqrt{\mu(\mathcal{S}_1)} - 1, & \text{if } \mu(\mathcal{S}_1) \leq 38, \\ \sqrt{29} \sqrt{\mu(\mathcal{S}_1)} + \frac{11}{2}, & \text{if } \mu(\mathcal{S}_1) \geq 39, \end{cases}$$

for any irreducible curve B with $B^2 = 0$ and $\dim |B|_a > 0$,

$$(4.5) \quad D - L - K_\Sigma \text{ is nef,}$$

$$(4.6) \quad D.L - 2g(L) \geq \begin{cases} \sqrt{\frac{207}{5}} \left(\sqrt{\mu(\mathcal{S}_1)} + \sqrt{\mu(\mathcal{S}_2)} \right) - 2, & \text{if } \mu(\mathcal{S}_1) \leq 38, \\ \sqrt{\frac{207}{5}} \sqrt{\mu(\mathcal{S}_1)} + \sqrt{29} \sqrt{\mu(\mathcal{S}_1)} + \frac{9}{2}, & \text{if } \mu(\mathcal{S}_1) \geq 39 \\ & \text{and } \mu(\mathcal{S}_2) \leq 38, \\ \sqrt{29} \left(\sqrt{\mu(\mathcal{S}_1)} + \sqrt{\mu(\mathcal{S}_2)} \right) + 11, & \text{if } \mu(\mathcal{S}_2) \geq 39, \end{cases}$$

$$(4.7) \quad D^2 \geq \frac{207}{5} \sum_{\mu(\mathcal{S}_i) \leq 38} \left(\sqrt{\mu(\mathcal{S}_i)} - \sqrt{\frac{5}{207}} \right)^2 + 29 \sum_{\mu(\mathcal{S}_i) \geq 39} \left(\sqrt{\mu(\mathcal{S}_i)} + \frac{11}{2\sqrt{29}} \right)^2,$$

then there is an irreducible reduced curve C in $|D|_l$ with r singular points of topological types $\mathcal{S}_1, \dots, \mathcal{S}_r$ as its only singularities.

Proof. This follows right away from Corollary 3.4, Theorem 4.3, and [Los98] Theorem 4.2. \square

Remark 4.5. One could easily simplify the above formulae by not distinguishing the cases $\mu(\mathcal{S}_i) \geq 39$ and $\mu(\mathcal{S}_i) \leq 38$. However, one would lose information.

On the other hand, knowing something more about the singularity type one could achieve much better results, applying the corresponding bounds for the $s(\mathcal{S}_i)$. We leave it to the reader to apply the bounds (cf. [Los98] Remarks 4.3, 4.8, and 4.15).

As we have already mentioned earlier the most restrictive of the above *sufficient* conditions is (4.3), which could be characterised as a condition of the type

$$\sum_{i=1}^r \mu(\mathcal{S}_i) \leq \alpha D^2 + \beta D.K + \gamma,$$

where K is some fixed divisor class, α, β and γ are some constants.

There are also *necessary* conditions of this type, e.g.

$$\sum_{i=1}^r \mu(\mathcal{S}_i) \leq D^2 + D.K_\Sigma + 2,$$

which follows from the genus formula.

See [Los98] Section 4.1 for considerations on the asymptotical properness of the constant α .

5. EXAMPLES

In this section we are going to examine the conditions in the vanishing theorem (Theorem 2.1) and in the corresponding existence results for various types of surfaces. In the classical case $\Sigma = \mathbb{P}_{\mathbb{C}}^2$, our results are much weaker than the previously known ones. We therefore start with the investigation of geometrically ruled surfaces.

Unless otherwise stated, $r \geq 1$ is a positive integer, and $m_1, \dots, m_r \in \mathbb{N}_0$ are non-negative, while at least one m_i is positive whenever we consider conditions for existence theorems.

5.a. Geometrically Ruled Surfaces. Let $\Sigma = \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} C$ be a geometrically ruled surface with normalised bundle \mathcal{E} (in the sense of [Har77] V.2.8.1). The Néron-Severi group of Σ is

$$\mathrm{NS}(\Sigma) = C_0\mathbb{Z} \oplus F\mathbb{Z},$$

with intersection matrix

$$\begin{pmatrix} -e & 1 \\ 1 & 0 \end{pmatrix},$$

where $F \cong \mathbb{P}_{\mathbb{C}}^1$ is a fibre of π , C_0 a section of π with $\mathcal{O}_{\Sigma}(C_0) \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, and $e = -\deg(\Lambda^2 \mathcal{E}) \geq -g$.⁹ For the canonical divisor we have

$$K_{\Sigma} \sim_a -2C_0 + (2g - 2 - e)F,$$

where $g = g(C)$ is the genus of the base curve C , and we can find very ample divisors L algebraically equivalent to $C_0 + lF$, with $l \geq \max\{e + 1, 2\}$ and $l = e + 1$, if $C_0 \cong \mathbb{P}_{\mathbb{C}}^1$. For such an L we have in particular $g(L) = g$.

Since the only irreducible curves $B \subset \Sigma$ with $B^2 = 0$ and $\dim |B|_a \geq 1$ are the fibres F and, maybe, curves B with $B \sim_a aC_0$ with $a \geq 1$, if $e = 0$, respectively $B \sim_a aC_0 + \frac{ae}{2}F$ with $a \geq 2$, if $e < 0$, we do understand condition (2.2) quite well in this situation.

Theorem 2.1a. *Given two integers $a, b \in \mathbb{Z}$ satisfying*

$$(2.1a) \quad a(b - (\frac{a}{2} - 1)e) \geq \sum_{i=1}^r (m_i + 1)^2,$$

$$(2.2a.i) \quad a > \max\{m_i \mid i = 1, \dots, r\},$$

$$(2.2a.ii) \quad b > \max\{m_i \mid i = 1, \dots, r\}, \text{ if } e = 0,$$

$$(2.2a.iii) \quad 2(b - (\frac{a}{2} - 1)e) > \max\{m_i \mid i = 1, \dots, r\}, \text{ if } e < 0, \text{ and}$$

$$(2.3a) \quad b \geq (a - 1)e, \text{ if } e > 0.$$

For $z_1, \dots, z_r \in \Sigma$ in very general position and $\nu > 0$

$$H^{\nu} \left(\text{Bl}_{\underline{z}}(\Sigma), (a - 2)\pi^*C_0 + (b - 2 + 2g)\pi^*F - \sum_{i=1}^r m_i E_i \right) = 0.$$

In order to obtain nice formulae we considered $D = (a - 2)C_0 + (b - 2 + 2g)F$ in the formulation of the vanishing theorem. For the existence theorems it turns out that the formulae look best if we work with $D = (a - 1)C_0 + (b + l + 2g - 2 - e)F$ instead. In the case of Hirzebruch surfaces this is just $D = (a - 1)C_0 + (b - 1)F$.

Corollary 3.4a. *Given integers $a, b \in \mathbb{Z}$ satisfying*

$$(3.9a) \quad a(b - \frac{a}{2}e) \geq \sum_{i=1}^r (m_i + 1)^2,$$

$$(3.10a.i) \quad a > \max\{m_i \mid i = 1, \dots, r\},$$

$$(3.10a.ii) \quad b > \max\{m_i \mid i = 1, \dots, r\}, \text{ if } e = 0,$$

$$(3.10a.iii) \quad 2(b - \frac{a}{2}e) > \max\{m_i \mid i = 1, \dots, r\}, \text{ if } e < 0, \text{ and}$$

$$(3.11a) \quad b \geq ae, \text{ if } e > 0,$$

then for $z_1, \dots, z_r \in \Sigma$ in very general position there is an irreducible reduced curve $C \in |(a - 1)C_0 + (b + l + 2g - 2 - e)F|_a$ with ordinary singularities of multiplicities m_i at the z_i as only singularities. Moreover, $V_{|C|}(\underline{m})$ is T -smooth at C .

⁹By [Nag70] Theorem 1 there is some section $D \sim_a C_0 + bF$ with $g \geq D^2 = 2b - e$. Since D is irreducible, by [Har77] V.2.20/21 $b \geq 0$, and thus $-g \leq e$.

With the same D and L as above the conditions in the existence theorem Corollary 4.4 reduce to

$$(4.3a) \quad a(b - \frac{a}{2}e) \geq \frac{207}{5} \sum_{\mu(\mathcal{S}_i) \leq 38} \mu(\mathcal{S}_i) + 29 \sum_{\mu(\mathcal{S}_i) \geq 39} \left(\sqrt{\mu(\mathcal{S}_i)} + \frac{13}{2\sqrt{29}} \right)^2,$$

$$(4.4a.i) \quad a > \begin{cases} \sqrt{\frac{207}{5}} \sqrt{\mu(\mathcal{S}_1)} - 1, & \text{if } \mu(\mathcal{S}_1) \leq 38, \\ \sqrt{29} \sqrt{\mu(\mathcal{S}_1)} + \frac{11}{2}, & \text{if } \mu(\mathcal{S}_1) \geq 39, \end{cases}$$

$$(4.4a.ii) \quad b > \begin{cases} \sqrt{\frac{207}{5}} \sqrt{\mu(\mathcal{S}_1)} - 1, & \text{if } \mu(\mathcal{S}_1) \leq 38, \\ \sqrt{29} \sqrt{\mu(\mathcal{S}_1)} + \frac{11}{2}, & \text{if } \mu(\mathcal{S}_1) \geq 39, \end{cases} \quad \text{if } e = 0,$$

$$(4.4a.iii) \quad 2(b - \frac{a}{2}e) > \begin{cases} \sqrt{\frac{207}{5}} \sqrt{\mu(\mathcal{S}_1)} - 1, & \text{if } \mu(\mathcal{S}_1) \leq 38, \\ \sqrt{29} \sqrt{\mu(\mathcal{S}_1)} + \frac{11}{2}, & \text{if } \mu(\mathcal{S}_1) \geq 39, \end{cases} \quad \text{if } e < 0, \text{ and}$$

$$(4.5a) \quad b \geq ae, \text{ if } e > 0.$$

5.b. Products of Curves. Let C_1 and C_2 be two smooth projective curves of genera $g_1 \geq 1$ and $g_2 \geq 1$ respectively. The surface $\Sigma = C_1 \times C_2$ is naturally equipped with two fibrations $\text{pr}_i : \Sigma \rightarrow C_i$, $i = 1, 2$, and by abuse of notation we denote two generic fibres $\text{pr}_2^{-1}(p_2) = C_1 \times \{p_2\}$ resp. $\text{pr}_1^{-1}(p_1) = \{p_1\} \times C_2$ again by C_1 resp. C_2 .

One can show that for a generic choice of the curves C_1 and C_2 the Neron-Severi group $\text{NS}(\Sigma) = C_1\mathbb{Z} \oplus C_2\mathbb{Z}$ of Σ is two-dimensional and $K_\Sigma \sim_a (2g_2 - 2)C_1 + (2g_1 - 2)C_2$. In this situation again, as in the case of trivially ruled surfaces, the only irreducible curves of self-intersection zero with a positive-dimensional algebraic equivalence class are the fibres C_1 and C_2 . Moreover, we may choose a very ample divisor L algebraically equivalent to $lC_1 + lC_2$ with $l \geq 3$.

In the case that C_1 and C_2 are both elliptic curves, *generic* just means that C_1 and C_2 are not isogenous, and we may choose $l = 3$. Taking into account that an algebraic class $aC_1 + bC_2$ is nef as soon as a and b are non-negative, we get the following results.

Theorem 2.1b. *Let C_1 and C_2 be two generic curves with $g(C_i) = g_i \geq 1$, $i = 1, 2$, and let $a, b \in \mathbb{Z}$ be integers satisfying*

$$(2.1b) \quad (a - 2g_2 + 2)(b - 2g_1 + 2) \geq \sum_{i=1}^r (m_i + 1)^2, \text{ and}$$

$$(2.2b) \quad (a - 2g_2 + 2), (b - 2g_1 + 2) > \max\{m_i \mid i = 1, \dots, r\},$$

then for $z_1, \dots, z_r \in \Sigma = C_1 \times C_2$ in very general position and $\nu > 0$

$$H^\nu \left(\text{Bl}_{\mathbb{Z}}(\Sigma), a\pi^*C_1 + b\pi^*C_2 - \sum_{i=1}^r m_i E_i \right) = 0.$$

In the existence theorem Corollary 3.4, the conditions (3.11), (3.12) and (3.13) become obsolete, while (3.9) and (3.10) take the form

$$(3.9b) \quad (a - l - 2g_2 + 2)(b - l - 2g_1 + 2) \geq \sum_{i=1}^r (m_i + 1)^2, \text{ and}$$

$$(3.10b) \quad (a - l - 2g_2 + 2), (b - l - 2g_1 + 2) > \max\{m_i \mid i = 1, \dots, r\}.$$

That is, under these hypotheses there is an irreducible curve in $|D|_l$, for any $D \sim_a aC_1 + bC_2$, with precisely r ordinary singular points of multiplicities m_1, \dots, m_r .

From these considerations we at once deduce the conditions for the existence of an irreducible curve in $|D|_l$, $D \sim_a aC_1 + bC_2$, with prescribed singularities of arbitrary type, i. e. the conditions in Corollary 4.4. They come down to

$$(4.3b) \quad (a-l-2g_2+2)(b-l-2g_1+2) \geq \frac{207}{5} \sum_{\mu(\mathcal{S}_i) \leq 38} \mu(\mathcal{S}_i) + 29 \sum_{\mu(\mathcal{S}_i) \geq 39} \left(\sqrt{\mu(\mathcal{S}_i)} + \frac{13}{2\sqrt{29}} \right)^2,$$

$$(4.4b) \quad (a-l-2g_2+2), (b-l-2g_1+2) > \begin{cases} \sqrt{\frac{207}{5}} \sqrt{\mu(\mathcal{S}_1)} - 1, & \text{if } \mu(\mathcal{S}_1) \leq 38, \\ \sqrt{29} \sqrt{\mu(\mathcal{S}_1)} + \frac{11}{2}, & \text{if } \mu(\mathcal{S}_1) \geq 39. \end{cases}$$

5.c. **Surfaces in $\mathbb{P}_{\mathbb{C}}^3$.** A smooth projective surface Σ in $\mathbb{P}_{\mathbb{C}}^3$ is given by a single equation $f = 0$ with $f \in \mathbb{C}[w, x, y, z]$ homogeneous, and by definition the degree of Σ , say n , is just the degree of f .

In general the Picard number $\rho(\Sigma)$ of a surface in $\mathbb{P}_{\mathbb{C}}^3$ may be arbitrarily large, but the Néron-Severi group always contains a very special member, namely the class $H \in \text{NS}(\Sigma)$ of a hyperplane section with $H^2 = n$. And the class of the canonical divisor is then just $(n-4)H$. Moreover, if the degree of Σ is at least four, that is, if Σ is not rational, then it is likely that $\text{NS}(\Sigma) = H\mathbb{Z}$. More precisely, if $n \geq 4$, Noether's Theorem says that $\{\Sigma \mid \rho(\Sigma) = 1, \deg(\Sigma) = n\}$ is a very general subset of the projective space of projective surfaces in $\mathbb{P}_{\mathbb{C}}^3$ of fixed degree n (cf. [Har75] Corollary 3.5 or [IS96] p. 146). The following considerations thus give a full answer for the “general case” of a surface in $\mathbb{P}_{\mathbb{C}}^3$.

Theorem 2.1c. *Let $\Sigma \subset \mathbb{P}_{\mathbb{C}}^3$ be a surface in $\mathbb{P}_{\mathbb{C}}^3$ of degree n , $H \in \text{NS}(\Sigma)$ be the algebraic class of a hyperplane section, and d an integer satisfying*

$$(2.1c) \quad n(d-n+4)^2 \geq 2 \sum_{i=1}^r (m_i+1)^2, \text{ and}$$

$$(2.2c) \quad (d-n+4) \cdot H.B > \max\{m_i \mid i = 1, \dots, r\} \\ \text{for any irreducible curve } B \\ \text{with } B^2 = 0 \text{ and } \dim |B|_a \geq 1, \text{ and}$$

$$(2.3c) \quad d \geq n-4,$$

then for $z_1, \dots, z_r \in \Sigma$ in very general position and $\nu > 0$

$$H^\nu \left(\text{Bl}_{\underline{z}}(\Sigma), d\pi^*H - \sum_{i=1}^r m_i E_i \right) = 0.$$

Remark 5.5. (i) If $\text{NS}(\Sigma) = H\mathbb{Z}$, then (2.2c) is redundant, since there are no irreducible curves B with $B^2 = 0$. Otherwise we would have $B \sim_a kH$ for some $k \in \mathbb{Z}$ and $k^2n = B^2 = 0$ would imply $k = 0$, but then $H.B = 0$ in contradiction to H being ample.

(ii) However, a quadric in $\mathbb{P}_{\mathbb{C}}^3$ or the K3-surface given by $w^4 + x^4 + y^4 + z^4 = 0$ contain irreducible curves of self-intersection zero.

(iii) In the existence theorems the condition depending on curves of self-intersection will vanish, even in the general case.

As for Corollary 3.4, with $L = H$, an easy calculation shows that

$$(3.9c) \quad n(d-n+3)^2 \geq 2 \sum_{i=1}^r (m_i+1)^2,$$

$$(3.12c) \quad d - \binom{n-1}{2} \geq m_i + m_j \text{ for all } i \neq j$$

ensure the existence of an irreducible curve $C \sim_a dH$ with precisely r ordinary singular points of multiplicities m_1, \dots, m_r and $h^1(\Sigma, \mathcal{J}_{X(\underline{m}; \underline{z})/\Sigma}(dH)) = 0$. With the aid of this result the conditions of Corollary 4.4 for the existence of an irreducible curve $C \sim_a dH$ with prescribed singularities \mathcal{S}_i therefore reduce to

$$(4.3c) \quad n(d - n + 3)^2 \geq \frac{414}{5} \sum_{\mu(\mathcal{S}_i) \leq 38} \mu(\mathcal{S}_i) + 58 \sum_{\mu(\mathcal{S}_i) \geq 39} \left(\sqrt{\mu(\mathcal{S}_i)} + \frac{13}{2\sqrt{29}} \right)^2,$$

and

$$(4.6c) \quad d - \binom{n-1}{2} \geq \begin{cases} \sqrt{\frac{207}{5}} \left(\sqrt{\mu(\mathcal{S}_1)} + \sqrt{\mu(\mathcal{S}_2)} \right) - 2, & \text{if } \mu(\mathcal{S}_1) \leq 38, \\ \sqrt{\frac{207}{5}} \sqrt{\mu(\mathcal{S}_1)} + \sqrt{29} \sqrt{\mu(\mathcal{S}_1)} + \frac{9}{2}, & \text{if } \mu(\mathcal{S}_1) \geq 39 \\ & \& \mu(\mathcal{S}_2) \leq 38, \\ \sqrt{29} \left(\sqrt{\mu(\mathcal{S}_1)} + \sqrt{\mu(\mathcal{S}_2)} \right) + 11, & \text{if } \mu(\mathcal{S}_2) \geq 39. \end{cases}$$

5.d. K3-Surfaces. We note that if Σ is a K3-surface then the Néron-Severi group $\text{NS}(\Sigma)$ and the Picard group $\text{Pic}(\Sigma)$ of Σ coincide, i.e. $|D|_a = |D|_l$ for every divisor D on Σ . Moreover, an irreducible curve B has self-intersection $B^2 = 0$ if and only if the arithmetical genus of B is one. In that case $|B|_l$ is a pencil of elliptic curves without base points endowing Σ with the structure of an elliptic fibration over $\mathbb{P}_{\mathbb{C}}^1$ (cf. [Mér85] or Proposition 2.4). However, a generic K3-surface does not possess an elliptic fibration, and so the following version of Theorem 2.1 applies for generic K3-surfaces (cf. [FM94] I.1.3.7).

Theorem 2.1d. *Let Σ be a K3-surface which is not elliptic, and let D be a divisor on Σ satisfying*

$$(2.1d) \quad D^2 \geq 2 \sum_{i=1}^r (m_i + 1)^2, \text{ and}$$

$$(2.3d) \quad D \text{ nef},$$

then for $z_1, \dots, z_r \in \Sigma$ in very general position and $\nu > 0$

$$H^\nu \left(\text{Bl}_{\underline{z}}(\Sigma), \pi^* D - \sum_{i=1}^r m_i E_i \right) = 0.$$

In view of equation (3.14) the conditions in Corollary 3.4 in this situation reduce to

$$(3.9d) \quad (D - L)^2 \geq 2 \sum_{i=1}^r (m_i + 1)^2,$$

$$(3.11d) \quad D - L \text{ nef, and}$$

$$(3.12d) \quad D.L - 2g(L) \geq m_i + m_j \text{ for all } i, j,$$

and, analogously, the conditions in Corollary 4.4 reduce to (4.6),

$$(4.3d) \quad (D - L)^2 \geq \frac{414}{5} \sum_{\mu(\mathcal{S}_i) \leq 38} \mu(\mathcal{S}_i) + 58 \sum_{\mu(\mathcal{S}_i) \geq 39} \left(\sqrt{\mu(\mathcal{S}_i)} + \frac{13}{2\sqrt{29}} \right)^2, \text{ and}$$

$$(4.5d) \quad D - L \text{ nef.}$$

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