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KATĚTOV'S PROBLEM

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ABSTRACT. In 1948 Miroslav Katětov showed that if the cube X^3 of a compact space X satisfies the separation axiom T_5 then X must be metrizable. He asked whether X^3 can be replaced by X^2 in this metrization result. In this note we prove the consistency of this implication.

1. Introduction

In his 1948 paper [9], Miroslav Katětov proved that if the cube of a compact space is T_5 (also called completely normal or hereditarily normal)[1] then the space must be metrizable. He asked if the cube can be replaced by the square in this statement. It has been known for some time that MA_{ω_1} and CH each give counterexamples to the implication, and more recently, a considerable weakening of CH was proved to be sufficient for a counterexample (see [19, 4, 18]). In this note we show the consistency of Katětov's metrization statement: Every compact space with T_5 square is metrizable. Our research in this area centers around a combinatorial analysis of forcing axioms compatible with the existence of a particular type of Souslin tree (related work appears in [1, 15, 16, 12, 13, 21, 27]). It turns out that for solving Katětov's problem it suffices to analyze the maximal amount of MA_{ω_1} compatible with the existence of this type of tree. Since MA_{ω_1} is by far the most familiar forcing axiom (see [2]), in order to make this paper accessible to a wider audience we have decided to extract only this part of our analysis from the more general picture that will be dealt with in subsequent papers.

It has been known for some time that the metrizability of a compactum X whose square is T_5 is closely tied with two seemingly contradictory statements, the assumption that there are more sets of countable ordinals than reals (which has been used quite successfully in analyzing the class of separable first countable normal spaces ever since Burton Jones's work [8] in 1937), and the assumption that a sufficient fragment of MA_{ω_1} holds, in particular the fragment that would ensure the separability of perfectly normal compacta (see [11, 4, 26]). Finding the right form of Jones's hypothesis and the right form of MA_{ω_1}, consistent with each other yet strong enough to give enough information about T_5 spaces to solve Katětov's problem, is the difficulty which has kept the problem open for so long. It

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[1] Recall that the separation axiom T_5 says that every two separated sets A and B (i.e., $\bar{A} \cap B = \bar{B} \cap A = \emptyset$) can be separated by disjoint open sets.

is reasonable to expect that a continuation of this work will eventually result in a much better understanding of the whole class of compact T_5 spaces. For example, the knowledge already accumulated suggests that separable — or even c.c.c. — spaces from this class must be closely tied to the unit interval, almost as closely tied, for instance, as the split interval $[0,1] \times \{0,1\}$. We also hope that this work will motivate further study of fragments of forcing axioms surviving mild forcing extensions. In our case, the extension is by a Souslin tree. Equally attractive are forcing extensions by measure algebras, as these are also quite relevant to the structure of compact T_5 spaces, as shown for example in [26], §4. We refer the reader to the works [20, 14, 25, 17, 6] for more on the preservation of forcing axioms under measure algebra extensions.

2. Notation

We consider ω_1-trees in this paper as subsets of $\omega^{<\omega_1}$, ordered by extension. If S is an ω_1-tree and $s, t \in S$, then $s \subset t$ denotes that s is an initial segment of t, which corresponds to $s \geq_S t$ in the corresponding forcing order. The length of $s \in S$, equivalently its level in S, is denoted by $l(s)$. If a is a finite subset of S, then $\inf(a)$ is the longest common initial segment of the members of a, and if the members of a are all pairwise compatible, then $\max(a)$ is the node given by $\bigcup a$. The set of unordered n-tuples (finite sets) from a set X is denoted $[X]^n$ ($[X]^{<\omega}$). For a function f and a subset a of the domain of f, $f \restriction a$ is the function with domain a which agrees with f on all points in a. If A is a subset of a topological space X, $\bar{A}$ is the closure of A in X.

We use the usual forcing terminology (see, e.g., [10]). Every object in an extension by a forcing $\mathbb{P}$ is represented by a $\mathbb{P}$-name in the ground model. The class of $\mathbb{P}$-names is defined recursively, each name being a set of pairs (p, τ) where p is a condition in $\mathbb{P}$ and τ is a $\mathbb{P}$-name which has already been defined. The pair (p, τ) being a member of a name σ means that the condition p forces that the realization of τ will be a member of the realization of σ. For each set x in the ground model, $\check{x}$ is the canonical name for x, i.e., the set of pairs $(1_{\mathbb{P}}, \check{a})$ for $a \in x$, where $1_{\mathbb{P}}$ is the empty condition in $\mathbb{P}$. The names that we refer to in this paper are all names for subsets of sets from the ground model, and as such are composed of pairs of the form $(p, \check{x})$.

All topological spaces are assumed to be T_3.

3. Souslin's axiom

A *coherent* tree is a downward closed subtree S of $\omega^{<\omega_1}$ with the property that

$$s\Delta t = \{\xi \in \mathrm{dom}(s) \cap \mathrm{dom}(t) : s(\xi) \neq t(\xi)\}$$

is finite for all $s, t \in S$. In this paper we work with coherent Souslin trees, which are Souslin trees given by a coherent family of functions in $\omega^{<\omega_1}$ closed under finite modifications. For S a coherent Souslin tree and s, t on the same (ηth) level of S, there is a canonical isomorphism π^S_{st} between the cones above s and t, defined by letting $\pi^S_{st}(v)(\alpha)$ be $t(\alpha)$ if $\alpha < \eta$ and $v(\alpha)$ otherwise, for each $v \supset s$.

We let SA_{ω_1} (where SA stands for "Souslin's Axiom") denote the statement that there is a coherent Souslin tree S such that for all posets $\mathbb{P}$ with $\mathbb{P} \times S$ c.c.c., and any collection D_ξ ($\xi < \omega_1$) of dense open subsets of $\mathbb{P}$, there is a filter $G \subset \mathbb{P}$ such that $G \cap D_\xi \neq \emptyset$ for all $\xi < \omega_1$. In our discussion of SA_{ω_1} we let S refer to such a coherent tree.

To obtain a model of SA_{ω_1}, start with a coherent Souslin tree S and a cardinal θ such that $\theta^{\omega_1} = \theta$, and build a finite support iteration of length θ of c.c.c. posets of size $\aleph_1$. This is worked out in [1, 12]. Stronger versions of the axiom appear in [15, 16, 12, 13, 21].

In the following section we shall prove that under SA_{ω_1} a considerable amount of MA_{ω_1} is true after forcing with S. It turns out that along with a certain failure of MA_{ω_1} which necessarily holds after forcing with a Souslin tree, this amount is sufficient to give the metrizability of any compact space whose square is T_5.

4. Rectangularly refinable partitions

Recall the notion of a c.c.c. partition $[\omega_1]^2 = K_0 \cup K_1$ from [24]: the partition is said to be c.c.c. if for all sequences $a_\xi \in [\omega_1]^{<\omega}$ $(\xi < \omega_1)$ either some $[a_\xi]^2 \not\subset K_0$ or there is a pair $\xi_0 \neq \xi_1$ such that $[a_{\xi_0} \cup a_{\xi_1}]^2 \subset K_0$. Let $\mathcal{K}_2$ be the statement that every c.c.c. partition of the pairs from ω_1 has an uncountable homogeneous subset. The statement $\mathcal{K}_2$ is clearly a consequence of MA_{ω_1}, and it is well known that $\mathcal{K}_2$ implies many of the consequences of MA_{ω_1}. In fact, it is still unknown whether $\mathcal{K}_2$ implies MA_{ω_1}, and it is this question which has motivated us to investigate this subject (see [24], §7; [13]). In this section we show that forcing with the coherent Souslin tree S over a model of SA_{ω_1} gives $\mathcal{K}_2$ for a class of c.c.c. partitions on $[\omega_1]^2$ which is especially relevant to Katětov's problem.

Definition 4.1. A partition $[\omega_1]^2 = K_0 \cup K_1$ satisfies the *rectangle refining property* if for all uncountable $A, B \subset \omega_1$ there are uncountable $A' \subset A$, $B' \subset B$ such that $\{\{\alpha, \beta\} : \alpha \in A', \beta \in B', \alpha < \beta\} \subset K_0$. Equivalently, for all uncountable families $\mathcal{A}, \mathcal{B} \subset [\omega_1]^{<\omega}$ of pairwise disjoint subsets of ω_1 there are uncountable $\mathcal{A}' \subset \mathcal{A}$, $\mathcal{B}' \subset \mathcal{B}$ such that $\{\{\alpha, \beta\} : \alpha \in a, \beta \in b\} \subset K_0$ for all $a \in \mathcal{A}'$, $b \in \mathcal{B}'$.

Partitions satisfying the rectangle refining property are easily seen to be c.c.c. We let $\mathcal{K}_2(rec)$ denote the statement that every partition of $[\omega_1]^2$ satisfying the rectangle refining property has an uncountable homogeneous set.

Theorem 4.2 (SA_{ω_1})**.** *The Souslin tree S forces $\mathcal{K}_2(rec)$.*

Given an ω_1-tree S and a set $C \subset \omega_1$, let S_C denote the set of $s \in S$ such that $l(s) \in C$, with the inherited ordering. Note that an uncountable $C \subset \omega_1$ and a subset K of the set of pairs from S_C induce an S-name τ_K for a subset of $[\omega_1]^2$, letting $t \Vdash \{\alpha, \beta\} \in \tau_K$ if and only if $\{s, r\} \in K$, where s and r are the predecessors of t on the αth and βth levels of S_C respectively. Conversely, if S is a Souslin tree and τ is an S-name for a set of pairs from ω_1, we can define an uncountable $C \subset \omega_1$ and a subset K of the pairs from S_C such that τ_K gives rise to the same set as τ. To define K, first note that we can find a strictly increasing function $f \colon \omega_1 \to \omega_1$ such that for all $\alpha < \omega_1$, every node on the $f(\alpha)$th level of S decides $\tau \cap [\alpha+1]^2$. Let $C = f[\omega_1]$, and define $K \subset [S_C]^2$ by letting $\{s, t\} \in K$ if and only if s and t are incompatible or $t \subset s$ and $s \Vdash \{\alpha, \beta\} \in \tau$, where $l(s) = f(\alpha)$ and $l(t) = f(\beta)$. Then τ_K is as desired.

So now let $C \subset \omega_1$ be uncountable, and let K be a subset of $[S_C]^2$ such that the partition associated to τ_K is forced to have the rectangle refining property. Let $\mathcal{P}_K$ be the set of finite K-homogeneous subsets of S_C, ordered by inclusion. We would like to see that $S \times \mathcal{P}_K$ is c.c.c., since by SA_{ω_1} this would give an uncountable homogeneous set for K, giving in turn an S-name for an uncountable homogeneous set for τ_K. Now if S is a coherent Souslin tree, then the induced ordering on S_C also gives a coherent Souslin tree, and so to establish Theorem 4.2 it suffices to prove the following lemma.

Lemma 4.3. *Say that S is a coherent Souslin tree and $K \subset [S]^2$ is such that $\{s,t\} \in K$ for all incomparable s,t, and such that the partition associated to τ_K is forced to have the rectangle refining property. Then if $\mathcal{P}_K$ is the set of finite K-homogeneous subsets of S, ordered by inclusion, then $S \times \mathcal{P}_K$ is c.c.c.*

Proof. Let $\{(t_\alpha, a_\alpha) : \alpha < \omega_1\}$ be an uncountable subset of $S \times \mathcal{P}_K$. We wish to find a pair $\alpha < \beta$ such that t_α and t_β are compatible in S and $a_\alpha \cup a_\beta \in \mathcal{P}_K$. We may fix an integer n and assume that each a_α has size n. Since moving any given t_α further up in S makes our job harder, we may assume that for all $\alpha < \omega_1$, $l(t_\alpha) > l(s)$ for all $s \in a_\alpha$. Since S is coherent, the sets $t_\alpha \triangle s$, $s \in a_\alpha$, are finite, so by n applications of the Δ-system lemma we may fix an $\bar{\alpha} < \omega_1$ and an increasing function $f : \omega_1 \to \omega_1$ such that, for all $\alpha < \omega_1$,

$$\{l(t_\alpha)\} \cup \{l(s) : s \in a_\alpha\} \subset (f(\alpha), f(\alpha+1)),$$

$$\bigcup\{t_\alpha \Delta s : s \in a_\alpha\} \subset \bar{\alpha} \cup (f(\alpha), f(\alpha+1).$$

Note that this reduction involves throwing out those $s \in a_\alpha$ with $l(s) < \bar{\alpha}$.

We can fix distinct nodes $v_0, ..., v_{m-1}$, for some integer $m \leq n$, on the $\bar{\alpha}$th level of S such that $\{t_\alpha : \alpha < \omega_1\}$ is a dense subset of the cone above above v_0, and such that every member of every a_α is above some v_i. We may also assume that the relationships among the nodes $\{\inf(b) \mid b \subset \{t_\alpha\} \cup a_\alpha\}$ are the same for all $\alpha < \omega_1$. For each $\alpha < \omega_1$, let $t^i_\alpha = \pi^S_{v_0,v_i}(t_\alpha)$, and let $a^i_\alpha = \{s \in a_\alpha \mid v_i \subset s\}$. Our application of the Δ-system lemma gives us the following density property: for any uncountable set $A \subset \omega_1$ and any $s \in S$, if $\{t_\alpha : \alpha \in A\}$ is dense above s, then so is

$$\{\inf(\bigcup_{i<m} \pi^S_{v_i v_0}[a^i_\alpha]) : \alpha \in A\}.$$

For each $\alpha < \omega_1$, let $\{c^{i,j}_\alpha \mid i < m, j < k_i\}$ enumerate the maximal pairwise compatible subsets of a_α, with each $c^{i,j}_\alpha \subset a^i_\alpha$ and each $c^{i,0}_\alpha = \{s \in a_\alpha \mid s \subset t^i_\alpha\}$. Note that if $\bigcup_{i<m} c^{i,0}_\alpha$ is empty for any (all) $\alpha < \omega_1$, then we are done.

Let $A, B = \omega_1$. We will now refine the families

$$\{(t_\alpha, a_\alpha) : \alpha \in A\}, \qquad \{(t_\beta, a_\beta) : \beta \in B\}$$

successively in $\sum_{i<m} k_i$ many steps, where in each step we replace A and B with uncountable subsets A' and B', and we replace each remaining t_α ($\alpha \in A'$) with a $t'_\alpha \supset t_\alpha$. Each refinement corresponds to a pair (i,j), $i < m$, $j < k_i$, after which for all $\alpha \in A'$, $\beta \in B'$, if $t'_\alpha \subset t'_\beta$ and

$$\{\pi^S_{v_0 v_i}(t'_\alpha)\} \cup c^{i,0}_\alpha \cup c^{i,j}_\beta$$

is an S-chain, then $c^{i,0}_\alpha \cup c^{i,j}_\alpha \in \mathcal{P}_K$. Each refinement also maintains the fact that the families

$$\{t_\alpha : \alpha \in A\}, \qquad \{t_\beta : \beta \in B\}$$

are both dense above v_0.

Now to do one step, fix A, B, i, j. By the density condition above, the sets $c^{i,0}_\alpha$ ($\alpha \in A$) and $c^{i,j}_\beta$ ($\beta \in B$) define S-names ρ, σ for uncountable sets of finite pairwise disjoint subsets of ω_1, where each t^i_α forces $\{l(s) : s \in c^{i,0}_\alpha\}$ into ρ and each $\max(c^{i,j}_\beta)$ forces $\{l(s) : s \in c^{i,j}_\beta\}$ into σ. By the rectangle refining property for τ_K, there exist names ρ_0, σ_0 for uncountable subsets of ρ, σ such that $[a \cup b]^2 \subset \tau_K$ for all $a \in \rho_0$, $b \in \sigma_0$. Now let

$$A' = \{\alpha \in A \mid \exists t \supset t^i_\alpha \text{ s.t. } t \Vdash \{l(s) : s \in \check{c}^{i,0}_\alpha\} \in \rho_0\}$$

and

$$B' = \{\beta \in B \mid \max(c^{i,j}_\beta) \nVdash \{l(s) : s \in \check{c}^{i,j}_\beta\} \notin \sigma_0\}.$$

For each $\alpha \in A'$, replace t_α with $t'_\alpha = \pi^S_{v_i v_0}(t^*)$, where t^* is any extention of t^i_α forcing $\{l(s) : s \in c^{i,0}_\alpha\}$ into ρ_0. By the choice of ρ_0 and σ_0, we see that A', B' and the t'_α's are as desired.

Having completed our refinements, we pick any $\alpha \in A$. By the density property we maintained, we may pick a $\beta \in B$ such that

$$t_\alpha \subset \inf(\bigcup_{i<m} \pi^S_{v_i v_0}[a^i_\beta]).$$

Then by the property of our refinements, $t_\beta \supset t_\alpha$ and $a_\alpha \cup a_\beta \in \mathcal{P}_K$, so α and β are as desired. □

The motivating question for our research in this area remains the following.

Question 4.4 (SA_{ω_1})**.** Does S force $\mathcal{K}_2$?

5. Subspaces of first countable compact spaces

The purpose of this section is to prove a result which shows that $\mathcal{K}_2(rec)$ imposes considerable structure on compact spaces with T_5 squares. It is this result that will move the difficulty of Katětov's problem to some questions about the real line which we consider in the next section.

Theorem 5.1 ($\mathcal{K}_2(rec)$)**.** *The following are equivalent for every space X which has a first countable compactification γX.*

(1) *For every family $\mathcal{U}$ of open subsets of X there is a countable subfamily $\mathcal{U}_0 \subset \mathcal{U}$ such that $\bigcup \mathcal{U}_0 = \bigcup \mathcal{U}$.*
(2) *For every subset Y of X there is a countable $Y_0 \subset Y$ such that $\bar{Y}_0 = \bar{Y}$.*

Proof. To prove the implication from (1) to (2), let Y be a given uncountable subset of X. If there is no countable $Y_0 \subset Y$ such that $\bar{Y}_0 = \bar{Y}$, then by shrinking Y we may assume that Y comes with a wellordering $<_w$ of ordertype ω_1 such that each $y \in Y$ is not in $\overline{\{x \in Y \mid x <_w y\}}$. For each $y \in Y$ pick an open set U_y such that $y \in U_y$ and $\bar{U}_y \cap \overline{\{x \in Y \mid x <_w y\}} \neq \emptyset$. Consider the following partition $[Y]^2 = K_0 \cup K_1$:

$$\{x, y\} \in K_0 \text{ iff } y \notin U_x \wedge x \notin U_y.$$

Since we have $\mathcal{K}_2(rec)$ at our disposal, the proof of the implication (1) ⇒ (2) is finished once we show that this partition satisfies the rectangle refining property. So let A and B be uncountable subsets of Y. By (1), there is a complete accumulation point z of B such that $z \in B$. Since z has a countable neighborhood base and since $z \notin \bar{U}_x$ for all $x \in Y$ with $x >_w z$, there is a neighborhood V of z such that

$$A_0 = \{x \in A : V \cap \bar{U}_x = \emptyset\}$$

is uncountable. Let $B_0 = B \cap V$. By the assumption that z is a complete accumulation point of B, B_0 is uncountable. Then $\{x, y\} \in K_0$ for all $x \in A_0$ and $y \in B_0$ with $x <_w y$.

To prove the implication from (2) to (1), let $\mathcal{U}$ be a given collection of open subsets of X such that $\bigcup \mathcal{U}_0 \neq \bigcup \mathcal{U}$ for all countable $\mathcal{U}_0 \subset \mathcal{U}$. Then we can fix a set $Y \subset X$, a wellordering $<_w$ of Y of ordertype ω_1, and for each $x \in Y$ a member $U_x \in \mathcal{U}$ such that $x \in U_x$ but $y \notin U_x$ for all $y >_w x$. Let γX be a first countable compact space containing X. We can replace U_x with an open subset of γX whose trace on Y is the same as the trace of our original U_x, thus assuring that U_x is open in γX. For each $x \in Y$ choose a set V_x open in γX such that $x \in V_x \subset \bar{V}_x \subset U_x$, and consider $[Y]^2 = K_0 \cup K_1$ as follows:

$$\{x, y\} \in K_0 \text{ iff } x \notin V_y \wedge y \notin V_x.$$

Again it remains only to check that this partition satisfies the rectangle refining property, so let A and B be uncountable subsets of Y. Let $z \in \gamma X$ be a condensation point of A. Note that $z \notin \bar{V}_y$ for any $y \in Y$, so, since by our assumption z has a countable neighborhood base in γX, we can find an open neighborhood V of z such that

$$B_0 = \{y \in B \mid V \cap \bar{V}_y = \emptyset\}$$

is uncountable. Let $A_0 = V \cap A$. By the assumption that z is a complete accumulation point of A, the set A_0 is uncountable, and

$$\{x, y\} \in K_0 \quad \text{for all } x \in A_0, y \in B_0, x <_w y.$$

This completes the proof. □

Remark 5.2. Note that in the above proof all we need to assume is that the ambient space γX is first countable and Lindelöf.

6. Subspaces of the real line

Katětov [9] showed that if the square of a compact space X is T_5 then X is *perfect*, i.e., closed subsets of X are G_δ. Note that every such X is first countable and satisfies hypothesis (1) of Theorem 5.1. So under $\mathcal{K}_2(rec)$ every compactum X whose square is T_5 satisfies (2). In particular, X is separable. We shall need the following classical facts.

Lemma 6.1 ([22], see also [3])**.** *A compact space is metrizable if and only its diagonal is a G_δ subset of its square.*

Proof. To prove the nontrivial direction, suppose we are given a compact space X^2 whose diagonal Δ can be written as the intersection $\bigcap_{n=0}^{\infty} G_n$ of open subsets of X^2. Since X^2 is normal, we may assume that in fact $\Delta = \bigcap_{n=0}^{\infty} \bar{G}_n$. For each n choose $f_n \colon X^2 \to [0, 1]$ such that $f_n[\Delta] = \{0\}$ and $f_n[X^2 \setminus G_n] \subset \{1\}$. Finally, define $d \colon X^2 \to [0, 1]$ by

$$d(x, y) = \sum_{n=0}^{\infty} \frac{\max\{|f_n(x, z) - f_n(y, z)| : z \in X\}}{2^n}.$$

It is easily checked that d is a metric on X which generates the original topology on X. □

Lemma 6.2 ([8, 5])**.** *If a separable, first countable normal space has an uncountable closed discrete subspace, then there is an uncountable set A of reals such that every subset of A is relatively G_δ.*

Proof. Let Z be a normal first countable space that can be written as $C \cup D$, where C is a countable dense subset of Z, D is a closed discrete subset of Z and $C \cap D = \emptyset$. For each $x \in D$, choose a subset a_x of C which converges to x. It suffces to show that the almost disjoint family $A = \{a_x : x \in D\}$ has the property that for every $B \subset A$ there is a $b \subset C$ such that for every $x \in D$, $x \in B$ if and only if $a_x \cap b$ is infinite (see Remark 6.4). Given such a $B \subset A$, since Z is a normal space there is a continuous $f\colon Z \to [0, 1]$ such that $f(x) = 0$ for all $x \in D$ with $a_x \in B$, and $f(x) = 1$ for all $x \in D$ with $a_x \notin B$. Let $b = \{x \in C \mid f(x) < 1/2\}$. □

Let X be a given compact first countable space and assume that $\mathcal{K}_2(rec)$ holds. Note that if the diagonal of X is not G_δ, then X^2 fails to satisfy property (1) of Theorem 5.1. To see this, consider the family $\mathcal{U}$ of all sets of the form $X^2 \setminus F$, where F is a closed G_δ superset of the diagonal. Applying $\mathcal{K}_2(rec)$ to X^2, we conclude that there is a $Y \subset X^2$ such that $\bar{Y}_0 \neq \bar{Y}$ for all countable $Y_0 \subset Y$. Shrinking Y, we may assume that it comes with a wellordering $<_w$ of ordertype ω_1 such that $y \notin \overline{\{x \in Y \mid x <_w y\}}$ for all $y \in Y$. Applying $\mathcal{K}_2(rec)$ to Y, via Theorem 5.1 we conclude that Y fails to satisfy (1). Let $\mathcal{U}$ be a family of open subsets of Y such that $\bigcup \mathcal{U}_0 \neq \bigcup \mathcal{U}$ for all countable $\mathcal{U}_0 \subset \mathcal{U}$. Then we can find uncountable $Z \subset Y$ such that for each $x \in Z$ there is an element $U_x \in \mathcal{U}$ such that $y \notin U_x$ whenever $x <_w y$. It follows that Z is a discrete subspace of X^2. So if X^2 were T_5 the space X would be perfect by Katětov's result [9], and so it would satisfy the hypothesis (1), and therefore the hypothesis (2), of Theorem 5.1, as $\mathcal{K}_2(rec)$ holds. In particular, X, and therefore X^2, would be separable. Choosing a countable dense set C in X^2 disjoint from $\bar{Z}$ and applying Lemma 6.2 to the subspace $C \cup Z$, we obtain an uncountable set of reals A satisfying the conclusion of the lemma. As customary, let $\mathfrak{q} = \aleph_1$ [2] denote the statement that such a set of reals cannot be found, i.e., the statement that for every uncountable $A \subset \mathbb{R}$ there is a $B \subset A$ such that $B \neq G \cap A$ for any G_δ-set $G \subset \mathbb{R}$. So we have established the following fact.[3]

Theorem 6.3 ($\mathcal{K}_2(rec)$ & $\mathfrak{q} = \aleph_1$)**.** *A compact space is metrizable if and only if its square is* T_5*.*

Remark 6.4. The proof of Lemma 6.2 shows that if there is a separable, first countable normal space with an uncountable closed discrete subspace, then there is a family y_α ($\alpha < \omega_1$) of pairwise almost disjoint subset of ω such that for every set $A \subset \omega_1$ there is a $z \subset \omega$ such that

$$A = \{\alpha < \omega_1 \mid z \cap y_\alpha \text{ is infinite}\}.$$

The existence of such a family is an equivalent form of $\mathfrak{q} \neq \aleph_1$. This is the form used most often in set theory [7], and it is commonly used in topology as well (see, e.g., [23]). For example, it is essentially shown in [4] that $\mathfrak{q} \neq \aleph_1$ implies that there is an uncountable $E \subset [0, 1]$ such that the square of $S(E)$ is T_5.[4] Thus $\mathfrak{q} \neq \aleph_1$ implies the negative answer to Katětov's question.

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[2] $\mathfrak{q} = \min\{\theta \mid \forall A \in [\mathbb{R}]^\theta \exists B \subset A \forall G_\delta\text{-sets } G \subset \mathbb{R}(B \neq G \cap A)\}$.

[3] The referee points out that the argument for the proof of Theorem 6.3 is essentially the argument presented in Theorem 6.4 of [4].

[4] Here $S(E) = ([0, 1] \times \{0\}) \cup (E \times \{1\})$ is considered as an ordered compactum with the lexicographical ordering.

Now the consistency of the positive answer to Katětov's question follows from the following fact.

Lemma 6.5 ([13]). *$\mathfrak{q} = \aleph_1$ holds after forcing with a Souslin tree.*

Proof. Let S be a Souslin tree, and let τ_α ($\alpha < \omega_1$) be S-names for an ω_1-sequence of pairwise almost disjoint subsets of ω. Since S is c.c.c. and has height ω_1, and since forcing with S doesn't add reals, we may assume that there is a club set C of levels of S such that for $\alpha \in C$, the value of τ_α is decided before the α^+th level of S, where α^+ is the least member of C above α; in other words, every member of the α^+th level decides the value of τ_α.

There is an S-name σ for a subset of ω_1 such that for all $\alpha \in C$, "$\check{\alpha} \in \sigma$" is never decided until level $\alpha^+ + 1$. For instance, we can let σ be the set of pairs $(t, \check{\alpha}) \in T \times C$ such that $\alpha^+ \in \mathrm{dom}(t)$ and $t(\alpha^+) \neq 0$. Then σ is forced to be a subset of C, and t forces $\check{\alpha} \in \sigma$ if and only if α^+ is in the domain of t and $t(\alpha^+) \neq 0$. Then no real x from the ground model can code the realization of σ, since for each $\alpha \in C$, the statement "$\check{x} \cap \tau_\alpha$ is infinite" is decided before the statement "$\check{\alpha} \in \sigma$." Since forcing with S adds no reals, we are done. ☐

Corollary 6.6 (SA_{ω_1}). *The Souslin tree S forces that every compact space whose square is T_5 is metrizable.*

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