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TOPOLOGICAL DYNAMICS ON MODULI SPACES II

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ABSTRACT. Let M be an orientable genus $g > 0$ surface with boundary ∂M. Let Γ be the mapping class group of M fixing ∂M. The group Γ acts on $\mathcal{M}_{\mathcal{C}} = \mathrm{Hom}_{\mathcal{C}}(\pi_1(M), \mathrm{SU}(2))/\mathrm{SU}(2)$, the space of SU(2)-gauge equivalence classes of flat SU(2)-connections on M with fixed holonomy on ∂M. We study the topological dynamics of the Γ-action and give conditions for the individual Γ-orbits to be dense in $\mathcal{M}_{\mathcal{C}}$.

1. Introduction

Let M be an orientable surface of genus g with n boundary components (circles). Let

$$\{C_1, C_2, ..., C_n\} \subset \pi_1(M)$$

be elements in the fundamental group that correspond to these n boundary components.

The representation space $\mathrm{Hom}(\pi_1(M), \mathrm{SU}(2))$ has a natural topology [4, 5]. The group SU(2) acts on $\mathrm{Hom}(\pi_1(M), \mathrm{SU}(2))$ by conjugation. Define the resulting quotient space to be

$$\mathcal{M} = \mathrm{Hom}(\pi_1(M), \mathrm{SU}(2))/\mathrm{SU}(2).$$

A conjugacy class in SU(2) is determined by its trace in $[-2, 2]$. To each C_i, assign a trace $-2 \leq c_i \leq 2$ and let

$$\mathcal{C} = \{c_1, c_2, ..., c_n\}.$$

Definition 1.1. The moduli space with fixed holonomy $\mathcal{C}$ is

$$\mathcal{M}_{\mathcal{C}} = \{[\rho] \in \mathcal{M} : \mathrm{tr}(\rho(C_i)) = c_i, 1 \leq i \leq n\}.$$

The space $\mathcal{M}_{\mathcal{C}}$ is compact, but possibly singular. The set of smooth points of $\mathcal{M}_{\mathcal{C}}$ possesses a natural symplectic structure which gives rise to a finite measure μ on $\mathcal{M}_{\mathcal{C}}$ (see [4, 5]).

Let $\mathrm{Diff}(M, \partial M)$ be the group of diffeomorphisms fixing ∂M. The mapping class group Γ is defined to be $\pi_0(\mathrm{Diff}(M, \partial M))$. The group Γ acts on $\pi_1(M)$. This action induces an action on $\mathrm{Hom}(\pi_1(M), \mathrm{SU}(2))$: $\Gamma \times \mathrm{Hom}(\pi_1(M), \mathrm{SU}(2)) \longrightarrow \mathrm{Hom}(\pi_1(M), \mathrm{SU}(2))$. If $\gamma \in \Gamma$ and $\rho \in \mathrm{Hom}(\pi_1(M), \mathrm{SU}(2))$, then $(\gamma \circ \rho)(X) = \rho(\gamma^{-1}(X))$. This, in turn, gives an action $\Gamma \times \mathcal{M}_{\mathcal{C}} \longrightarrow \mathcal{M}_{\mathcal{C}}$.

Theorem 1.2 (Goldman [4])**.** *The mapping class group Γ acts ergodically on $\mathcal{M}_{\mathcal{C}}$.*

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One may also study the topological dynamics of the mapping class group action on $\mathcal{M}_{\mathcal{C}}$. The topological-dynamical problem is considerably more delicate. To begin with, not all orbits are dense in $\mathcal{M}_{\mathcal{C}}$. If G is a proper closed subgroup of $\mathrm{SU}(2)$, then the image of $\mathrm{Hom}(\pi_1(M), G)$ in $\mathcal{M}_{\mathcal{C}}$ is proper, closed and Γ-invariant. The case of the one-holed torus has been dealt with in [9]:

Theorem 1.3. *Suppose M is a torus with one boundary component, and that $\rho \in \mathrm{Hom}(\pi_1(M), \mathrm{SU}(2))$ is such that $\rho(\pi_1(M))$ is dense in $\mathrm{SU}(2)$. Then the Γ-orbit of the conjugacy class $[\rho] \in \mathcal{M}_{\mathcal{C}}$ is dense in $\mathcal{M}_{\mathcal{C}}$.*

This paper deals with the general case of $g > 0$ and proves:

Theorem 1.4. *Suppose M is an orientable surface with boundary having genus greater than zero. Let $\rho \in \mathrm{Hom}(\pi_1(M), \mathrm{SU}(2))$ be such that $\rho(\pi_1(M))$ is dense in $\mathrm{SU}(2)$. Then the Γ-orbit of the conjugacy class $[\rho] \in \mathcal{M}_{\mathcal{C}}$ is dense in $\mathcal{M}_{\mathcal{C}}$.*

The group $\mathrm{SU}(2)$ double covers the group $\mathrm{SO}(3)$. The group $\mathrm{SO}(3)$ contains closed subgroups isomorphic to $\mathrm{O}(2)$, as well as the symmetry groups of the regular polyhedra: T' (the tetrahedron), C' (the cube), D' (the dodecahedron), and their subgroups. The inverse images (by the covering map) of $\mathrm{O}(2)$, T', C', D' will be called $\mathrm{Pin}(2)$, T, C, D, respectively. The identity component of $\mathrm{Pin}(2)$ is called $\mathrm{Spin}(2)$. Let $\rho \in \mathrm{Hom}(\pi_1(M), \mathrm{SU}(2))$. Theorem 1.4 implies that if $\rho(\pi_1(M))$ is not contained in a group isomorphic to C, D, or $\mathrm{Pin}(2)$, then the Γ-orbit of the conjugacy class $[\rho] \in \mathcal{M}_{\mathcal{C}}$ is dense in $\mathcal{M}_{\mathcal{C}}$. Theorem 1.4 covers all moduli spaces except those of the n-holed spheres.

1.1. **Outline of the proof.** A pants decomposition $\mathcal{P}$ of M gives rise to a smooth open dense subset $\mathcal{M}_{\mathcal{P}}$ that is an integrable system (see [7]) inside the moduli space $\mathcal{M}_{\mathcal{C}}$. Hence, one obtains the following diagram [4]:

$$\begin{array}{ccc} \mathcal{M}_{\mathcal{P}} & \xrightarrow{i} & \mathcal{M}_{\mathcal{C}} \\ \downarrow f_{\mathcal{P}} & & \downarrow f_{\mathcal{P}} \\ P & \xrightarrow{i} & P' \end{array}$$

where $\mathcal{M}_{\mathcal{P}}$ is a torus bundle over P, and $P' \subset [-2,2]^N$, where $N = \frac{1}{2}\dim(\mathcal{M}_{\mathcal{C}})$. The subgroup $\Gamma_{\mathcal{P}} \subset \Gamma$ that preserves the fibres of $f_{\mathcal{P}}$ acts as rotations on each fibre with angles depending on the base coordinates [4]. Section 2 gives a brief outline of this integrable system and the decomposition of the Γ-action.

The proof of Theorem 1.4 involves two steps. Suppose that $[\rho]$ is a generic representation (i.e. $\rho(\pi_1(M))$ is dense in $\mathrm{SU}(2)$). Let $\Gamma([\rho])$ denote the Γ-orbit of $[\rho]$. The first step is to show that if $f_{\mathcal{P}}(\Gamma([\rho]))$ is dense in P, then $\Gamma([\rho])$ is dense in $\mathcal{M}_{\mathcal{C}}$ (Corollary 2.4). The second step involves proving the base density theorem, i.e., the density of $f_{\mathcal{P}}(\Gamma([\rho]))$ in P.

As the problem deals with arbitrary genus, the proof involves induction with the one-holed torus as the base case. For a generic representation ρ, one first shows that there is a one-holed torus T inside M such that the restriction of ρ to $\pi_1(T)$ is generic. This is a detailed combinatorial calculation which is outlined in Section 3.

After obtaining a generic handle, we proceed to demonstrate the base density theorem for the $(n+2g-2)$-holed torus. An analysis of the case of the four-holed sphere is required to get the induction process started. This is used to prove the result for the case of the two-holed torus. From there, the case of the three-holed torus is proven, which, in turn, is used to prove the case of the $(n+2g-2)$-holed torus.

To complete the proof, the $2g-2$ holes of the $(n+2g-2)$-holed torus are grouped in pairs, and each pair is glued along their boundary to obtain the original surface M with genus g and n boundary components.

1.2. **Some definitions.** Fix a surface M with genus $g > 0$ and n boundary components. Then M may be described as an n-holed $2g$-gon, with appropriate identifications. More precisely, the fundamental group $\pi_1(M, O)$ is generated by $S = \{A_i\}_{i=1}^{2g+n}$, subject to the relation

$$(\prod_{i=1}^{g}[A_i, A_{i+g}])(\prod_{i=2g+1}^{2g+n} A_i) = e.$$

Definition 1.5.

1. A representation ρ into SU(2) is generic if the image of ρ is dense in SU(2).
2. A handle (A, B) consists of two simple loops $A, B \in \pi_1(M, O)$ crossing at O, but otherwise disjoint (see [3]).
3. A handle (A, B) is generic with respect to a representation ρ if the image of $\langle A, B\rangle \subset \pi_1(M)$ is dense in SU(2) (cf. Theorem 1.3).
4. Suppose $G \subset$ SU(2). A representation ρ is said (resp. not) to be G if $Im(\rho)$ is (resp. not) contained in some (resp. any) isomorphic copy of G in SU(2).
5. Associated to each simple loop $A \in \pi_1(M, O)$ is the Dehn twist in A, represented in Γ by a diffeomorphism of M. The action of the Dehn twist amounts to cutting M at A, twisting one of the resulting boundary circles once, and then re-identifying the two circles.
6. With a fixed representation ρ, $X \in \pi_1(M, O)$, and $\gamma \in \Gamma$, we write X for $\rho(X)$ and $\gamma(X)$ for $\gamma(\rho)(X)$ when there is no ambiguity. A small letter will be used to denote the trace of the matrix represented by the corresponding capital letter. For example, we use x to denote $\mathrm{tr}(\rho(X))$. In this setting, $\gamma(x)$ denotes $\mathrm{tr}(\gamma(\rho)(X))$.
7. Let $\langle V, \mathrm{d}\rangle$ be a metric space. For any $\epsilon > 0$, a subset $U \subset V$ is ϵ-dense in V if for each $v \in V$, there exists a point $u \in U$ such that $\mathrm{d}(u, v) < \epsilon$.

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2. Moduli Spaces as Tori Bundles

We begin by giving a brief description of the integrable system on the moduli space [5]. Suppose M has genus $g \geq 1$ and $n \geq 0$ boundary components. Let

$$\mathcal{C} = \{c_1, ..., c_n\}$$

be a fixed set of conjugacy classes with $c_i \neq \pm 2$ for all i. Then the real dimension of $\mathcal{M}_{\mathcal{C}}$ is $6g - 6 + 2n$. Since the case of the torus is well understood [9], we assume throughout the remainder of the paper that $g > 1$ or $n > 0$.

There is a map

$$f_{\mathcal{P}} : \mathcal{M}_{\mathcal{C}} \longrightarrow [-2, 2]^N$$

that arises from a pants decomposition $\mathcal{P}$ of M, where $N = 3g-3+n$ (see [4]). Fix a pants decomposition $\mathcal{P}$ of M. This provides $3g-3+n$ loops $B_1, ..., B_{3g-3+n} \in \pi_1(M, O)$. Let $[\rho] \in \mathcal{M}_{\mathcal{C}}$ be given. Then

$$f_{\mathcal{P}}([\rho]) = (b_1, ..., b_{3g-3+n})$$

is the desired map, where $b_i = \mathrm{tr}(\rho(B_i))$. Let $\beta = (b_1, ..., b_{3g-3+n})$, let P' be the image of $f_{\mathcal{P}}$ and let $P = P' \setminus \partial P'$. The map $f_{\mathcal{P}}$ restricted to $\mathcal{M}_{\mathcal{P}} = f_{\mathcal{P}}^{-1}(P)$ is a submersion [4]. Denote by $\Gamma_{\mathcal{P}} \subset \Gamma$ the stabilizer of the fibres of $f_{\mathcal{P}}$.

Proposition 2.1. *The set P' consists of all $\beta \in [-2,2]^{3g-3+n}$ that simultaneously satisfy the $2g-2+n-1$ inequalities*

$$b_i^2 + b_j^2 + b_k^2 - b_i b_j b_k \leq 4,$$

where the three curves B_i, B_j and B_k bound a triply punctured sphere in the decomposition $\mathcal{P}$ of M (this includes the possibility of B_i being a boundary curve in ∂M). Moreover, for $\beta \in P$, there is a $\Gamma_{\mathcal{P}}$-equivariant homeomorphism

$$h : f_{\mathcal{P}}^{-1}(\beta) \to T^{3g-3+n}$$

such that for all $\xi \in f_{\mathcal{P}}^{-1}(\beta)$ and $(n_1, ..., n_{3g-3+n}) \in \mathbb{Z}^{3g-3+n}$

$$h : \tau_1^{n_1} ... \tau_{3g-3+n}^{n_{3g-3+n}} \xi \mapsto \begin{bmatrix} e^{in_1\theta_1} h_1 \\ \vdots \\ e^{in_{3g-3+n}\theta_{3g-3+n}} h_{3g-3+n} \end{bmatrix},$$

where

$$h(\xi) = \begin{bmatrix} h_1 \\ \vdots \\ h_{3g-3+n} \end{bmatrix},$$

$\theta_j = \cos^{-1}(b_j/2)$, and τ_i is the action of the Dehn twist in B_i.

Proof. See [4]. □

In short, $(\mathcal{M}_{\mathcal{P}}, f_{\mathcal{P}})$ is an integrable system (see [7] for details). The real dimensions of P and $\mathcal{M}_{\mathcal{P}}$ are $3g-3+n$ and $6g-6+2n$, respectively. Denote by $\mathcal{M}_{\mathcal{C}}^s$ the subset of irreducible representations of $\mathcal{M}_{\mathcal{C}}$.

Lemma 2.2. *$\mathcal{M}_{\mathcal{C}}^s$ is smooth, open, and dense in $\mathcal{M}_{\mathcal{C}}$.*

Proof. See [4, 5]. □

Proposition 2.3. *The subset $\mathcal{M}_{\mathcal{P}}$ is open and dense in $\mathcal{M}_{\mathcal{C}}^s$.*

Proof. A direct calculation from Proposition 2.1 shows that $\mathcal{M}_{\mathcal{C}}^s \setminus \mathcal{M}_{\mathcal{P}}$ is a real algebraic subvariety with positive codimension. The result then follows from the fact that $\mathcal{M}_{\mathcal{C}}^s$ is smooth and has dimension $6g-6+2n$. □

Together, Propositions 2.1 and 2.3 imply:

Corollary 2.4. *Let $[\rho] \in \mathcal{M}_{\mathcal{C}}$, and let $\Gamma([\rho])$ be the Γ-orbit of $[\rho]$. If $f_{\mathcal{P}}(\Gamma([\rho]))$ is dense in P, then $\Gamma([\rho])$ is dense in $\mathcal{M}_{\mathcal{C}}$.*

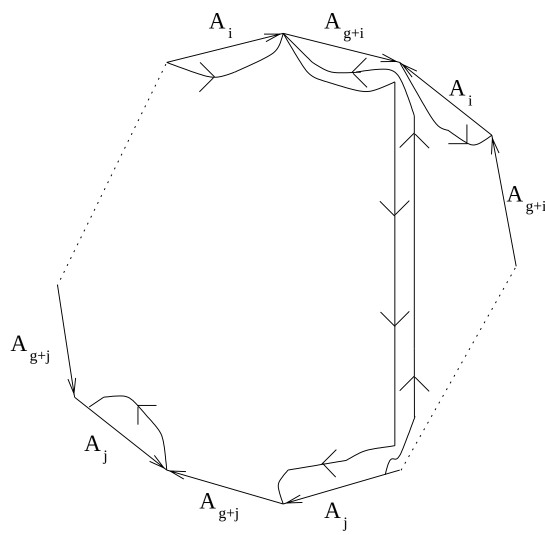

FIGURE 1. $(A_i, A_{g+i}A_j)$ is a handle.

3. Generic Representations and Handles

Here we adapt an idea in [3] to find a generic handle (one-holed torus) inside M.

Proposition 3.1. *For any generic* $\rho \in \mathrm{Hom}(\pi_1(M,O), \mathrm{SU}(2))$*, there exists a generic handle* (A, B).

The proof of Proposition 3.1 is highly computational. One should first consult Section 5 and [9], as the proof involves the moduli spaces of one-holed tori and uses many ideas pertaining to those spaces.

The proof starts with a generic representation ρ, which when restricted to each handle $\langle A_i, A_{g+i}\rangle$ is Spin(2), but not Spin(2) on M. One shows that there exists j such that $\langle A_i, A_{g+i}A_j\rangle$ is not Spin(2), with $\langle A_i, A_{g+i}A_j\rangle$ forming a handle (see Figure 1).

From there, one applies the same technique to the groups Pin(2), T, C, and finally to D. This process amounts to numerous routine computations, which we omit here but which are available at `http://vortex.bd.psu.edu/~jpp/td2/` in the form of MAPLE worksheets with accompanying text.

4. The Three-Holed Sphere

Suppose M is a three-holed sphere. Then $\pi_1(M)$ has a presentation:

$$\langle A, B, C : ABC = I\rangle,$$

where A, B, and C represent the homotopy classes of the three boundaries of M. A direct computation, together with results from [4] and [9], shows

Proposition 4.1.

1. *A representation* ρ *on a three-holed sphere is a* Spin(2)*-representation if and only if* $a^2 + b^2 + c^2 - abc - 4 = 0$.

2. *A representation* ρ *on a three-holed sphere is* $\mathrm{Pin}(2)$ *and not* $\mathrm{Spin}(2)$ *if and only if* $a^2+b^2+c^2-abc-4 \neq 0$ *and at least two of the three values* a, b, c *are zero.*

5. The One-Holed Torus

We briefly summarize some relevant results that appear in [4] and [9]. Suppose that M is a one-holed torus. The fundamental group $\pi_1(M)$ has a presentation

$$\pi_1(M) = \langle X, Y, K : K = XYX^{-1}Y^{-1}\rangle,$$

where K represents the element corresponding to the boundary component. Let

$$E = \mathrm{Hom}(\pi_1(M), \mathrm{SU}(2))/\,\mathrm{SU}(2).$$

A representation class $[\rho] \in E$ is determined by

$$x = \mathrm{tr}(\rho(X)), \quad y = \mathrm{tr}(\rho(Y)), \quad z = \mathrm{tr}(\rho(XY)).$$

Therefore, there is a global coordinate chart:

$$[\rho] \overset{F}{\longmapsto} (x, y, z).$$

In addition, $k = \mathrm{tr}(\rho(K))$ is given by the formula

$$k = \mathrm{tr}(\rho(K)) = x^2 + y^2 + z^2 - xyz - 2.$$

Let

$$E_k = \{(x, y, z) \in [-2, 2]^3 : x^2 + y^2 + z^2 - xyz - 2 = k\};$$

then

$$E = \bigcup_{k\in[-2,2]} E_k.$$

For $-2 < k < 2$, the set E_k is a smooth two-sphere; the set E_2 is a singular sphere, and $E_{-2} = (0, 0, 0)$.

The mapping class group Γ is generated by the maps τ_X and τ_Y induced by the Dehn twists in $X, Y \in \pi_1(M)$. These act on the fundamental group as

$$\tau_X(X) = X \text{ and } \tau_X(Y) = YX^{-1}$$

and

$$\tau_Y(X) = XY^{-1} \text{ and } \tau_Y(Y) = Y.$$

The effect on a representation ρ is as follows:

$$(\tau_X \circ \rho)(X) = \rho(X) \text{ and } \tau_X(\rho(Y)) = \rho(YX)$$

and

$$(\tau_Y \circ \rho)(X) = \rho(XY) \text{ and } \tau_Y(\rho(Y)) = \rho(Y).$$

Remark 5.1. The reader should be aware that when Y actually stands for $\rho(Y)$ (Definition 1.5.6), $\tau_X(Y) = YX$.

The induced action of Γ on E preserves E_k. The actions can be described explicitly [4]:

$$\tau_X(x, y, z) = (x, z, xz - y),$$

$$\tau_Y(x, y, z) = (z, y, yz - x).$$

The action of τ_X fixes x and k, and preserves the ellipse

$$X_k(x) = \{x\} \times \{(y,z) : \frac{2-x}{4}(y+z)^2 + \frac{2+x}{4}(y-z)^2 = 2+k-x^2\}.$$

A change of coordinates transforms $X_k(x)$ into the circle

$$X_k(x) = \{x\} \times \{(\tilde{y},\tilde{z}) : \tilde{y}^2 + \tilde{z}^2 = 2+k-x^2\}$$

(see [4, 9]). In this new coordinate system, τ_X acts as a rotation by $\cos^{-1}(x/2)$. In short, the sphere E_k is the union of circles

$$E_k = \bigcup_x X_k(x),$$

and τ_X rotates (up to a coordinate transformation) each level set $X_k(x)$ by an angle of $\cos^{-1}(x/2)$. Similarly, there is a coordinate transformation so that τ_Y acts as a rotation of $Y_k(y)$ by an angle of $\cos^{-1}(y/2)$.

Proposition 5.2. *The space of* $\mathrm{Spin}(2)$ *representation classes consists precisely of* E_2*. The* $\mathrm{Pin}(2)$ *representation classes consist of* E_2 *and the intersections of the three coordinate axes with* E*. For each* $-2 < k < 2$*, there are exactly six points corresponding to* $\mathrm{Pin}(2)$ *representation classes in* E_k*. Moreover, a representation class* $(x,y,z) \in E_k$ *with* $-2 < k < 2$ *and* $x \neq 0$ *is* $\mathrm{Pin}(2)$ *if and only if* $k = x^2 - 2$*.*

Proof. See [9]. ☐

Remark 5.3. This is the first explicit example of Proposition 2.1 with $g = 1$ and $n = 1$. One obtains a pair of pants by cutting along X (resp. Y). The important property is that if $X_k(x)$ (resp. $Y_k(y)$) is a non-degenerate circle, then τ_X (resp. τ_Y) acts on the fibre $X_k(x)$ (resp. $Y_k(y)$) as a rotation with an angle depending solely on x (resp. y) and is independent of either k or y (resp. x). In particular, if a representation ρ is not $\mathrm{Pin}(2)$, then neither τ_X nor τ_Y fixes $[\rho]$.

Remark 5.4. Let ρ be generic. By Theorem 1.3, the $\langle \tau_X, \tau_Y \rangle$-orbit $\mathcal{O}$ of $[\rho]$ is dense in E_k. Hence, there is a number $r > 0$ such that the set $\{(x,y) : (x,y,z) \in \mathcal{O}\}$ is dense in $R = [-r,r]^2$. In particular, by Dehn twisting in τ_X and τ_Y, one can always assume that x and y are simultaneously off of any finite set of values and are arbitrarily close to zero.

Remark 5.5. In addition to Theorem 1.3, it was shown in [9] that the k-values for the surjective T, C, D representations are in the set

$$\mathcal{S} = \{\frac{1 \pm \sqrt{5}}{2}, 0, 1\}.$$

In particular, if (x,y,z) is not $\mathrm{Pin}(2)$ and $k \notin \mathcal{S}$, then (x,y,z) is generic.

6. The Four-Holed Sphere

We first review some results that appear in [1] and [4]. Suppose M is a four-holed sphere. Then the fundamental group $\pi_1(M, O)$ admits a presentation

$$\langle A, B, C, D : ABCD = I \rangle.$$

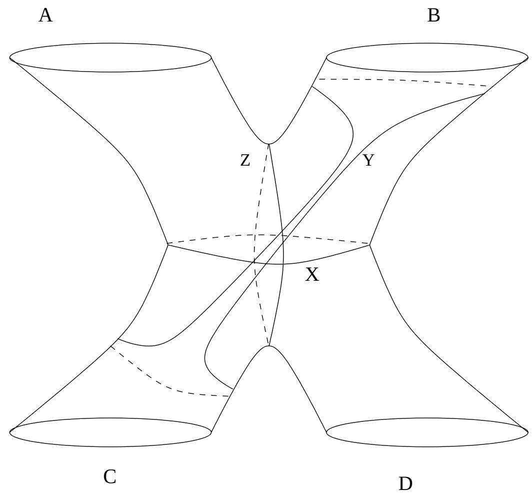

FIGURE 2. The four-holed sphere

6.1. **The moduli space $\mathcal{M}$.** If one of $a, b, c, d = \pm 2$, the moduli space can be identified with a moduli space of a three-holed sphere, so we assume that $a, b, c, d \neq \pm 2$. The moduli space $\mathcal{M}$ for the four-holed sphere is six-dimensional. Let $g : \mathcal{M} \longrightarrow [-2, 2]^4$ be the map defined by $g([\rho]) = (a, b, c, d)$.

Set $X = AB, Y = BC$, and $Z = CA$ (see Figure 2). With the (x, y, z)-global coordinates, the moduli space $E_{(a,b,c,d)} = g^{-1}(a, b, c, d)$ is the level set defined by the equation

$$\begin{aligned} x^2 + y^2 + z^2 + xyz = (ab + cd)x + (ad + bc)y + (ac + bd)z \\ - (a^2 + b^2 + c^2 + d^2 + abcd - 4). \end{aligned}$$

Generically, $E_{(a,b,c,d)}$ is a topological two-sphere.

Let

$$I_{a,b} = \left[\frac{ab - \sqrt{(a^2-4)(b^2-4)}}{2}, \frac{ab + \sqrt{(a^2-4)(b^2-4)}}{2} \right].$$

For any $\kappa = (a, b, c, d) \in (-2, 2)^4$ with $x \in I_{a,b} \cap I_{c,d}$, the x-level sets $X_\kappa(x) \subset E_\kappa$ are ellipses (possibly degenerate) in y and z given by

$$\frac{2+x}{4}\left[(y+z) - \frac{(a+b)(d+c)}{2+x}\right]^2 + \frac{2-x}{4}\left[(y-z) - \frac{(a-b)(d-c)}{2-x}\right]^2$$

$$= \frac{(x^2 - abx + a^2 + b^2 - 4)(x^2 - cdx + c^2 + d^2 - 4)}{4 - x^2}.$$

If $x \notin I_{a,b} \cap I_{c,d}$, then the x-level sets are empty. There are similar descriptions for the y- and z-level sets $Y_\kappa(y)$ and $Z_\kappa(z)$ respectively (see [4]).

For x in the interior of $I_{a,b} \cap I_{c,d}$ the level set $X_\kappa(x)$ is an ellipse centered at

$$(1) \qquad \begin{cases} y_c(x) = \frac{2[2(ad+bc) - x(ac+bd)]}{4-x^2}, \\ z_c(x) = \frac{2[2(ac+bd) - x(ad+bc)]}{4-x^2}. \end{cases}$$

Note that

$$E_\kappa = \bigcup_{x \in I_{a,b} \cap I_{c,d}} X_\kappa(x)$$

(see Proposition 2.1 and [1]). By symmetry, similar constructs exist for the y- and z-coordinates.

6.2. **The mapping class action.** The Dehn twist along X acts on $\pi_1(M)$ as follows:

$$\tau_X(A) = A, \quad \tau_X(B) = B, \quad \tau_X(C) = X^{-1}CX, \quad \tau_X(D) = X^{-1}DX.$$

This action effects the representation ρ as

$$\tau_X(\rho(A)) = \rho(A), \quad \tau_X(\rho(B)) = \rho(B), \quad \tau_X(\rho(C)) = \rho(XCX^{-1}),$$

$$\tau_X(\rho(D)) = \rho(XDX^{-1}).$$

We define τ_Y and τ_Z similarly. In local coordinates, this implies that the actions of τ_X, τ_Y, τ_Z are

$$\begin{bmatrix} y \\ z \end{bmatrix} \overset{\tau_X}{\longmapsto} \begin{bmatrix} ad + bc - x(ac + bd - xy - z) - y \\ ac + bd - xy - z \end{bmatrix},$$

$$\begin{bmatrix} z \\ x \end{bmatrix} \overset{\tau_Y}{\longmapsto} \begin{bmatrix} bd + ca - y(ba + cd - yz - x) - z \\ ba + cd - yz - x \end{bmatrix},$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \overset{\tau_Z}{\longmapsto} \begin{bmatrix} cd + ab - z(cb + ad - zx - y) - x \\ cb + ad - zx - y \end{bmatrix}.$$

These actions preserve the ellipses $X_\kappa(x) \subset E_\kappa, Y_\kappa(y) \subset E_\kappa$, and $Z_\kappa(z) \subset E_\kappa$, respectively. After coordinate transformations, these are rotations by the angles $2\cos^{-1}(x/2), 2\cos^{-1}(y/2)$, and $2\cos^{-1}(z/2)$, respectively [4].

Remark 6.1. This is the second explicit example of Proposition 2.1 with $g = 0$ and $n = 4$. One obtains two pairs of pants by cutting along X (resp. Y or Z). The important property is that if $X_\kappa(x)$ (resp. $Y_\kappa(y)$ or $Z_\kappa(z)$) is a non-degenerate circle, then τ_X (resp. τ_Y or τ_Z) acts on the fibre $X_\kappa(x)$ (resp. $Y_\kappa(x)$ or $Z_\kappa(z)$) as a rotation with an angle depending only on x (resp. y or z) and is independent of κ.

Let d be the metric on $\mathbb{R}^3$ given by

$$\mathrm{d}((x_1, y_1, z_1), (x_2, y_2, z_2)) = max\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}.$$

The metric d generates the usual topology on $\mathcal{M}_\mathcal{C}$. For a fixed κ, the coordinates provide an embedding of E_κ into $\mathbb{R}^3$.

6.3. **Filtration on the level sets.** We introduce a filtration that is analogous to the one introduced in [9] for the one-holed torus. The Dehn twist τ_Y acts on the (transformed) subsets $Y_\kappa(y)$ via a rotation of angle $2\cos^{-1}(y/2)$. Thus there is a filtration of the y-coordinates that yields finite orbits under τ_Y as follows:

Let $Y_n \subset (-2, 2)$ be such that $y \in Y_n$ if and only if the τ_Y-action on non-fixed points $(x, y, z) \in E_\kappa$ is periodic with period less than or equal to n. This gives a filtration

$$\{0\} = Y_2 \subset Y_3 \subset ... \subset Y_n \subset ... .$$

For example: $Y_2 = \{0\}$, $Y_3 = \{0, 1, -1\}$, $Y_4 = \{0, 1, -1, \sqrt{2}, -\sqrt{2}\}$, etc. Note that the filtration is independent of the choice of κ and that Y_n is a finite set for every n. By symmetry, there are similar filtrations X_n and Z_n, with $X_n = Y_n = Z_n$ as sets. The following lemmas, though proven for the Y_n filtration, apply equally to the other filtrations.

Lemma 6.2. *For $\epsilon > 0$ there exists $N(\epsilon) > 0$ so that if $y \notin Y_{N(\epsilon)}$, then the τ_Y-orbit of (x, y, z) is ϵ-dense in $Y_\kappa(y)$ for any (x, y, z) in any E_κ (see part 7 of Definition 1.5).*

Proof. Since the ellipses $Y_\kappa(y)$ are (possibly degenerate) of uniformly bounded circumferences, there exists $N(\epsilon) > 1$ such that for any $y \notin Y_{N(\epsilon)}$, the τ_Y-orbit is ϵ-dense in $Y_\kappa(y)$. □

Throughout the remainder of the paper, the moduli spaces of four-holed spheres with $\kappa = (a, b, c, d)$, having small $|c|$ and $|d|$, play an important role. We first analyze the case of E_κ for $\kappa = (a, b, 0, 0)$.

Lemma 6.3. *Suppose $(a, b, 0, 0) = \kappa \in (-2, 2)^4$. Then $X_\kappa(x) \subset E_\kappa$ is an ellipse (possibly degenerate) centered at $(x, 0, 0)$. Thus, $Y_\kappa(0)$ and $Z_\kappa(0)$ intersect every ellipse $X_\kappa(x)$.*

Proof. This result follows directly from equation (1). □

Lemma 6.4. *Let $a, b \in (-2, 2)$ and $\epsilon > 0$. Then there exists $\delta > 0$ so that for any $\kappa = (a, b, c, d)$ with $|c|, |d| < \delta$ the following hold:*

1. *The set*
$$\bigcap_{|y|<\delta} \{x : (x, y, z) \in Y_\kappa(y)\}$$
is ϵ-dense inside $\{x : (x, y, z) \in E_\kappa\}$, i.e., for all $|y| < \delta$, the set of x-coordinates of $Y_\kappa(y)$ is ϵ-dense in the set of all x-coordinates of points in E_κ.
2. *For all $x \in \{x : (x, y, z) \in E_\kappa\}$, either $\{y : (x, y, z) \in X_\kappa(x)\} \subset [-\delta, \delta]$ or $[-\frac{\delta}{2}, \frac{\delta}{2}] \subset \{y : (x, y, z) \in X_\kappa(x)\}$, i.e., the set of y-coordinates of any given $X_\kappa(x)$ either contains $[-\frac{\delta}{2}, \frac{\delta}{2}]$ or is contained in $[-\delta, \delta]$;*
3. *For all $x \in \{x : (x, y, z) \in E_\kappa\}$, either $\{z : (x, y, z) \in X_\kappa(x)\} \subset [-\delta, \delta]$ or $[-\frac{\delta}{2}, \frac{\delta}{2}] \subset \{z : (x, y, z) \in X_\kappa(x)\}$.*

Proof. The result holds by the continuous dependence of E_κ on c and d, by the continuous dependence of $X_\kappa(x)$ on x, by the geometry of E_κ for $\kappa = (a, b, 0, 0)$ as described in Lemma 6.3, and by the fact that $a, b \neq \pm 2$. □

7. The Two-Holed Torus

In this section, we prove Theorem 1.4 for the case of $g = 1$ and $n = 2$. The fundamental group $\pi_1(M, O)$ has a presentation
$$\langle X, Y, A, B : XYX^{-1}Y^{-1} = BA \rangle,$$
where A and B represent the boundary components. Let
$$K = XYX^{-1}Y^{-1}, \quad W = AX, \quad W' = XB, \quad Z = XY$$
(see Figure 3).

Cutting M along K yields a one-holed torus with boundary K and a three-holed sphere with boundaries A, B, K. If $a = \pm 2$ (resp. $b = \pm 2$), then $A = \pm I$ (resp.

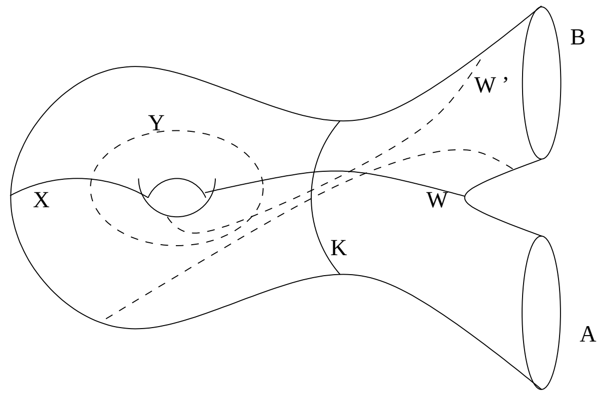

FIGURE 3. Two-holed torus.

$B = \pm I$). In this case the moduli space for M can be identified with the moduli space of a one-holed torus (with $K = BA$), and so Theorem 1.3 applies. Thus we may assume that $a, b \neq \pm 2$.

Assume that $\rho \in \mathrm{Hom}(\pi_1(M, O), \mathrm{SU}(2))$ is generic. By Proposition 3.1, we may assume without loss of generality that $\langle X, Y \rangle$ is a generic handle. If we cut M along X (resp. Y), then we obtain a four-holed sphere. For $\kappa = (b, a, x, x)$ (resp., $\kappa = (b, a, y, y)$), we obtain the moduli space E_κ of a four-holed sphere.

The defining equation of the moduli space of this 4-holed sphere, E_κ, is

$$w^2 + (w')^2 + k^2 + kww'$$
$$= k(ab + x^2) + wx(a + b) + w'x(a + b) - a^2 - b^2 - 2x^2 - abx^2 + 4,$$

and the formulas for the actions $\tau_K, \tau_W, \tau_{W'}$ of the Dehn twists on the associated moduli space of this four-holed sphere become

$$\begin{bmatrix} w \\ w' \end{bmatrix} \overset{\tau_K}{\longmapsto} \begin{bmatrix} x(a+b) - k(x(a+b) - kw - w') - w \\ x(a+b) - kw - w' \end{bmatrix},$$

$$\begin{bmatrix} w' \\ k \end{bmatrix} \overset{\tau_W}{\longmapsto} \begin{bmatrix} x(a+b) - w(x^2 + ab - ww' - k) - w' \\ x^2 + ab - ww' - k \end{bmatrix},$$

$$\begin{bmatrix} k \\ w \end{bmatrix} \overset{\tau_{W'}}{\longmapsto} \begin{bmatrix} ab + x^2 - w'(x(a+b) - w'k - w) - k \\ x(a+b) - w'k - w \end{bmatrix}.$$

Note that $I_{x,x} = [x^2 - 2, 2]$ and that the actions τ_W and $\tau_{W'}$ do not (in general) preserve y or z (see Figure 3), while the action τ_K preserves x, y, and z.

Lemma 7.1. *For the representation ρ, there exists a $\gamma \in \Gamma$ such that at least one of the following has non-zero trace: $\gamma(AX)$, $\gamma(XB)$, $\gamma(AXY)$, $\gamma(XYB)$, $\gamma(AY)$, $\gamma(YB)$, $\gamma(AYX)$, $\gamma(YXB)$, with $\langle \gamma(X), \gamma(Y) \rangle$ generic.*

Proof. Suppose that AX, XB, AXY, XYB, AY, YB, AYX, and YXB all have zero trace. If τ_K preserves $w = w' = 0$, then $x(a+b) = 0$ (if not, note that the twist in τ_K preserves x, y and k, so $\langle \tau_K(X), \tau_K(Y) \rangle$ remains generic). By Remark 5.4, x can be assumed non-zero. Hence $a = -b$. The equation for $E_{(-a,a,x,x)}$ is

$$k^2 + ka^2 + 2a^2 - 4 = x^2(k - 2 + a^2). \tag{2}$$

By Remark 5.4 and the fact that the left-hand side of equation (2) is invariant under $\langle \tau_X, \tau_Y \rangle$, it must be that $k - 2 - a^2 = 0$. The formula $\operatorname{tr}(AB^{-1}) + \operatorname{tr}(AB) = \operatorname{tr}(A)\operatorname{tr}(B)$ (see [6, 8]) and the fact that $K = BA$ give $\operatorname{tr}(AB^{-1}) = -2$. This implies that $A = -B$ as matrices.

Suppose $A = -B$. Conjugating by an element in SU(2), we may assume that

$$X = \begin{bmatrix} x_1 + y_1 i & 0 \\ 0 & x_1 - y_1 i \end{bmatrix}, \quad Y = \begin{bmatrix} x_2 + y_2 i & z_2 \\ -z_2 & x_2 - y_2 i \end{bmatrix},$$

and

$$A = \begin{bmatrix} a_1 + b_1 i & c_1 + d_1 i \\ -c_1 + d_1 i & a_1 - b_1 i \end{bmatrix}.$$

The equations $\operatorname{tr}(AX) = 0$, $\operatorname{tr}(AY) = 0$, $\operatorname{tr}(AXY) = 0$, $\operatorname{tr}(AYX) = 0$, together with the matrix equation $K = -A^2 = XYX^{-1}Y^{-1}$ and $\det(A) = \det(X) = \det(Y) = 1$ lead to one of the following: $x_1 = 0, \pm 1$, $a_1 = 0$ or $a_1 = \pm\sqrt{1 - x_1^2}$. By Remark 5.4, we have $a_1 \neq \sqrt{1 - x_1^2}$ and $x_1 \neq 0, \pm 1$. This leaves us with the case $a_1 = 0$, which implies that $K = I$, contradicting the fact that $\langle X, Y \rangle$ is generic. □

Combining Proposition 5.2 and Lemma 7.1, one obtains

Corollary 7.2. *For a generic ρ, we may assume that not all of $\langle AX, Y \rangle$, $\langle XB, Y \rangle$, $\langle AY, X \rangle$, and $\langle YB, X \rangle$ are* Pin(2).

By Remark 5.3 and Corollary 7.2, we assume without loss of generality that the action τ_Y does not fix $w = \operatorname{tr}(AX)$.

Proposition 7.3. *Suppose $g = 1$, $n = 2$, and ρ is a generic representation. Then the Γ-orbit $\Gamma([\rho])$ is dense in $\mathcal{M}_{\mathcal{C}}$.*

Proof. Let $\epsilon > 0$. Let ρ be a generic representation. By Proposition 3.1, M has a generic handle $\langle X, Y \rangle$ (we adopt the notation presented in Figure 3). By Proposition 2.3, it is enough to show that for any $[\rho_0] \in \mathcal{M}_{\mathcal{P}}$, there exists $\gamma \in \Gamma$ such that $\mathrm{d}(\gamma([\rho]), [\rho_0]) < \epsilon$. Let $x_0 = \operatorname{tr}(\rho_0(X))$, $k_0 = \operatorname{tr}(\rho_0(K))$, $w_0 = \operatorname{tr}(\rho_0(W))$, etc.

Cutting along K and X gives a pants decomposition of M. Hence, by Corollary 2.4, we only need to show that there exists $\gamma \in \Gamma$ such that $\gamma([\rho])$ satisfies $|\gamma(k) - k_0| < \epsilon$ and $|\gamma(x) - x_0| < \epsilon$.

The strategy is summarized as follows: Obtain a τ_Y-orbit with a sufficiently large number of points that have the following properties:

1. x-coordinates that are sufficiently close to zero so that $E_{(b,a,x,x)}$ (the moduli space of the four-holed sphere obtained by cutting M at X) contains points with k-coordinates equal to k_0;
2. each of the points has sufficiently many points on its respective τ_W-orbits (in $E_{(b,a,x,x)}$) so that one of these points can further be moved via τ_K and τ_W to get a point with k-coordinate within ϵ of k_0.

Along the way, there will be a certain finite list of values to avoid, e.g. $\mathcal{S}$, to guarantee that the handle $\langle X, Y \rangle$ remains generic.

One potential obstacle is the possibility that the points on the τ_Y-orbit themselves have τ_W-actions that fix $(k, w, w') \in E_{(b,a,x,x)}$. If this occurs, we will show that we can resort to the $\tau_{W'}$-action. This amounts to showing that $\langle Y, W' \rangle$ itself is not Pin(2) and that the corresponding $\tau_{W'}$-actions will not fix $(k, w, w') \in E_{(b,a,x,x)}$.

Lemma 7.4. *Suppose that $x, y, w \neq 0$ and $\tau_W(k, w, w') = (k, w, w') \in E_{(b,a,x,x)}$. Then we have the following:*

1. *If $\langle W', Y\rangle$ is* Pin(2), *then*
$$2k = ab + x^2.$$
2. *If $\tau_{W'}(k, w, w') = (k, w, w') \in E_{(b,a,x,x)}$, then*
$$x^2 = \frac{2k - ab}{1 - \left(\frac{a+b}{k+2}\right)^2}.$$

Proof. Case 1: $\langle W', Y\rangle$ is Pin(2). This implies that $w' = 0$. Hence,

$$2k = ab + x^2. \tag{3}$$

Case 2: Suppose that $\tau_{W'}(k, w, w') = (k, w, w') \in E_{(b,a,x,x)}$. Then

$$\begin{cases} 2k &= ab + x^2 - ww', \\ 2w' &= x(a+b) - wk, \\ 2w &= x(a+b) - w'k. \end{cases}$$

This leads to $2(w' - w) = k(w' - w)$. Since $k \neq \pm 2$, we have $w = w' = \frac{x(a+b)}{k+2}$. Moreover, since $w \neq 0$, we have $a \neq -b$. This implies that

$$x^2\left[1 - \left(\frac{a+b}{k+2}\right)^2\right] = 2k - ab. \tag{4}$$

If $a + b = \pm(k + 2)$ and $k = \frac{ab}{2}$, then either $a = \pm 2$ or $b = \pm 2$. This contradicts our assumptions on a and b. So, we must have

$$x^2 = \frac{2k - ab}{1 - \left(\frac{a+b}{k+2}\right)^2}. \tag{5}$$

□

Since $\langle X, Y\rangle$ is generic, Lemma 7.1 implies that we may readily find points with τ_Y-orbits having coordinates that satisfy $w \neq 0$ and $y \neq 0$, while violating both equations (3) and (5).

The following lemma follows immediately from Remark 5.4, Corollary 7.2 and the fact that the τ_X and τ_Y actions fix k.

Lemma 7.5. *For any integer $J > 0$, there is $\gamma \in \Gamma$ such that the τ_Y-orbit of $\gamma(\rho)$ has at least J points satisfying the following conditions:*

1. *The x-coordinates of these J points have $|x|$ small enough so that $k_0 \pm \frac{\epsilon}{2} \in I_{a,b} \cap I_{x,x}$ and $|x| \leq \delta$, where δ is provided in Lemma 6.4 for $E_{(b,a,x,x)}$.*
2. *These J points do not belong to the real subvarieties defined by equations (3), (5), $y = 0$, and $w = 0$.*

Consider the handle $\langle Y, W\rangle$. Choose J sufficiently large in Lemma 7.5 so that one of the J points on the τ_Y-orbit of $\gamma(\rho)$, denoted $\gamma_1(\rho)$, has a τ_W-orbit (resp. $\tau_{W'}$-orbit) with at least

$$N\left(\frac{\delta}{N(\frac{\epsilon}{2}) + 3}\right) + 3$$

points having distinct k-coordinates inside $E_{(b,a,\gamma_1(x),\gamma_1(x))}$.

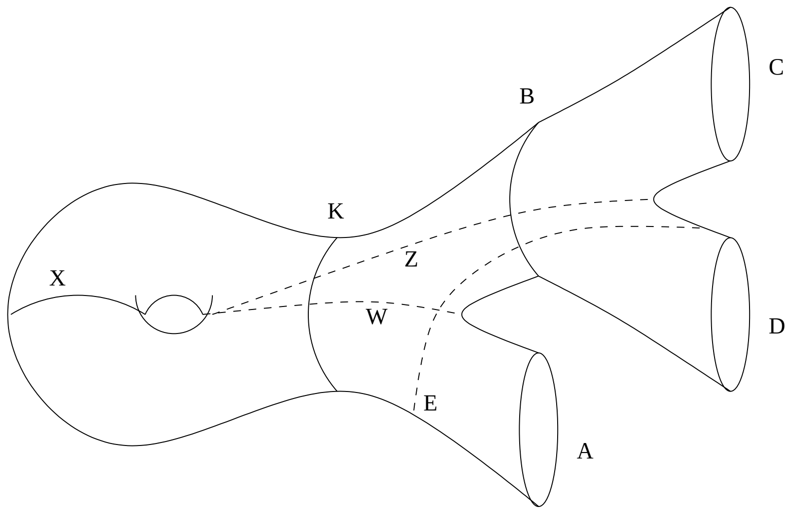

FIGURE 4. Three-holed torus.

Since $|\gamma_1(x)| \le \delta$ and $k_0 \pm \epsilon \in I_{a,b} \cap I_{\gamma_1(x),\gamma_1(x)}$, the sphere E_κ (with $\kappa = (b, a, \gamma_1(x), \gamma_1(x))$) has points with k-coordinates inside $[k_0 - \frac{\epsilon}{2}, k_0 + \frac{\epsilon}{2}]$. Since the τ_W-orbit (resp. $\tau_{W'}$) of $\gamma_1(\rho)$ has at least $N(\frac{\delta}{N(\frac{\epsilon}{2})+3}) + 3$ points with distinct k-coordinates, One such point $\gamma_2(\rho)$, has k-coordinate not in $K_{N(\frac{\delta}{N(\frac{\epsilon}{2})+3})}$ (where K_n denotes the filtration in K) with non-degenerate ellipse $K_\kappa(\gamma_2(k)) \subset E_\kappa$. Thus, the τ_K orbit of $\gamma_2(w)$ is $\frac{\delta}{N(\frac{\epsilon}{2})+3}$-dense. The τ_K-orbit of $\gamma_2(\rho)$ has at least $N(\frac{\epsilon}{2}) + 3$ points with w-coordinates inside $(-\delta, \delta)$. One such point, $\gamma_3(\rho)$ has w-coordinate not in $W_{N(\frac{\epsilon}{2})}$ with non-degenerate ellipse $W_\kappa(\gamma_3(w))$. Thus, the τ_W-orbit of $\gamma_3(\rho)$ is $\frac{\epsilon}{2}$-dense in $W_\kappa(\gamma_3(w))$. This fact, together with the properties of δ provided in Lemma 6.4, implies that at least one point, $\gamma_4([\rho])$, in the τ_W-orbit of $\gamma_3(\rho)$ has k-coordinate $\gamma_4(k)$ that comes within ϵ of k_0.

The restriction to the one-holed torus $\langle \gamma_4(X), \gamma_4(Y) \rangle$ is generic so long as $\gamma_4(k) \notin \mathcal{S}$ and $\gamma_4(k) \neq (\gamma_4(x))^2 - 2$ (see Proposition 5.2 and Remark 5.5). This can be accomplished by replacing ϵ by $\frac{\epsilon}{20}$ at the start of the argument. Since $\langle \gamma_4(X), \gamma_4(Y) \rangle$ is generic, we next apply τ_X and τ_Y to obtain $\gamma_5 \in \Gamma$ so that the x-coordinate $\gamma_5(x)$ of $\gamma_5([\rho])$ is within ϵ of x_0. Note that both τ_X and τ_Y fix $\gamma_4(k)$. Thus $\langle \gamma_5(X), \gamma_5(Y) \rangle$ remains generic. The result now follows from Corollary 2.4. □

8. THE n-HOLED TORUS

The conjugacy classes that correspond to $c_i = \pm 2$ are central. Thus, for $n > 2$, one may assume that $C_i \neq \pm I$ for all i, or that $n = 1$ and $C_1 = \pm I$.

Proposition 8.1. *Let M be an $n-$holed torus and ρ a generic representation with generic handle $\langle X, Y \rangle$, which is part of a pants decomposition $\mathcal{P}$, i.e., both $K = XYX^{-1}Y^{-1} \in \mathcal{P}$ and $X \in \mathcal{P}$. Then there is $\gamma \in \Gamma$ so that $\gamma(P) \neq \pm I$ for all $P \in \mathcal{P}$. Moreover, the handle $\langle \gamma(X), \gamma(Y) \rangle$ is generic.*

Proof. We first treat the case of $n = 3$. Let C and D be two boundary loops separated from $\langle X, Y \rangle$ by $B = CD$. Let A be the remaining boundary loop (see

Figure 4). Since $\langle X, Y\rangle$ is generic, we have that $K, X \neq \pm I$. The goal, therefore, is to find $\gamma \in \Gamma$ such that $\gamma(B) \neq \pm I$.

Suppose $B = \pm I$. Then $C = \pm D^{-1}$ and $A = \pm K^{-1}$. If $AD = E \neq \pm I$, then we may apply Proposition 7.3 to the two-holed torus bounded by E and C. Hence, if $E \neq \pm I$, there exists an element $\gamma \in \Gamma$ that fixes a with $\gamma(k) \neq \pm a$ such that $\langle \gamma(X), \gamma(Y)\rangle$ is generic. This implies $\gamma(B) \neq \pm I$. The same conclusion can be drawn if $AC \neq \pm I$. Thus, we assume that $AC = \pm I$ and $AD = \pm I$. Since $A = \pm K^{-1}$, we have that $C, D = \pm K$. Hence, $B = CD = \pm K^2 = \pm I$. This is only possible if $k = 0$ or $K = \pm I$. The latter case is ruled out by the generic assumption on $\langle X, Y\rangle$. Hence $k = 0$. We will show that in this special case, we can obtain $\gamma(B) \neq \pm I$.

Suppose that X commutes with K. If $\tau_Y(X) = XY$ also commutes with K, then X, Y, and K all belong to the same one-parameter subgroup of SU(2). Hence X and Y commute, contradicting the fact that $\langle X, Y\rangle$ is generic. Suppose that $XKX^{-1}K^{-1} = -I$ (X and K anti-commute). Then, $X = K(-X)K^{-1}$, which implies that $x = 0$. Again, since $\langle X, Y\rangle$ is generic, we may assume that $x \neq 0$. To summarize, since $\langle X, Y\rangle$ is generic, one may arrange that X neither commutes nor anti-commutes with K (i.e. $XKX^{-1}K^{-1} \neq \pm I$).

Now consider the curve $Z = XC$. The Dehn twist in Z preserves $E = \pm I$ and c. Hence it fixes $k = 0$ (consider the pants bounded by C, E, and K). On the four-holed sphere bounded by C, D, X and W, the action of τ_Z on the matrix B is $\tau_Z(B) = CZDZ^{-1} = \pm CXCDC^{-1}X^{-1}$. Since $C, D = \pm K$ and $K^2 = -I$, $\tau_Z(B) = \pm KXK^{-1}X^{-1} \neq \pm I$. Hence there exists $\gamma \in \Gamma$ with $\gamma(B) \neq \pm I$. By Remark 5.4, we may assume that x is not in $\{0, \pm 1, \pm\frac{1\pm\sqrt{5}}{2}, \pm\sqrt{2}\}$. Since τ_Z preserves k and x, the handle $\langle \tau_Z(X), \tau_Z(Y)\rangle$ is generic, as $k = 0$ and $x \notin \{\pm\sqrt{2}, 0\}$. This proves the case $n = 3$.

The above argument may be repeated iteratively, starting with loops in $\mathcal{P}$ that bound two boundary loops and working inward towards K. We demonstrate using the case $n = 5$, with notation provided in Figure 5.

First consider the three-holed torus bounded by E, F, and $H = ACD$. Since $\langle X, Y\rangle$ is generic, if $B_1 = \pm I$, then $ACD \neq \pm I$ (otherwise $K = \pm I$). Use the previous argument for $n = 3$ to arrange $B_1 \neq \pm I$. Next, the three-holed torus bounded by B_1, D, and AC is used to make $B_2 \neq \pm I$, etc. □

Proposition 8.2. *Suppose that $g = 1$, $n = 3$ and ρ is a generic representation. Then the Γ-orbit $\Gamma([\rho])$ is dense in $\mathcal{M}_{\mathcal{C}}$.*

Proof. The proof is similar in nature to the proof of Proposition 7.3. Consider the notation suggested in Figure 4.

Cutting along X, K and B yields a pants decomposition. Hence, by Proposition 7.3 and Corollary 2.4, we need to show that there exists $\gamma \in \Gamma$ such that $\gamma([\rho])$ satisfies $|\gamma(b) - b_0| < \epsilon$, with $\langle \gamma(X), \gamma(Y)\rangle$ generic. We first move x and w near zero so that the moduli space of the four-holed sphere bounded by X, W, C, and D contains b-coordinates near b_0. We use Propositions 8.1 and 7.3 to do this.

As in the proof of Proposition 7.3, we wish to obtain a τ_Y-orbit with a sufficiently large number of points that have the following properties:

1. x- and w-coordinates that are sufficiently near zero so that the moduli space of the four-holed sphere bounded by X, W, C, and D contains points with b-coordinate equal to b_0;

2. each of these points has sufficiently many points on their respective τ_Z-orbits (in $E_{(x,w,c,d)}$) so that one such point can further be moved via τ_B and τ_Z on $E_{(x,w,c,d)}$ to get a point with a b-coordinate within ϵ of b_0.

For this strategy to work, we must first have that $\langle Y, Z\rangle$ or $\langle Y, Z'\rangle$ is not Pin(2), where $Z' = DX$. If both $\langle Y, Z\rangle$ and $\langle Y, Z'\rangle$ are Pin(2), then $z = z' = 0$ and $\mathrm{tr}(YZ') = \mathrm{tr}(YZ) = 0$. Note that the application of τ_B fixes the moduli space of the two-holed torus bounded by A and B, i.e., τ_B fixes a, b, x, y, k, and w. If the Dehn twist τ_B fixes (b, z, z') with $z = z' = 0$, then we have

$$wd + xc = 0 \tag{6}$$

and

$$xd + wc = 0. \tag{7}$$

In the special case where $c = d = 0$, the defining equation for $E_{(x,w,c,d)}$ yields

$$b^2 = xwb - x^2 - w^2 + 4. \tag{8}$$

However, by Proposition 7.3 and the fact that b, c, d are fixed by mapping class actions on the two-holed torus bounded by B and A, we may assume that the x, w-coordinates do not satisfy any of the above equations. Hence, we may assume that $\langle Y, Z\rangle$ is not Pin(2).

We consider the possibility that the points on the τ_Y-orbit themselves have τ_Z-actions that fix $(b, z, z') \in E_{(x,w,c,d)}$. As before, in such cases we will rely on the $\tau_{Z'}$-action.

If τ_Z fixes $(b, z, z') \in E_{(x,w,c,d)}$ and $\langle Y, Z'\rangle$ is Pin(2), then

$$2b = xw + cd. \tag{9}$$

If neither $\langle Y, Z\rangle$ nor $\langle Y, Z'\rangle$ is Pin(2) but both τ_Z and $\tau_{Z'}$ fix (b, z, z'), then

$$\begin{cases} 2b &= xw + cd - zz', \\ 2z &= xc + wd - bz, \\ 2z' &= xd + wc - bz'. \end{cases}$$

This implies that τ_B also fixes (b, z, z'). As neither $\langle Y, Z\rangle$ nor $\langle Y, Z'\rangle$ is Pin(2), we may assume that $z, z' \neq \pm 2$. Hence $B_\kappa(b)$, $Z_\kappa(z)$ and $Z'_\kappa(z')$ are single points. This leads to $I_{x,w} \cap I_{c,d} = \{b\}$. However, as $b \neq \pm 2$, one can use Proposition 7.3 to guarantee that the x, w-coordinates on sufficiently many points of a τ_Y-orbit do not satisfy equation (9) and are close enough to zero so that b is not an endpoint of $I_{x,w}$.

The argument now follows similarly to the one presented in Proposition 7.3 to produce $\gamma \in \Gamma$ satisfying $|\gamma(b) - b_0| < \epsilon$. ∎

Proposition 8.3. *Suppose that M is a genus one surface with $n \geq 1$ boundary components. Let ρ be a generic representation. Then the Γ-orbit $\Gamma([\rho])$ is dense in $\mathcal{M}_{\mathcal{C}}$.*

Proof. The cases $n = 1, 2$, and 3 have previously been established. For $n \geq 4$, the argument follows by an induction process similar to that in the proof of Proposition 8.1.

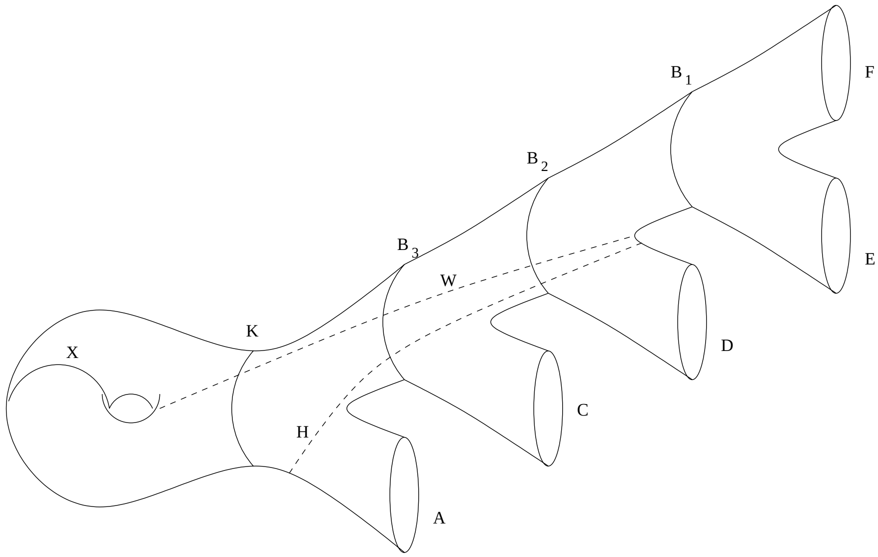

FIGURE 5. Getting rid of $\pm I$ on $\mathcal{P}$ for $n = 5$.

Let ρ be a generic representation. By Proposition 3.1, M has a generic handle $\langle X, Y\rangle$. We demonstrate how to proceed in the case $n = 5$ (see Figure 5). Let $[\rho_0] \in \mathcal{M}_{\mathcal{P}}$.

By Proposition 8.1, we have that $B_1 \neq \pm I$. Assume that Proposition 8.3 is true for the four-holed torus bounded by B_1, A, C and D. We use this four-holed torus to arrange for x and w (see Figure 5) to have near zero traces. This ensures that the b_1-coordinate of $[\rho_0]$ is accessible. We then arrange $H \neq \pm I$. Next, cut M at H and use Proposition 8.2 to get the b_1-coordinate of $\gamma([\rho])$ within ϵ of $(b_1)_0$. Next, we get the b_2-coordinate of $\gamma([\rho])$ within ϵ of $(b_2)_0$, by using the four-holed torus obtained by cutting at B_1. Finally, Proposition 8.2 and Corollary 2.4 yield the result. The situation is identical for any $n > 3$. □

9. GENUS g WITH n BOUNDARY COMPONENTS

Suppose M is a surface with $g > 1$ and $n \geq 0$. Again, we assume that $C_i \neq \pm I$ for all boundary curves $C_i \in \partial M$, unless $n = 1$ and $C_1 = \pm I$.

Proposition 9.1. *Let M have generic $\langle X, Y\rangle$ that is part of a pants decomposition $\mathcal{P}$. Then there is $\gamma \in \Gamma$ so that $\gamma(\rho(P)) \neq \pm I$, for all $P \in \mathcal{P}$ with $\langle\gamma(X), \gamma(Y)\rangle$ generic. (In the special case $n = 1$ and $C_1 = \pm I$, we have $\gamma(\rho(P)) \neq \pm I$ for all $P \in \mathcal{P}$ except C_1.) Furthermore, $\langle\gamma(P), \gamma(Q)\rangle$ is generic, for all $P \in \mathcal{P}$ and $PQP^{-1}Q^{-1} = R \in \mathcal{P}$ that bound a three-holed sphere which forms a one-holed torus in M.*

Proof. We first show that there is $\gamma \in \Gamma$ so that $\langle\gamma(P), \gamma(Q)\rangle$ is generic for all $P \in \mathcal{P}$ and $PQP^{-1}Q^{-1} = R \in \mathcal{P}$ that bound a three-holed sphere which forms a one-holed torus in M. Consider P, Q, and R as in Figure 6, where $P, R \in \mathcal{P}$ bound a pants that forms a one-holed torus in M.

Case 1: Suppose that $R \neq \pm I$. In this case, at least one of p, q or $\mathrm{tr}(PQ)$ is not in $\{0, \pm 2\}$. Without loss of generality, assume that $p \notin \{0, \pm 2\}$. Thus, we

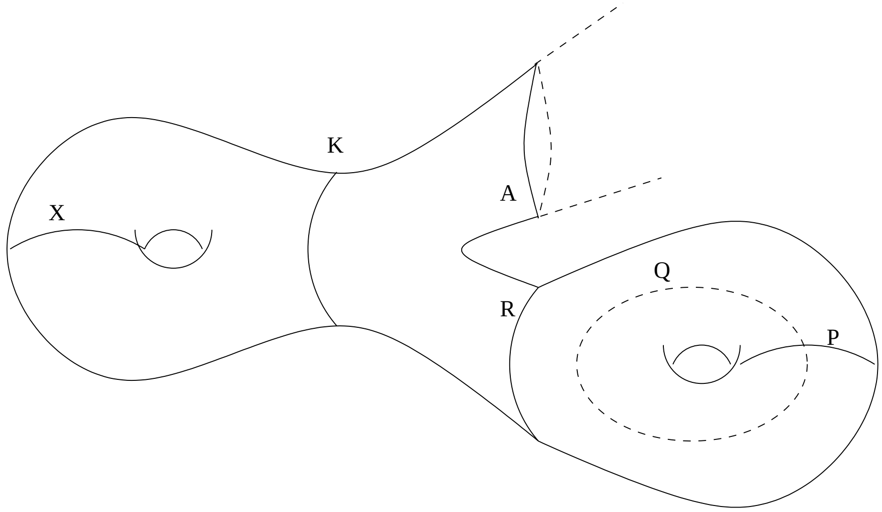

FIGURE 6. A punctured genus two inside M.

may cut M at P and A and apply Proposition 8.2 to the resulting three-holed torus to obtain $r \notin \mathcal{S}$ with $r \neq p^2 + 2$. Note that in the special case $A = \pm I$, the corresponding three-holed torus can be identified to that of a two-holed torus, to which Proposition 7.3 can be applied. Thus the handle $\langle \gamma(P), \gamma(Q) \rangle$ is generic.

Case 2: Suppose that $R = -I$. Then both P and Q are not $\pm I$. Moreover, $A \neq \pm I$ since $K \neq \pm I$. Thus, we may cut M at P and A, and then apply Proposition 8.2 to reduce the situation to Case 1.

Case 3: Suppose that $R = I$. Then, necessarily, $A = \pm K^{-1} \neq \pm I$, since $\langle X, Y \rangle$ is generic. If either $P \neq \pm I$ or $Q \neq \pm I$, we may cut at P (or Q) and apply Proposition 8.2 as in case 2. If both $P = \pm I$ and $Q = \pm I$, then $\tau_{PA}(Q) = Q(PA) = \pm K \neq \pm I$. Note that the action of τ_{PA} does not affect the generic handle $\langle X, Y \rangle$. We now cut at Q and apply Proposition 8.2 as in case 2. This argument can be applied independently to each of the $g - 1$ handles of M.

Having obtained that all handles are generic, we now apply Proposition 8.1 to the remaining curves in $\mathcal{P}$ that are interior to the $(n + g - 1)$-holed torus obtained by cutting off each of the $g - 1$ handles. In the special case $n = 1$ and $C_1 = \pm I$, Proposition 8.1 applies to those curves of $\mathcal{P}$ that are separated from C_1 by the curve KC_1. Moreover, $KC_1 \neq \pm I$ since $\langle X, Y \rangle$ is assumed generic. □

Now we prove Theorem 1.4. Let ρ be a generic representation. By Proposition 3.1, M has a generic handle $\langle X, Y \rangle$ (we adopt the notation presented in Figure 7). By Proposition 2.3, without loss of generality we may take $[\rho_0] \in \mathcal{M}_{\mathcal{P}}$ and show that there exists $\gamma \in \Gamma$ such that $\mathrm{d}(\gamma([\rho]), [\rho_0]) < \epsilon$.

By Proposition 9.1, we may assume that $\langle P_i, Q_i \rangle$ is generic and that $p_i \notin \{0, \pm 2\}$ for all $1 \leq i \leq g - 1$.

Dehn twist each generic handle $\langle P_i, Q_i \rangle$ to obtain p_i arbitrarily close to zero. Since x and $z_i = \mathrm{tr}(R_i X)$ can be made arbitrarily close to zero by Proposition 8.3 (one can do this also in the exceptional case $n = 1$, $C_1 = \pm I$, and $g > 1$), we are ensured that the target value r_i-coordinate of ρ_0 will be inside the $\epsilon-$neighborhood

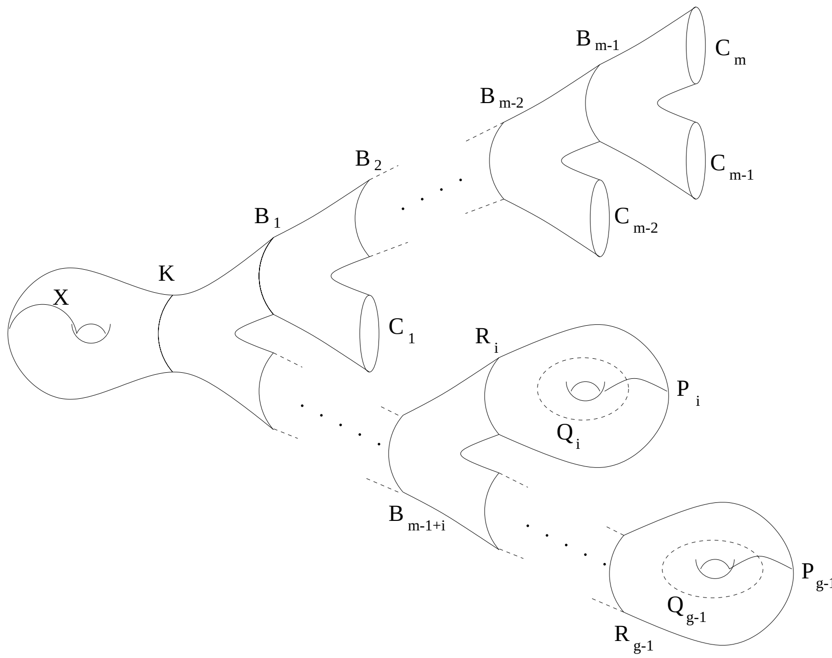

FIGURE 7. Decomposition of M.

of $I_{p_i,p_i} \cap I_{x,z_i}$. Thus, there are points on the moduli space of the four-holed sphere $E_{(p_i,p_i,x,z_i)}$ with r_i-coordinates within ϵ of $(r_i)_0$, the r_i-coordinate of ρ_0.

We next apply Proposition 8.3 to the $(n+2g-2)$-holed torus obtained by cutting M at each of the P_i to move each r_i within ϵ of $(r_i)_0$, keeping $\langle\gamma(P_i), \gamma(Q_i)\rangle$ generic. Again, this is also possible in the special case of $n = 1$, $C_1 = \pm I$, and $g > 1$, as the moduli space of the $(1+2g-2)$-holed torus reduces to that of a generic $(2g-2)$-holed torus.

Next, we twist on each of the $g-1$ generic handles to get p_i within ϵ of $(p_i)_0$. Finally, we cut off each handle at R_i and apply Proposition 8.3 to the resulting $(n+g-1)$-holed torus with generic $\langle\gamma(X), \gamma(Y)\rangle$.

Theorem 1.4 now follows from Corollary 2.4.

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