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REGULARIZED ORBITAL INTEGRALS FOR REPRESENTATIONS OF SL(2)

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ABSTRACT. Given a finite-dimensional representation of $\mathbf{SL}(2, F)$, on a vector space V defined over a local field F of characteristic zero, we produce a regularization of orbital integrals and determine when the resulting distribution is non-trivial.

0. INTRODUCTION

It is a well-known result of Ranga-Rao and Deligne (see [7]) that the invariant measure on each conjugacy class in a reductive algebraic group extends to a tempered distribution on the group. In this paper, we concern ourselves only with the linear picture: here F is a local field, G is the set of F-rational points of a reductive algebraic group, which acts on its Lie algebra $\mathfrak{g}$—an F-vector space—via the adjoint action. Then there is a canonical (up to scalar) G-invariant measure $\mu_{\mathfrak{o}}$ on each orbit $\mathfrak{o}$ in $\mathfrak{g}$, and it extends to a tempered G-invariant distribution on $\mathfrak{g}$ given by the absolutely convergent integral

$$f \mapsto \int_{\mathfrak{o}} f \, d\mu_{\mathfrak{o}}. \tag{0.1}$$

In the more general setting where G acts on a finite-dimensional F-vector space V via a linear representation, some orbits in V do not have G-invariant measures, and for some of the rest, the integral (0.1) diverges for general f. As an example, consider the four-dimensional irreducible representation of $G = \mathbf{SL}(2, F)$; we can think of V (see [8]) as the space of binary cubic forms. Here every orbit has a G-invariant measure. If we set $\mathfrak{o}$ to be the orbit of the form x^2y, then the integral (0.1) diverges for general functions on V. On the other hand, Shintani in [8] (for $F = \mathbb{R}$) and Datskovsky and Wright in [2] (for other local fields) produced a distribution Σ_1 that agrees with (0.1) on a large space of functions on V, and then proved that this distribution is invariant. This distribution was then used in the calculation of the Shintani zeta function for the space of binary cubic forms.

In this paper, we examine all finite-dimensional rational representations of $G = \mathbf{SL}(2, F)$. In this case, for each orbit $\mathfrak{o}$, there exists a canonical (up to scalar) G-invariant measure $\mu_{\mathfrak{o}}$ on $\mathfrak{o}$. If F is non-archimedean, we show that the integral (0.1) does converge for functions f that vanish on the boundary of $\mathfrak{o}$ in the usual topology on V. Our main result for non-archimedean F is as follows:

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Theorem 0.1. *Suppose F is non-archimedean and $G = \mathbf{SL}(2, F)$ acts on a finite-dimensional F-vector space. There exists an invariant distribution that agrees with (0.1) at all functions that vanish on the (topological) boundary of an orbit $\mathfrak{o}$ if the normalizer of some maximal F-split torus fixes the boundary of $\mathfrak{o}$.*

A similar, somewhat more complicated, result holds for archimedean fields. We believe that the converse of Theorem 0.1 is also true. As evidence, we have the following result, valid for $F = \mathbb{R}$.

Theorem 0.2(=2.6). *Suppose that $G = \mathbf{SL}(2, \mathbb{R})$ acts on a finite-dimensional real vector space V, and $\mathfrak{o}$ is an orbit whose boundary is not fixed by the normalizer of any maximal split torus. Then the integral (0.1) converges for functions f that vanish on the boundary of $\mathfrak{o}$, but no invariant distribution on V agrees with (0.1) at all such f.*

Verifying the invariance of distributions produced in such a context has been typically done by one of three methods: using global arguments from the Shintani zeta function, explicit computation of the Fourier transform in terms of known invariant distributions, and in the case of the adjoint representation, using (G, M)-families. In this paper we use a different method, based on analytic continuation and the effect of translation on the Iwasawa decomposition.

The motivation of this paper was to explain locally the invariance of certain distributions that appear in the zeta function of the space of binary quartic forms, without resorting to the explicit computations required in the methods used in [8] and [2]. This is done in Theorem 2.5(b).

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1. Building local invariant distributions

Let $\mathbf{G}$ be the group $\mathbf{SL}(2)$. Suppose we are given a finite-dimensional vector space $\mathbf{V}$ and a linear action of $\mathbf{G}$ on $\mathbf{V}$, both defined over F. Write G, V for the F-points of $\mathbf{G}, \mathbf{V}$, and $v \mapsto g \cdot v,\ v \in V$ for the action of $g \in G$ on V. If v is any vector in V, write G_v for the connected component of the identity in the stabilizer $\mathrm{Stab}_G\, v$ of v in G. If G_v is unimodular, then there is a canonical, up to constant multiple, left-invariant measure $d\bar{g}$ on G/G_v, or equivalently, an invariant measure $\mu_{\mathfrak{o}}$ on the orbit $G \cdot v$.

Call a vector $v \in V$ semisimple if its G-orbit is closed, and nilpotent if the closure of its G-orbit contains 0. It is well-known that if v is semisimple, then G_v is reductive, hence unimodular, and the integral

$$\int_{G/G_v} f(\bar{g} \cdot v) d\bar{g} = \int_{\mathfrak{o}} f \, d\mu_{\mathfrak{o}} \tag{1.1}$$

converges absolutely for every f in the space of smooth compactly supported functions on V, and defines a distribution on V, whose support is the orbit $G \cdot v$. Let us call this distribution $I_{G,V}(v)$.

If the space V is $sl(2)$, with G-action given by conjugation, the theorem of Ranga Rao and Deligne [7] implies that the integral (1.1) defines a tempered distribution, supported on the closure of the orbit $G \cdot v$, for every element $v \in V$. This is not the case for general V; the integral (1.1) will often diverge even for f of compact support.

By the Hilbert-Mumford theorem (see [4]), given any non-semisimple vector in V, there exists an F-split torus $\mathbf{A}$ in $\mathbf{G}$ and a unique choice of a simple root for ($\mathbf{G}$,$\mathbf{A}$) so that the vector lies in the direct sum of the weight spaces with respect to $\mathbf{A}$ of V corresponding to non-negative weights, and the component of the vector in the 0-weight space is semisimple.

We will first consider the case that $\mathbf{A}(F)$ is the group

$$A = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} = d(x) \mid x \in F^* \right\}$$

of diagonal matrices, and that the positive root of ($\mathbf{G}$,$\mathbf{A}$) is $\alpha \in X^*(\mathbf{A})$ defined by $\alpha(d(x)) = x^2$—since any two F-split tori are conjugate in G, every non-semisimple point is in the G-orbit of a point leading to this case. For the remainder of this paper, we will take our weight spaces with respect to this $\mathbf{A}$. Write N for the group

$$N = \left\{ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = n(y) \mid y \in F \right\}$$

of strictly upper triangular matrices. The modular character δ_P on the minimal parabolic subgroup $P = AN$ is given by $\delta_P(an) = |\alpha(a)|$, where $|\cdot|$ is the norm on F normalized as in [9]. We will also write

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Pick a decomposition $V = \bigoplus_\pi V_\pi$ of V into irreducible subrepresentations, and decompose each V_π into its (1-dimensional) weight spaces V_π^q, $q \in \mathbb{Z}$, consisting of those vectors in V_π on which $d(x)$ acts as multiplication by x^q for all $x \in F^\times$. Choose a nonzero vector inside each nontrivial subspace V_π^q; these vectors constitute a basis of V. When we refer to coordinates on V we will mean coordinates with respect to this basis. Given any $v \in V$, $q \in \mathbb{Z}$, and π, write v_π^q for the component of v in V_π^q. Similarly, write $V^q = \bigoplus_\pi V_\pi^q$, $v^q = \sum_\pi v_\pi^q$.

Lemma 1.1. *Suppose that v is not semisimple, and lies in the direct sum of non-negative weight spaces.*

(a) G_v is either $\{I\}$ or N.

(b) $g \cdot v$ lies in the direct sum of non-negative weight spaces if and only if $g \in P$.

Proof. (a) The vector v^0, the component of v in the 0-weight space, is also the component of $n \cdot v$ in the 0-weight space for any $n \in N$. Since $\mathbf{G} \cdot v^0$ is closed and $\mathbf{G} \cdot v$ is not, we see that $n \cdot v$ has a non-trivial component in the positive weight space for any $n \in N$. Any element of $wP \cdot v$ has a non-zero component in a negative weight space; the lowest such component is preserved under the action of N, hence $v \notin NwP \cdot v$. By the Bruhat decomposition $G = P \cup NwP$, we see then that $\operatorname{Stab}_G v = \operatorname{Stab}_P v$.

If, for any $y \in F^\times$, the element $n(y)$ stabilizes v, then so does the infinite group $\langle n(y) \rangle$ it generates, and hence the Zariski closure N of $\langle n(y) \rangle$. Therefore $\operatorname{Stab}_N v$ is either $\{I\}$ or N.

Now, suppose that

$$\begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} \cdot v = v.$$

Then there exists a $u \in F$ such that $d(x)$ fixes $n(u) \cdot v$—if $x \neq \pm 1$ take $u = y/(x - x^{-1})$ so that $n(u) \left(\begin{smallmatrix} x & y \\ 0 & x^{-1} \end{smallmatrix}\right) = d(x)n(u)$, and if $x = \pm 1$, take $u = 0$ (in the case $x=-1$, notice that $\left(\begin{smallmatrix}-1 & y\\ 0 & -1\end{smallmatrix}\right)^2=\left(\begin{smallmatrix}1 & -2y\\ 0 & 1\end{smallmatrix}\right)$ stabilizes v). Since $n(u)\cdot v$ has a non-trivial component in a positive weight space, x must be a root of unity of order less than $\dim V$. Furthermore, if y is not uniquely determined by x, for example if $\left(\begin{smallmatrix}x & y\\ 0 & x^{-1}\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}x & y'\\ 0 & x^{-1}\end{smallmatrix}\right)$ both stabilize v with $y\neq y'$, then

$$\begin{pmatrix}x & y\\ 0 & x^{-1}\end{pmatrix}\begin{pmatrix}x & y'\\ 0 & x^{-1}\end{pmatrix}^{-1}=\begin{pmatrix}1 & x(y-y')\\ 0 & 1\end{pmatrix},$$

and hence by the previous paragraph N stabilizes v. Therefore $\mathrm{Stab}_G\, v$ is the product of a finite set and $\mathrm{Stab}_N\, v$. This tells us that $G_v=\mathrm{Stab}_N\, v$ is either $\{I\}$ or N.

Part (b) is proven similarly. □

Remark. Lemma 1.1(a) implies that for every point x in every finite-dimensional representation of G, the group G_x is unimodular. For higher-rank groups, this is not true—see [1]—and hence some orbits in some representations will not have invariant measures.

Let K be the usual maximal compact subgroup of G, that is, $\mathbf{SL}(2,R)$ if F is non-archimedean with ring of integers R, $SO(2,\mathbb{R})$ if $F=\mathbb{R}$, and $SU(2,\mathbb{R})$ if $F=\mathbb{C}$. Pick a Haar measure dk on K, and write $f^K\in\mathcal{S}(V)$ for the function

$$f^K(x)=\int_K f(k\cdot x)dk,$$

if f is in the space $\mathcal{S}(V)$ of Schwartz-Bruhat functions on V; one can verify that the map $f\mapsto f^K$ is continuous on $\mathcal{S}(V)$. Take Haar measures $d^\times x$ on $F^\times$ and dy on F, and transfer them to $A\cong F^\times$ and $N\cong F$ using the isomorphisms

$$x\mapsto d(x)=\begin{pmatrix}x & 0\\ 0 & x^{-1}\end{pmatrix},\quad y\to n(y)=\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}.$$

It is known that

$$\int_K\int_N\int_A \phi(kna)da\,dn\,dk=\int_K\int_A\int_N|\alpha(a)|\phi(kan)dn\,da\,dk,$$

ϕ a function on G, defines a Haar measure dg on G. Clearly then the quotient measure $d\bar{g}=dg/dn$ on G/N is given by

$$\int_{G/N}\phi(\bar{g})d\bar{g}=\int_K\int_A|\alpha(a)|\phi(ka)da\,dk,$$

for ϕ a function on G/N (extended in the obvious way to G).

We are still considering v as in Lemma 1.1. If the group G_v equals N, then (1.1) equals

$$\begin{aligned}\int_{G/N} f(\bar{g}\cdot v)d\bar{g}&=\int_A|\alpha(a)|f^K(a\cdot v)da\\ &=\int_{F^\times}|x|^2 f^K\big(d(x)\cdot v\big)\,d^\times x.\end{aligned}$$

Because of the restrictions on v, the action of $d(x)$ on v is polynomial in x, rather than in x and x^{-1}, so that the function ϕ on $F^\times$ given by

$$\phi(x)=f^K\big(d(x)\cdot v\big)$$

extends to a Schwartz-Bruhat function on F, that we will still call ϕ. Then (1.1) is nothing but the local Tate zeta function $L(2,\phi)$, which converges. (Since we did not normalize our Haar measures, this local Tate zeta function may be a non-zero constant multiple of the local zeta function given in Tate's thesis [9].) It is not hard to show further that in this case the formula (1.1) defines a tempered distribution. Let us call it $I_{G,V}(v)$.

In the other case, when G_v is the trivial group, (1.1) is, at least formally, the integral

$$\int_K \int_N \int_A f(kna \cdot v) da\, dn\, dk = \int_{F^\times} \psi(x) d^\times x, \tag{1.2}$$

where ψ is the function on $F^\times$ given by the absolutely convergent integral

$$\psi(x) = \int_F f^K(n(y)d(x) \cdot v) dy, \quad x \in F^\times. \tag{1.3}$$

We will see that the integral (1.2) will diverge for general f. On the other hand, we do have the following result.

Lemma 1.2. *Given $g \in G$, let $\xi(g) = |x|$ where $x \in F^\times$ is any number satisfying $g \in KNd(x)$. Suppose that v is not semisimple, is contained in the direct sum of the non-negative weight spaces, and that $G_v = 1$. Then the integral*

$$L(s,\psi) = \int_{F^\times} |x|^s \psi(x) d^\times x = \int_G f(g \cdot v) \xi(g)^s dg$$

converges absolutely for $s \in \mathbb{C}$ in some right half-plane, and extends to a meromorphic function on $\mathbb{C}$, where ψ is given by (1.3).

Remark. If F is non-archimedean or complex, then an $x \in F^\times$ such that $g \in KNd(x)$ is not uniquely determined by $g \in G$, but its absolute value is, so that ξ is well-defined on G.

Proof. We first introduce some notation. Notice that for $x \in F^\times$, $y \in F$, and $q \in \mathbb{Z}$, the coordinates of $(n(y)d(x) \cdot v)^q$ are polynomials in x and y, and the only monomials that can appear are of the form $x^{q-2k}y^k$, with k a non-negative integer—the coordinates are polynomials since v lies in $\bigoplus_{q \geq 0} V^q$. Among all pairs (i,j) such that the monomial $x^i y^j$ appears in the expression for any of the coordinates of $n(y)d(x) \cdot v$, consider those pairs that minimize i/j, and from these choose the one with j smallest. We have $j \geq 1$ for such a pair since N acts non-trivially on v. If $i = 0$, our rule gives $j = 1$.

Pick a finite set $\mathcal{A}$ of representatives of $F^\times/(F^\times)^j$, and let $\alpha \in \mathcal{A}$. Then by the change of variables $y \mapsto x^{-i}y$, we see that for $x \in F^\times$, $\psi(\alpha x^j)$ equals $|x|^{-i}$ times

$$\int_F f^K\big(n(x^{-i}y)d(\alpha x^j) \cdot v\big)\, dy. \tag{1.5}$$

Let us write $(x,y) \cdot v$ for the point $n(x^{-i}y)d(\alpha x^j) \cdot v$; by our choice of i and j, the coordinates of the point $(x,y) \cdot v$ are polynomials in x and y. The only monomials that can appear in its coordinates in the subspace V^{i+2j} are of the form $x^{(i+2j)k}y^{j-k}$, k a non-negative integer, and the monomial y^j does appear in at least one of these coordinates with a non-zero coefficient. Therefore $(x,y) \cdot v$ will lies outside a fixed compact set in V for $|y|$ sufficiently large, *uniformly* for x in any bounded set in F (though originally defined only for $x \in F^\times$). If we extend (1.5) to $x = 0$ in the natural way, we conclude that (1.5) converges absolutely for all $x \in F$, and furthermore defines a smooth function on F. Let us call this function $\phi_\alpha(f)$ or ϕ_α; recall that we have shown that $\psi(\alpha x^j) = |x|^{-i}\phi_\alpha(x)$, $x \in F^\times$, $\alpha \in \mathcal{A}$.

To show that ϕ_α is a Schwartz-Bruhat function it suffices to show that ψ is rapidly decreasing; we will now prove that. Notice first that the integral (1.3) defining ψ does not change if v is replaced by $n \cdot v$, $n \in N$.

Let q be the smallest positive integer with $v^q \neq 0$. If for all $n \in N$, $(n \cdot v)^q = v^q$, then the component of $n(y)d(x) \cdot v$ in the q-weight space is $x^q v^q$ and hence ψ is rapidly decreasing. Now suppose otherwise. Then N acts non-trivially on v^0 and hence for some *non-trivial* irreducible subrepresentation π, we have $v^0_\pi \neq 0$. This implies that for a unique $n \in N$, $(n \cdot v)^2_\pi = 0$; since translating v by n does not change the function ψ, we may assume without loss of generality that $n = 1$. Then the coefficient in V^2_π of $n(y)d(x) \cdot v$ is a fixed multiple of y. (Fixed here means independent of x, y, f.) On the other hand, since v is not semisimple, it has a non-trivial component in some positive weight space $V^p_{\pi'}$, $p > 2$. The coordinate of $n(y)d(x) \cdot v$ in $V^p_{\pi'}$ is of the form $cx^p + yP(x, y)$, with c a fixed constant and P a fixed polynomial in x and y, of degree relative to x at most $p - 2$. If F is non-archimedean, then since f^K is a Schwartz-Bruhat function, there exists $M > 0$ such that $f^K(n(y)d(x) \cdot v)$ will vanish if either $|y|$ or $|cx^q + yP(x, y)|$ becomes larger than M, and hence will vanish for all y if $|x|$ is sufficiently large. This proves that in the non-archimedean case, ψ is rapidly decreasing. In the archimedean case the idea is similar; details of a similar argument can be found in the proof of Lemma 1.8.

We now know that ϕ_α is a Schwartz-Bruhat function. Make the change of variables $x \mapsto \alpha x^j$, $F^\times \to \mathcal{A} \times F^\times$ (see [10]), to obtain that

$$\begin{aligned}\int_{F^\times} |x|^s \psi(x) d^\times x &= \frac{|j|_F}{w_j} \sum_{\alpha \in \mathcal{A}} |\alpha|^s \int_{F^\times} |x|^{sj} \psi(\alpha x^j) d^\times x \\ &= \frac{|j|_F}{w_j} \sum_{\alpha \in \mathcal{A}} |\alpha|^s \int_{F^\times} |x|^{sj-i} \phi_\alpha(x) d^\times x \\ &= \frac{|j|_F}{w_j} \sum_{\alpha \in \mathcal{A}} |\alpha|^s L(sj - i, \phi_\alpha),\end{aligned}$$

where w_j denotes the number of jth roots of unity in F, and $L(s, \phi)$ denotes the local Tate zeta function. Each ϕ_α is Schwartz-Bruhat, so that each $L(sj - i, \phi_\alpha)$ is meromorphic in s. Since $\mathcal{A}$ is finite, we are done. □

Notice that we proved somewhat more than is stated in Lemma 1.2. We actually have the following.

Corollary 1.3. *Suppose that v is not semisimple, is contained in the direct sum of the non-negative weight spaces, and that $G_v = 1$. Let $i \geq 0$ and $j \geq 1$ be as in the proof of Lemma 1.2. Then*

$$L(sj - i)^{-1} \int_G f(g \cdot v) \xi(g)^s dg \tag{1.4}$$

converges absolutely for $\mathrm{Re}(s) > i/j$ *and analytically continues to an entire function of s, with $L(s)$ the standard local zeta function corresponding to the character $|\cdot|^s$,* *that is*

$$L(s) = \begin{cases} \pi^{-s/2}\Gamma(s/2) & \text{if } F \text{ is real,} \\ (2\pi)^{1-s}\Gamma(s) & \text{if } F \text{ is complex,} \\ \frac{\mathcal{N}\mathfrak{d}^{s-\frac{1}{2}}}{1-\mathcal{N}\mathfrak{p}^{-s}} & \text{if } F \text{ is } p\text{-adic with prime } \mathfrak{p} \text{ and absolute different } \mathfrak{d}. \end{cases}$$

Remark. Since $i \geq 0$, and $\phi_\alpha(0) \neq 0$ for general $f \in \mathcal{S}(V)$, we see that (1.2) will diverge for general f. In fact, for general f, the continuation of $L(s,\psi)$ will have a pole at $s = 0$ if $L(sj - i)$ does, that is, if $F = \mathbb{R}$ and i is even, or $F = \mathbb{C}$ and i is integral, or F is non-archimedean and $i = 0$.

Lemma 1.2 gives us an obvious regularization of the integral (1.2), namely the value at $s = 0$ of the analytic continuation of (1.4). Let us write $I_{G,V}(v)$ for the map sending $f \in \mathcal{S}(V)$ to this limit.

Let us now consider the general case of non-semisimple v. Lemma 1.1(a) implies that G_v is either $\{I\}$ or a conjugate of N. In the latter case one sees as before that (1.1) converges and defines a tempered distribution, which we call $I_{G,V}(v)$. In the former case, let $v' \in G \cdot v$ be in the direct sum of the non-negative weight spaces, and write $I_{G,V}(v)$ for $I_{G,V}(v')$. A simple calculation, using Lemma 1.1(b), shows that $I_{G,V}(v)$ does not depend on the choice of v', so is well-defined. Let us summarize the definition of the maps $I_{G,V}(v)$, $v \in V$.

Definition 1.4. If $v \in V$ is semisimple or if $G_v \neq \{I\}$, write $I_{G,V}(v)$ for the distribution sending $f \in \mathcal{S}(V)$ to the absolutely convergent integral (1.1). Otherwise, let $v' \in G \cdot v$ be in the direct sum of the non-negative weight spaces, and write $I_{G,V}(v)$ for the map sending $f \in \mathcal{S}(V)$ to the value at $s = 0$ of the (entire) analytic continuation of

$$L(sj - i)^{-1} \int_G f(g \cdot v')\xi(g)^s dg. \tag{1.6}$$

Recall that a distribution D on V is said to be invariant if $D(f^u) = D(f)$ for every $u \in G$, where $f^u(v) = f(u \cdot v)$.

Theorem 1.5. *(a) For any $v \in V$, $I_{G,V}(v)$ defines an invariant tempered distribution.*

(b) If F is non-archimedean, and v is either semisimple or nilpotent, then (1.1) converges absolutely for all functions f that vanish on the boundary of $G \cdot v$, and its value equals $I_{G,V}(v, f)$.

Proof. If v is nilpotent and $G_v \neq \{I\}$, then $I_{G,V}(v)$ is the orbital integral (1.1), which is invariant. Temperedness of $I_{G,V}(v)$ is straightforward.

For semisimple v, we make use of the following notations and lemma.

Definition 1.6. A semisimple $v \in V$ is said to be F-*elliptic* if it does not lie in the 0-weight space of any F-split torus in G.

Note that if v is F-elliptic, then for every $g \in G$, the vector $g \cdot v$ has non-trivial components in both the positive and negative weight spaces with respect to A—otherwise the 0-weight space component would lie in $\overline{G \cdot v} = G \cdot v$.

Lemma 1.7. *If v is semisimple and not elliptic, then G_v is either G or a conjugate of A.*

Proof. Since v is not elliptic, there is a $g \in G$ with $g \cdot v \in V^0$. Replacing v by $g \cdot v$ conjugates the stabilizer by g, so without loss of generality we may assume that $g = 1$. If v lies in the subspace of V isotypic to the trivial representation, then G_v clearly equals G. Regularized orbital integrals Suppose otherwise. Then N acts non-trivially on v (since N acts non-trivially on every non-zero element in the 0-weight space of every non-trivial irreducible representation of $\mathbf{SL}(2)$). Since $n(y)d(x) \cdot v = n(y) \cdot v$ we conclude that $\mathrm{Stab}_P\, v = A$.

Suppose now that $g \cdot \gamma = \gamma$, with $g \in G \setminus P$, and write $g = nwn'a$, with $n, n' \in N$ and $a \in A$. Then $wn' \cdot \gamma = n^{-1} \cdot \gamma$. The weights of $wn' \cdot \gamma$ are all non-positive, while the weights of $n^{-1} \cdot \gamma$ are all non-negative, hence both lie in V^0. Since the component of $n^{-1} \cdot \gamma$ in the 0-weight space is γ, the previous paragraph tells us that $n = 1$, and $n' \cdot \gamma = w^{-1} \cdot \gamma$. Since w^{-1} normalizes A, the vector $w^{-1} \cdot \gamma$ lies in the 0-weight space, and we conclude that $n' = 1$. Therefore $\mathrm{Stab}_G\, \gamma$ is either $A \cup wA$ or A, depending on whether or not $w \cdot \gamma = \gamma$, and hence $G_v = A$. This proves Lemma 1.7. ◻

Now, if v is elliptic, then there exists $c > 0$ such that for any $k \in K$ and any $x \in F$ with $|x| \geq 1$, some coordinate of $d(x)k \cdot v$ has absolute value at least $c|x|$. Since

$$\int_G f(g \cdot v)\, dg = \int_{\substack{F^\times \\ |x| \geq 1}} D(x) \int_K f^K(d(x)k \cdot v)\, dk\, d^\times x, \tag{1.7}$$

with D a positive continuous function on $\{\, x \in F^\times \mid |x| \geq 1 \,\}$ that is bounded by a constant multiple of $|x|^2$, it is not hard to show that (1.7) defines a tempered invariant distribution, and (1.1) is $(\mathrm{vol}\, G_v)^{-1}$ times that.

If v is semisimple and $G_v = G$, then (1.1) is just evaluation at v, which is clearly tempered and invariant. In the remaining case for v semisimple, G_v is a split torus A'. We will assume that $A' = A$ as the general case requires only additional notation. Then

$$\int_{G/G_v} f(\bar{g} \cdot v)\, d\bar{g} = \int_F f^K(n(y) \cdot v) dy,$$

and since for some irreducible subrepresentation π the formula for the coordinate of $n(y) \cdot v$ in V_π^2 is cy with $c \neq 0$, we see that (1.1) defines an invariant tempered distribution on V.

In all these cases we have proven (a) and (b). In the remaining case, v is not semisimple, $G_v = \{I\}$, and $I_{G,V}(v)$ is given by the continuation to $s = 0$ of (1.6). Without loss of generality, we assume that $v' = v$. Let i and j be as in Corollary 1.3. Then for $v \in G$, $I_{G,V}(v, f^u)$ is the continuation to $s = 0$ of $L(sj - i)^{-1}$ times

$$\begin{aligned}\int_G f(ug \cdot v)\xi(g)^s dg &= \int_G f(g \cdot v)\xi(u^{-1}g)^s dg \\ &= \int_K \int_F \int_{F^\times} f(kn(y)d(x) \cdot v)\xi(u^{-1}kn(y)d(x))^s d^\times x dy dk, \end{aligned} \tag{1.8}$$

an absolutely convergent integral whenever $\mathrm{Re}(s) > i/j$. Since $\xi(u^{-1}kn(y)d(x)) = |x|\xi(u^{-1}k)$ we see that (1.8) equals

$$\int_K \xi(u^{-1}k)^s \int_F \int_{F^\times} f(kn(y)d(x) \cdot v)|x|^s d^\times x dy dk.$$

As in the proof of Lemma 1.2, we can write

$$\int_F \int_{F^\times} f(kn(y)d(x)\cdot v)|x|^s d^\times x dy = \frac{|j|_F}{w_j} \sum_{\alpha\in\mathcal{A}} |\alpha|^s L(sj-i,\phi_{k,\alpha})$$

with

$$\phi_{k,\alpha}(x) = \int_F f(kn(x^{-i}y)d(\alpha x^j)\cdot v)dy.$$

Therefore $I_{G,V}(v,f^u)$ is the continuation to $s=0$ of

$$\frac{|j|_F}{w_j}\int_K \xi(u^{-1}k)^s \sum_{\alpha\in\mathcal{A}} |\alpha|^s \frac{L(sj-i,\phi_{k,\alpha})}{L(sj-i)} dk. \tag{1.9}$$

We will prove in the following lemma that the function $k\mapsto \phi_{k,\alpha},\ K\to\mathcal{S}(F)$ is continuous. The function

$$(\phi,s)\mapsto \frac{L(s,\phi)}{L(s)} = L(s)^{-1}\int_{F^\times}\phi(x)|x|^s d^\times x, \quad \phi\in\mathcal{S}(F),\ \mathrm{Re}(s)>0$$

is continuous on $\mathcal{S}(F)\times\{\,s\in\mathbb{C}\mid \mathrm{Re}(s)>0\,\}$. Normalize the Fourier transform $\phi\mapsto\hat{\phi}$ on $\mathcal{S}(F)$ so that the functional equation of the local Tate zeta functions takes the form

$$\frac{L(s,\phi)}{L(s)} = \frac{L(1-s,\hat{\phi})}{L(1-s)}.$$

Since the map $\phi\mapsto\hat{\phi}$, $\mathcal{S}(F)\to\mathcal{S}(F)$, is continuous, we see that the function $(\phi,s)\mapsto L(s,\phi)/L(s) = L(1-s,\hat{\phi})/L(1-s)$ is also continuous on $\mathcal{S}(F)\times\{\,s\in\mathbb{C}\mid \mathrm{Re}(s)<1\,\}$ and hence on all $\mathcal{S}(F)\times\mathbb{C}$. Given any $s_0\in\mathbb{C}$, let T be any triangle with s_0 inside, that lies either to the right of $\mathrm{Re}(s)=i/j$ or to the left of $\mathrm{Re}(s)=(1+i)/j$. Then the integrand of

$$\int_T\int_K \xi(u^{-1}k)^s \sum_{\alpha\in\mathcal{A}} |\alpha|^s \frac{L(sj-i,\phi_{k,\alpha})}{L(sj-i)} dkds \tag{1.10}$$

is a continuous function on $T\times K$, so Fubini's theorem allows us to switch the integrals. For each $k\in K$, the integrand of (1.10) is an entire function in s, so by the Cauchy integral formula its integral over T is 0. Therefore (1.10) equals 0 for any $s_0\in\mathbb{C}$ and any sufficiently small triangle T containing s_0, hence by Morera's theorem (1.9) defines an entire function, and

$$I_{G,V}(v,f^u) = \lim_{s\to 0}\frac{|j|_F}{w_j}\int_K \xi(u^{-1}k)^s \sum_{\alpha\in\mathcal{A}} |\alpha|^s \frac{L(sj-i,\phi_{k,\alpha})}{L(sj-i)} dk.$$

The integrand is continuous in s and k, hence is bounded on $\{\,s\in\mathbb{C}\mid |s|\le 1\,\}\times K$, so by dominated convergence we can switch the limit and the integral, simplify, and switch back to obtain

$$\lim_{s\to 0}\frac{|j|_F}{w_j}\int_K \sum_{\alpha\in\mathcal{A}} |\alpha|^s \frac{L(sj-i,\phi_{k,\alpha})}{L(sj-i)} dk.$$

Now, for $\mathrm{Re}(s) > i/j$, we can switch integrals as before to obtain the identity

$$\begin{aligned}\int_K \sum_{\alpha\in\mathcal{A}} |\alpha|^s \frac{L(sj-i,\phi_{k,\alpha})}{L(sj-i)} dk &= L(sj-i)^{-1} \int_K \sum_{\alpha\in\mathcal{A}} |\alpha|^s \int_{F^\times} \phi_{k,\alpha}(x)|x|^{sj-i} d^\times x dk \\ &= L(sj-i)^{-1} \int_{F^\times} \sum_{\alpha\in\mathcal{A}} |\alpha|^s \Big(\int_K \phi_{k,\alpha}(x) dk\Big) |x|^{sj-i} d^\times x \\ &= \sum_{\alpha\in\mathcal{A}} |\alpha|^s \frac{L(sj-i,\phi_\alpha)}{L(sj-i)}, \qquad (1.11)\end{aligned}$$

with ϕ_α as in the proof of Lemma 1.2. The left- and right-hand sides of (1.11) are entire, so the equality holds for all $s \in \mathbb{C}$, hence

$$I_{G,V}(v, f^u) = \lim_{s\to 0} \frac{|j|_F}{w_j} \sum_{\alpha\in\mathcal{A}} |\alpha|^s \frac{L(sj-i,\phi_\alpha)}{L(sj-i)} = I_{G,V}(v,f).$$

We have proven invariance.

The map $f \mapsto \phi_\alpha(f)$ from $\mathcal{S}(V) \to \mathcal{S}(F)$ can be shown to be continuous in a fashion similar to the proof of the following lemma. The functional equation of local Tate zeta functions implies that

$$I_{G,V}(v,f) = \lim_{s\to 0} \frac{|j|_F}{w_j} \sum_{\alpha\in\mathcal{A}} |\alpha|^s \frac{L(sj-i,\phi_\alpha)}{L(sj-i)} = \frac{|j|_F}{w_j} \sum_{\alpha\in\mathcal{A}} \frac{L(1+i,\widehat{\phi_\alpha})}{L(1+i)},$$

with $\widehat{\phi_\alpha}$ the Fourier transform (properly normalized) of ϕ_α. The Fourier transform is a continuous automorphism of $\mathcal{S}(F)$. We conclude that $I_{G,V}(v)$ is a tempered distribution.

If v is nilpotent and $G_v = \{1\}$, then the smallest integer q with $v^q \neq 0$ is positive, and for every $y \in F$, the component of the point

$$(0,y)\cdot v = \lim_{x\to 0} n(x^{-i}y)d(\alpha x^j)\cdot v$$

in the weight space V^q is 0. Using Lemma 1.1(b), we see that $(0,y)\cdot v \notin G\cdot v$, hence $(0,y)\cdot v$ lies in the boundary of $G\cdot v$. Therefore if f vanishes on the boundary of $G\cdot v$, then $\phi_\alpha(0) = 0$ for each α. Since F is non-archimedean, ϕ_α vanishes in a neighbourhood of 0, so that the integral defining $L(sj-i,\phi_\alpha)$ converges absolutely for all $s\in\mathbb{C}$, and working backwards through the derivation in the proof of Lemma 1.2, we see that (1.1) converges absolutely and its value equals $I_{G,V}(v,f)$. □

Lemma 1.8. *The function $k \mapsto \phi_{k,\alpha}$, $K \to \mathcal{S}(F)$, in the proof of Theorem 1.5 is continuous.*

Proof. We will prove this for $F = \mathbb{R}$—the complex case is similar and the non-archimedean case is considerably easier. Since left-translating k amounts to changing f, we may without loss of generality prove continuity at $k = 1$.

First notice that the coordinates of $kn(x^{-i}y)d(\alpha x^j)\cdot v$ are polynomials in x and y, with coefficients continuous in k. We saw in the proof of Lemma 1.2 that we can say something about the coordinates of

$$n(x^{-i}y)d(\alpha x^j)\cdot v; \qquad (1.12)$$

there were two possibilities, and we will show how to deal with one of them. The other is similar. In the case we choose, we have that the coordinate of (1.12) in some direction V_π^{i+2j} is a polynomial of the form

$$\sum_{\ell=0}^{j} c_\ell x^{(i+2j)\ell} y^{j-\ell}, \quad c_0 \neq 0,$$

and its coordinate in some direction $V_{\pi'}^q$, $q > 0$, is cx^{qj}, $c \neq 0$. We must prove that given $\varepsilon > 0$ and non-negative integers n_1, n_2, then for all $k \in K$ sufficiently close to 1, we have

$$\sup_{x\in\mathbb{R}} (1+|x|)^{n_1} \left| \frac{d^{n_2}}{dx^{n_2}} (\phi_{k,\alpha} - \phi_{1,\alpha}) \right| < \varepsilon. \tag{1.13}$$

Now,

$$\frac{d^{n_2}}{dx^{n_2}} \phi_{k,\alpha} = \int_{\mathbb{R}} \frac{d^{n_2}}{dx^{n_2}} f(kn(x^{-i}y)d(\alpha x^j) \cdot v) dy.$$

The integrand on the right-hand side of this equality is the sum of $(\dim V)^{n_2}$ terms, each the product of $Df(kn(x^{-i}y)d(\alpha x^j) \cdot v)$, with Df an n_2th-order derivative of f, and a polynomial $P(x,y)$ in x and y, with coefficients continuous in k, of total degree bounded by a number depending only on n_2 and v. We may therefore consider the term of (1.13) corresponding to a fixed n_2th-order derivative Df of f. Since f is a Schwartz-Bruhat function on V and K is compact, given any $\ell_1, \ell_2 > 0$, there exist $M_1, M_2 > 1$ such that if the coordinate x_1 of a vector in the direction V_π^{i+2j} and its coordinate x_2 in the direction $V_{\pi'}^q$ satisfy $|x_1| > M_1$ or $|x_2| > M_2$, then the absolute value of Df evaluated at any K-translate of the vector is less than $(1+|x_1|)^{-\ell_1}|x_2|^{-\ell_2}$.

We first consider those x such that $|cx^{qj}| > M_2$. For any $y \in \mathbb{R}$ with

$$|y| > |x|^{i+2j} \max_{0\le\ell\le j} \frac{(j+1)c_\ell}{c_0},$$

the coordinate x_1 of (1.12) in the direction V_π^{i+2j} satisfies $|x_1| > \frac{c_0}{j+1}|y|^j$. Therefore both $d^{n_2}\phi_{k,\alpha}/d^{n_2}$ and $d^{n_2}\phi_{1,\alpha}/d^{n_2}$ are bounded by

$$\begin{aligned} &\int_{|y|\le|x|^{i+2j}\max_{0\le\ell\le j}(j+1)\frac{c_\ell}{c_0}} |cx^{qj}|^{-\ell_2} |P(x,y)| dy \\ &\quad + \int_{|y|>|x|^{i+2j}\max_{0\le\ell\le j}(j+1)\frac{c_\ell}{c_0}} \left| \frac{c_0}{j+1} y^j \right|^{-\ell_1} |cx^{qj}|^{-\ell_2} |P(x,y)| dy. \end{aligned} \tag{1.14}$$

The supremum over x with $|cx^{qj}| > M_2$ of the product of $(1+|x|)^{n_1}$ and (1.14) can be made arbitrarily small by taking ℓ_1 and ℓ_2 sufficiently large.

There exists $M > 1$ such that if $|y| > M$ then for all $x \in \mathbb{R}$ with $|cx^{qj}| \le M_2$,

$$\left| \sum_{\ell=0}^{j} c_k x^{(i+2j)\ell} y^{j-\ell} \right| > \left| \frac{c_0 y^j}{2} \right| > M_1.$$

Therefore for any such x,

(1.15)

$$(1+|x|)^{n_1}\left|\frac{d^{n_2}}{dx^{n_2}}(\phi_{k,\alpha}-\phi_{1,\alpha})\right| \leq 2(1+|x|)^{n_1}\int_{|y|>M}\left|\frac{c_0y^j}{2}\right|^{-\ell_1}|P(x,y)|dy$$
$$+\,(1+|x|)^{n_1}\int_{-M}^{M}\left|Df(kn(x^{-i}y)d(\alpha x^j)\cdot v)-Df(n(x^{-i}y)d(\alpha x^j)\cdot v)\right||P(x,y)|dy.$$

The first integral on the right-hand side of (1.15) can be made arbitrarily small by taking ℓ_1 sufficiently small. At this point we choose ℓ_1 and ℓ_2 satisfying all the above bounds. This fixes M_1, M_2, and M.

Since the set $\{\, n(x^{-i}y)d(\alpha x^j) \mid |cx^q| \leq M_2,\ |y| \leq M \,\} \subset G$ is compact, and the functions $G\times G \to G$, $G\times V \to V$, $V \to \mathbb{C}$ given by group multiplication, group action, and the function Df respectively, are all continuous, we can make the second integral in (1.15) arbitrarily small by taking k sufficiently close to 1. We have therefore shown that for all $k \in K$ sufficiently close to 1, we have the inequality (1.13), hence the map $k \mapsto \phi_{k,\alpha}$, $K \to \mathcal{S}(\mathbb{R})$ is continuous. □

2. The role of weighted orbital integrals

We keep the notation of the previous section. Fix, throughout this section, a vector v in the direct sum of the non-negative weight spaces, with $G_v = \{I\}$ and $i = 0$. Let γ be the component of v in the 0-weight space.

Proposition 2.1. *(a)* $\mathrm{Stab}_G\,\gamma$ *is either A or $A \cup wA$, where* $w = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right)$. *In either case* $G_\gamma = A$.

(b) $I_{G,V}(v) = I_{G,V}(\gamma)$.

Proof. (a) For $y \in F$, the action of $n(y)$ on γ is given by substituting $x = 0$ into the coordinates of $n(y)d(x)\cdot v$. Since by assumption the monomial y appears with non-zero coefficient in some component of $n(y)d(x)\cdot v$, we see that N acts non-trivially on γ. The rest follows from the proof of Lemma 1.7.

(b) For $f \in \mathcal{S}(V)$,

$$I_{G,V}(v,f) = \lim_{s\to 0}\frac{L(s,\phi(f))}{L(s)} = \frac{L(1,\widehat{\phi(f)})}{L(1)} = \int_F \widehat{\phi(f)}(a)\,da = \phi(f,0).$$

The point γ is semisimple, and by (a) we have $G_\gamma = A$, so that

$$I_{G,V}(v,f) = \int_K\int_N f(kn\cdot\gamma)\,dn\,dk = \int_{G/G_\gamma} f(\bar g\cdot\gamma)\,d\bar g = I_{G,V}(\gamma,f).$$

□

In other words, the invariant distribution we have associated to the orbit of v is actually the orbital integral at γ. In this section, we will determine the existence of other invariant distributions whose support is contained in the closure of the orbit $G\cdot v$.

Let us abuse notation by writing $d(0)\cdot v$ for γ, so that $d(x)\cdot v$ is defined for all $x \in F$, and its coordinates are polynomial in x. We will use this abuse of notation also in expressions of the form $kn\,d(0)\cdot v$, with $k \in K$, $n \in N$. Define, for $x \in F$,

$f \in \mathcal{S}(V)$, $k \in K$,

$$\phi_k(f,x) = \int_N f(kn\, d(x)\cdot v)\, dn,$$
$$\phi(f,x) = \int_K \phi_k(f,x)\, dk = \int_N \int_K f(kn\, d(x)\cdot v)\, dk\, dn.$$

These are the functions $\phi_{\alpha,k}$, ϕ_α from section 1, since in the notation of that section, $i = 0$ and $j = 1$. As before, we see that $\phi_k(f)$, $\phi(f)$ are Schwartz-Bruhat functions on $x \in F$, for each $f \in \mathcal{S}(V)$, $k \in K$.

Lemma 2.2. *Write $\mathcal{S}_0(V)$ for the set of $f \in \mathcal{S}(V)$ that vanish on $G \cdot \gamma$, with the subspace topology. Then the function $J_{G,V}(v)$ on $\mathcal{S}_0(V)$ defined by*

$$f \mapsto J_{G,V}(v,f) = \int_{F^\times} \phi(f,x) d^\times x = \int_G f(g\cdot v) dg, \quad f \in \mathcal{S}_0(V), \tag{2.1}$$

is given by an absolutely convergent integral, and is continuous and G-invariant.

Proof. If $f \in \mathcal{S}_0(V)$, then $\phi(f,0) = 0$. That (2.1) converges absolutely then follows from the first formulation of the integral. That (2.1) is G-invariant follows immediately from the second formulation of the integral (the equality of these two formulations follows from the decomposition $G = KNA$). The continuity of (2.1) follows from the continuity of $\phi\colon \mathcal{S}(V) \to \mathcal{S}(F)$, whose proof is somewhat tedious but elementary. □

Now, suppose that we had a continuous map $f \mapsto f_0$, $\mathcal{S}(V) \to \mathcal{S}_0(V)$, such that $(f - f_0)(kn\, d(x)\cdot v) = \phi_0(x) f(kn\cdot\gamma)$, $x \in F^\times$, $n \in N$, $k \in K$, where $\phi_0 \in \mathcal{S}(F)$ is the function such that $L(s,\phi_0) = L(s)$. (We will later indicate the construction of such a map in the easy case when $\mathrm{Stab}_G(\gamma) = A$; in the other case when $\mathrm{Stab}_G(\gamma) = N_G(A)$, the construction of such a map is more involved but will not be required.) We could then define a distribution $J_{G,V}(v)$ on $\mathcal{S}(V)$ by $J_{G,V}(v,f) = J_{G,V}(v,f_0)$, that is

$$J_{G,V}(v,f) = \int_{F^\times} \big(\phi(f,x) - \phi(f,0)\phi_0(x)\big) d^\times x = c_F \frac{d}{ds}\left(\frac{L(s,\phi(f))}{L(s)}\right)\Big|_{s=0}, \tag{2.2}$$

where the last equality follows from a simple calculation. The constant c_F is given by $c_F = \lim_{s\to 0} sL(s)$ and is $2e^{-\gamma}$ if $F = \mathbb{R}$, $2\pi e^{-\gamma}$ if $F = \mathbb{C}$ (γ here is Euler's constant), and $1/\log q$ if F is non-archimedean with residue field of order q. Since ϕ is continuous, (2.2) can be shown to be a distribution on $\mathcal{S}(V)$ even without assuming the existence of the map $f \mapsto f_0$ above.

Lemma 2.3. *If $J_{G,V}(v)$ is defined as in (2.2), then*
(a) If $f \in \mathcal{S}_0(V)$, then $J_{G,V}(v,f)$ equals (2.1).
(b) For $u \in G$,

$$J_{G,V}(v,f^u) - J_{G,V}(v,f) = -c_F \int_K H(k^{-1}u)\phi_k(f,0) dk,$$

where $H(g) = \ln|x|$ if $g \in d(x)NK$. In particular, this extension of the distribution (2.1) to $\mathcal{S}(V)$ is not invariant.

Proof. (a) If $f \in \mathcal{S}_0(V)$, then $\phi(f,0) = 0$, and the conclusion is obvious.

(b) As in the proof of Theorem 1.5, we see that $J_{G,V}(v, f^u)$ is c_F times the derivative at $s = 0$ of (1.9), that is

$$c_F \frac{d}{ds} \left(\int_K \xi(u^{-1}k)^s \frac{L(s, \phi_k(f))}{L(s)} dk \right) \Big|_{s=0} .$$

The dominated convergence theorem lets us differentiate and take limits under the integral sign to obtain

$$c_F \int_K \frac{d}{ds}(\xi(u^{-1}k)^s) \Big|_{s=0} \frac{L(s, \phi_k(f))}{L(s)} \Big|_{s=0} dk \quad + \quad c_F \int_K \frac{d}{ds}\Big(\frac{L(s, \phi_k(f))}{L(s)}\Big) \Big|_{s=0} dk$$
$$= c_F \int_K \ln(\xi(u^{-1}k))\phi_k(f, 0) dk \quad + \quad J_{G,V}(v, f).$$

Since $H(g) = -\ln(\xi(g^{-1}))$ we have proven the desired result.

Since a non-negative f can be chosen to vanish on $kn \cdot \gamma$ if k lies outside a given open set, and for any $u \notin K$, the continuous map $k \mapsto H(k^{-1}u)$ does not vanish, the distribution $J_{G,V}(v)$ is not invariant. □

We now introduce another new distribution, whose motivation comes from global theory, and is defined analogously to the weighted orbital integrals that first appeared in the (global) Selberg-Arthur trace formula—see for example [3].

Definition 2.4. (a) Define $\lambda \colon N \to \mathbb{R}$ by $\lambda(n) = H(wn)$.

(b) Define the tempered distribution $J_{M,V}(\gamma)$ on $\mathcal{S}(V)$ by

$$J_{M,V}(\gamma, f) = \int_K \int_N \lambda(n) f(kn \cdot \gamma)\, dn\, dk = \int_{G/A} \big(H(w\bar{g}^{-1}) + H(\bar{g}^{-1})\big) f(\bar{g} \cdot \gamma)\, d\bar{g}.$$

Notice that the function $g \mapsto H(wg^{-1}) + H(g^{-1})$ on G is right A-invariant. Similarly, for any $y \in G$, the function $g \mapsto H(g^{-1}y) - H(g^{-1})$ on G is right NA-invariant, so if $g = kna$, $k \in K$, $n \in N$, $a \in A$, we have $H(g^{-1}y) - H(g^{-1}) = H(k^{-1}y)$.

We calculate the non-invariance of $J_{M,V}(\gamma)$ by the standard calculation for the non-invariance of weighted orbital integrals:

$$\begin{aligned} J_{M,V}(\gamma, f^y) - J_{M,V}(\gamma, f) &= \int_{G/A} \big(H(\bar{g}^{-1}y) - H(\bar{g}^{-1})\big) f(\bar{g} \cdot \gamma)\, d\bar{g}, \\ &\quad + \int_{G/A} \big(H(w\bar{g}^{-1}y) - H(w\bar{g}^{-1})\big) f(\bar{g} \cdot \gamma)\, d\bar{g}, \\ &= \int_{G/A} \big(H(\bar{g}^{-1}y) - H(\bar{g}^{-1})\big)\big(f(\bar{g} \cdot \gamma) + f(\bar{g} \cdot (w \cdot \gamma))\big) d\bar{g} \\ &= \int_K H(k^{-1}y) \int_N \big(f(kn \cdot \gamma) + f(knw \cdot \gamma)\big) dn\, dk. \end{aligned} \tag{2.3}$$

The absolute convergence of (2.3) is clear. Since each of the other integrals in this derivation can be put into the same form as (2.3), their absolute convergence is also clear, justifying each step of the derivation.

We therefore have the following result.

Theorem 2.5. *Write c_F for $\lim_{s \to 0} sL(s)$.*

(a) Suppose that v' lies in the direct sum of the non-negative weight spaces, and that the component of v' in the 0-weight space is $w \cdot \gamma$. Then the tempered

distribution

$$J_{G,V}(v,f)+J_{G,V}(v',f)+c_F J_{M,V}(\gamma,f)$$

is invariant, and its support is the closure of $G\cdot v\cup G\cdot v'$.

(b) Suppose that $w\cdot\gamma=\gamma$*. Then the tempered distribution* $I_{M,V}(v)$ *defined by*

$$I_{M,V}(v,f)=J_{G,V}(v,f)+\frac{c_F}{2}J_{M,V}(\gamma,f)$$

is invariant, supported on the closure of $G\cdot v$*, and is an extension to* $\mathcal{S}(V)$ *of (2.1).*

Remarks. 1. Theorem 0.1 follows from Theorems 1.5 and 2.5(b).

2. The global analogue of $I_{M,V}$ appears in the Shintani zeta function for the space of binary quartic forms, see [5]. In that example V is the irreducible 5-dimensional representation of $\mathbf{SL}(2)$.

To complement Theorem 2.5, we present the following result.

Theorem 2.6. *Assume that* $F=\mathbb{R}$*. If* $w\cdot\gamma\neq\gamma$*, then there is no invariant distribution on* V *that agrees with (2.1) on* $\mathcal{S}_0(V)$*.*

Remark. An example is given in [6, pp. 128-129] of an invariant distribution on an orbit that cannot be extended to the whole vector space. That example is easier to prove since in that case the closure of the orbit contains just the additional point 0, and the space of distributions on a single point is particularly simple.

We will assume for the remainder of the paper that $w\cdot\gamma\neq\gamma$, so by Proposition 2.1, $\mathrm{Stab}_G\,\gamma=A$. We begin the proof of Theorem 2.6 with some lemmas, valid for F a local field as before.

Lemma 2.7. *The closure* $\mathrm{cl}(G\cdot v)$ *of* $G\cdot v$ *is* $G\cdot v\cup G\cdot\gamma$*.*

Proof. First we replace v with $n\cdot v$, where $n\in N$ is given in the following paragraph. This replacement does not change $G\cdot v$ or γ.

We choose n as follows: Pick a decomposition of V into irreducible representations under the action of G. Then for some non-trivial irreducible subrepresentation V_π of $G=\mathbf{SL}(2)$ in this decomposition, the component x of v in its 0-weight space V_π^0 is non-zero. If y is the component of v in the α-weight space V_π^2, we set $n=\left(\begin{smallmatrix}1 & -y/x\\ 0 & 1\end{smallmatrix}\right)$. After replacing v with $n\cdot v$, we have that the component of v in the V_π^2 is 0, and its component in V_π^0 is $x\neq 0$.

Suppose that v' is in $\mathrm{cl}(G\cdot v)$. Then there is a sequence $\{g_i\}_{i\in\mathbb{N}}$ in G such that $g_n\cdot v$ has a limit. Write $g_i=k_in_ia_i$, with $k_i\in K$, $n_i\in N$, $a_i\in A$. Since K is compact, a subsequence of $\{k_i\}_{i\in\mathbb{N}}$ converges to some $k\in K$. Replacing $\{g_i\}$ with the corresponding subsequence, we obtain that $n_ia_i\cdot v$ converges. Now, the component of $\left(\begin{smallmatrix}1 & z\\ 0 & 1\end{smallmatrix}\right)d(t)\cdot v$ in V_π^2 is zx, so the convergence of $n_ia_i\cdot v$ implies that the sequence n_i converges to some $n\in N$. Therefore, the sequence $a_i\cdot v$ converges. Write $a_i=d(t_i)$, with $t_i\in F^\times$. Since the action of $d(t_i)$ on v is given by a polynomial of positive degree in t_i, and $d(t_i)\cdot v$ converges, we conclude that the set $\{|t_i|\mid i\in\mathbb{N}\}$ is bounded in F. If there is any (infinite) subsequence of $\{t_i\}_{i\in\mathbb{N}}$ contained in a compact set of $F^\times$, then a subsequence of $\{a_i\}_{i\in\mathbb{N}}$ converges to some $a\in A$, so that $v'=kna\cdot v\in G\cdot v$. If $v'\notin G\cdot v$, we must have that $x_i\to 0$, so that $a_i\cdot v\to\gamma$, and so $v'=kn\cdot\gamma\in G\cdot\gamma$. □

Definition 2.8. Let $C_c^\infty(G\cdot\gamma)$ be the space of compactly supported functions f^0 on $G\cdot\gamma$ such that the function $\widetilde{f^0}\colon g\mapsto f^0(g\cdot\gamma)$ on G is smooth. Put on $C_c^\infty(G\cdot\gamma)$

the topology determined by the following: a sequence f_n^0, $n \in \mathbb{N}$, in $C_c^\infty(G\cdot\gamma)$ tends to $f_0 \in C_c^\infty(G\cdot\gamma)$ if and only if $\widetilde{f_n^0}$ converges to $\widetilde{f^0}$ in $C^\infty(G)$ and there is a compact set $\mathcal{C}$ in G such that for all n, $\operatorname{supp}\widetilde{f_n^0} \subset \mathcal{C}A$.

Lemma 2.9. *Given $\phi_0 \in C_c^\infty(F)$, there is a continuous map $\Psi\colon f^0 \mapsto f$, $\Psi\colon C_c^\infty(G\cdot\gamma) \to C_c^\infty(V)$, such that $f(kn\,d(x)\cdot v) = \phi_0(x)f^0(kn\cdot\gamma)$, $k \in K$, $n \in N$, $x \in F$.*

Proof. The derivative of the action of $G = \mathbf{SL}(2,F)$ on V gives an action of $sl(2,F)$ on V. Define, for $g \in G$,

$$T_{g\cdot v} = \operatorname{span}\langle \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\cdot(g\cdot v), \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\cdot(g\cdot v), \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\cdot(g\cdot v)\rangle.$$

Pick a non-degenerate inner product $(\cdot,\cdot)\colon V\times V \to F$ on V, and write, for $\epsilon > 0$ and $g \in G$,

$$N_{g\cdot v} = T_{g\cdot v}^\perp,\ N_{g\cdot v}(\epsilon) = \{\, x \in N_{g\cdot v} \mid \|x\| < \epsilon \,\}.$$

Notice that for each $g \in G$, there is a neighbourhood U of g and an $\epsilon > 0$ such that if $g', g'' \in U$ are distinct, then $g'\cdot v + N_{g'\cdot v}(\epsilon)$ and $g''\cdot v + N_{g''\cdot v}(\epsilon)$ do not intersect.

Write $X_{g\cdot v} \in V$ for the term in $gd(x)\cdot v$ of lowest positive degree in x. We then define, for $g \in G$,

$$T_{g\cdot\gamma} = \operatorname{span}\langle \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\cdot(g\cdot v), \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\cdot(g\cdot v), X_{g\cdot v}\rangle;$$

notice that $T_{g\cdot\gamma}$ depends not on $g \in G$ but only on the point $g\cdot\gamma$. Define also for $\epsilon > 0$,

$$N_{g\cdot\gamma} = T_{g\cdot\gamma}^\perp,\ N_{g\cdot\gamma}(\epsilon) = \{\, x \in N_{g\cdot\gamma} \mid \|x\| < \epsilon \,\}.$$

Notice that for each $kn \in KN$ there is a neighbourhood $U_1 \subset KN$ of kn, a neighbourhood $U_2 \subset F$ of 0, and an $\epsilon > 0$ such that if either $k'n', k''n'' \in U_1$ or $x', x'' \in U_2$ are distinct, then $k'n'd(x')\cdot v + N_{k'n'd(x')\cdot v}(\epsilon)$ and $k''n''d(x'')\cdot v + N_{k''n''d(x'')\cdot v}(\epsilon)$ do not intersect.

The support S of $\phi_0(x)f^0(kn\cdot\gamma)$ in $KN \times F$ is compact, so $S\cdot v \subset V$ is too. The map from $KN\times F$ to V given by the action on v is open, so we can conclude from a standard argument that there exists $\epsilon > 0$ such that if $kn\,d(x)\cdot v \neq k'n'\,d(x')\cdot v$, where $k, k' \in K$, $n, n' \in N$, $x, x' \in F$ satisfy $(kn, x), (k'n', x') \in S$, then $kn\,d(x)\cdot v + N_{kn\,d(x)\cdot v}(\epsilon)$ and $k'n'\,d(x')\cdot v + N_{k'n'\,d(x')\cdot v}(\epsilon)$ do not intersect.

Given f^0, define f by

$$f(X) = \begin{cases} f^0(kn\cdot\gamma)\phi_0(x)\phi_1(\|X - kn\,d(x)\cdot v\|/\epsilon) & \text{if } X \in kn\,d(x)\cdot v \\ & \quad +N_{kn\,d(x)\cdot v}(\epsilon), \\ 0 & \text{otherwise,} \end{cases}$$

where $\phi_1 \in C_c^\infty(\mathbb{R})$ is a function that satisfies $\operatorname{supp}\phi_1 \subseteq [-1,1]$, $\phi_1(0) = 1$, $\phi_1(-x) = \phi_1(x)$. Then $f^0 \mapsto f$ is the desired map. □

Remark. The map $f \mapsto f_0 = f - \Psi(f|_{G\cdot\gamma})$, $C_c^\infty(V) \to \mathcal{S}_0(V)$ is the map mentioned in the discussion preceding Lemma 2.3, or rather its restriction to $C_c^\infty(V)$. This map can be extended to a continuous map on $\mathcal{S}(V)$.

Proof of Theorem 2.6. Any distribution extending (2.1) is, by Lemma 2.3(a), of the form $J_{G,V}(v) + c_F D$, with D vanishing on $\mathcal{S}_0(V)$. Suppose then that such a distribution is invariant. Then by Lemma 2.3(b), D is non-invariant, and its non-invariance is given by the formula

$$D(f^g) - D(f) = \int_K H(k^{-1}g) \int_N f(kn \cdot \gamma) dndk, \quad g \in G. \tag{2.4}$$

We are assuming that $F = \mathbb{R}$, so that $K \cap NA = \{ \pm I \}$. Given $f_1 \in C^\infty(K)$ satisfying $f_1(-k) = f_1(k)$, and $f_2 \in C_c^\infty(N)$, we can define f^0 on $G \cdot \gamma$ by $f^0(kn \cdot \gamma) = f_1(k) f_2(n)$. The function f^0 lies in $C_c^\infty(G \cdot \gamma)$, and Lemma 2.9 then gives us a function $f \in \mathcal{S}(V)$. The map sending (f_1, f_2) to f is continuous, and if $k \in K$, then (f_1^k, f_2) is sent to f^k. By (2.4), for each $f_2 \in C_c^\infty(N)$, the function on $C^\infty(K/\pm I) = \{ f_1 \in C^\infty(K) \mid f_1(-k) = f_1(k) \}$ sending f_1 to $D(f)$ is K-invariant, and hence is a constant multiple of

$$\int_K f_1(k) dk.$$

Write $D_2(f_2)$ for this constant; then D_2 lies in the dual of $C_c^\infty(N)$.

Given $k \in K$ and $f \in C_c^\infty(V)$, write $f_k(n) = f(kn \cdot \gamma)$. It is straightforward that the the map $k \mapsto f_k$, $K \to \mathcal{S}(N)$, and hence the function $D_2(f_k)$ on K, is continuous. Since the span of functions of the form $kn \cdot \gamma \mapsto f_1(k) f_2(n)$ (with f_1, f_2 as above) is dense in $C_c^\infty(G \cdot \gamma)$, continuity and the dominated convergence theorem imply that

$$D(f) = \int_K D_2(f_k) dk.$$

Since $F = \mathbb{R}$, we can decompose any element $g \in G$ uniquely as $\kappa(g)\nu(g)\mu(g)$ with $\kappa(g) \in K$, $\nu(g) \in N$, and $\mu(g) \in A^0$, the connected component of the identity in A, and furthermore these maps are all smooth. Notice that $\mu(g) = d(\xi(g))$, so that $\delta(\mu(g))^{-1} = \xi(g)^{-2}$. Given $g \in G$, the change of variables $k \mapsto \kappa(g^{-1}k)$ on K has Jacobian $\xi(g^{-1}k)^{-2} = e^{2H(k^{-1}g)}$; this is well-known but a proof follows anyway. Take any $F \in C^\infty(K)$. Choose $h \in C_c(NA^0)$ to be everywhere non-negative and to satisfy

$$\int_N \int_{A^0} h(na) dnda = \frac{1}{2} = \frac{1}{\mathrm{vol}(A/A^0)},$$

and define the function $\tilde{F} \in C_c(G)$ on G by $\tilde{F}(kna) = F(k)h(na)$, $k \in K$, $na \in NA^0$. Then

$$\begin{aligned}
\int_K F(k) dk &= \int_G \tilde{F}(x) dx = \int_G \tilde{F}(g^{-1}x) dx \\
&= \mathrm{vol}(A/A^0) \int_K \int_N \int_{A^0} \tilde{F}(g^{-1}kna) da\, dn\, dk \\
&= 2 \int_K \int_N \int_{A^0} \tilde{F}(\kappa(g^{-1}k)\nu(g^{-1}k)\mu(g^{-1}k)na) da\, dn\, dk \\
&= 2 \int_K \delta(\mu(g^{-1}k))^{-1} \int_N \int_{A^0} F(\kappa(g^{-1}k)) h(na) da\, dn\, dk \\
&= \int_K \xi(g^{-1}k)^{-2} F(\kappa(g^{-1}k)) dk.
\end{aligned}$$

Therefore for $f \in \mathcal{S}(V)$ and $g \in G$,

$$D(f^g) = \int_K D_2((f^g)_k)dk = \int_K e^{2H(k^{-1}g)} D_2((f^g)_{\kappa(g^{-1}k)})dk.$$

Now, for a given $n \in N$,

$$(f^g)_{\kappa(g^{-1}k)}(n) = f(g\kappa(g^{-1}k)n \cdot \gamma) = f\big(k(\nu(g^{-1}k)\mu(g^{-1}k))^{-1}n \cdot \gamma\big) = f(k\Phi(n) \cdot \gamma),$$

where Φ is the automorphism on N defined by

$$\Phi(n) = [\nu(g^{-1}k)\mu(g^{-1}k)]^{-1}n\mu(g^{-1}k).$$

Thus if f arises from (f_1, f_2) as above, and $g \in G$, then

$$\begin{aligned} D(f^g) &= \int_K f_1(k)\Big(D_2(f_2) + H(k^{-1}g)\int_N f_2(n)dn\Big)dk \\ &= \int_K f_1(k)e^{2H(k^{-1}g)} D_2(f_2 \circ \Phi)dk, \end{aligned} \tag{2.5}$$

The first equality here is the statement of the non-invariance of D.

For a given f_2, this equality holds for all $f_1 \in C_c^\infty(K/\pm I)$. Since each factor of the integrands in (2.5) is invariant under $k \mapsto -k$, we conclude that

$$D_2(f_2) + H(k^{-1}g)\int_N f_2(n)dn = e^{2H(k^{-1}g)} D_2(f_2 \circ \Phi) \tag{2.6}$$

for almost all $k \in K$. But both sides of (2.6) are continuous in k, so that the identity holds for all $k \in K$.

Given any $y \in \mathbb{R}$, let $a = 1/\sqrt{y^2+1}$, and set

$$g = \begin{pmatrix} a & -ya \\ 0 & a^{-1} \end{pmatrix}, \quad k = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then we have

$$g^{-1}k = \begin{pmatrix} ya & -a \\ a & ya \end{pmatrix}\begin{pmatrix} 1 & -y \\ 0 & 1 \end{pmatrix}, \text{ with } \begin{pmatrix} ya & -a \\ a & ya \end{pmatrix} \in K,$$

so this latter matrix is $\kappa(g^{-1}k)$, Φ is translation by $n(y)$, and $H(k^{-1}g) = 0$. Specializing (2.6) to this choice of g and k gives us that $D_2(f_2) = D_2(f_2 \circ \Phi)$. Since $y \in \mathbb{R}$ and $f_n \in C_c^\infty(N)$ were arbitrary, this means that the distribution D_2 on $C_c^\infty(N)$ is left N-invariant, and hence is of the form

$$D_2(f_2) = c\int_N f_2(n)dn,$$

for some constant n. But then the general formula for D is

$$D(f) = c\int_K\int_N f(kn \cdot \gamma)dndk,$$

which is invariant, in contradiction to the non-invariance formula. Therefore D cannot exist. □

Remark. Notice that if $F \neq \mathbb{R}$, then $K \cap A$ is an infinite group and it becomes difficult to decompose a function $f \in \mathcal{S}(V)$ in terms of functions like f_1 and f_2 as in the proof of Theorem 2.6.

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