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A NON-FIXED POINT THEOREM FOR HAMILTONIAN LIE GROUP ACTIONS

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Abstract. We prove that, under certain conditions, if a compact connected Lie group acts effectively on a closed manifold, then there is no fixed point. Because two of the main conditions are satisfied by any Hamiltonian action on a closed symplectic manifold, the theorem applies nicely to such actions. The method of proof, however, is cohomological; and so the result applies more generally.

1. Introduction

In the cohomology theory of transformation groups (based on a study of the Borel construction) most theorems concern torus actions, and results about non–abelian Lie group actions are scarce. In this paper we shall give two results about compact connected Lie group actions: the first gives a condition under which a compact connected Lie group G cannot act effectively on a certain type of space; and the second, which is a small variation on the first, gives a condition under which G cannot act effectively with a fixed point.

Both results apply to Hamiltonian actions on closed symplectic manifolds; but, since the proof is purely cohomological, they apply rather more generally. The main condition imposed on the space X, on which G acts, concerns the degrees of the relations in an algebra presentation of the rational or real cohomology algebra of X. This kind of information is often well known for symplectic manifolds, especially for toric varieties: see, e.g., [Audin], Chapter VI, Theorem 4.4.3. It is also known for some symplectic quotients: see [Tolman,Weitsman].

One of our main results is the following (Corollary (4.8) below).

Theorem. *Let G be a compact connected Lie group and let M be a closed symplectic manifold. For a field k of characteristic 0, suppose $H^*(M;k)$ has a k–algebra presentation*

$$0 \longrightarrow J \longrightarrow k[x_1,\dots,x_g] \otimes \wedge(y_1,\dots,y_h) \longrightarrow H^*(M;k) \longrightarrow 0\,,$$

where x_i, $1 \le i \le g$, and y_i, $1 \le i \le h$, are generators of even degree, respectively, odd degree; and the ideal $J = (f_1,\dots,f_m,e_1,\dots,e_n)$, where f_i, $1 \le i \le m$, and e_i, $1 \le i \le n$, are relations of even degree, respectively, odd degree. Let $H^*(BG;k) = k[v_1,\dots,v_r]$.

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Suppose that

$$\max\{\deg(f_i); 1 \leq i \leq m\} \leq \max\{\deg(v_i); 1 \leq i \leq r\}\,.$$

Suppose that G is acting on M in an effective Hamiltonian way. Then there is no fixed point.

2. Preliminaries

Throughout this paper k will denote a field of characteristic 0. Recall the following notation and terminology used in [Allday,Puppe].

Definitions (2.1). (1) If G is a compact Lie group, then G^0 denotes the connected component of the identity.

(2) Let G be a compact Lie group and let X be a paracompact G–space. Then X is said to satisfy Condition (LT) if either

(a) X is compact; or

(b) the set of conjugacy classes of identity components of isotropy subgroups, $\{[G_x^0]; x \in X\}$, is finite, and

(i) the cohomological dimension of X over $k, cd_k(X)$, is finite, or

(ii) X is finitistic.

(For some comments on the finitistic (Swan) condition see [Allday,Puppe]. For some information on cohomological dimension see [Bredon2] or [Quillen]. Note that spaces of finite covering dimension, such as finite–dimensional CW–complexes and finite–dimensional manifolds, are finitistic. Note too that, by a theorem of [Mann], the set of conjugacy classes of isotropy subgroups, $\{[G_x]; x \in X\}$, is finite whenever G is a compact Lie group and X is an orientable topological manifold such that $H_*(X;\mathbb{Z})$ is finitely generated.)

(3) Let G be a compact Lie group and let X be a G–space. Let $X \to^{i_G} X_G \to^p BG$ be the fibre bundle of the Borel construction. Then X is said to be totally non–homologous to zero (TNHZ) in X_G with respect to $H^*(-;k)$ if the Leray–Serre spectral sequence of p in $H^*(-;k)$ collapses, i.e. $E_2 = E_\infty$, or, equivalently,

$$i_G^* : H_G^*(X;k) \longrightarrow H^*(X;k)$$

is surjective.

Next we need to recall two standard variants on the notion of effectiveness.

Definitions (2.2). Let G be a compact connected Lie group and let X be a G–space.

(1) The action of G on X is said to be almost–effective if the ineffective kernel, $\bigcap_{x\in X} G_x$, is finite.

(2) The action of G on X is said to be cohomologically effective (with coefficients in k) if the restriction homomorphism

$$H^*(X;k) \longrightarrow H^*(X^K;k)$$

is not a monomorphism for any subcircle $K \subseteq G$. Otherwise the action is said to be cohomologically ineffective. (In [Allday,Puppe], Definition (4.7.3), the condition is that the restriction homomorphism is not an isomorphism for any subcircle $K \subseteq G$. This is equivalent to the monomorphism condition used here if X satisfies Condition (LT).)

(3) The action of G on X is said to be cohomologically trivial if the restriction homorphism

$$H^*(X;k) \longrightarrow H^*(X^G;k)$$

is an isomorphism.

Remarks (2.3). (1) Any non–trivial action of a simple compact connected Lie group is almost–effective.

(2) If G is a compact connected Lie group and X is a closed orientable manifold, then an action of G on X is cohomologically effective if and only if it is almost–effective. More generally, this holds if X is a compact orientable cohomology manifold over k. (See [Borel et al.], Chapter I, Corollary 4.6, and Chapter V, Theorem 3.2.)

For more details on the Borel construction see, for example, [Allday,Puppe], [Atiyah,Bott], [Borel et al.], [Bredon3], [Guillemin,Sternberg], [Hsiang] or [Quillen].

3. Hamiltonian and Cohomologically Hamiltonian Actions

As before, k will denote a field of characteristic 0.

Definitions (3.1). Let X be a space, and let G be a compact connected Lie group. Suppose that X satisfies Poincaré duality with coefficients in k, and suppose that the formal dimension of X, $fd(X)$, is $2n$: i.e., $H^i(X;k) = 0$ for $i > 2n$, $H^{2n}(X;k) \cong k$, $\dim_k H^i(X;k) < \infty$, for all i, and the cup product pairing

$$H^i(X;k) \times H^{2n-i}(X;k) \longrightarrow H^{2n}(X;k) \cong k$$

is non–singular for all i.

(1) X is said to be cohomologically symplectic (c–symplectic) if there is a class $w \in H^2(X;k)$ such that $w^n \neq 0$. w is called the c–symplectic class.

(2) If G is acting on X, then the action is said to be cohomologically Hamiltonian (c–Hamiltonian) if

$$w \in im[i_G^* : H_G^*(X;k) \longrightarrow H^*(X;k)],$$

where, as before, $i_G : X \longrightarrow X_G$ is the inclusion of the fibre in the Borel construction bundle $X_G \longrightarrow BG$.

Remarks (3.2). (1) A closed symplectic manifold is c–symplectic (over $\mathbb{R}$) with $w = [\omega]$, the class of the symplectic form.

(2) For an action of a connected compact Lie group G on a c–symplectic space X, it always holds that $g^*w = w$, for all $g \in G$; and so any action of G is said to be c–symplectic.

(3) If M is a closed symplectic manifold, G is a compact connected Lie group, G is acting on M, and the action is symplectic, then the action is Hamiltonian if and only if it is c–Hamiltonian. This follows directly from the Cartan model for computing equivariant cohomology: see [Guillemin,Sternberg]. See also [Audin], Chapter V, Proposition 3.1.1.

(4) See [Allday2] or [Lupton,Oprea] for more about actions on c–symplectic spaces.

(5) If X satisfies Poincaré duality over k with $fd(X) = 2n$, and if there are classes $w_1, \ldots, w_n \in H^2(X;k)$ such that $w_1 \ldots w_n \neq 0$, then X is c–symplectic. (This follows because char $k = 0$.) In particular, suppose that X is a topological

manifold, or, more generally, a rational cohomology manifold, and suppose that $\dim_{\mathbb{Q}} H^i(X;\mathbb{Q}) < \infty$ for all i. Then X is c–symplectic over a field k of characteristic 0 if and only if X is c–symplectic over $\mathbb{Q}$.

The next lemma is important for our main results.

Lemma (3.3). *Let X be a c–symplectic space satisfying Condition (LT)– see Definitions (2.1)(2). Let $G = S^1$, the circle group, and suppose that G is acting on X in a c–Hamiltonian and cohomologically effective way. Then the fixed point set X^G is non–empty and it has at least two connected components.*

See [Allday2], Proposition (6.7) and Remark (6.8) for the easy proof; and see Theorem (6.10), there, for the generalization to torus actions. The result is, of course, easy for truly Hamiltonian actions since the moment map must have a maximum and a minimum.

We also want the following theorem of Frankel. Since Frankel's proof was ostensibly only for circle actions, we shall indicate how to deduce the theorem as stated here.

Theorem (3.4) ([Frankel])**.** *Let M be a closed symplectic manifold, let G be a compact connected Lie group, and suppose that G is acting on M in a Hamiltonian way. Then M is TNHZ in $M_G \longrightarrow BG$ with respect to $H^*(-;k)$.*

When $G = S^1$, Frankel uses Morse theory to prove that

$$\dim_k H^*(M;k) = \dim_k H^*(M^G;k).$$

By a standard result in the cohomology theory of transformation groups, this is equivalent to M being TNHZ in $M_G \longrightarrow BG$ with respect to $H^*(-;k)$. See, e.g., [Allday,Puppe], Theorem (3.10.4). The following proposition, then, shows how to generalize Frankel's theorem to all compact connected Lie groups. The proof is an easy matter of comparing spectral sequences.

Proposition (3.5). *Let G be a compact connected Lie group, and let X be a G–space. Then the following statements are equivalent.*

(1) *X is TNHZ in $X \to^{i_G} X_G \longrightarrow BG$ with respect to $H^*(-;k)$.*

(2) *For any compact connected Lie group K and homomorphism $\varphi : K \longrightarrow G, X$ is TNHZ in $X \to^{i_K} X_K \longrightarrow BK$ with respect to $H^*(-;k)$, where X is made into a K–space via φ.*

(3) *For every subcircle $L \subseteq G, X$ is TNHZ in $X \to^{i_L} X_L \longrightarrow BL$ with respect to $H^*(-;k)$.*

(4) *X is TNHZ in $X \to^{i_T} X_T \longrightarrow BT$ with respect to $H^*(-\,;k)$ for at least one maximal torus $T \subseteq G$.*

Remarks (3.6). (1) Frankel's theorem does not hold in general for c–Hamiltonian actions, even with the weak Lefschetz condition. See [Allday1] for an example. By an old result of Blanchard, however, the theorem holds for any compact connected Lie group action on a c–Kähler space, i.e., a c–symplectic space satisfying the strong (or hard) Lefschetz condition. See [Blanchard] or [Borel et al.], Chapter XII, Section 6.

(2) Since TNHZ implies c–Hamiltonian, the converse of Frankel's theorem is valid by Remarks (3.2)(3).

(3) By Remarks (3.2)(3) and Proposition (3.5), if M is a closed symplectic manifold and G is a compact connected Lie group acting on M in a symplectic way, then the following statements are equivalent.

(i) The action of G is Hamiltonian.
(ii) The action of at least one maximal subtorus $T \subseteq G$ is Hamiltonian.
(iii) The action of every maximal subtorus $T \subseteq G$ is Hamiltonian.
(iv) The action of every subcircle $L \subseteq G$ is Hamiltonian.

(4) Since the condition TNHZ is equivalent to the vanishing of the differential in the appropriate minimal Hirsch–Brown model (see [Allday,Puppe], Definitions (3.5.6) et seq.), one can restate part of Proposition (3.5) as follows.

The differential in the minimal Hirsch–Brown model $H^*(BG;k) \otimes H^*(X;k)$ is zero if and only if the differential in the minimal Hirsch–Brown model $H^*(BL;k) \otimes H^*(X;k)$ is zero for every subcircle $L \subseteq G$.

4. The Main Theorem

In the following let $R_G = H^*(BG;k)$, where, as always here, k is a field of characteristic zero. Furthermore, k is considered as an R_G–module via the standard augmentation homomorphism $R_G \longrightarrow k$.

We begin with an easy generalization of a result of T. Chang ([Chang]). Chang's proof works without change here.

Lemma (4.1). *Let X be a space, and suppose that $H^*(X;k)$ has a k–algebra presentation*

$$0 \longrightarrow J \longrightarrow k[x_1,\dots,x_g] \otimes \wedge(y_1,\dots,y_h) \longrightarrow H^*(X;k) \longrightarrow 0,$$

where $x_1,\dots,x_g$ are generators of positive even degree, $y_1,\dots,y_h$ are generators of odd degree, and the ideal of relations $J = (f_1,\dots,f_m,e_1,\dots,e_n)$, where $f_1,\dots,f_m$ are relations of even degree and $e_1,\dots,e_n$ are relations of odd degree.

Let G be a compact connected Lie group, and suppose that G is acting on X so that X is TNHZ in $X_G \longrightarrow BG$ with respect to $H^*(-;k)$.

Then $H^*_G(X;k)$ has an R_G–algebra presentation as follows, and there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J_G & \longrightarrow & R_G[X_1,\dots,X_g] \otimes \wedge(Y_1\dots,Y_h) & \longrightarrow & H^*_G(X;k) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle i^*_G} & & \\
0 & \longrightarrow & J & \longrightarrow & k[x_1,\dots,x_g] \otimes \wedge(y_1,\dots,y_h) & \longrightarrow & H^*(X;k) & \longrightarrow & 0\,,
\end{array}$$

where $J_G = (F_1,\dots,F_m,E_1,\dots,E_n), \varphi(X_i) = x_i$ for $1 \le i \le g, \varphi(Y_i) = y_i$ for $1 \le i \le h, \varphi(F_i) = f_i$ for $1 \le i \le m$ and $\varphi(E_i) = e_i$ for $1 \le i \le n$.

Next we need a version of the Evaluation Theorem, [Allday,Puppe], Theorem (3.5.1). Let X be a connected space, k a field of characteristic 0, and T a torus, and suppose that T is acting on X so that X is TNHZ in $X_T \longrightarrow BT$. Let $R_T = H^*(BT;k) = k[t_1,\dots,t_r]$, where $r =$rank (T) and $\deg(t_i) = 2$, for $1 \le i \le r$.

Let $\mathfrak{p} \subseteq R_T$ be a homogeneous prime ideal. Then there is a field E extending k, and $\alpha_1,\dots,\alpha_r \in E$, such that the evaluation homomorphism $\alpha : R_T \longrightarrow E[t]$, defined by $\alpha(t_i) = \alpha_i t$ for $1 \le i \le r$, has kernel $\mathfrak{p}$: i.e., $\ker(\alpha) = \mathfrak{p}$. As in [Allday,Puppe], Section 3.5, we define the subtorus $K(\alpha) \subseteq T$ to be the smallest subtorus of T with $PK(\alpha) \subseteq \mathfrak{p}$, where, for a subtorus $K \subseteq T, PK$ is the kernel of the restriction homomorphism $R_T \longrightarrow R_K$. So $K(\alpha)$ is non–trivial provided that $\mathfrak{p} \neq (t_1,\dots,t_r)$, the maximal homogeneous ideal in R_T.

We have the following version of [Allday,Puppe], Theorem (3.5.1).

Lemma (4.2). *Let X, k, T and the action be as above. (In particular, X is TNHZ in $X_T \longrightarrow BT$.) Suppose that X satisfies Condition (LT): see Definitions (2.1)(2). Let $\mathfrak{p}, \alpha$ and $K(\alpha)$ be as above. Then α factors through $R_{K(\alpha)} = H^*(BK(\alpha); k)$ to give $\overline{\alpha} : R_{K(\alpha)} \longrightarrow E[t]$. (So α is the composition of $\overline{\alpha}$ with the epimorphism $R_T \longrightarrow R_{K(\alpha)}$.) Also, there are isomorphisms*

$$E[t] \otimes_\alpha H_T^*(X; k) \cong E[t] \otimes_{\overline{\alpha}} H_{K(\alpha)}^*(X; k),$$

$$E[t, t^{-1}] \otimes_\alpha H_T^*(X; k) \cong E[t, t^{-1}] \otimes H^*(X^{K(\alpha)}; k).$$

Proof. Since X is TNHZ in $X_T \longrightarrow BT, H_T^*(X; k)$ is a free R_T–module; and so, from the pull–back diagram

$$\begin{array}{ccc} X_K & \longrightarrow & X_T \\ \downarrow & & \downarrow \\ BK & \longrightarrow & BT \end{array}$$

for any subtorus $K \subseteq T$, using either the Leray–Serre spectral sequence or the Eilenberg–Moore spectral sequence,

$$H_K^*(X; k) \cong R_K \otimes_{R_T} H_T^*(X; k).$$

The first, unlocalized, part above follows immediately.

The rest follows as in the proof of [Allday,Puppe], Theorem (3.5.1), noting that $S(K(\alpha)) \cap \ker(\alpha) = \emptyset$, where $S(K(\alpha))$ is the multiplicative subset of R_T generated by all homogeneous linear polynomials not in $PK(\alpha)$, and, hence, not in $\ker(\alpha)$. □

Remark (4.3). The proof of Lemma (4.2) shows that

$$E[t] \otimes_\alpha H_T^*(X; k) \cong E[t] \otimes_{\overline{\alpha}} H_K^*(X; k),$$

for any subtorus $K \subseteq T$ such that $K(\alpha) \subseteq K$. Furthermore, Condition (LT) is not needed for this isomorphism. But, if $K(\alpha) \underset{\neq}{\subset} K$, then $S(K) \cap \ker(\alpha) \neq \emptyset$; and the localized part of Lemma (4.2) is not valid.

Corollary (4.4). *With the notation and conditions of Lemma (4.2), suppose, in addition, that $E[t] \otimes_\alpha H_T^*(X; k)$ is generated, as an $E[t]$–algebra, by nilpotent elements. Then X^K is connected for every subtorus $K \subseteq K(\alpha)$.*

Proof. Since the action is uniform (see [Allday,Puppe], Corollary (3.6.19)), it is enough to prove the result when $K = K(\alpha)$. Suppose that X^K is not connected: let $F_1 \ldots, F_s$ be its components. Let $e = (1, 0, \ldots, 0) \in \bigoplus_{i=1}^{s} H^0(F_i; k) = H^0(X^K; k)$, and let $e' = 1 - e = (0, 1, \ldots, 1)$. By Lemma (4.2), there is a non–negative integer m, and there are elements $u, u' \in E[t] \otimes_\alpha H_T^*(X; k)$, such that $\varphi^*(u) = t^m e$ and $\varphi^*(u') = t^m e'$, where φ^* is the standard homomorphism

$$E[t] \otimes_\alpha H_T^*(X; k) \xrightarrow{\sim} E[t] \otimes_{\overline{\alpha}} H_K^*(X; k) \longrightarrow E[t] \otimes_{\overline{\alpha}} H_K^*(X^K; k).$$

(Note that $E[t] \otimes_{\overline{\alpha}} H_K^*(X^K; k) \cong E[t] \otimes H^*(X^K; k)$.)

Since $\varphi^*(uu') = 0$, it follows from Lemma (4.2) that $uu' = 0$. But u and u' are non–nilpotent, since e and e' are non–nilpotent. So $uu' = 0$ contradicts the assumption that $E[t] \otimes_\alpha H_T^*(X; k)$ is generated by nilpotent elements as an $E[t]$–algebra. □

In the following, for a Noetherian ring $A, \dim(A)$ means the Krull dimension.

Corollary (4.5). *Assume the notation and conditions of Lemma (4.1), suppose that $G = T$, a torus, and suppose that X satisfies Condition (LT). Consider $f_i = 1 \otimes f_i$, for $1 \leq i \leq m$, as an element in $R_T[X_1, \ldots, X_g] \otimes \wedge(Y_1, \ldots, Y_h)$ via the standard lifting. Let $\mathfrak{a} \subseteq R_T$ be the ideal generated by the coefficients (in R_T) of the polynomials $F_i - f_i$ for $1 \leq i \leq m$. Suppose that $\dim(R_T/\mathfrak{a}) > 0$. Then there is a subcircle $K \subseteq T$ such that X^K is connected.*

Proof. Since $\dim(R_T/\mathfrak{a}) > 0$, there is a minimal prime ideal $\mathfrak{p}$ of $\mathfrak{a}$ such that $\mathfrak{p} \subsetneq (t_1, \ldots, t_r)$, the maximal homogeneous ideal of R_T. Let α be as in Lemma (4.2). So $K(\alpha)$ is non–trivial.

Since X satisfies Condition (LT), $H^*(X; k)$ is generated by nilpotent elements. But, since $\mathfrak{a} \subseteq \mathfrak{p} = \ker(\alpha)$,

$$E[t] \otimes_\alpha H_T^*(X; k) \cong E[t][X_1, \ldots, X_g] \otimes \wedge(Y_1, \ldots, Y_h)/J_\alpha,$$

where $J_\alpha = E[t] \otimes_\alpha J_T$. So the even generators of J_α are $f_1, \ldots, f_m$. Thus it follows that $E[t] \otimes_\alpha H_T^*(X; k)$ is generated by nilpotent elements as an $E[t]$–algebra. Hence X^K is connected for any subtorus $K \subseteq K(\alpha)$. □

The following is our main technical result.

Theorem (4.6). *Assume the notation and conditions of Lemma (4.1), and suppose that X satisfies Condition (LT) for a maximal torus $T \subseteq G$.*

Let $R_G = H^*(BG; k) = k[v_1, \ldots, v_r]$, where $r = rank\ (G)$. Suppose either

(1) $\max\{\deg(f_i); 1 \leq i \leq m\} < \max\{\deg(v_i); 1 \leq i \leq r\}$; *or*
(2) $X^G \neq \emptyset$ *and*

$$\max\{\deg(f_i); 1 \leq i \leq m\} \leq \max\{\deg(v_i); 1 \leq i \leq r\}.$$

Then there is a subcircle $K \subseteq T$ such that X^K is connected.

Proof. By a standard result in equivariant cohomology, see, e.g., [Allday,Puppe], Theorem (3.9.3)(2),

$$H_T^*(X; k) \cong R_T \otimes_{R_G} H_G^*(X; k).$$

Thus $H_T^*(X; k)$ has an algebra presentation with exactly the same generators and relations as $H_G^*(X; k)$.

Let $\mathfrak{a}$ be the ideal defined in Corollary (4.5).

Now assume condition (1) on the degrees, and suppose that $v_1, \ldots, v_r$ are ordered so that $\deg(v_1) \leq \cdots \leq \deg(v_r)$. So $\mathfrak{a} \subseteq (v_1, \ldots, v_{r-1})$.

It follows from Krull's Theorem (see, e.g., [Allday,Puppe], Theorem (A.1.5)) that $\dim(R_T/\mathfrak{a}) > 0$. So the result in this case follows from Corollary (4.5).

Now assume condition (2). Let $\sigma : BG \longrightarrow X_G$ be a section of the bundle $X_G \longrightarrow BG$ coming from a fixed point. Let $\overline{X}_i$ be the image of X_i in $H_G^*(X; k)$ for $1 \leq i \leq g$. Subtracting elements of R_G if necessary, we can choose X_i so that $\sigma^*(\overline{X}_i) = 0$ for $1 \leq i \leq g$.

Let $\epsilon : R_G[X_1, \ldots, X_g] \otimes \wedge(Y_1, \ldots, Y_h) \longrightarrow R_G$ be the standard augmentation given by $\epsilon(X_i) = 0$ for $1 \leq i \leq g$. Since $\sigma^*\left(\overline{X}_i\right) = 0$ for $1 \leq i \leq g, \epsilon = \sigma^* q$, where $q : R_G[X_1, \ldots, X_g] \otimes \wedge(Y_1, \ldots, Y_h) \longrightarrow H_G^*(X; k)$ is the quotient homomorphism. Clearly $\epsilon(f_i) = 0$ for $1 \leq i \leq m$; and, of course, $q(F_i) = 0$ for $1 \leq i \leq m$. (Here f_i and F_i are as in Corollary (4.5).) So $\epsilon(F_i - f_i) = 0$ for $1 \leq i \leq m$. Hence the coefficients of $F_i - f_i$ have degrees less than $\deg(F_i - f_i)$. Thus condition (2) implies that $\mathfrak{a} \subseteq (v_1, \ldots, v_{r-1})$. The result now follows as before. □

We shall now give three corollaries and some examples.

Corollary (4.7). *Let G be a compact connected Lie group and let M be a c–symplectic G–space. (M is c–symplectic with respect to a given field k of characteristic 0.) Suppose that M satisfies Condition (LT) for a maximal torus $T \subseteq G$.*

Suppose that $H^*(M;k)$ has a k–algebra presentation

$$0 \longrightarrow J \longrightarrow k[x_1,\dots,x_g] \otimes \wedge(y_1,\dots,y_h) \longrightarrow H^*(M;k) \longrightarrow 0,$$

where x_i has positive even degree for $1 \le i \le g$, y_i has odd degree for $1 \le i \le h$, and the ideal of relations $J = (f_1,\dots,f_m,e_1,\dots,e_n)$, where f_i is a relation of even degree for $1 \le i \le m$, and e_i is a relation of odd degree for $1 \le i \le n$.

Let $R_G = H^*(BG;k) = k[v_1,\dots,v_r]$, where $r = rank(G)$.

Suppose that M is TNHZ in $M_T \longrightarrow BT$ (or, equivalently, in $M_G \longrightarrow BG$), with respect to $H^*(-;k)$, and suppose that the action of T on M is c–effective. Finally, suppose that

$$\max\{\deg(f_i); 1 \le i \le m\} \le \max\{\deg(v_i); 1 \le i \le r\}.$$

Then $M^G = \emptyset$.

Proof. If $M^G \neq \emptyset$, then, by Theorem (4.6), there is a subcircle $K \subseteq T$ such that M^K is connected. This contradicts Lemma (3.3). □

Corollary (4.8). *Let G be a compact connected Lie group and let M be a closed symplectic manifold. For a field k of characteristic 0, suppose $H^*(M;k)$ has a k–algebra presentation*

$$0 \longrightarrow J \longrightarrow k[x_1,\dots,x_g] \otimes \wedge(y_1,\dots,y_h) \longrightarrow H^*(M;k) \longrightarrow 0,$$

where $J = (f_1,\dots,f_m,e_1,\dots,e_n)$, and $x_i, 1 \le i \le g, y_i, 1 \le i \le h, f_i, 1 \le i \le m$, and $e_i, 1 \le i \le n$, are as in Corollary (4.7).

Let $R_G = H^*(BG;k) = k[v_1,\dots,v_r]$, and suppose that

$$\max\{\deg(f_i); 1 \le i \le m\} \le \max\{\deg(v_i); 1 \le i \le r\}.$$

Suppose that G is acting on M in an almost–effective Hamiltonian way. Then $M^G = \emptyset$.

Proof. The action is c–effective by Remarks (2.3)(2), and M is TNHZ in $M_G \longrightarrow BG$ by Frankel's Theorem (3.4). So the result follows from Corollary (4.7). □

The next corollary is an explicit application to certain homogeneous spaces which are Kählerian.

Corollary (4.9). *Let G be a compact connected Lie group, and let $M = G/Z(K)$, where $K \subseteq G$ is a (not necessarily maximal) subtorus, and $Z(K)$ is the centralizer of K in G. Suppose that G is acting on M in any continuous almost–effective way. Then $M^G = \emptyset$.*

Proof. Let $f_1,\dots,f_m,v_1,\dots,v_r$ be as in Corollary (4.7). From the fibre bundle $G \longrightarrow M \longrightarrow BZ(K)$, it follows that $m = r$ and

$$H^*(M;\mathbb{Q}) \cong \mathbb{Q}[u_1,\dots,u_r]/(f_1,\dots,f_r),$$

where $\mathbb{Q}[u_1,\dots,u_r] = H^*(BZ(K);\mathbb{Q})$ and $\deg(f_i) = \deg(v_i)$ for $1 \le i \le r$.

By a result of Borel (see [Tralle,Oprea], Chapter 5, Section 2), M is a Kähler manifold. Hence M is TNHZ in $M_G \longrightarrow BG$ (which also follows easily since $H^*(M;\mathbb{Q})$ is zero in odd degrees). So the result follows from Corollary (4.7). □

Examples (4.10). (1) Let

$$M = U(n)/U(n_1) \times \cdots \times U(n_s) \approx SU(n)/S(U(n_1) \times \cdots \times U(n_s)),$$

where $n_1 + \cdots + n_s = n$. Then no non–trivial action of $SU(n)$ on M has a fixed point. This follows from Corollary (4.9). (See Remarks (2.3)(1).)

(2) Let $M = S^2 \times S^2 \times \cdots \times S^2 = (S^2)^n$, or, more generally, let M be any space satisfying Condition (LT) and having rational cohomology isomorphic to that of $(S^2)^n$. Let G be any non–abelian compact connected Lie group. If G acts almost–effectively (or, in the general case, cohomologically effectively) on M, then $M^G = \emptyset$.

Since G is non–abelian, $H^*(BG; \mathbb{Q})$ has a generator of degree 4. So this example follows from Corollary (4.7).

5. Generalizations

In Corollaries (4.7) and (4.8), the crucial point is that effective Hamiltonian circle actions do not have connected fixed point sets. (See Lemma (3.3).) In this section we briefly consider a generalization through the notion of minimal formal dimension and the related notion of minimal grading (cf. [Hauschild]). The definitions are as follows.

Definitions (5.1). Let k be a field, and let A^* be a connected commutative graded k–algebra. Suppose that the total dimension of A^*, as a k–vector space, is finite. Let $A^{(*)}$ denote the same algebra viewed as $\mathbb{Z}_2$–graded: i.e., all elements of even degree are considered to have degree 0, and all elements of odd degree are considered to have degree 1. Let

$$\mathcal{A} = \left\{B^*; B^* \text{is a connected commutative graded } k-\text{algebra and } B^{(*)} \cong A^{(*)}\right\}.$$

(1) A^* is said to have minimal grading if, given any $B^* \in \mathcal{A}$,

$$\sum_{i=0}^{\infty} \dim_k B^{m+2i} \leq \sum_{i=0}^{\infty} \dim_k A^{m+2i} \text{ for all } m \geq 0$$

implies that

$$\dim_k B^j = \dim_k A^j \text{ for all } j.$$

(2) Suppose that A^* satisfies Poincaré duality over k. Then A^* is said to have minimal formal dimension if $fd(A^*) \leq fd(B^*)$ for all $B^* \in \mathcal{A}$.

Remarks (5.2). (1) A Poincaré duality algebra has minimal grading if it has minimal formal dimension.

(2) If X is a c–symplectic space (over a field k of characteristic 0), then $H^*(X; k)$ has minimal formal dimension (and hence also minimal grading).

(3) For $B^* \in \mathcal{A}$, define the dimension vector of B^* to be

$$d(B^*) = (\dim_k B^\circ, \dim_k B^1, \dots),$$

i.e., the sequence of coefficients in the Poincaré series of B^*. Suppose that $d(A^*)$ is maximal among all $d(B^*)$ for $B^* \in \mathcal{A}$ with respect to the lexicographic order: i.e., for any $B^* \in \mathcal{A}$, either

(i) $\dim_k B^j = \dim_k A^j$ for all $j \geq 0$, or

(ii) there is a positive integer j such that $\dim_k B^i = \dim_k A^i$ for all $i < j$, and $\dim_k B^j < \dim_k A^j$.

Then A^* has minimal grading.

An algebra with minimal grading, however, need not have a maximal dimension vector: consider, for example, the rational cohomology of the connected sum $\mathbb{C}P^4\#(S^4 \times S^4)$ compared to that of $\mathbb{C}P^4\#(S^2 \times S^6)$.

In Remarks (2.3)(2), it was pointed out that the notions of almost–effective and cohomologically effective (over a field k of characteristic 0) are equivalent for compact orientable cohomology manifolds over k (or, more simply, for closed orientable topological manifolds). The next theorem relates cohomological effectiveness directly to formal dimension: it is due to Bredon ([Bredon1]) and Chang and Skjelbred ([Chang,Skjelbred1]). The assertion in the last paragraph is not given explicitly in these two references; but it is an easy consequence of the first paragraph and the Localization Theorem.

Theorem (5.3). *Let T be a torus, X a T–space and k a field of characteristic 0. Suppose that X satisfies Condition (LT), and that $X^T \neq \emptyset$. Suppose that X satisfies Poincaré duality over k. Let F be a component of X^T. Then F satisfies Poincaré duality over k, and $fd(F) \leq fd(X)$.*

Furthermore, if $fd(F) = fd(X)$, then $F = X^T$, and the restriction homomorphism $H^*(X;k) \longrightarrow H^*(F;k)$ is an isomorphism: i.e., the action is cohomologically trivial (see Definitions (2.2)(3)).

We can now state the following partial generalization of Corollary (4.7). It is necessary here to include the relations of odd degree as well as those of even degree. In many applications, however, there are no generators of odd degree (and hence no relations of odd degree).

Theorem (5.4). *Assume the notation and conditions of Lemma (4.1), and suppose that X satisfies Condition (LT) for a maximal torus $T \subseteq G$. Suppose further that either (1) $H^*(X;k)$ satisfies Poincaré duality with minimal formal dimension, or (2) $H^*(X;k)$ has minimal grading.*

Let $R_G = H^*(BG;k) = k[v_1, \ldots, v_r]$, where $r = rank(G)$; and suppose that

$$\max\{\deg(f_i); 1 \leq i \leq m\} \leq \max\{\deg(v_i); 1 \leq i \leq r\}\,,$$
$$\max\{\deg(e_i); 1 \leq i \leq n\} \leq \max\{\deg(v_i); 1 \leq i \leq r\}\,.$$

Suppose that the action of G on X is cohomologically effective.

Then $X^G = \emptyset$.

Proof. Suppose that $X^G \neq \emptyset$. As in Corollary (4.5),

$$E[t] \otimes_\alpha H_T^*(X;k) \cong E[t][X_1, \ldots, X_g] \otimes \wedge(Y_1, \ldots, Y_h)/J_\alpha,$$

where now $J_\alpha = (f_1 \ldots, f_m, e_1, \ldots, e_n)$, where in e_i the variables x_j, y_j have been replaced by X_j, Y_j. So

$$E[t] \otimes_\alpha H_T^*(X;k) \cong E[t] \otimes H^*(X;k)$$

as $E[t]$–algebras.

Putting $t = 1$ in Lemma (4.2), we now see that

$$H^{(*)}(X;k) \cong H^{(*)}(X^{K(\alpha)};k)$$

as $\mathbb{Z}_2$–graded k–algebras.

In case (1), minimal formal dimension, the result now follows (by contradiction) from Theorem (5.3).

Now consider case (2), minimal grading. Let $K = K(\alpha)$, and let $\varphi : X^K \longrightarrow X$ be the inclusion. By the dimensional inequalities (see, e.g., [Allday,Puppe], Corollary (3.1.14) or Theorem (3.10.13)),

$$\sum_{i=0}^{\infty} \dim_k H^{m+2i}(X^K; k) \leq \sum_{i=0}^{\infty} \dim_k H^{m+2i}(X; k),$$

for any $m \geq 0$.

Thus the minimal grading condition implies that

$$\dim_k H^j(X^K; k) = \dim_k H^j(X; k),$$

for all $j \geq 0$. And it now follows from the Localization Theorem that $\varphi^* : H^*(X; k) \longrightarrow H^*(X^K; k)$ is an isomorphism, which contradicts the cohomological effectiveness. □

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