COMMENSURABILITY OF
1-CUSPED HYPERBOLIC 3-MANIFOLDS

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ABSTRACT. We give examples of non-fibered hyperbolic knot complements in homology spheres that are not commensurable to fibered knot complements in homology spheres. In fact, we give many examples of knot complements in homology spheres where every commensurable knot complement in a homology sphere has non-monic Alexander polynomial.

1. INTRODUCTION

For over 20 years, progress in 3-manifold theory has been stimulated by:

1.1. Thurston's Virtual Fibration Conjecture. *Let M be a finite volume hyperbolic 3-manifold. Then M has a finite cover which is a surface bundle over S^1.*

Little progress has been made towards the resolution of this conjecture since it was proposed [Thu]. In fact, there are few 3-manifolds which do not fiber over S^1 but are known to have finite covers which do fiber (see the references in [Kir, Prob. 3.51]). Moreover, Boileau and Wang [BW] produced infinitely many examples of closed hyperbolic 3-manifolds for which no solvable cover fibers over S^1. However, fundamental groups of hyperbolic 3-manifolds have huge numbers of finite index subgroups with a variety of quotients [Lub], and many 3-manifolds fiber over S^1 in many different ways. Thus it is likely that more complicated classes of covers provide numerous examples.

A relatively tractable class of 3-manifolds are knot complements in S^3, or more generally, knot complements in rational homology spheres. If such a 3-manifold fibers over S^1, the fiber is a minimal genus Seifert surface of the knot. Complements of knots in rational homology spheres rarely cover each other, but much more frequently they share a common finite cover—that is, they are *commensurable*. For this class of manifolds, the natural analogue of Thurston's question is:

1.2. Question. *Let M be a knot complement in a rational homology sphere. When is M commensurable with a fibered knot complement in a rational homology sphere?*

Here, we give conditions on a knot complement which ensure that it is not commensurable with a fibered knot complement. These conditions are satisfied in

many examples, including the complements of a large number of 2-bridge knots in S^3. These conditions are somewhat subtle, but we give examples showing that the subtleties are essential.

Recall the basic:

1.3. Criterion. *Let M be a knot complement in a rational homology sphere. If the Alexander polynomial Δ_M of M is not monic, then M does not fiber over S^1.*

Our main idea is to combine this criterion with the fact that the roots of Δ_M are related to eigenvalues of reducible $\mathrm{PSL}_2\mathbb{C}$-representations of $\pi_1(M)$.

We'll now state the main result. A 1-cusped hyperbolic 3-manifold M is *generic* if it is not arithmetic and its commensurator orbifold has a flexible cusp. The latter condition holds if the cusp shape of M is not in $\mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-3})$. Our condition concerns reducible representations $\rho\colon \pi_1(M) \to \mathrm{PSL}_2\mathbb{C}$ whose image is non-abelian. A representation $\rho\colon \pi_1(M) \to \mathrm{PSL}_2\mathbb{C}$ is called *integral* if the trace of $\rho(\gamma)$ is an algebraic integer for all $\gamma \in \pi_1(M)$. Otherwise, ρ is *non-integral*. As discussed in Section 5, whether M has a non-abelian reducible representation which is non-integral is closely related to whether Δ_M has a non-integral root. When M fibers, its Alexander polynomial Δ_M is *monic*, that is, has lead coefficient ± 1, and so all the roots are integral. Using the connection between Δ_M and reducible representations, it is easy to show that every non-abelian reducible representation of $\pi_1(M)$ is integral.

Let $X_0(M)$ denote the geometric component of the $\mathrm{PSL}_2\mathbb{C}$-character variety of $\pi_1(M)$ (see Section 3). We will show:

6.1. Theorem. *Let M_1 be a generic hyperbolic knot complement in a $\mathbb{Z}/2\mathbb{Z}$-homology sphere. Suppose that the geometric component $X_0(M_1)$ contains the character of a non-integral reducible representation. Then M_1 is not commensurable to a fibered knot complement in a $\mathbb{Z}/2\mathbb{Z}$-homology sphere.*

Suppose M_1 and M_2 are commensurable manifolds. Given a representation $\rho_1\colon \pi_1(M_1) \to \mathrm{PSL}_2\mathbb{C}$ one cannot in general induce a representation $\rho_2\colon \pi_1(M_2) \to \mathrm{PSL}_2\mathbb{C}$ which is *compatible*, that is, agrees with ρ_1 on the fundamental group of the common cover of M_1 and M_2. However, when M_1 and M_2 are generic commensurable 1-cusped hyperbolic 3-manifolds one knows more. Here, the existence of a commensurator [Bor], together with Thurston's Hyperbolic Dehn Surgery Theorem [Thu], gives a natural birational isomorphism between the geometric components of the character varieties of M_1 and M_2 (this is due to the first author (unpublished) and [LR]). Moreover, representations of $\pi_1(M_1)$ coming from its geometric component $X_0(M_1)$ correspond to compatible representations of $\pi_1(M_2)$ coming from its geometric component $X_0(M_2)$. The key to Theorem 6.1 is showing that for a reducible representation of $\pi_1(M_1)$ in $X_0(M_1)$, the corresponding compatible representation of $\pi_1(M_2)$ is also reducible. Then if $X_0(M_1)$ contains the character of a non-integral reducible representation, there is a corresponding reducible representation of $\pi_1(M_2)$. This representation has to be non-integral as well, and so M_2 can't be fibered.

We end this section with an outline of the rest of the paper. In Section 2, we give basic topological restrictions on when fibered and non-fibered 1-cusped manifolds can be commensurable. We also provide constructions of pairs of commensurable

1-cusped manifolds satisfying these restrictions. Section 3 contains background material about character varieties. Section 4 discusses representations of commensurable 3-manifolds. Section 5 discusses the Alexander polynomial and its connection to reducible representations. Section 6 is devoted to the proof of Theorem 6.1.

In Section 7, we show that Theorem 6.1 applies to the complements of many 2-bridge knots, in particular, to all non-fibered 2-bridge knots $K(p, q)$ where $q < p < 40$.

In Section 8, we give examples of pairs of commensurable 1-cusped hyperbolic 3-manifolds exactly one of which fibers. These illustrate the necessity and subtlety of the hypotheses of Theorem 6.1.

2. Commensurability of 1-cusped 3-manifolds

In this section, we'll discuss some basic necessary conditions for a 1-cusped non-fibered 3-manifold to be commensurable with a 1-cusped fibered 3-manifold. We'll also describe some constructions of pairs of commensurable manifolds which show that these necessary conditions are satisfied in many examples.

Let M be a 3-manifold. Given A in $H^1(M; \mathbb{Z})$ we can think of it in several ways: as a homomorphism of $\pi_1(M)$ to $\mathbb{Z}$, as a homotopy class of maps from M to S^1, or as a surface representing a class in $H_2(M, \partial M)$ (via Lefschetz duality). Thought of as a class of maps from M to S^1, it makes sense to say that A is *representable by a fibration over S^1*.

We'll begin with the question: Suppose M is a 3-manifold which does not fiber over S^1; when does a finite cover of M fiber? The fundamental fact here is:

2.1. Lemma (Stallings)**.** *Suppose M is an orientable 3-manifold which does not fiber over S^1. Let N be a finite cover which does fiber over S^1, with $\phi\colon N \to M$ the covering map. If A is a class in $H^1(M; \mathbb{Z})$, then the pullback $\phi^*(A)$ in $H^1(N; \mathbb{Z})$ cannot represent a fibration over S^1. In particular,*

$$\operatorname{rank} H^1(N) > \operatorname{rank} H^1(M).$$

Proof. Let A be in $H^1(M, \mathbb{Z})$. Stallings showed that A can be represented by a fibration over S^1 if and only if the kernel of the map $A\colon \pi_1(M) \to \mathbb{Z}$ is finitely generated [Sta]. So as M does not fiber, the kernel of the $A\colon \pi_1(M) \to \mathbb{Z}$ is not finitely generated. As $\pi_1(N)$ has finite index in $\pi_1(M)$, it follows that the kernel of the restricted map $\phi^*(A)\colon \pi_1(N) \to \mathbb{Z}$ is also not finitely generated. So $\phi^*(A)$ cannot represent a fibration. $\qquad\square$

Now suppose two manifolds M_1 and M_2 are *commensurable*, that is, they have a common finite sheeted cover N. The following theorem gives a restriction on when a non-fibered 1-cusped 3-manifold can be commensurable to a fibered one:

2.2. Theorem. *Let M_1 and M_2 be two commensurable hyperbolic knot complements in rational homology spheres. Suppose M_2 fibers over S^1 but M_1 does not. Then a common regular cover must have at least 2 boundary components.*

Proof. First, let's make some basic observations. Throughout, all (co)homology will have coefficients in $\mathbb{Z}$. Suppose $\phi\colon N \to M$ is a *regular* finite cover of 3-manifolds. Let $G = \pi_1(M)/\pi_1(N)$ be the covering group. The homomorphism $\phi^*\colon H^1(M) \to H^1(N)$ is injective, and $\phi^*(H^1(M))$ is exactly the G-invariant cohomology.

Now let's prove the theorem. Suppose N is a common regular cover of M_1 and M_2. Let $\phi_i\colon N \to M_i$ be the covering maps, and $G_i = \pi_1(M_i)/\pi_1(N)$ the covering groups. By Mostow rigidity, we can assume that the covering groups G_i act via isometries of some fixed hyperbolic metric on N. As the isometry group of N is finite, so is the group $G = \langle G_1, G_2 \rangle$.

From now on, assume that N has only one cusp. We will show there is a non-zero G-invariant class A in $H^1(N)$. This gives a contradiction for the following reason. Every non-zero class in $\phi_2^*(H^1(M_2))$ can be represented by a fibration, while by Lemma 2.1 no non-zero class in $\phi_1^*(H^1(M_1))$ can be represented by a fibration. But because A is G-invariant, A is in $\phi_i^*(H^1(M_i))$ for both i, which is impossible.

Let S be a Seifert surface for M_2 which is a fiber, and let $F = \phi_2^{-1}(S)$ be the lift to N. The surface F represents a non-trivial class in $H_2(N, \partial N)$. Moreover, since $[\partial S]$ in $H_1(\partial M_2)$ is nontrivial, so is $[\partial F]$ in $H_1(\partial N)$. Look at the the class in $H_2(N, \partial N)$ which is

$$X = \sum_{g \in G} g_*([F]).$$

Consider the restricted coverings $\phi_i\colon \partial N \to \partial M_i$. Now the covering group G_i acts freely on the torus ∂N. Hence ϕ_i induces a rational isomorphism on H_1, and G_i acts identically on $H_1(\partial N; \mathbb{Z})$. Therefore G acts identically on $H_1(\partial N; \mathbb{Z})$. Thus

$$\partial X = \sum_{g \in G} g_*([\partial F]) = |G| \cdot [\partial F] \neq 0.$$

So X is non-zero. If A is the dual class in $H^1(N)$, then A is the non-zero G-invariant class we sought. □

2.3. Remark. Suppose M_1 and M_2 have a common manifold quotient M with a torus cusp. In this case, there are Seifert surfaces generating $H_2(M_i, \partial M_i)$ which are pullbacks of a single Seifert surface in M. In particular, if M_2 fibers but M_1 does not, the commensurator cannot be a manifold, but must be an orbifold.

2.4. Example. Suppose M_1 and M_2 have a common finite regular cover N with covering deck groups G_1 and G_2. If the M_i are not hyperbolic, it is not always true that there are actions G_i' isotopic to G_i so that $G = \langle G_1', G_2' \rangle$ is finite. For example, let M_1 be the unit tangent bundle of the $(2, 4, 4)$ Euclidean triangle orbifold, and M_2 the unit tangent bundle of the $(2, 3, 6)$ Euclidean triangle orbifold. These manifolds have T^3 as a common regular cover, with deck groups $G_1 \cong \mathbb{Z}/4\mathbb{Z}$ and $G_2 \cong \mathbb{Z}/6\mathbb{Z}$ respectively. But the action of $\langle G_1, G_2 \rangle$ on $H_1(T^3; \mathbb{Z})$ generates a group isomorphic to $SL(2, \mathbb{Z})$. Thus, we can't isotope the G_i so that together they generate a finite group.

In light of Theorem 2.2, it is worth producing examples of commensurable 1-cusped manifolds whose common covers have multiple cusps.

2.5. Example. Let $N_0 = T^2 \times I$. There are two orientation-preserving involutions ϕ_1, ϕ_2 of N_0 defined by

$$\phi_1(x, y, t) = (-x, y, 1 - t), \quad \phi_2(x, y, t) = (x, -y, 1 - t)$$

where (x, y) are angular coordinates on T^2 and t is the coordinate on $I = [0, 1]$.

Now, the union of the fixed point sets of ϕ_1 and ϕ_2 is a graph $\Gamma \subset T^2 \times 1/2$. A regular neighborhood $N(\Gamma)$ of Γ is a genus 5 handlebody. Let $N_1 = N_0 - N(\Gamma)$, and let N_2 be the double of N_1. Then N_2 has two sets of cusps, $\{C_1, C_2\}$ and $\{D_1, D_2\}$, where C_1 and C_2 are the original pair of cusps from N_0. The involutions ϕ_1 and ϕ_2 extend to fixed-point–free involutions of N_2 which interchange C_1 with C_2 and D_1 with D_2. Notice that

$$\phi_2 \phi_1^{-1} : T^2 \times 0 \to T^2 \times 0$$

is just the involution $(x, y) \to (-x, -y)$. It follows that these involutions descend to the manifold obtained by equivariantly Dehn filling D_1 and D_2. Let N_3 be obtained from N_2 by Dehn filling on both D_1 and D_2, so that ϕ_1 and ϕ_2 both act on N_3. We can also equivariantly surger N_3 along some collection of links to destroy any "accidental" additional symmetry to get N_3', so that the quotients $M_1 = N_3'/\phi_1$ and $M_2 = N_3'/\phi_2$ are not homeomorphic, and have no common regular cover with fewer than two cusps. This equivariant surgery can also be used to kill off rational homology and ensure N_3 is hyperbolic, so that M_1 and M_2 can be chosen to be knot complements in rational homology spheres.

These examples have the property that for S a Seifert surface in M_2, the class $[\phi_1 \phi_2^{-1}(S)]$ is trivial in $H_2(M_1, \partial M_1)$. So the proof of Theorem 2.2 does not apply here, and we cannot conclude anything about whether M_1 and M_2 mutually fiber or do not fiber over S^1.

These examples cannot be chosen to be knot complements in $\mathbb{Z}/2\mathbb{Z}$-homology spheres because of the very existence of a 2-fold cover. A modification of this construction gets around this difficulty.

2.6. Example. Let L be the unlink in S^3 on n components. Arrange these symmetrically so that there is a rotation r with axis α permuting the components of L. Let s be a rotation of order n fixing each component, translating α, and fixing another axis β which links each component of L. Let $\phi_1 = rs$ and $\phi_2 = rs^k$ for some $k > 1$ coprime with n. Then $M_i = (S^3 - L)/\phi_i$ is a knot complement in a lens space which is a $\mathbb{Z}/2\mathbb{Z}$-homology sphere for n odd. By equivariant surgery, we can make the M_i hyperbolic knot complements in $\mathbb{Z}/2\mathbb{Z}$-homology spheres whose smallest common cover has n cusps.

3. CHARACTER VARIETIES

Here, we review the part of the theory of character varieties of 3-manifolds that we will need for Theorem 6.1. For details, see [CS, Sha]. For the technicalities of the $\mathrm{PSL}_2\mathbb{C}$ as opposed to $\mathrm{SL}_2\mathbb{C}$ case see [BZa].

3.1. Definition. For M a compact 3-manifold, we let

$$R(M) = \mathrm{Hom}(\pi_1(M), \mathrm{PSL}_2\mathbb{C})$$

denote the $\mathrm{PSL}_2\mathbb{C}$-representation variety of M.

Further, let

$$X(M) = \bigcup_{\rho \in R(M)} \mathrm{tr}_\rho^2 \subseteq \mathbb{C}^{\pi_1(M)}$$

denote the $\mathrm{PSL}_2\mathbb{C}$-character variety of M.

Since $\pi_1(M)$ is finitely generated, $R(M)$ is an affine algebraic variety over $\mathbb{C}$. For each $\gamma \in \pi_1(M)$, the function $\tau_\gamma : R(M) \to \mathbb{C}$ defined by

$$\tau_\gamma(\rho) = \operatorname{tr}_\rho^2(\gamma)$$

is a regular function on $R(M)$. As $\pi_1(M)$ is finitely generated, the ring of functions generated by the τ_γ is finitely generated. It follows that $X(M)$ in $\mathbb{C}^{\pi_1(M)}$ projects isomorphically to an algebraic subvariety of $\mathbb{C}^G$ for some finite subset $G \subset \pi_1(M)$. Therefore $X(M)$ has the structure of an affine algebraic variety over $\mathbb{C}$.

Away from the reducible locus, the action of $\operatorname{PSL}_2\mathbb{C}$ on $R(M)$ by conjugation is transverse, and the natural projection $R(M)/\operatorname{PSL}_2\mathbb{C} \to X(M)$ is injective on a Zariski open set. Let $t \colon R(M) \to X(M)$ be the projection. For a character χ in $X(M)$, $t^{-1}(\chi)$ either consists solely of the conjugates of a single irreducible representation, or $t^{-1}(\chi)$ consists of reducible representations. In the latter case, the reducible representations in $t^{-1}(\chi)$ may not all be conjugate (it is easy to see that the closure of the orbit of a non-abelian reducible representation contains an abelian reducible representation).

We will need the following, which is Proposition 6.2 of [CGLS].

3.2. Lemma. *Let X be an irreducible component of $X(M)$ which contains the character of an irreducible representation. Let $\chi \in X$ be the character of a reducible representation. Then there is a non-abelian reducible representation with character χ.*

The lemma follows from the fact that the fibers of t over X are all at least 3-dimensional whereas the orbit under conjugation of an abelian reducible representation is 2-dimensional.

If M is an orientable finite volume hyperbolic 3-manifold, there is a unique discrete faithful representation

$$\rho_\delta : \pi_1(M) \to \operatorname{PSL}_2\mathbb{C}$$

up to conjugacy in $O(3,1) \cong \operatorname{PSL}_2\mathbb{C} \rtimes \mathbb{Z}/2\mathbb{Z}$. Up to conjugacy in $\operatorname{PSL}_2\mathbb{C}$, there are a pair of such representations which differ by complex conjugation, and their characters may occur in different irreducible components of $X(M)$. Fixing an orientation of M fixes a $\operatorname{PSL}_2\mathbb{C}$-conjugacy class of discrete faithful representations. We will assume our manifolds are oriented in what follows, and therefore that ρ_δ is well-defined up to conjugacy in $\operatorname{PSL}_2\mathbb{C}$.

3.3. Definition. Let M be a finite volume hyperbolic 3-manifold. Let $X_0(M)$ denote the irreducible component of $X(M)$ containing the character of the discrete faithful representation ρ_δ. The component $X_0(M)$ is called the *geometric component* of $X(M)$.

4. Representations of commensurable manifolds

Let M_1 and M_2 be commensurable hyperbolic 3-manifolds with common finite cover N. Two representations $\rho_i \colon \pi_1(M_i) \to \operatorname{PSL}_2\mathbb{C}$ are said to be *compatible* if they agree on $\pi_1(N)$. For instance, if ρ_1 is a discrete faithful representation for M_1, then Mostow rigidity implies that there is a discrete faithful representation ρ_2 of $\pi_1(M_2)$ which is compatible with ρ_1. The property of having compatible representations extends to other representations whose characters are in $X_0(M_1)$.

4.1. Proposition. *Suppose M_1 and M_2 are generic commensurable orientable hyperbolic 3-manifolds with one cusp. Let χ_1 be a character in the geometric component $X_0(M_1)$. Then there exist compatible representations ρ_i of $\pi_1(M_i)$ such that ρ_1 has character χ_1 and the character of χ_2 lies in a geometric component $X_0(M_2)$.*

The reason for weaseling around with χ_1 instead of just starting with ρ_1 is that for characters of reducible representations, there can be different conjugacy classes of representations with that character. It turns out that you can't always specify ρ_1, but only χ_1, in this case.

Proof. As the M_i are non-arithmetic, they cover a common orientable commensurator orbifold Q [Bor]. Let $p_i \colon M_i \to Q$ be the (orbifold) covering maps. The inclusion of $\pi_1(M_i)$ into $\pi_1(Q)$ induces a map $p_{i*} \colon X_0(Q) \to X_0(M_i)$ via restriction of representations. Because the M_i are generic, Q has a flexible cusp, and the variety $X_0(Q)$ is also a complex curve. In fact, p_{i*} is a birational isomorphism, though we will not need this [LR]. The main step is:

4.2. Lemma. *The map $p_{1*} \colon X_0(Q) \to X_0(M_1)$ is onto.*

Proof of Lemma. The map p_{1*} is a non-constant map of irreducible affine algebraic curves over $\mathbb{C}$. Let $\bar{X}_0(Q)$ denote the smooth projective model of $X_0(Q)$. The curve $\bar{X}_0(Q)$ is the normalization of $X_0(Q)$ compactified by adding an ideal point for each end of $X_0(Q)$ [CGLS, §1.5]. Similarly, let $\bar{X}_0(M_1)$ be the smooth projective model of $X_0(M_1)$. The map p_{1*} induces a regular map of the same name between $\bar{X}_0(Q)$ and $\bar{X}_0(M_1)$ (this map is just a branched covering of closed Riemann surfaces). Let χ_1 be a point in $\bar{X}_0(M_1)$ which corresponds to a character—that is, not an ideal point. As the map from $\bar{X}_0(Q)$ to $\bar{X}_0(M_1)$ is surjective, choose χ_0 in $\bar{X}_0(Q)$ with $p_{1*}(\chi_0) = \chi_1$. We need to show that χ_0 is not an ideal point. Suppose that χ_0 is an ideal point. By Proposition 1.4.4 of [CS] there is some γ in $\pi_1(Q)$ for which $\mathrm{tr}^2_\gamma(\chi_0) = \infty$. That is, there is some element of $\pi_1(Q)$ which acts by a hyperbolic isometry on the simplicial tree associated to the ideal point χ_0. Now for any $n > 0$, γ^n also acts by a hyperbolic isometry on the tree and so $\mathrm{tr}^2_{\gamma^n}(\chi_0) = \infty$. As $\pi_1(M_1)$ is of finite index in $\pi_1(Q)$, we can choose n so that γ^n is in $\pi_1(M_1)$. But then $\mathrm{tr}^2_{\gamma^n} \chi_0 = \mathrm{tr}^2_{\gamma^n} \chi_1 = \infty$, contradicting that χ_1 is the character of a representation. So χ_0 is not an ideal point and hence $p_{1*} \colon X_0(Q) \to X_0(M_1)$ is onto. ☐

Now to finish the proof of the theorem, let $\chi_1 \in X_0(M_1)$. By the lemma, there is some character χ_0 in $X_0(Q)$ with $p_{1*}(\chi_0) = \chi_1$. Let ρ_0 be a representation with character χ_0. Then the restrictions of ρ_0 to the subgroups $\pi_1(M_i)$ give a pair of compatible representations with the required properties. ☐

5. The Alexander polynomial and reducible representations

Let M be a knot complement in a rational homology sphere. Let N denote the universal free abelian cover of M. That is, set $H = H_1(M;\mathbb{Z})/(\text{torsion})$ and take the covering corresponding to the kernel of the natural homomorphism $\pi_1(M) \to H$. Then N is a regular covering of M, and the group $\pi_1(M)/\pi_1(N) = H \cong \mathbb{Z}$ acts on N by deck transformations. It follows that $H_1(N;\mathbb{Z})$ has the natural structure of a $\mathbb{Z}[H]$-module. If t denotes the generator of H, then $H_1(N;\mathbb{Z})$ is a $\mathbb{Z}[t,t^{-1}]$-module. The *Alexander polynomial* Δ_M of M is an invariant of this module. In the case that $H_1(N;\mathbb{Z})$ is cyclic, that is $H_1(N;\mathbb{Z}) = \mathbb{Z}[t,t^{-1}]/p(t)$, the polynomial Δ_M is just $p(t)$. In general, Δ_M is the greatest common divisor of the 0-th elementary ideal of a matrix which presents $H_1(N;\mathbb{Z})$ as a $\mathbb{Z}[t,t^{-1}]$-module. The Laurent polynomial

Δ_M is only defined up to multiplication by a unit $\pm t^n$ in $\mathbb{Z}[t, t^{-1}]$. A key property for us is that the Alexander polynomial is symmetric, that is $\Delta_M(t^{-1}) = \pm t^n \Delta_M(t)$ [Tur]. For more on the Alexander polynomial see [Rol].

If M is a surface bundle over S^1 with fiber F and monodromy $\phi : F \to F$, then $N = F \times \mathbb{R}$ and the action of t on $H_1(N; \mathbb{Z})$ is exactly equal to the action of ϕ_* on $H_1(F; \mathbb{Z}) = H_1(N; \mathbb{Z})$. In this case, Δ_M is the characteristic polynomial of ϕ_*. Since ϕ is a homeomorphism, ϕ_* is an automorphism, and Δ_M is *monic*.

Now consider a non-abelian reducible representation $\rho : \pi_1(M) \to \mathrm{SL}_2\mathbb{C}$. Conjugate ρ so that its image is upper-triangular. Given $\gamma \in \pi_1(M)$ we will say that the *primary eigenvalue* of $\rho(\gamma)$ is the $(1, 1)$ entry of the matrix $\rho(\gamma)$. This is well-defined for the following reason. Since ρ has non-abelian image, there is a unique line L in $\mathbb{C}^2$ which is left invariant by all $\rho(\pi_1(M))$. The primary eigenvalue of $\rho(\gamma)$ is just the eigenvalue of $\rho(\gamma)$ with eigenspace L.

A reducible representation into $\mathrm{SL}_2\mathbb{C}$ has meta-abelian (two-step solvable) image. The Alexander polynomial of M is an invariant of the maximal meta-abelian quotient of $\pi_1(M)$, so it's not surprising that it is related to non-abelian reducible representations of $\pi_1(M)$ into $\mathrm{SL}_2\mathbb{C}$. For knots in $\mathbb{Z}$-homology spheres, the statement is:

5.1. Theorem (de Rham). *Let M be a knot complement in a $\mathbb{Z}$-homology sphere. Let μ in $\pi_1(\partial M)$ be a meridian. The following are equivalent:*

- *There is a non-abelian reducible representation*

$$\rho : \pi_1(M) \to \mathrm{SL}_2\mathbb{C}$$

  such that $\rho(\mu)$ has primary eigenvalue m.
- *m^2 is a root of $\Delta_M(t)$.*

More generally, for knots in $\mathbb{Q}$-homology spheres there is a similar connection that is a bit harder to state. Let $\rho : \pi_1(M) \to \mathrm{SL}_2\mathbb{C}$ be a non-abelian reducible representation. Then ρ acts on $\mathbb{C}\mathrm{P}^1$ and has a unique fixed point. Translating that point to ∞, ρ can be interpreted as a homomorphism from $\pi_1(M)$ into the (complex) affine group of $\mathbb{C}$:

$$\mathrm{Affine}(\mathbb{C}) = \{\text{maps } z \mapsto az + b\} \cong \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \right\} \quad \text{where } a \in \mathbb{C}^\times \text{ and } b \in \mathbb{C}.$$

Define a homomorphism $x_\rho : \pi_1(M) \to \mathbb{C}^\times$ by setting $x_\rho(\gamma) = a$, where a is the homothety part of $\rho(\gamma)$ thought of as an element of $\mathrm{Affine}(\mathbb{C})$. Note that $x_\rho(\gamma)$ is just the square of the primary eigenvalue of $\rho(\gamma)$ regarded as being in $\mathrm{SL}_2\mathbb{C}$. The map x_ρ is often called the character of ρ but to prevent confusion we'll avoid this practice. The case of knots in $\mathbb{Q}$-homology spheres is more complicated than the $\mathbb{Z}$ case because not every homomorphism $x : \pi_1(M) \to \mathbb{C}^\times$ factors through the *free* abelianization of $\pi_1(M)$. For those homomorphisms that do, Theorem 5.1 in this context is just:

5.2. Theorem. *Let M be a knot complement in a $\mathbb{Q}$-homology sphere. Let H be the free abelianization of $\pi_1(M)$. Let $x : H \to \mathbb{C}^\times$ be a homomorphism. Then the following are equivalent:*

- *There is a non-abelian reducible representation*

$$\rho : \pi_1(M) \to \mathrm{SL}_2\mathbb{C}$$

  with $x_\rho = x$.

- $x'(\Delta_M) = 0$, where x' is the map x induces from $\mathbb{Z}[H]$ to $\mathbb{C}$.

For a proof, see e.g. [McM, §3].

5.3. **Non-integral reducible representations.** From the introduction, recall that a representation ρ of $\pi_1(M)$ into $\mathrm{PSL}_2\mathbb{C}$ is *integral* if the trace of every $\rho(\gamma)$ is an algebraic integer. Otherwise ρ is *non-integral*. Let's reformulate this a little. Consider a matrix $A \in \mathrm{PSL}_2\mathbb{C}$. An eigenvalue λ of A (well-defined up to sign) is a root of the monic polynomial $x^2 \pm \mathrm{tr}(A)x + 1$. Therefore if $\mathrm{tr}(A)$ is integral, the eigenvalues of A are also algebraic integers. The converse is clearly true, so A has integral trace if and only if *both* eigenvalues of A are integral. Now if λ is an eigenvalue of A, then the other eigenvalue is $1/\lambda$. Thus we see that A has integral trace if and only if the eigenvalues of A are algebraic units. In the terminology of the last section, a reducible representation ρ is integral if and only if the primary eigenvalues of all the $\rho(\gamma)$ are algebraic units. Since the primary eigenvalue of $\rho(\gamma^{-1})$ is the inverse of the primary eigenvalue of $\rho(\gamma)$, we see that a reducible representation is integral if and only if every primary eigenvalue is an algebraic integer. If we think of ρ as a homomorphism to $\mathrm{Affine}(\mathbb{C})$, we see that ρ is integral if and only if the homothety of each $\rho(\gamma)$ is integral.

For knots in $\mathbb{Z}$-homology spheres, the Alexander polynomial determines the existence of non-abelian reducible representations which are non-integral:

5.4. Proposition. *Let M be a knot complement in a $\mathbb{Z}$-homology sphere. Then M has a non-abelian reducible representation into $\mathrm{PSL}_2\mathbb{C}$ which is non-integral if and only if Δ_M is not monic.*

Proof. As M is a knot complement in a $\mathbb{Z}/2\mathbb{Z}$-homology sphere, the cohomology group $H^2(M; \mathbb{Z}/2\mathbb{Z})$ vanishes and every representation into $\mathrm{PSL}_2\mathbb{C}$ lifts to $\mathrm{SL}_2\mathbb{C}$, so we're free to think about $\mathrm{SL}_2\mathbb{C}$-representations instead. Consider a reducible representation ρ. Since $H_1(M) = \mathbb{Z}$ is generated by μ, we see that the primary eigenvalue of any $\rho(\gamma)$ is a power of the primary eigenvalue of $\rho(\mu)$. Thus, it follows from Theorem 5.1 that M has a non-integral reducible representation if and only if Δ_M has a root which is not an algebraic unit.

Suppose that Δ_M is not monic. Then Δ_M has a non-integral root provided that Δ_M is not an integer multiple of a monic integer polynomial. As we're in the $\mathbb{Z}$-homology sphere case, $\Delta_M(1) = \pm 1$, and this can't happen. So Δ_M has a non-integral root, and thus a non-abelian reducible representation which is non-integral.

Now suppose that Δ_M is monic. Then all the roots of Δ_M are algebraic integers. Let α be a root of Δ_M. Because Δ_M is symmetric, $1/\alpha$ is also a root of Δ_M and so is integral. Thus all the roots of Δ_M are algebraic units. So all the non-abelian reducible representations of $\pi_1(M)$ are integral, completing the proof of the theorem. □

In the general $\mathbb{Z}/2\mathbb{Z}$-homology sphere case, there isn't an easy statement like this because Theorem 5.2 only applies to representations coming from certain homomorphisms to $\mathbb{C}^\times$. It is true that if M has a non-integral representation, then Δ_M is non-monic (to prove this, a nice point of view is the theory of BNS invariants [Dun, BNS, Bro]). However, if Δ_M is non-monic, M need not have a non-integral reducible representation (e.g. the SnapPea census manifold $m261$). Nor is the above proposition true for the $\mathbb{Z}/2\mathbb{Z}$-homology sphere case if one replaces the non-monic hypothesis with "has a non-integral root" (to see that the "only if" direction is false, take the complement of a fibered knot in S^3 connected sum with a lens space).

Regardless, the following proposition, which in the $\mathbb{Z}$-homology sphere case follows immediately from Proposition 5.4 and Criterion 1.3, is easy to prove in general.

5.5. Lemma. *Let M be a knot complement in a $\mathbb{Z}/2\mathbb{Z}$-homology sphere. If M fibers over S^1 then every non-abelian reducible representation of $\pi_1(M)$ into $\mathrm{PSL}_2\mathbb{C}$ is integral.*

Proof. Let $\rho\colon \pi_1(M) \to \mathrm{Affine}(\mathbb{C})$ be a lift of a given non-abelian reducible $\mathrm{PSL}_2\mathbb{C}$-representation. As M fibers over S^1, the universal abelian cover of M is of the form $F \times \mathbb{R}$ where F is a compact surface (here F is some finite abelian cover of a fiber in the fibration of M). As $\pi_1(F)$ is the commutator subgroup of $\pi_1(M)$, the representation ρ takes $\pi_1(F)$ to a finitely generated abelian subgroup G consisting of translations. The subgroup G is non-trivial as ρ is non-abelian. For each $\gamma \in \pi_1(M)$ we need to show that if $A = \rho(\gamma) = (z \mapsto az+b)$ then the homothety a is an algebraic integer. The action of A by conjugation on the normal subgroup G takes an element $(z \mapsto z + \tau)$ to $(z \mapsto z + a\tau)$. So A is a group automorphism of the lattice $G \cong \mathbb{Z}^n$. Thought of as an element of $\mathrm{SL}_n\mathbb{Z}$, the map A satisfies its characteristic polynomial $f(t)$, which is a monic polynomial with integer coefficients. Let $B = (z \mapsto z + \tau)$ be a non-identity element of G. If we act on B via $f(A)$ we get that $f(a)\tau = 0$. Thus $f(a) = 0$ and a is an algebraic integer. So ρ is integral. $\square$

6. Inducing reducible representations of commensurable manifolds

This section is devoted to proving:

6.1. Theorem. *Let M_1 be a generic hyperbolic knot complement in a $\mathbb{Z}/2\mathbb{Z}$-homology sphere. Suppose that the geometric component $X_0(M_1)$ contains the character of a non-integral reducible representation. Then M_1 is not commensurable to a fibered knot complement in a $\mathbb{Z}/2\mathbb{Z}$-homology sphere.*

Proof. Suppose that M_1 is commensurable to another knot complement M_2 in a $\mathbb{Z}/2\mathbb{Z}$-homology sphere. Call the common finite cover N. We will show that M_2 has a non-abelian reducible representation which is non-integral, and so cannot fiber.

Let χ_1 in $X_0(M_1)$ be the character of a non-integral reducible representation. As M_1 is generic, by Proposition 4.1, there are representations $\rho_i\colon \pi_1(M_i) \to \mathrm{PSL}_2\mathbb{C}$ which agree on $\pi_1(N)$ where the character of ρ_1 is equal to χ_1. In particular, ρ_1 is reducible and non-integral (ρ_1 may be abelian, because we don't get to pick ρ_1, just χ_1). Also, the character of ρ_2 is in $X_0(M_2)$.

We will show

6.2. Claim. *The representation ρ_2 of $\pi_1(M_2)$ is reducible.*

Assuming the claim, let's prove that M_2 is not fibered. Pick γ in $\pi_1(M_1)$ such that $\rho_1(\gamma)$ has non-integral trace. Then for any $n > 0$, the matrix $\rho_1(\gamma^n)$ also has non-integral trace as its eigenvalues are powers of those of $\rho_1(\gamma)$. Since $\pi_1(N)$ is of finite index in $\pi_1(M_1)$, choose an n such that γ^n is in $\pi_1(N)$. But then γ^n is in $\pi_1(M_2)$ as well, and so $\rho_2(\gamma^n) = \rho_1(\gamma^n)$ has non-integral trace. Thus ρ_2 is non-integral. Since the character of ρ_2 is in $X_0(M_2)$, by Lemma 3.2 there is a non-abelian reducible representation ρ_2' which has the same character as ρ_2. As ρ_2' has the same character as ρ_2, it is non-integral. By Lemma 5.5, M_2 does not fiber over S^1. This completes the proof of the theorem modulo the claim.

Now let's go back and prove Claim 6.2. Let $\Gamma = \pi_1(M_2)$ and $\Gamma' = \pi_1(N)$. Now ρ_2 restricted to Γ' is the same as ρ_1 restricted to Γ', and ρ_1 is reducible. Thus ρ_2

is reducible on Γ'. The subgroup Γ' is of finite index in Γ, so we can replace it by a finite index normal subgroup of Γ. Let $G = \rho_2(\Gamma)$ and $G' = \rho_2(\Gamma')$, two subgroups of $\mathrm{PSL}_2\mathbb{C}$. Note that Γ' is not the trivial subgroup because ρ_1 is non-trivial, in fact non-integral, on any finite index subgroup of $\pi_1(M_1)$.

Now suppose that ρ_2 is irreducible, that is, the fixed point set of G acting on $\partial\mathbb{H}^3$ is empty. As G' is reducible, $\mathrm{fix}(G')$ is either 1 or 2 points. As G' is normal in G, the set $\mathrm{fix}(G')$ is G-invariant. So if $\mathrm{fix}(G')$ consisted of a single point, G would be reducible as well. So $\mathrm{fix}(G')$ is 2 points. Look at the homomorphism $h\colon G \to \mathbb{Z}/2\mathbb{Z}$ where $\mathbb{Z}/2\mathbb{Z}$ is thought of as the symmetric group on $\mathrm{fix}(G')$. The homomorphism h is non-trivial as G is irreducible. Any $A \in G$ leaves invariant the geodesic g joining the two points $\mathrm{fix}(G')$. If $h(A) = 1$ then A acts on g by an orientation reversing isometry, and A has order 2. Note that G is meta-abelian, as the kernel of h is abelian because it consists of isometries which fix the pair of points $\mathrm{fix}(G')$.

To finish the proof of the claim, we look at $H = \rho_2(\pi_1(\partial M_2))$. We claim that H is finite. Let μ_2 in $\pi_1(\partial M_2)$ be a meridian, that is, Dehn filling in along μ_2 yields a $\mathbb{Z}/2\mathbb{Z}$-homology sphere. Let λ_2 in $\pi_1(\partial M_2)$ be a longitude, that is, a generator of the kernel $H_1(\partial M_2, \mathbb{Z}) \to H_1(M_2, \mathbb{Z})$. If M were the complement of a knot in a $\mathbb{Z}$-homology sphere, (μ_2, λ_2) would be a basis of $\pi_1(\partial M_2)$. In general, (μ_2, λ_2) generate a finite index subgroup of $\pi_1(\partial M_2)$. As μ_2 generates $H_1(M_2, \mathbb{Z}_2)$, we must have $h \circ \rho_2(\mu_2) = 1$ and $\rho_2(\mu_2)$ has order two.

We claim that since M_2 is a knot complement in a $\mathbb{Z}/2\mathbb{Z}$-homology sphere, if K is the kernel of the unique surjection $\pi_1(M_2) \to \mathbb{Z}/2\mathbb{Z}$, then $\lambda_2 \in [K, K]$. Consider a Seifert surface S for M. The surface S has $[\partial S]$ equal to $[\lambda_2]$ in $H_1(\partial M)$. We can explicitly construct the cover M_2' corresponding to K by gluing together two copies of M_2 cut along S. Thus we see that S lifts to M_2'. This shows that the boundary of S, namely $[\lambda_2]$, is 0 in $H_1(M_2', \mathbb{Z})$. Thus λ_2 is in $[K, K]$. Therefore, as $\rho_2(K)$ is an abelian group of isometries fixing $\mathrm{fix}(G')$, we have $\rho_2(\lambda_2) = I$. So the subgroup of H generated by the images of (μ_2, λ_2) is finite, in fact has order 2. Thus H itself is finite.

Now we'll argue that H is infinite, yielding a contradiction. Look at M_1 and in particular at $\rho_1(\pi_1(\partial M_1))$. Let μ_1 be a meridian in $\pi_1(\partial M_1)$. As ρ_1 is non-integral, it is easy to see from the homomorphism $x_{\rho_1}\colon \pi_1(M_1) \to \mathbb{C}^\times$ that the γ in $\pi_1(M_1)$ with non-integral trace are exactly those γ which are non-zero in $H_1(M_1, \mathbb{Z})/(\mathrm{torsion})$. Therefore, $\rho(\mu_1)$ has non-integral trace. In particular, $\rho_1(\mu_1)$ has infinite order, and hence $\rho_1(\pi_1(\partial M_1))$ is infinite. As $\pi_1(\partial M_1)$ shares a finite index subgroup with $\pi_1(\partial M_2)$, H shares a finite index subgroup with $\rho_1(\pi_1(\partial M_1))$. Thus H is infinite. But we've already shown that H is finite. This is a contradiction, and so ρ_2 must be reducible. This proves Claim 6.2 and thus the theorem. $\qquad\square$

7. 2-BRIDGE KNOTS TO WHICH THE THEOREM APPLIES

Theorem 6.1 applies to many 2-bridge knots in S^3. A 2-bridge knot is determined by a pair of relatively prime odd integers (p, q) with $0 < q < p$ (for background see [BZb, §12], [HT]). In this section, we describe computations which show:

7.1. Theorem. *Let $K(p, q)$ be a 2-bridge knot such that $p < 40$. Let M be the exterior of $K(p, q)$. If M does not fiber over the circle, then M satisfies the hypotheses of Theorem 6.1, and so M is not commensurable to a fibered knot complement in a $\mathbb{Z}/2\mathbb{Z}$-homology sphere.*

Let $K(p, q)$ be a 2-bridge knot, and M be its exterior. We will follow [HLM], where Hilden, Lozano, and Montesinos, building on work of Burde and Riley, give a simple method for computing the $\mathrm{PSL}_2\mathbb{C}$-character variety $X(M)$.

The standard presentation of $\pi_1(M)$ has as generators two elements a and b, each of which is a meridian at the top of one of the two bridges. As a and b are conjugate, we can take coordinates on $X(M)$ to be $x = \mathrm{tr}_{a^2}$ and $z = \mathrm{tr}_{ab}$ (the latter makes sense even in $\mathrm{PSL}_2\mathbb{C}$ because a and b are conjugate). Thus $X(M)$ is a plane curve. There is a polynomial with integer coefficients $f(x, z)$ such that $X(M)$ is the set of points in $\mathbb{C}^2$ satisfying $f(x, z) = 0$. Section 5 of [HLM] gives a simple recursive procedure for computing this polynomial.

Let M be the complement of a 2-bridge knot which does not fiber. Because 2-bridge knots are alternating, this is equivalent to Δ_M being non-monic (see e.g. [BZb, §13.C]). To decide if M satisfies the hypotheses of Theorem 6.1, we first need to factor the polynomial $f(x, z)$ into irreducible factors over $\mathbb{C}$, and determine which component is $X_0(M)$. Let $f_0(x, z)$ be the polynomial defining $X_0(M)$. It is easy to check that a character in $X(M)$ comes from a reducible representation if and only if $x = z$. So $X_0(M)$ contains a non-integral reducible representation if and only if the polynomial $g(x) = f_0(x, x)$ has a non-integral root.

So the hard part of checking whether Theorem 6.1 applies is determining the factor f_0 of f. First, since $f(x, z)$ has rational coefficients, there is an algorithm for factoring it over $\mathbb{C}$. This is because one can a priori determine a finite extension k of $\mathbb{Q}$ such that the irreducible factors of f over k are the same as those of f over $\mathbb{C}$. Take a rational line L in $\mathbb{C}^2$ which has simple intersections with the algebraic set $V = \{f = 0\}$ and such that L and V don't intersect at infinity in $\mathbb{C}P^2$. Then take k to be $\mathbb{Q}$ adjoin the coordinates of $L \cap V$. The factoring of a multivariable polynomial with coefficients in a number field is a well-studied problem (for surveys see [Kal1, Kal2]). It is worth mentioning that f sometimes factors into more components over $\mathbb{C}$ than over $\mathbb{Q}$, the lexicographically smallest example being $K(45, 29)$. The computations for Theorem 7.1 were done using the computer algebra system `Maple` [Wat], which has a built in procedure for factoring polynomials over $\mathbb{C}$.

For most of the 2-bridge knots included in Theorem 7.1, every factor f_i of f contained a non-integral reducible representation, and so it was not necessary to determine which f_i defined $X_0(M)$. In the exceptional cases $\{(15, 11), (27, 5), (27, 11), (27, 17), (27, 19), (33, 23), (35, 29)\}$, we used the result of Section 6.4 of [HLM], who determined f_i for $p < 40$ ([HLM] gives an algorithm for determining f_0 in general, but it is quite involved).

Finally, to apply Theorem 6.1 we have to check that M is generic. Reid [Rei, §4] showed that the only arithmetic knot complement in S^3 is the figure-8 complement, which fibers. So M is non-arithmetic. We also need to check that the cusp of the commensurator is non-rigid. We did this by checking that the cusp shape is not in $\mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-3})$. Let $\Gamma \subset \mathrm{PSL}_2\mathbb{C}$ be the image of the discrete faithful representation of $\pi_1(M)$. Conjugate Γ so that the meridian generators are:

$$a = \left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \quad \text{and} \quad b = \left(\begin{array}{cc} 1 & 0 \\ u & 1 \end{array} \right)$$

for some $u \in \mathbb{C}$. Riley showed that u is always an algebraic integer [Ril, §3]. Thus Γ consists solely of matrices with algebraic integer entries. A longitudein the same

copy of $\pi_1(\partial M)$ as a has the form

$$\begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}.$$

The cusp shape of M is τ, and so the cusp shape of M is always an algebraic *integer*. The integers in $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{-3})$ are discrete, and so it's easy to check numerically using SnapPea [W] that the cusp shape of M is not in $\mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-3})$, and hence that M is generic.

It would have been nicer to prove that every non-fibered 2-bridge knot satisfies the hypothesis of Theorem 6.1, but this seems a difficult thing to do—in some cases there are components of $X(M)$ which contain no reducible representations, and it is hard to see any special property $X_0(M)$ that would prevent this from happening there.

7.2. *Remark.* Actually, worrying about how f splits up over $\mathbb{C}$ as opposed to $\mathbb{Q}$ is not really necessary. The character varieties and all the maps between them in the proof of Theorem 6.1 are all defined over $\mathbb{Q}$. Thus we could weaken the hypothesis of having a non-integral reducible representation in $X_0(M_1)$ to having such a representation in the $\mathbb{Q}$-irreducible component of $X(M)$ containing $X_0(M_1)$.

8. Examples of fibered and non-fibered pairs

8.1. **The dodecahedral knots.** The two dodecahedral knots D_f and D_s were introduced by Aitchison and Rubinstein in [AR]. They are a pair of knots in S^3. The complements M_f and M_s are hyperbolic and both are quotients of $\mathbb{H}^3$ by subgroups of the symmetry group of the tiling of $\mathbb{H}^3$ by $\{5,3,6\}$-ideal dodecahedra. Thus M_f and M_s are commensurable. In [AR], they show that M_f is fibered. On the other hand, M_s is not fibered because its Alexander polynomial is non-monic:

$$\Delta_{M_s} = 25t^4 - 250t^3 + 1035t^2 - 2300t + 2981$$
$$- 2300t^{-1} + 1035t^{-2} - 240t^{-3} + 25t^{-4}.$$

The commensurator of M_f and M_s has a rigid cusp, and so D_f and D_s are not generic (though they are non-arithmetic). So the hypotheses of Theorem 6.1 are not satisfied by M_s.

8.2. **Small volume examples.** Here is an example of two 1-cusped manifolds which are commensurable where one is fibered and the other not. The two manifolds are $M_1 = m035$ and $M_2 = m039$ from the Callahan-Hildebrand-Weeks census [CHW]. These manifolds have the same volume, 3.177293278..., and same first homology group, $\mathbb{Z}/4\mathbb{Z}\oplus\mathbb{Z}$. Weeks' computer program SnapPea [W], checks that M_1 and M_2 have a common 2-fold cover. Presentations for the manifolds' fundamental groups are:

$$\pi_1(M_1) = \langle a,b \mid ab^3a^{-2}b^3ab^{-2} \rangle \quad \text{and} \quad \pi_1(M_2) = \langle a,b \mid ab^4ab^{-1}a^{-1}b^2a^{-1}b^{-1} \rangle.$$

An easy calculation shows that the Alexander polynomials are:

$$\Delta_{M_1} = 3t - 2 + 3t^{-1} \quad \text{and} \quad \Delta_{M_2} = t - 6 + t^{-1}.$$

Because of the lead coefficient of Δ_{M_1}, the manifold M_1 does not fiber over S^1. On the other hand, M_2 is the punctured torus bundle over S^1 with monodromy $+L^4R$.

Goodman's program Snap [G], calculates that these manifolds are not arithmetic and that the cusp field is a cubic extension of $\mathbb{Q}$. Thus they are generic. It is not

too hard to check that there are non-integral reducible representations on $X_0(M_1)$. Thus this example shows that the hypothesis of Theorem 6.1 requiring knot complements in a $\mathbb{Z}/2\mathbb{Z}$-homology sphere is necessary. Some further examples among the census manifolds are:

- The pair $(m037, m040)$ have a common 2-fold cover and $m037$ doesn't fiber but $m040$ does. This pair is also commensurable with $m035$ and $m039$ via 4-fold covers.
- The pair $(m139, m140)$ have a common 4-fold cover, and $m139$ doesn't fiber but $m140$ does. Both of these manifold are arithmetic and so not generic.

None of these examples are knot complements in $\mathbb{Z}/2\mathbb{Z}$-homology spheres.

The strategy for finding these examples was this. First, we used the data provided with Snap to get a list of census manifolds grouped by commensurability invariants. Then we used Lackenby's taut ideal triangulations [Lac] to identify many census manifolds which fiber over S^1. From this, we selected pairs of manifolds whose trace field and cusp density were the same, one of which fibered and the other of which did not appear to fiber. Most of the census manifolds fiber, making examples rare.

8.3. **Surgeries on the Whitehead link.** Let W be the complement of the Whitehead link in S^3. Let $W(a, b)$ denote the 1-cusped manifold obtained by filling in one of the two cusps of W via (a, b) Dehn filling. Hodgson, Meyerhoff, and Weeks gave a very elegant construction of a family of fibered/non-fibered pairs which are fillings of W [HMW]. They showed that

8.4. Theorem. *Let $m \in \mathbb{Z}$ be a multiple of 4, $m \notin \{0, 4\}$. Then*

$$M_1 = W(m, -1 - (m/2)) \quad and \quad M_2 = W(m, -1)$$

are a pair of 1-cusped hyperbolic 3-manifolds such that:

- *M_1 and M_2 have a common 2-fold cover with two cusps.*
- *M_1 does not fiber over S^1 because its Alexander polynomial is not monic.*
- *M_2 fibers over S^1, as it is the punctured torus bundle with monodromy $\pm RL^m$.*

These examples overlap with the ones in the preceding section. Namely, the pairs $(m035, m039)$ and $(m037, m040)$ are of this type. The manifold M_1 doesn't satisfy the hypotheses of Theorem 6.1 because $H_1(M_1; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$. As m is divisible by 4, M_1 is not a knot complement in a $\mathbb{Z}/2\mathbb{Z}$-homology sphere.

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