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COMPLEX CROWNS OF RIEMANNIAN SYMMETRIC SPACES AND NON-COMPACTLY CAUSAL SYMMETRIC SPACES

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ABSTRACT. In this paper we define a distinguished boundary for the complex crowns $\Xi \subseteq G_{\mathbb{C}}/K_{\mathbb{C}}$ of non-compact Riemannian symmetric spaces G/K. The basic result is that affine symmetric spaces of G can appear as a component of this boundary if and only if they are non-compactly causal symmetric spaces.

INTRODUCTION

Let $X = G/K$ be a semisimple non-compact Riemannian symmetric space. We may assume that $G \subseteq G_{\mathbb{C}}$ with $G_{\mathbb{C}}$ the universal complexification of G. Then we have $X \subseteq X_{\mathbb{C}} := G_{\mathbb{C}}/K_{\mathbb{C}}$. Note that G does not act properly on $X_{\mathbb{C}}$. Write $G = NAK$ for an Iwasawa decomposition of G. The *complex crown* $\Xi \subseteq X_{\mathbb{C}}$ of X (cf. [6]) was first considered in [1]. It can be defined by

$$\Xi := G \exp(i\Omega) K_{\mathbb{C}}/K_{\mathbb{C}},$$

where Ω is a polyhedral convex domain in $\mathfrak{a} := \mathrm{Lie}(A)$ defined by

$$\Omega := \{X \in \mathfrak{a} : (\forall \alpha \in \Sigma)\ |\alpha(X)| < \frac{\pi}{2}\}\ .$$

Here Σ denotes the restricted root system with respect to $\mathfrak{a}$. Note that G acts properly on Ξ (cf. [1]).

The domain Ξ is interesting in several ways. It accumulates many crucial geometrical and analytical properties of X. For example, it was shown in [9] that Ξ parametrizes the compact cycles in complex flag domains. From the point of harmonic analysis the domain Ξ is universal in the sense that all eigenfunctions on X for the algebra $\mathbb{D}(X)$ of G-invariant differential operators extend holomorphically to Ξ (cf. [14]). It was conjectured in [1] that the domain Ξ is Stein; this has been proved for different cases by different authors during the last year (cf. [7] for references and a more detailed account on the domain Ξ).

In this paper we investigate the boundaries of Ξ (see also [5] and [2]). Write $\partial\Xi$ for the boundary of Ξ in $X_{\mathbb{C}}$. Apparently, the boundary will be a union of G-orbits, but the stratification of these orbits can be intricate. First, not all orbits on the

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boundary intersect $\exp(i\mathfrak{a})K_{\mathbb{C}}/K_{\mathbb{C}}$ – but in this paper we focus only on such orbits. Moreover, we are only interested in very special orbits of such type which are in some sense minimal. Let us define the *distinguished boundary* $\partial_d\Xi$ of Ξ as the union of the following G-orbits in $\partial\Xi$:

$$\partial_d\Xi := G\exp(i\partial_e\Omega)K_{\mathbb{C}}/K_{\mathbb{C}},$$

where $\partial_e\Omega$ the set of extreme points in the polyhedral compact convex set $\overline{\Omega}$. Note that

$$\partial_e\Omega = \mathcal{W}(Y_1) \amalg \ldots \amalg \mathcal{W}(Y_n)$$

is a finite union of Weyl group orbits, where $\mathcal{W}$ is the Weyl group of Σ.

The distinguished boundary is a geometrically complicated object. Usually it is a disconnected set. Nevertheless, we show that it is minimal from some analytical points of view and that it features properties expected from a Shilov boundary. In particular,

Theorem A. *Write $\mathcal{A}(\Xi)$ for the algebra of bounded holomorphic functions which continuously extend to the boundary of Ξ. Then*

$$(\forall f \in \mathcal{A}(\Xi)) \qquad \sup_{z\in\Xi}|f(z)| = \sup_{z\in\partial_d\Xi}|f(z)| \ .$$

Further, $\partial_d\Xi \subseteq \partial\Xi$ is minimal in a certain plurisubharmonic sense as explained in Section 1.

We now describe the distinguished boundary $\partial_d\Xi$ in more detail. Write $z_j' := \exp(iY_j)K_{\mathbb{C}} \in \partial_d\Xi$ for all $1 \leq j \leq n$. Denote by H_j the stabilizer of z_j' in G. Then, under the assumption that $G_{\mathbb{C}}$ is simply connected, it follows that we have a G-isomorphism

$$\partial_d\Xi \simeq G/H_1 \amalg \ldots \amalg G/H_n \ .$$

In [6] it was conjectured that non-compactly causal symmetric spaces appear in the "Shilov boundary" of the complex crowns Ξ. We establish this conjecture (in a more exact form); namely, we prove

Theorem B. *If one of the components G/H_j in $\partial_d\Xi$ is a symmetric space, then it is a non-compactly causal symmetric space. Moreover, every non-compactly causal symmetric space occurs as a component of the distinguished boundary of some complex crown Ξ.*

Let us say a few words about the motivation of this conjecture. On Riemannian symmetric spaces we have an elliptic analysis, and on non-compactly causal symmetric spaces we have a hyperbolic analysis. It is known in mathematical physics that in many important cases elliptic and hyperbolic theories can be "connected" through complex domains (Laplacians and wave equations, Euclidean and Minkowski field theories, etc.). Theorem B implies a connection between Riemannian and non-compactly causal symmetric spaces through the complex crowns Ξ. It shows that the phenomenon described above has a non-trivial generalization to symmetric spaces.

Write $\mathfrak{g} := \mathrm{Lie}(G)$ for the Lie algebra of G. We call $\mathfrak{g}$ *non-compactly causal* if $\mathfrak{g}$ admits an involution such that $(\mathfrak{g}, \tau)$ is a non-compactly causal symmetric Lie algebra. Theorem B then tells us that $\mathfrak{g}$ has to be non-compactly causal in order for $\partial_d\Xi$ to contain symmetric spaces. The class of non-compactly causal Lie algebras is very rich; for example it contains all classical Lie algebras except $\mathfrak{su}(p,q)$ for $p \neq q$, $\mathfrak{sp}(p,q)$ for $p \neq q$ and $\mathfrak{so}^*(2n)$ for n odd (cf. [11] or the table in Remark 3.3(a) below).

In Theorem 3.25 we give a complete classification of the distinguished boundary $\partial_d\Xi$ for $\mathfrak{g}$ non-compactly causal. It turns out that $\partial_d\Xi$ is a union of non-compactly causal symmetric spcaes except for $\mathfrak{g} = \mathfrak{so}(p,q)$ with $3 \leq p < q$ or $\mathfrak{g} = \mathfrak{so}(2n+3,\mathbb{C})$, where $\partial_d\Xi \cong G/H_1 \amalg G/H_2$ with G/H_1 non-compactly causal and H_2 a non-symmetric subgroup of G. Another consequence of the classification is the following result:

Theorem C. *Let $\mathfrak{g}$ be a non-compactly causal Lie algebra and $\partial_d\Xi \simeq \coprod_{j=1}^n G/H_j$ the decomposition of $\partial_d\Xi$ into G-orbits. Then a boundary component G/H_j of $\partial_d\Xi$ is totally real if and only if G/H_j is symmetric.*

In the case of groups G of Hermitian type, X is is a Hermitian symmetric space and Ξ is G-equivariantly biholomorphic to $X \times \overline{X}$, where $\overline{X}$ refers to X equipped with the opposite complex structure (cf. [3], [9] or [15]). This example can be seen in a more general framework. Suppose that X is the real form of a Hermitian symmetric space. This means that there is a Hermitian group S containing G with maximal compact subgroup $U \supseteq K$ such that the inclusion $G/K \hookrightarrow S/U$ realizes X as a totally real submanifold of S/U. For example, if G is of Hermitian type, we take $S = G \times G$ and $U = K \times K$. The existence of S/U is guaranteed in many cases. In particular, S/U exists for all classical groups G in the sense that we have to replace G sometimes with $G \times \mathbb{R}$ (e.g. $\mathrm{Sl}(n,\mathbb{R})$ with $\mathrm{GL}(n,\mathbb{R})$). By the results of [3] or [15] there exists a generic G-invariant subdomain Ξ_0 of Ξ which is G-biholomorphic to S/U. Moreover, the equality $\Xi_0 = \Xi$ holds if and only if Σ is of type C_n or BC_n for $n \geq 2$ or $G = \mathrm{SO}(1,n)$ with $S = \mathrm{SO}(2,n)$. For the cases where $\Xi_0 \subsetneq \Xi$, for example if $G = \mathrm{SO}(p,q)$ $(p,q > 2)$, the geometric structure of Ξ becomes very complicated, and the boundary of Ξ can be especially complicated.

In the above-mentioned cases when $\Xi_0 = \Xi$ the situation is simpler. We can realize the Hermitian domain $D = S/U$ via the Cartan embedding in the compact Hermitian symmetric space Y dual to S/U. Here it has a compact boundary and a compact Shilov boundary $\partial_c D$. Further, in this situation the Stein manifold $X_{\mathbb{C}}$ can be realized as a Zariski open part of Y, which will contain Ξ biholomorphically equivalent to D, and $\partial_d\Xi$ will be only a Zariski open part of the compact manifold $\partial_c D$. Let us emphasize that the Shilov boundary of D essentially depends on the realization of D.

In Sections 4 and 5 we compare the dinstinguished boundary of Ξ with the distinguished boundary of Ξ_0 for all those groups G for which Ξ_0 exists and $\Xi_0 \subsetneq \Xi$. These are precisely the structure groups of Euclidean Jordan algebras and the special orthogonal groups $G = \mathrm{SO}(p,q)$ and $G = \mathrm{SO}(n,\mathbb{C})$.

This paper serves as the geometric foundation for the forthcoming work of the authors with Gestur Ólafsson towards the definition of a Hardy space on Ξ which realizes the most-continuous part $L^2(G/H)_{\mathrm{mc}}$ of $L^2(G/H)$ for a non-compactly causal symmetric space G/H (cf. [8]).

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1. Notation

Let G be a semisimple Lie group sitting inside its universal complexification $G_{\mathbb{C}}$. We denote by $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{C}}$ the Lie algebras of G and $G_{\mathbb{C}}$, respectively. Let $K < G$ be a maximal compact subgroup and $\mathfrak{k}$ its Lie algebra.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition attached to $\mathfrak{k}$. Take $\mathfrak{a} \subseteq \mathfrak{p}$ a maximal abelian subspace and let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a}) \subseteq \mathfrak{a}^*$ be the corresponding root system. Related to this root system is the root space decomposition according to the simultaneous eigenvalues of $\mathrm{ad}(H), H \in \mathfrak{a}$:

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}^{\alpha};$$

here $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ and $\mathfrak{g}^{\alpha} = \{X \in \mathfrak{g} : (\forall H \in \mathfrak{a})\ [H, X] = \alpha(H)X\}$. For the choice of a positive system $\Sigma^+ \subseteq \Sigma$ one obtains the nilpotent Lie algebra $\mathfrak{n} = \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}^{\alpha}$. Then one has the Iwasawa decomposition on the Lie algebra level

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}\ .$$

We write A, N for the analytic subgroups of G corresponding to $\mathfrak{a}$ and $\mathfrak{n}$. For these choices one has for G the Iwasawa decomposition, namely, the multiplication map

$$N \times A \times K \to G, \quad (n, a, k) \mapsto nak$$

In particular, every element $g \in G$ can be written uniquely as $g = n(g)a(g)\kappa(g)$ with each of the maps $\kappa(g) \in K$, $a(g) \in A$, $n(g) \in N$ depending analytically on $g \in G$. The last piece of structure theory we shall recall is the little Weyl group. We denote by $\mathcal{W} = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ the *Weyl group* of $\Sigma(\mathfrak{a}, \mathfrak{g})$.

Next we define a domain using the restricted roots. We set

$$\Omega = \{X \in \mathfrak{a} : (\forall \alpha \in \Sigma)\ |\alpha(X)| < \frac{\pi}{2}\}\ .$$

Clearly, Ω is convex and $\mathcal{W}$-invariant.

If L is a Lie group, then we write L_0 for the connected component containing $\mathbf{1}$. If L is a group and σ is an involution on L, then we write L^{σ} for the σ-fixed points in L. Similarly, if $\mathfrak{l}$ is a Lie algebra and σ an involution on $\mathfrak{l}$, then we write $\mathfrak{l}^{\sigma}$ for the subspace of $\mathfrak{l}$ which is pointwise fixed under σ.

The domain Ξ. Write $A_{\mathbb{C}}$, $N_{\mathbb{C}}$ and $K_{\mathbb{C}}$ for the complexifications of the groups A, N and K realized in $G_{\mathbb{C}}$. Define the domain

$$\Xi := G\exp(i\Omega)K_{\mathbb{C}}/K_{\mathbb{C}},$$

which will be in the center of attention throughout this paper. Write $\partial\Xi$ for the topological boundary of Ξ in $G_{\mathbb{C}}/K_{\mathbb{C}}$. Recall from [1] that the piece of the boundary of Ξ which intersects $\exp(i\mathfrak{a})K_{\mathbb{C}}/K_{\mathbb{C}}$ is given through

$$\partial_a\Xi := G\exp(i\partial\Omega)K_{\mathbb{C}}/K_{\mathbb{C}}\ .$$

Note the following properties of Ξ:

- Ξ is open in $G_{\mathbb{C}}/K_{\mathbb{C}}$ (cf. [1]).
- Ξ is connected and G-invariant.
- G acts properly on Ξ (cf. [1]).
- The domain Ξ is Stein ([3, Th. 10]).
- One has $\Xi \subseteq N_{\mathbb{C}}A\exp(i\Omega)K_{\mathbb{C}}/K_{\mathbb{C}}$ (special case of [7, Lemma 3.1]).

2. The distinguished boundary of Ξ

Write $\overline{\Xi}$ for the closure of Ξ in $G_{\mathbb{C}}/K_{\mathbb{C}}$. Note that $\overline{\Xi} = \Xi \amalg \partial\Xi$. Since $\overline{\Xi}$ is not compact, there is no standard definition of a Shilov boundary for Ξ. However, it is possible to define a natural distinguished boundary $\partial_d\Xi$ which features many properties of a Shilov boundary, as we will explain below.

We write $\mathcal{A}(\Xi)$ for the algebra of bounded continuous functions on $\overline{\Xi}$ which are holomorphic when restricted to Ξ:

$$\mathcal{A}(\Xi) := \{f \in C(\overline{\Xi}) : f|_{\Xi} \in \mathcal{O}(\Xi),\ \|f\| := \sup_{z\in\Xi} |f(z)| < \infty\} \ .$$

It is easy to check that $\mathcal{A}(\Xi)$ equipped with the supremum norm is a commutative Banach algebra with identity.

There is a natural action of G on $\mathcal{A}(\Xi)$ by left translation in the arguments:

$$G \times \mathcal{A}(\Xi) \to \mathcal{A}(\Xi), \quad (g, f) \mapsto \lambda_g(f);\ \lambda_g(f)(z) = f(g^{-1}z). \tag{2.1}$$

The domain Ξ is special in the sense that it admits a maximal abelian flat subdomain:

$$T_\Omega = A\exp(i\Omega) \subseteq A_{\mathbb{C}} \ .$$

Write $\overline{T_\Omega} = A\exp(i\overline{\Omega})$ for the closure of T_Ω in $A_{\mathbb{C}}$. Note that T_Ω is biholomorphic to the tube domain $\mathfrak{a} + i\Omega \subseteq \mathfrak{a}_{\mathbb{C}}$ over Ω via the mapping

$$\mathfrak{a} + i\Omega \to T_\Omega, \quad Z \mapsto \exp(Z) \ .$$

Observe that $\overline{\Omega}$ is a compact polyhedron in $\mathfrak{a}$. Denote by $\partial_e\Omega$ the extreme points of $\overline{\Omega}$. Notice that $\partial_e\Omega$ is a finite $\mathcal{W}$-invariant set, hence a finite union of $\mathcal{W}$-orbits:

$$\partial_e\Omega = \mathcal{W}(Y_1) \amalg \ldots \amalg \mathcal{W}(Y_n) \ .$$

Remark 2.1. If the restricted root system Σ is of type C_n or BC_n, then $\partial_e\Omega$ consists of a single $\mathcal{W}$-orbit (see our discussion in Section 3).

Set

$$\partial_d T_\Omega = A\exp(i\partial_e\Omega) \ .$$

Similarily as before, we define the Banach algebra $\mathcal{A}(T_\Omega)$. Then we have the following elementary lemma:

Lemma 2.2. *For all $f \in \mathcal{A}(T_\Omega)$ we have*

$$\sup_{z\in T_\Omega} |f(z)| = \sup_{z\in\partial_d T_\Omega} |f(z)| \ .$$

Proof. Recall that T_Ω is isomorphic to the tube domain $\mathfrak{a} + i\Omega$ with polyhedral bounded base Ω. For $\dim\mathfrak{a} = 1$ this lemma is just the Phragmén-Lindelöf theorem. An easy iteration of this argument gives a proof in the general case. □

Note that the mapping

$$\overline{T_\Omega} \hookrightarrow \overline{\Xi}, \quad a \mapsto aK_{\mathbb{C}}$$

defines a continuous embedding which is holomorphic when restricted to T_Ω. If $x_0 := K_{\mathbb{C}}$ denotes the base point in $G_{\mathbb{C}}/K_{\mathbb{C}}$, then

$$GT_\Omega(x_0) = \Xi, \quad G\overline{T_\Omega}(x_0) \subseteq \overline{\Xi}, \quad G\partial T_\Omega(x_0) \subseteq \partial\Xi. \tag{2.2}$$

Finally we define

$$\partial_d\Xi = G\exp(i\partial_e\Omega)K_{\mathbb{C}}/K_{\mathbb{C}} \subseteq \partial\Xi \ .$$

We call $\partial_d\Xi$ the *distinguished boundary of* Ξ *in* $G_{\mathbb{C}}/K_{\mathbb{C}}$. This notion is justified by the following result:

Theorem 2.3. *For all* $f \in \mathcal{A}(\Xi)$ *we have*

$$\sup_{z\in\Xi} |f(z)| = \sup_{z\in\partial_d\Xi} |f(z)| \ .$$

Proof. Recall from (2.1) the natural action of G on $\mathcal{A}(\Xi)$. With Lemma 2.2 and (2.2) we therefore obtain that

$$\begin{aligned}\sup_{z\in\Xi}|f(z)| &= \sup_{g\in G}\sup_{z\in T_\Omega} |f(g^{-1}z(x_0))| = \sup_{g\in G}\sup_{z\in T_\Omega} |\lambda_g(f)(z(x_0))| \\ &= \sup_{g\in G}\sup_{z\in\partial_d T_\Omega} |\lambda_g(f)(z(x_0))| = \sup_{g\in G}\sup_{z\in\partial_d T_\Omega} |f(g^{-1}z(x_0))| \\ &= \sup_{z\in\partial_d\Xi} |f(z)|.\end{aligned}$$

This proves the theorem. □

Remark 2.4. If $\partial_d\Xi$ is connected, i.e., if $\partial_e\Omega$ consists of a single $\mathcal{W}$-orbit (cf. Remark 2.1), then $\partial_d\Xi$ is in fact minimal in the sense of Theorem 2.3. Hence it makes perfect sense to call $\partial_d\Xi$ the *Shilov boundary* of Ξ in this case.

Minimality of the distinguished boundary. The whole boundary $\partial\Xi$ is a rather complicated stratified G-space. Because of the complicated structure of $\partial\Xi$, we were not able to prove that the distinguished boundary $\partial_d\Xi$ is minimal in $\partial\Xi$ in the sense of Theorem 2.3. However, we can obtain the minimality of $\partial_d\Xi$ if we replace the algebra $\mathcal{A}(\Xi)$ by the convex cone $\mathcal{P}(\Xi)$ of bounded continuous plurisubharmonic functions on Ξ:

$$\mathcal{P}(\Xi) := \{f \in C(\Xi) : f \text{ plurisubharmonic}, \quad \|f\| := \sup_{z\in\Xi}|f(z)| < \infty\} \ .$$

For $0 \le t < 1$ define

$$\partial_{d,t}\Xi = G\exp(it\partial_e\Omega)K_{\mathbb{C}}/K_{\mathbb{C}} \ .$$

and note that $\lim_{t\to1}\partial_{d,t}\Xi = \partial_d\Xi$ set-theoretically.

Theorem 2.4. *The following assertions hold:*

1. *For all* $f \in \mathcal{P}(\Xi)$ *we have*

$$\sup_{z\in\Xi}|f(z)| = \lim_{t\to1}\big(\sup_{z\in\partial_{d,t}\Xi}|f(z)|\big) \ .$$

2. $\partial_d\Xi$ *is minimal in* $\partial_a\Xi$ *with respect to the property in 1.*

Proof. 1. This is proved in the same way as Theorem 2.3.

2. Recall the partition $\partial_e\Omega = \mathcal{W}(Y_1) \amalg \ldots \amalg \mathcal{W}(Y_n)$. Further, set $\partial_{d,t,j}\Xi := G\exp(itY_j)K_{\mathbb{C}}/K_{\mathbb{C}}$ for every $1 \le j \le n$ and $0 \le t < 1$.

Suppose that $\partial_d\Xi$ is not minimal with respect to the property in 1. Since a minimal set is necessarily G-invariant, we hence find a Y_j such that

$$\lim_{t\to1}\big(\sup_{z\in\partial_{d,t,j}\Xi}|f(z)|\big) \le \lim_{t\to1}\max_{k\neq j}\big(\sup_{z\in\partial_{d,t,k}\Xi}|f(z)|\big) \tag{2.3}$$

for all $f \in \mathcal{P}(\Xi)$.

To conclude the proof of 2, we have to recall some facts from [7, Sect. 3]. There exists a unique holomorphic surjective mapping

$$\Xi \to T_\Omega \subseteq A_\mathbb{C}, \quad z \mapsto a(z)$$

such that $z \in N_\mathbb{C} a(z) K_\mathbb{C}/K_\mathbb{C}$. Furthermore, if $z = g\exp(iX)K_\mathbb{C}$ for $g \in G$ and $X \in \Omega$, then

$$\operatorname{Im}\log a(z) \subseteq \operatorname{conv}(\mathcal{W}(X)) \ .$$

Recall that T_Ω is biholomorphic to $\mathfrak{a} + i\Omega$. Thus we can find an A-invariant $F \in \mathcal{P}(T_\Omega)$ which peaks at $\exp(iY_j)$. Now the function

$$f : \Xi \to \mathbb{C}, \quad z \mapsto F(a(z))$$

is a well defined element in $\mathcal{P}(\Xi)$. By construction f does not satisfy (2.3), concluding the proof of 2. □

3. Determination of the distinguished boundary

In this section we will show that the distinguished boundary $\partial_d\Xi$ can only be a union of symmetric spaces under the geometric assumption that G is a non-compactly causal group. Further, we will determine the distinguished boundary for all non-compactly causal groups.

We first have to recall some facts on causal symmetric Lie algebras.

Causal symmetric Lie algebras. The standard reference for the facts collected below is the book [11].

As before, $\mathfrak{g}$ denotes a semisimple real Lie algebra and θ a Cartan involution on $\mathfrak{g}$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\tau : \mathfrak{s} \to \mathfrak{s}$ be an involution on $\mathfrak{g}$ which we may assume to commute with θ. Write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ for the τ-eigenspace decomposition corresponding to the τ-eigenvalues $+1$ and -1.

The symmetric Lie algebra $(\mathfrak{g}, \tau)$ is called *irreducible* if the only τ-invariant ideals of $\mathfrak{g}$ are $\{0\}$ and $\mathfrak{g}$.

Remark 3.1. If $(\mathfrak{g}, \tau)$ is irreducible, then either $\mathfrak{g}$ is simple or $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}$ with $\mathfrak{h}$ simple, and τ is the flip $\tau(X, Y) = (Y, X)$ ("group case").

If $(\mathfrak{g}, \tau)$ is a symmetric Lie algebra, then we define its *c-dual* by $\mathfrak{g}^c = \mathfrak{h} \oplus i\mathfrak{q}$. Denote the restriction of the complex linear extension of τ to $\mathfrak{g}^c$ also by τ. Then $(\mathfrak{g}^c, \tau)$ is called the symmetric Lie algebra *c-dual* to $(\mathfrak{g}, \tau)$.

Definition 3.2. Let $(\mathfrak{g}, \tau)$ be an irreducible semisimple symmetric Lie algebra.

(a) We call $(\mathfrak{g}, \tau)$ *compactly causal* if $\mathfrak{z}(\mathfrak{k}) \cap \mathfrak{q} \neq \{0\}$.

(b) We call $(\mathfrak{g}, \tau)$ *non-compactly causal* if its c-dual is compactly causal, or, equivalently, if there exists a $0 \neq Y_0 \in \mathfrak{q} \cap \mathfrak{p}$ which is fixed under $\mathfrak{h} \cap \mathfrak{k}$.

(c) We call $(\mathfrak{g}, \tau)$ of *Cayley type* if it is both compactly and non-compactly causal.

Remark 3.3. (a) Non-compactly causal symmetric Lie algebras have been classified. The complete list is as follows (cf. [11, Th. 3.2.8]):

$\mathfrak{g}$	$\mathfrak{h}$
$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{gl}(n,\mathbb{R})$
$\mathfrak{su}(n,n)$	$\mathfrak{sl}(n,\mathbb{C})\times\mathbb{R}$
$\mathfrak{so}^*(4n)$	$\mathfrak{sl}(n,\mathbb{H})\times\mathbb{R}$
$\mathfrak{so}(p,q)$	$\mathfrak{so}(1,p-1)\times\mathfrak{so}(1,q-1)$
$\mathfrak{so}(n,n)$	$\mathfrak{so}(n,\mathbb{C})$
$\mathfrak{sp}(n,n)$	$\mathfrak{sp}(n,\mathbb{C})$
$\mathfrak{sl}(n,\mathbb{R})$	$\mathfrak{so}(q,n-q)\ \ (1\leq q<n)$
$\mathfrak{sl}(n,\mathbb{H})$	$\mathfrak{sp}(q,n-q)\ \ (1\leq q<n)$
$\mathfrak{so}(2n,\mathbb{C})$	$\mathfrak{so}^*(2n)$
$\mathfrak{so}(n+2,\mathbb{C})$	$\mathfrak{so}(2,n)$
$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{sp}(n,\mathbb{R})$
$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{su}(q,n-q)\ (1\leq q<n)$
$\mathfrak{e}_{6(6)}$	$\mathfrak{sp}(2,2)$
$\mathfrak{e}_{6(-26)}$	$\mathfrak{f}_{4(-20)}$
$\mathfrak{e}_6$	$\mathfrak{e}_{6(-14)}$
$\mathfrak{e}_{7(7)}$	$\mathfrak{su}^*(8)$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-26)}\times\mathbb{R}$
$\mathfrak{e}_7$	$\mathfrak{e}_{7(-25)}$

(b) If $(\mathfrak{g},\tau)$ is of Cayley type, then $\mathfrak{g}$ has to be simple Hermitian and of tube type. Conversely, if $\mathfrak{g}$ is simple Hermitian and of tube type, then there exists up to conjugation only one involution τ on $\mathfrak{g}$ (the square of the Cayley transform) turning $(\mathfrak{g},\tau)$ into a Cayley type symmetric Lie algebra (cf. [11]). The Cayley type spaces are the following:

$\mathfrak{g}$	$\mathfrak{h}$
$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{gl}(n,\mathbb{R})$
$\mathfrak{su}(n,n)$	$\mathfrak{sl}(n,\mathbb{C})\times\mathbb{R}$
$\mathfrak{so}^*(4n)$	$\mathfrak{sl}(n,\mathbb{H})\times\mathbb{R}$
$\mathfrak{so}(2,n)$	$\mathfrak{so}(1,n-1)\times\mathbb{R}$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-26)}\times\mathbb{R}$

We call a Lie algebra $\mathfrak{g}$ *non-compactly causal* if there exists an involution τ on $\mathfrak{g}$ turning $(\mathfrak{g},\tau)$ into a non-compactly causal symmetric Lie algebra. A Lie group G is called *non-compactly causal* if its Lie algebra $\mathfrak{g}$ is non-compactly causal.

General results on G-orbits in the distinguished boundary. Let $\partial_d\Xi = G\exp(i\partial_e\Omega)K_{\mathbb{C}}/K_{\mathbb{C}}$ be the distinguished boundary of Ξ. Let

$$\partial_e\Omega = \mathcal{W}(Y_1)\amalg\ldots\amalg\mathcal{W}(Y_n)$$

be the partition of Weyl group orbits with $Y_j\in\partial_e\Omega$.

For $Y\in\partial_e\Omega$ we set $z:=\exp(iY)$ and $z':=zK_{\mathbb{C}}\in\partial_d\Xi$. Notice that

$$\partial_d\Xi=\bigcup_{j=1}^n G(z_j')\ . \tag{3.1}$$

If we write $G_{z_j'}$ for the isotropy subgroup of G in z_j', then we have $G(z_j')\simeq G/G_{z_j'}$.

We also write θ for the holomorphic extension of the Cartan involution from G to $G_{\mathbb{C}}$.

If we assume that $G_{\mathbb{C}}^{\theta} = K_{\mathbb{C}}$ (this is always satisfied if $G_{\mathbb{C}}$ is simply connected), then we can characterize $G_{z'_j}$ through the following equivalences:

$$\begin{aligned} g \in G_{z'_j} &\iff z_j^{-1} g z_j \in K_{\mathbb{C}} \\ &\iff (z_j^{-1} g z_j)\theta(z_j^{-1} g z_j)^{-1} = \mathbf{1} \\ &\iff g z_j^2 \theta(g)^{-1} = z_j^2 \\ &\iff z_j^2 \theta(g) z_j^{-2} = g. \end{aligned} \tag{3.2}$$

Note that we always have $(G_{\mathbb{C}}^{\theta})_0 = K_{\mathbb{C}}$. If $G_{\mathbb{C}}^{\theta} \neq K_{\mathbb{C}}$, then the computation (3.2) still gives us the connected component $(G_{z'_j})_0$ of $G_{z'_j}$, namely

$$(G_{z'_j})_0 = \{g \in G : z_j^2 \theta(g) z_j^{-2} = g\}_0 \ .$$

Summarizing our discussions from above, we have proved:

Lemma 3.4. *Let $Y \in \partial_e \Omega$, $z = \exp(iY)$ and $z' = zK_{\mathbb{C}} \in \partial_d \Xi$. Then the following assertions hold:*

1. *The connected component of the isotropy subgroup in z' is given by*
$$(G_{z'})_0 = \{g \in G : z^2 \theta(g) z^{-2} = g\}_0 \ .$$
2. *If $G_{\mathbb{C}}$ is simply connected, then*
$$G_{z'} = \{g \in G : z^2 \theta(g) z^{-2} = g\} \ .$$
3. *If $G_{\mathbb{C}}$ is simply connected and if $\tau(g) = z^2 \theta(g) z^{-2}$ is an involution on G, then*
$$g \in G_{z'} \iff g \in G^{\tau} \ .$$

In general we cannot expect that the automorphism τ in Lemma 3.4.3 preserves G. This can only happen under the geometric assumption that G is a non-compactly causal group.

Theorem 3.5. *Let $Y \in \partial_e \Omega$ and $z := \exp(iY)$. Consider the automorphism*
$$\tau : G_{\mathbb{C}} \to G_{\mathbb{C}}, \quad g \mapsto z^2 \theta(g) z^{-2} \ .$$
Then the following assertions hold:

1. *If $\tau := d\tau(\mathbf{1})$ preserves $\mathfrak{g}$, then τ is an involution and $(\mathfrak{g}, \tau)$ is a non-compactly causal symmetric Lie algebra.*
2. *If τ preserves G, then G/G^{τ} is non-compactly causal and we have a G-isomorphism*
$$G/G^{\tau} \simeq G(z') \subseteq \partial_d \Xi \ .$$

Proof (following a suggestion of Gestur Ólafsson). 1. Suppose that τ defines an automorphism of $\mathfrak{g}$. Note that $\tau(X) = e^{i2 \operatorname{ad} Y}(X)$ for $X \in \mathfrak{g}$.

Since $Y \in \overline{\Omega}$, we have $\operatorname{Spec} \operatorname{ad}(Y) \subseteq [-\frac{\pi}{2}, \frac{\pi}{2}]$. Further, the fact that $\tau(X) \in \mathfrak{g}$ for all $X \in \mathfrak{g}$ means precisely that $\operatorname{Spec}(\operatorname{ad} Y) \subseteq \mathbb{Z}\frac{\pi}{2}$. Hence we obtain that $\operatorname{Spec}(\operatorname{ad} Y) = \{-\frac{\pi}{2}, 0, \frac{\pi}{2}\}$ by the symmetry of the spectrum. In particular, τ is an involution on $\mathfrak{g}$, and it follows from [11, Th. 3.2.4] that $(\mathfrak{g}, \tau)$ is non-compactly causal.

2. This follows from 1. $\square$

We conclude our general discussion on G-orbits in $\partial_d\Xi$ by addressing the question of when the union in (3.1) is actually disjoint.

We now simplify notation and set $H_j := G_{z_j'}$, and denote by $\mathfrak{h}_j$ the Lie algebra of H_j.

Theorem 3.6. *Suppose that $G_\mathbb{C}$ is simply connected. Then we have $G(z_j') = G(z_k')$ if and only if $j = k$. In particular,*

$$\partial_d\Xi = \coprod_{j=1}^n G(z_j') \simeq \coprod_{j=1}^n G/H_j \ .$$

Proof. Write σ for the complex conjugation in $G_\mathbb{C}$ with respect to the real form G.

Notice that $G(z_j') = G(z_k')$ means that there exist a $g \in G$ and a $k \in K_\mathbb{C}$ such that

$$gz_jk = z_k \ . \tag{1}$$

Then it follows from (1) that

$$\sigma(gkz_j)^{-1}gkz_j = \sigma(z_k)^{-1}z_k,$$

which is equivalent to $\sigma(k)^{-1}z_j^2k = z_k^2$, or

$$z_j^2k = \sigma(k)z_k^2 \ . \tag{2}$$

Write $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ and U for the analytic subgroup of $G_\mathbb{C}$ with Lie algebra $\mathfrak{u}$. Note that U is a maximal compact subgroup of $G_\mathbb{C}$ which is simply connected. Now write $k = k_0 \exp(Z)$ for some $k_0 \in K$ and $Z \in i\mathfrak{k}$. Then it follows from (2) that

$$z_j^2k_0 \exp(Z) = k_0z_k^2 \exp(-\operatorname{Ad}(z_k^{-2})Z),$$

and so $z_j^2k_0 = k_0z_k^2$ by the polar decomposition $G_\mathbb{C} \simeq U \times i\mathfrak{u}$ of $G_\mathbb{C}$. Thus we obtain

$$z_j^2 = k_0z_k^2k_0^{-1} \ . \tag{3}$$

Notice that (3) is actually an identity in the simply connected compact group U. Since $z_l^2 = \exp(i2Y_j)$ and $Y_l \in \overline{\Omega}$ for every $1 \leq l \leq n$, it hence follows from [10, Th. 8.6(ii)] that Y_j is conjugate to Y_k under $\mathcal{W}$. Hence $Y_j = Y_k$, since $\mathcal{W}(Y_j)\cap\mathcal{W}(Y_k) = \emptyset$ for $k \neq j$ by definition. □

In view of Theorem 3.5, we can only expect that symmetric spaces are contained in the distinguished boundary $\partial_d\Xi$ if G is a non-compactly causal group. Before we start to classify the distinguished boundaries related to all such groups, we first discuss the case where G is an arbitrary complex semisimple Lie group.

The case where G is complex. We will recall some material on complexifications of semisimple complex Lie algebras and certain complex homogeneous spaces.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{k} < \mathfrak{g}$ a maximal compact subalgebra. Then $\mathfrak{k}$ is a compact real form of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$ is a Cartan decomposition of $\mathfrak{g}$. We write $X \mapsto \overline{X}$ for the complex conjugation of $\mathfrak{g}$ with respect to the real form $\mathfrak{k}$.

Write $\mathfrak{g}_\mathbb{R}$ for $\mathfrak{g}$ considered as a real Lie algebra, and define $\mathfrak{g}_\mathbb{C}$ as the complexification of $\mathfrak{g}_\mathbb{R}$. Denote by J the multiplication by i in $\mathfrak{g}_\mathbb{C}$. Finally, we write $\mathfrak{g}^{\rm opp}$ for $\mathfrak{g}$ equipped with the opposite complex structure.

The mapping

$$\mathfrak{g}_\mathbb{C} \to \mathfrak{g} \oplus \mathfrak{g}^{\rm opp}, \quad X + JY \mapsto (X + iY, X - iY)$$

is an isomorphism of the complex Lie algebra $(\mathfrak{g}_{\mathbb{C}}, J)$ with $\mathfrak{g} \oplus \mathfrak{g}^{\mathrm{opp}}$. Now

$$\mathfrak{g}^{\mathrm{opp}} \to \mathfrak{g}, \quad X \mapsto \overline{X}$$

establishes a complex Lie algebra isomorphism. Hence we get the complex Lie algebra isomorphism

$$\mathfrak{g}_{\mathbb{C}} \to \mathfrak{g} \oplus \mathfrak{g}, \quad X + JY \mapsto (X + iY, \overline{X} + i\overline{Y}) \ .$$

Note the following realizations of subalgebras in $\mathfrak{g}_{\mathbb{C}}$ inside of $\mathfrak{g} \oplus \mathfrak{g}$:

$$\mathfrak{g} \simeq \{(X, \overline{X}) : X \in \mathfrak{g}\},$$

$$\mathfrak{k}_{\mathbb{C}} \simeq \Delta(\mathfrak{g}) := \{(X, X) : X \in \mathfrak{g}\},$$

$$\mathfrak{a}_{\mathbb{C}} \simeq \{(X, -X) : X \in \mathfrak{a} + i\mathfrak{a} \subseteq \mathfrak{g}\}.$$

Now let G be a Lie group with Lie algebra $\mathfrak{g}$. Then our observations from above imply the isomorphism

$$G_{\mathbb{C}}/K_{\mathbb{C}} \simeq (G \times G)/\Delta(G)$$

with $\Delta(G)$ the diagonal subgroup. Further, we have the canonical isomorphism

$$G \times G/\Delta(G) \to G, \quad (g, h)\Delta(G) \mapsto gh^{-1} \ .$$

Hence we have $G_{\mathbb{C}}/K_{\mathbb{C}} \simeq G$. Note that the morphism of $A_{\mathbb{C}}$ into $G_{\mathbb{C}}/K_{\mathbb{C}} = G$ is given by

$$A_{\mathbb{C}} \to G, \quad a \mapsto a^2,$$

the square mapping. Further, the action of G on $G_{\mathbb{C}}/K_{\mathbb{C}} = G$ is given by

$$G \times G_{\mathbb{C}}/K_{\mathbb{C}} \to G_{\mathbb{C}}/K_{\mathbb{C}}, \quad (g, x) \mapsto gx\overline{g}^{-1},$$

where we identified $G_{\mathbb{C}}/K_{\mathbb{C}}$ with G.

The summary of our discussion is now:

Proposition 3.7. *Assume that G is a semisimple complex Lie group. Then $G_{\mathbb{C}}/K_{\mathbb{C}}$ can be canonically identified with G. Moreover, if $Y \in \partial_e\Omega$, $z = \exp(iY)$ and $z' = zK_{\mathbb{C}}$, then the isotropy subgroup of G in z' is given by*

$$G_{z'} = \{g \in G : z^2\overline{g}z^{-2} = g\},$$

where $g \mapsto \overline{g}$ denotes the conjugation in G with respect to the compact real form K of G.

From now on we will assume that $\mathfrak{g}$ is a non-compactly causal Lie algebra. Depending on the type of the restricted root system Σ, we are are now going to determine the distinguished boundary of Ξ.

Note. Unless otherwise stated, we will assume that $G_{\mathbb{C}}$ is *simply connected*.

The cases with restricted root system of type C_n. Assume that $\mathfrak{g}$ is a non-compactly causal Lie algebra with restricted root system of type C_n. According to the list in Remark 3.3(a) this means that $\mathfrak{g}$ is one of the following list:

$$\mathfrak{sp}(n,\mathbb{R}) \quad \mathfrak{sp}(n,n) \quad \mathfrak{sp}(n,\mathbb{C}) \quad \mathfrak{su}(n,n) \quad \mathfrak{so}^*(4n) \quad \mathfrak{so}(2,n) \quad \mathfrak{e}_{7(-25)} \ .$$

Note that for all these cases there exists up to conjugation only one involution τ on $\mathfrak{g}$ which turns $(\mathfrak{g},\tau)$ into a non-compactly causal symmetric Lie algebra.

On the group level we take G to be the real form of a simply connected Lie group $G_\mathbb{C}$ with Lie algebra $\mathfrak{g}_\mathbb{C}$.

The restricted root system Σ is of type C_n, say

$$\Sigma = \{\frac{1}{2}(\pm\varepsilon_i \pm \varepsilon_j) : 1 \leq i,j \leq n\}\backslash\{0\} \ .$$

Define $Y_j \in \mathfrak{a}$ by $\varepsilon_i(Y_j) = \frac{\pi}{2}\delta_{ij}$. Then

$$\Omega = \bigoplus_{j=1}^n \,]-1,1[Y_j$$

and

$$\partial_e\Omega = \{\pm Y_1 \pm \ldots \pm Y_n\} \ .$$

Define

$$Y_0 := Y_1 + \ldots + Y_n$$

and note that

$$\partial_e\Omega = \mathcal{W}(Y_0),$$

since $\mathcal{W}$ consists of all permutations and sign changes. Observe that

$$\operatorname{Spec}(\operatorname{ad} Y_0) = \{-\frac{\pi}{2}, 0, \frac{\pi}{2}\} \ .$$

We have $\partial_d\Xi = G\exp(iY_0)K_\mathbb{C}/K_\mathbb{C}$. Set $z_0 := \exp(iY_0)$. Then $\partial_d\Xi = G(z_0)$, and so $\partial_d\Xi \simeq G/G_{z_0'}$ with $z_0' = z_0K_\mathbb{C} \in \partial_d\Xi$. Note that

$$\tau : G_\mathbb{C} \to G_\mathbb{C}, \quad g \mapsto z_0^2\theta(g)z_0^{-2}$$

restricts to an involution on G since $\operatorname{Spec}(\operatorname{ad} Y_0) = \{-\frac{\pi}{2}, 0, \frac{\pi}{2}\}$. Thus we obtain from Lemma 3.4 and Theorem 3.5 that:

Proposition 3.8. *Let G be a non-compactly causal group sitting inside a simply connected complex Lie group $G_\mathbb{C}$. Suppose that the restricted root system Σ is of type C_n. Then the distinguished boundary of Ξ is G-isomorphic to the (up to isomorphism unique) non-compactly causal symmetric space G/H associated to G:*

$$\partial_d\Xi \simeq G/H \ .$$

Examples with restricted root system of type A_n.

The example of $G = \mathrm{Sl}(n, \mathbb{R})$. Here we will encounter the situation that the distinguished boundary is not connected and a union of different non-compactly causal symmetric spaces. The example features the situation where the restricted root system is of type A_n.

We introduce some notation. Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ and choose $\mathfrak{k} = \mathfrak{so}(n, \mathbb{R})$ as a maximal compact subalgebra of $\mathfrak{g}$. Write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the corresponding Cartan decomposition. For $1 \leq j \leq n$ we introduce the diagonal matrix $e_j := (\delta_{ij})_{i,j}$. Then

$$\mathfrak{a} := \{\sum_{j=1}^{n} x_j e_j : x_j \in \mathbb{R}, \sum_{j=1}^{n} x_j = 0\}$$

is a maximal abelian subspace in $\mathfrak{p}$. Define linear functionals ε_j on the diagonal matrices by $\varepsilon_j(e_i) = \delta_{ij}$. Then

$$\Sigma = \{\varepsilon_i - \varepsilon_j : i \neq j\}$$

and $\mathcal{W}$ is the permutation group of $\{e_1, \dots, e_n\}$.

For every $1 \leq q \leq n-1$ we define elements $Y_q \in \mathfrak{a}$ by

$$Y_q := \frac{\pi}{2}\Big(\sum_{j=1}^{q} e_j - \frac{q}{n}\sum_{j=1}^{n} e_j\Big).$$

Then it follows from [14, Lemma 2.1] that

$$\partial_e \Omega = \coprod_{q=1}^{n-1} \mathcal{W}(Y_q). \tag{3.3}$$

The union in (3.3) is disjoint. For every $1 \leq q \leq n-1$ set $z_q := \exp(iY_q)$ and denote by $H_q < G$ the stabilizer in G of the element $z_q' = z_q K_{\mathbb{C}} \in \partial_d \Xi$. Since $G_{\mathbb{C}} = \mathrm{Sl}(n, \mathbb{C})$ is simply connected, it follows from Theorem 3.6 that

$$\partial_d \Xi \simeq \coprod_{q=1}^{n-1} G/H_q \ .$$

It remains to compute the isotropy subgroups H_q. We have

$$z_q = e^{-i\frac{\pi}{2}\frac{q}{n}} \,\mathrm{diag}(i, \dots, i, 1, \dots, 1)$$

with i on the diagonal q times. From (3.2) we obtain the following equivalences:

$$\begin{aligned} g \in H_q &\iff z_q^{-1} g z_q \in K_{\mathbb{C}} \\ &\iff (z_q^{-1} g z_q)(z_q^{-1} g z_q)^t = \mathbf{1} \\ &\iff g z_q^2 g^t = z_q^2 \\ &\iff g\,\mathrm{diag}(-1, \dots, -1, 1, \dots, 1) g^t = \mathrm{diag}(-1, \dots, -1, 1, \dots 1) \\ &\iff g \in \mathrm{SO}(q, n-q). \end{aligned}$$

All together we thus have shown:

Proposition 3.9. *For $G = \mathrm{Sl}(n, \mathbb{R})$ we have the following G-isomorphism of the distinguished boundary:*

$$\partial_d \Xi \simeq \coprod_{q=1}^{n-1} \mathrm{Sl}(n, \mathbb{R}) / \mathrm{SO}(q, n-q) \ .$$

The example of $G = \mathrm{Sl}(n, \mathbb{C})$. This is very similar to the case of $G = \mathrm{Sl}(n, \mathbb{R})$. We will use our results on complex groups. Let $G = \mathrm{Sl}(n, \mathbb{C})$ and $K = \mathrm{SU}(n)$. As before we can take $\mathfrak{a}$ to be the diagonal matrices in $\mathfrak{g}$. Since the root system is the same as for $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ and since $G_\mathbb{C} \simeq G \times G$ is simply connected, we conclude that

$$\partial_d \Xi = \coprod_{q=1}^{n-1} G(z_q')$$

with $z_q' = \exp(iY_q) K_\mathbb{C}$ and Y_q as before.

Now we are going to use the identification of $G_\mathbb{C}/K_\mathbb{C}$ with G as explained before Proposition 3.7. Then $z_q' \in G_\mathbb{C}/K_\mathbb{C}$ becomes identified with the point $z_q^2 := \exp(i2Y_q)$ in G. We have

$$z_q^2 = c \,\mathrm{diag}(1, \dots, 1, -1, \dots, -1)$$

with c a constant and $+1$ on the diagonal q times. Further, G acts on $G_\mathbb{C}/K_\mathbb{C} = G$ by $g(x) = gx\overline{g}^t$, where $\overline{g}$ denotes the complex conjugation with respect to $\mathrm{Sl}(n, \mathbb{R})$. Hence Proposition 3.7 implies that $G_{z_q'} \simeq \mathrm{SU}(q, n-p)$. Thus we have proved:

Proposition 3.10. *For $G = \mathrm{Sl}(n, \mathbb{C})$ we have the following G-isomorphism of the distinguished boundary:*

$$\partial_d \Xi \simeq \coprod_{q=1}^{n-1} \mathrm{Sl}(n, \mathbb{C}) / \mathrm{SU}(q, n-q) \ .$$

The example of $G = \mathrm{Sl}(n, \mathbb{H})$. Considerations very similar to the two other A_n-cases treated before yield the following result:

Proposition 3.11. *For $G = \mathrm{Sl}(n, \mathbb{H})$ we have the following G-isomorphism of the distinguished boundary:*

$$\partial_d \Xi \simeq \coprod_{q=1}^{n-1} \mathrm{Sl}(n, \mathbb{H}) / \mathrm{Sp}(q, n-q) \ .$$

Examples with restricted root systems of type B_n or D_n. The classical Lie algebras which have restricted root system of type B_n or D_n are $\mathfrak{so}(p, q)$ and $\mathfrak{so}(n, \mathbb{C})$. Before we can treat these cases we first have to recall some facts of root systems of types B_n and D_n.

Root systems of types B_n and D_n. In the standard notation the root systems of type B_n and D_n are given by

$$B_n = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i \neq j \leq n\} \amalg \{\pm\varepsilon_i : 1 \leq i \leq n\}$$

and

$$D_n = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i \neq j \leq n\} \ .$$

As usual we write $e_1, \ldots, e_n$ for the basis dual to $\varepsilon_1, \ldots, \varepsilon_n$. If Σ is a root system, then we also write $\Omega = \Omega(\Sigma)$ to make the relation to Σ clear whenever ambigious.

Lemma 3.12. *For the root systems B_n and D_n the following assertions hold:*

1. $\Omega(B_n) = \Omega(D_n)$. *In particular,* $\partial_e\Omega(B_n) = \partial_e\Omega(D_n)$.
2. *For $\Sigma = D_n$ and $n \geq 3$ we have*

$$\partial_e\Omega = \mathcal{W}(Y_1) \amalg \mathcal{W}(Y_2) \amalg \mathcal{W}(Y_3),$$

   a disjoint union with $Y_1 = \frac{\pi}{2}e_1$, $Y_2 = \frac{\pi}{4}(e_1 + \ldots + e_n)$ *and* $Y_3 = \frac{\pi}{4}(e_1 + \ldots + e_{n-1} - e_n)$.
3. *For $\Sigma = B_2$ we have $\partial_e\Omega = \mathcal{W}(Y_1)$ and for $n \geq 3$ we have*

$$\partial_e\Omega = \mathcal{W}(Y_1) \amalg \mathcal{W}(Y_2),$$

   a disjoint union with Y_1 and Y_2 as in 1.

Proof. See [14, Sect. 2]. □

The example of G locally $\mathrm{SO}(2n,\mathbb{C})$ *for* $n \geq 3$. In this case the restricted root system is of type D_n. Hence Lemma 3.12.2 implies that

$$\partial_e\Omega = \mathcal{W}(Y_1) \amalg \mathcal{W}(Y_2) \amalg \mathcal{W}(Y_3)$$

consists of three Weyl group orbits. Note that Y_2 and Y_3 are conjugate under an outer isomorphism κ induced from an outer isomorphism of the Dynkin diagram.

As a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$ we choose

$$\mathfrak{a} := \left\{ \begin{pmatrix} \begin{pmatrix} 0 & it_1 \\ -it_1 & 0 \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} 0 & it_n \\ -it_n & 0 \end{pmatrix} \end{pmatrix} : t_1, \ldots, t_n \in \mathbb{R} \right\} .$$

To compute the isotropic Lie algebras $\mathfrak{h}_j$ we may assume for a moment that $G = \mathrm{SO}(2n,\mathbb{C})$ (cf. Lemma 3.4.1). With $z_j := \exp(iY_j)$ and $z_j' = z_j K_{\mathbb{C}}$ for $j = 1, 2, 3$, we then get

$$z_1^2 = \mathrm{diag}(-1, -1, 1, \ldots, 1), \qquad z_2^2 = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix}$$

and $z_3^2 = \kappa(z_2^2)$. From Proposition 3.7 we have

$$G_{z_j'} = \{g \in G : z_j^2 \overline{g} z_j^{-2} = g\},$$

and so

$$G_{z_1'} \simeq \mathrm{SO}(2, 2n-2) \quad \text{and} \quad G_{z_2'} \simeq G_{z_3'} \simeq \mathrm{SO}^*(2n) .$$

Thus we have proved:

Proposition 3.13. *For G locally* $\mathrm{SO}(2n,\mathbb{C})$ *with $n \geq 3$ we have the following G-isomorphism of the distinguished boundary:*

$$\partial_d \Xi \simeq G/H_1 \amalg G/H_2 \amalg G/H_2$$

with $\mathfrak{h}_1 = \mathfrak{so}(2, 2n-2)$ *and* $\mathfrak{h}_2 = \mathfrak{so}^*(2n)$.

The example of G locally $\mathrm{SO}(2n+1,\mathbb{C})$. In this case the restricted root system is of type B_n and Lemma 3.12.3 implies that

$$\partial_e \Omega = \mathcal{W}(Y_1) \amalg \mathcal{W}(Y_2)$$

where the second term in the union is not existent for $n = 2$. As before we now obtain that:

Proposition 3.14. *For G locally* $\mathrm{SO}(2n+1,\mathbb{C})$ *we have the following G-isomorphism of the distinguished boundary:*

1. *For $n \geq 3$ one has $\partial_d \Xi \simeq G/H_1 \amalg G/H_2$ with*

$$\mathfrak{h}_1 = \mathfrak{so}(2, 2n-1) \qquad \textit{and} \qquad \mathfrak{h}_2 = \mathfrak{so}^*(2n)\ .$$

   In particular, the boundary component G/H_2 is not a symmetric space.
2. *For $n = 2$ one has $\partial_d \Xi \simeq G/H$ with $\mathfrak{h} := \mathrm{Lie}(H) = \mathfrak{so}(2,3)$.*

Some general remarks on $\mathfrak{g} = \mathfrak{so}(p,q)$ for $0 < p \leq q$. An appropriate maximal abelian subspace of $\mathfrak{p}$ in $\mathfrak{g} = \mathfrak{so}(p,q)$ is

$$\mathfrak{a} := \{\begin{pmatrix} 0_{pp} & I_{t_1,\dots,t_p} \\ I^t_{t_1,\dots,t_p} & 0_{qq} \end{pmatrix} : t_1, \dots, t_p \in \mathbb{R}\},$$

where

$$I_{t_1,\dots,t_p} = \begin{pmatrix} & & & t_1 \\ 0_{p,p-q} & & \cdot^{\cdot^{\cdot}} & \\ & t_p & & \end{pmatrix} \in M(p \times q; \mathbb{R})\ .$$

Let $n = p+q$. For all $1 \leq i,j \leq n$ define $E_{ij} \in M_n(\mathbb{R})$ by $E_{ij} = (\delta_{ki}\delta_{jl})_{k,l}$. Then $\mathfrak{a} = \bigoplus_{j=1}^p \mathbb{R} e_j$ with

$$e_j = E_{j,n+1-j} + E_{n+1-j,j}\ .$$

Define $\varepsilon_j \in \mathfrak{a}^*$ by $\varepsilon_j(e_k) := \delta_{jk}$. Then the root system $\Sigma = \Sigma(\mathfrak{a},\mathfrak{g})$ is given by

$$\Sigma = \begin{cases} \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i \neq j \leq p\} \amalg \{\pm\varepsilon_i : 1 \leq i \leq p\} & \text{for } 1 < p < q, \\ \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i \neq j \leq p\} & \text{for } 1 < p = q, \\ \{\pm\varepsilon_1\} & \text{for } p = 1. \end{cases}$$

The restricted root system is hence of type B_p, D_p or A_1.

The example of G locally $\mathrm{SO}(n,n)$ *for $n \geq 3$.* Here the root system is of type D_n, and we have

$$\partial_e \Omega = \mathcal{W}(Y_1) \amalg \mathcal{W}(Y_2) \amalg \mathcal{W}(Y_3)\ .$$

To compute the isotropy subalgebras $\mathfrak{h}_j$ we may assume for a moment that $G = \mathrm{SO}(n,n)$. In the coordinates introduced before we then have

$$z_1^2 = \mathrm{diag}(-1, 1, \dots, 1, -1), \qquad z_2^2 = \begin{pmatrix} 0_{nn} & iI_{1,\dots,1} \\ -iI_{1,\dots,1} & 0_{nn} \end{pmatrix}$$

and $z_3^2 = \kappa(z_2^2)$. From Lemma 3.4 we thus obtain that

$$(G_{z_1'})_0 = \mathrm{SO}_e(1, n-1) \times \mathrm{SO}_e(1, n-1) \quad \text{and} \quad (G_{z_2'})_0 = \mathrm{SO}(n, \mathbb{C}) \ .$$

In view of Theorem 3.6 we thus we have shown that:

Proposition 3.15. *For G locally $\mathrm{SO}(n,n)$ and $n \geq 3$ we have the following G-isomorphism of the distinguished boundary: $\partial_d \Xi \simeq G/H_1 \amalg G/H_2 \amalg G/H_2$ with*

$$\mathfrak{h}_1 = \mathfrak{so}(1, n-1) \times \mathfrak{so}(1, n-1) \qquad \textit{and} \qquad \mathfrak{h}_2 = \mathfrak{so}(n, \mathbb{C}) \ .$$

The example of G locally $\mathrm{SO}(p,q)$ for $1 \leq p < q$. Here the restricted root system is of type B_p for $p \geq 2$ and A_1 for $p = 1$. Considerations very similar to the previous ones lead us to the following result:

Proposition 3.16. *For G locally $\mathrm{SO}(p,q)$ and $1 \leq p < q$ the distinguished boundary $\partial_d \Xi$ is described as follows:*

1. *For $p = 1$ one has $\partial_d \Xi \simeq G/H$ with $\mathfrak{h} = \mathfrak{so}(1, q-1)$.*
2. *For $p = 2$ one has $\partial_d \Xi \simeq G/H$ with $\mathfrak{h} = \mathfrak{so}(1, q-1) \times \mathbb{R}$.*
3. *For $p \geq 3$ one has $\partial_d \Xi \simeq G/H_1 \amalg G/H_2$ with*

$$\mathfrak{h}_1 = \mathfrak{so}(1, p-1) \times \mathfrak{so}(1, q-1) \qquad \textit{and} \qquad \mathfrak{h}_2 = \mathfrak{so}(p, \mathbb{C}) \times \mathfrak{so}(q-p) \ .$$

In particular, the boundary component G/H_2 is not a symmetric space.

Exceptional cases. We first have to recall some facts about the root systems E_6 and E_7. Our references are the tables from [12]. Write $e_1, \ldots, e_n$ for the standard basis in Euclidean space $\mathbb{R}^n$ and write $\langle \cdot, \cdot \rangle$ for the standard inner product.

The system E_6. Let

$$V := \{v \in \mathbb{R}^8 : \langle v, e_6 - e_7 \rangle = \langle v, e_7 + e_8 \rangle = 0\} \ .$$

Then the root system E_6 can be defined by

$$\Sigma := \{\pm e_i \pm e_j : 1 \leq j < i \leq 5\} \amalg \{\frac{1}{2} \sum_{j=1}^{8} (-1)^{n(j)} e_j \in V : \sum_{j=1}^{8} n(j) \text{ even}\} \ .$$

A positive system is given by

$$\begin{aligned} \Sigma^+ = &\{e_i \pm e_j : 1 \leq j < i \leq 5\} \\ &\amalg \{\frac{1}{2}(e_8 - e_7 - e_6 + \sum_{j=1}^{5} (-1)^{n(j)} e_j) : \sum_{j=1}^{5} n(j) \text{ even}\} \ . \end{aligned}$$

Note that $\langle \alpha, \alpha \rangle = 2$ for all $\alpha \in \Sigma$, i.e., all roots have the same length. A basis for Σ^+ is given by

$$\begin{aligned} \Pi = \{\alpha_1, \ldots, \alpha_6\} = \{&\frac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 - e_3 - e_2 + e_1), e_1 + e_2, \\ &e_2 - e_1, e_3 - e_2, e_4 - e_3, e_5 - e_4\}. \end{aligned}$$

A simple calculation then shows that the fundamental weights are given by

$$\begin{aligned}
\omega_1 &= \frac{2}{3}(e_8 - e_7 - e_6),\\
\omega_2 &= \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5) + \frac{1}{2}(e_8 - e_7 - e_6),\\
\omega_3 &= \frac{1}{2}(-e_1 + e_2 + e_3 + e_4 + e_5) + \frac{5}{6}(e_8 - e_7 - e_6),\\
\omega_4 &= (e_3 + e_4 + e_5) + (e_8 - e_7 - e_6),\\
\omega_5 &= (e_4 + e_5) + \frac{2}{3}(e_8 - e_7 - e_6),\\
\omega_6 &= e_5 + \frac{1}{3}(e_8 - e_7 - e_6).
\end{aligned}$$

Note that $\omega_1, \dots, \omega_6$ is also the dual basis of Π, since $\langle \alpha, \alpha \rangle = 2$ for all $\alpha \in \Sigma$.

The highest root β in Σ^+ can be writtem as

$$\beta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 \ .$$

In the following we set $\mathfrak{a} := V$ and we identify $\mathfrak{a}$ with $\mathfrak{a}^*$ via the inner product on $\mathfrak{a}$.

If $E \subseteq \mathfrak{a}$ is a closed convex set, then we write $\operatorname{Ext}(E)$ for the extreme points of E.

Lemma 3.17. *Let Σ be an irreducible abstract root system in the Euclidean space $\mathfrak{a}$. Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be a basis of Σ and $\omega_1, \dots, \omega_n$ its dual basis. Further, let $\beta = \sum_{j=1}^n m_j \alpha_j$ be the highest root with respect to Π, and write C for the closed Weyl chamber with respect to Π. Then the following assertions hold:*

1. *Ω is $\mathcal{W}$-invariant.*
2. *$\partial_e \Omega \cap C \subseteq \operatorname{Ext}(\overline{\Omega} \cap C)$.*
3. *We have the following inclusion:*

$$\partial_e \Omega \subseteq \mathcal{W}(\frac{\pi}{2}\{\frac{\omega_1}{m_1}, \dots, \frac{\omega_n}{m_n}\}) \ .$$

Proof. 1 and 2 are trivial.

3. First note that 1 and 2 imply that

$$\partial_e \Omega = \operatorname{Ext}(\overline{\Omega}) \subseteq \mathcal{W}(\operatorname{Ext}(\overline{\Omega}) \cap C) \ .$$

Thus it is sufficient to show that

$$\partial_e \Omega \cap C \subseteq \frac{\pi}{2}\{\frac{\omega_1}{m_1}, \dots, \frac{\omega_n}{m_n}\} \ .$$

Note that

$$C = \{x \in \mathfrak{a} : (\forall j) \langle x, \alpha_j \rangle \geq 0\} = \bigoplus_{j=1}^n \mathbb{R}^+ \omega_j \ .$$

Thus from $m_j \in \mathbb{N}$ for all j we get

$$\partial \Omega \cap C = \{x = \sum_{j=1}^n \lambda_j \omega_j : \lambda_j \geq 0, \sum_{j=1}^n m_j \lambda_j = \frac{\pi}{2}\} \ .$$

From this the assertion follows. □

For any $\alpha \in \mathfrak{a}$ we denote by s_α the reflection with respect to the hyperplane $\alpha^\perp$.

Lemma 3.18. *For the root system E_6 we have*

$$\partial_e\Omega = \mathcal{W}(\frac{\pi}{2}\omega_1) \amalg \mathcal{W}(\frac{\pi}{2}\omega_6) \ .$$

Proof. First recall that E_6 admits a nontrivial outer automorphims κ, the reflection of the Dynkin diagram. Note that κ leaves $\overline{\Omega}$ invariant, $\kappa(\alpha_1) = \alpha_6$ and $\kappa(\alpha_3) = \alpha_5$.

In view of Lemma 3.17.3, it is hence sufficient to show that $\frac{\omega_2}{2}$, $\frac{\omega_4}{3}$, $\frac{\omega_5}{2}$ are not extremal in $\overline{\Omega}$. Now observe that

$$\frac{\omega_5}{2} = \frac{1}{2}(e_4+e_5) + \frac{1}{3}(e_8-e_7-e_6) = \frac{1}{2}(\omega_6) + \frac{1}{2}(s_{e_4-e_5}\omega_6),$$
$$\frac{\omega_4}{3} = \frac{1}{3}(e_3+e_4+e_5) + \frac{1}{3}(e_8-e_7-e_6) = \frac{2}{3}(\frac{\omega_5}{2}) + \frac{1}{3}(s_{e_3-e_5}\omega_6).$$

This shows that both $\frac{\omega_5}{2}$ and $\frac{\omega_4}{3}$ are non-trivial convex combinations of elements in $\overline{\Omega}$, hence are not extremal. It remains to show that $\frac{\omega_2}{2}$ is not extremal. From the definition of Σ it is easy to check that $\frac{\omega_2}{2} + t(e_1-e_2) \in \overline{\Omega}$ for $t \in \mathbb{R}$ and $|t|$ small. Thus $\frac{\omega_2}{2}$ is not extremal, concluding the proof of the lemma. □

The system E_7. Let

$$V := \{v \in \mathbb{R}^8 : \langle v, e_7+e_8\rangle = 0\} \ .$$

The root system E_7 is defined by

$$\Sigma := \{\pm e_i \pm e_j : 1 \le i < j \le 6\} \amalg \{\pm(e_7-e_8)\}$$
$$\amalg \{\frac{1}{2}\sum_{j=1}^{8} : (-1)^{n(j)}e_j \in V : \sum_{j=1}^{8} n(j) \text{ even}\} \ .$$

Further, a positive system is given by

$$\Sigma^+ := \{e_i \pm e_j : 1 \le j < i \le 6\} \amalg \{e_8-e_7\}$$
$$\amalg \{\frac{1}{2}(e_8-e_7+\sum_{j=1}^{6}(-1)^{n(j)}e_j) : \sum_{j=1}^{6} n(j) \text{ odd}\}.$$

Note that $\langle\alpha,\alpha\rangle = 2$ for all $\alpha \in \Sigma$, i.e., all roots have the same length. A basis for Σ^+ is given by

$$\Pi = \{\alpha_1, \dots, \alpha_6, \alpha_7\} = \{\frac{1}{2}(e_8-e_7-e_6-e_5-e_4-e_3-e_2+e_1), e_1+e_2,$$
$$e_2-e_1, e_3-e_2, e_4-e_3, e_5-e_4, e_6-e_5\}.$$

The fundamental weights are

$$\omega_1 = (e_8-e_7),$$
$$\omega_2 = \frac{1}{2}(e_1+e_2+e_3+e_4+e_5+e_6) + (e_8-e_7),$$
$$\omega_3 = \frac{1}{2}(-e_1+e_2+e_3+e_4+e_5+e_6) + \frac{3}{2}(e_8-e_7),$$
$$\omega_4 = (e_3+e_4+e_5+e_6) + 2(e_8-e_7),$$
$$\omega_5 = (e_4+e_5+e_6) + \frac{3}{2}(e_8-e_7),$$
$$\omega_6 = (e_5+e_6) + (e_8-e_7),$$
$$\omega_7 = e_6 + \frac{1}{2}(e_8-e_7).$$

The highest root β in Σ^+ has the expression

$$\beta = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 \ .$$

Lemma 3.19. *For a root system of type E_7 we have*

$$\partial_e \Omega = \mathcal{W}(\frac{\pi}{2}\omega_7) \ .$$

Proof. In view of Lemma 3.17.3, it is sufficient to show that $\frac{\omega_1}{2}$, $\frac{\omega_2}{2}$, $\frac{\omega_3}{2}$, $\frac{\omega_4}{4}$, $\frac{\omega_5}{3}$ and $\frac{\omega_6}{2}$ are not extremal in $\overline{\Omega}$. Simple verification shows that

$$\begin{aligned}
\frac{\omega_1}{2} &= \frac{1}{2}(\omega_7) + \frac{1}{2}(s_{e_5-e_6}s_{e_5+e_6}\omega_7),\\
\frac{\omega_3}{3} &= \frac{2}{3}(\frac{\omega_2}{2}) + \frac{1}{3}(s_{e_5-e_6}s_{e_5+e_6}\omega_7),\\
\frac{\omega_4}{4} &= \frac{3}{4}(\frac{\omega_5}{3}) + \frac{1}{4}(s_{e_3-e_6}\omega_7),\\
\frac{\omega_5}{3} &= \frac{1}{3}(\omega_7) + \frac{2}{3}(\frac{\omega_6}{2}),\\
\frac{\omega_6}{2} &= \frac{1}{2}(\omega_7) + \frac{1}{2}(s_{e_5-e_6}\omega_7).
\end{aligned}$$

Finally, it is easy to show that $\frac{\omega_3}{2} + t(e_1 - e_2) \in \overline{\Omega}$ for $t \in \mathbb{R}$ small. □

Having this information, we can now handle all the exceptional non-compactly causal Lie algebras. We write E_6 and E_7 for simply connected complex Lie groups with complex exceptional Lie algebra $\mathfrak{e}_6$, resp. $\mathfrak{e}_7$. By $E_{6(*)}$ and $E_{7(*)}$ we denote the real forms of E_6 and E_7 with real forms $\mathfrak{e}_{6(*)}$, resp. $\mathfrak{e}_{7(*)}$.

The case of $E_{6(6)}$. Let$\mathfrak{g} = \mathfrak{e}_{6(6)}$. Note that $\mathfrak{g}$ is a normal real form of $\mathfrak{g}_{\mathbb{C}}$, i.e., $\mathfrak{a}$ is a Cartan algebra of $\mathfrak{g}$. In particular, Σ is of type E_6.

From Lemma 3.18 we obtain that

$$\partial_e \Omega = \mathcal{W}(\frac{\pi}{2}\omega_1) \amalg \mathcal{W}(\frac{\pi}{2}\omega_6) \ .$$

From the structure of Σ (in particular from the formula for the highest root) we obtain that

$$\mathrm{Spec}(\mathrm{ad}\,\frac{\pi}{2}\omega_1) = \mathrm{Spec}(\mathrm{ad}\,\frac{\pi}{2}\omega_6) = \{-\frac{\pi}{2}, 0, \frac{\pi}{2}\} \ .$$

Hence the prescriptions

$$\sigma_1(X) = \mathrm{Ad}(\exp(i\pi\omega_1)(X) \qquad \text{and} \qquad \sigma_2(X) = \mathrm{Ad}(\exp(i\pi\omega_6))(X)$$

define two involutions on $\mathfrak{g}$. Then $\tau_1 := \sigma_1 \circ \theta$ and $\tau_2 := \sigma_2 \circ \theta$ are involutions on $\mathfrak{g}$. Note that both involutions are conjugate under the outer automorphism κ.

Recall from the list in Remark 3.3(a) that there is up to conjugation a unique involution τ on $\mathfrak{g}$ which turns $(\mathfrak{g}, \tau)$ into a non-compactly causal symmetric Lie algebra. Thus Lemma 3.4, Theorem 3.5 and Theorem 3.6 imply that:

Proposition 3.20. *If $\mathfrak{g} = \mathfrak{e}_{6(6)}$, then*

$$\partial_d \Xi \simeq E_{6(6)}/E_{6(6)}^{\tau_1} \amalg E_{6(6)}/E_{6(6)}^{\tau_2}$$

with $\mathfrak{g}^{\tau_1} \simeq \mathfrak{g}^{\tau_2} \simeq \mathfrak{sp}(2,2)$.

The case of E_6. This case is completely analogous to the $e_{6(6)}$-case. One obtains that:

Proposition 3.21. *If $\mathfrak{g} = \mathfrak{e}_6$, then*

$$\partial_d \Xi \simeq E_6/E_{6(-14)} \amalg E_6/E_{6(-14)}.$$

The case of $E_{6(-26)}$. Let $\mathfrak{g} = \mathfrak{e}_{6(-26)}$. In this case Σ is of type A_2, and we can use our results from our discussions from the root systems A_n. In particular, we have

$$\partial_e \Omega = \mathcal{W}(Y_1) \amalg \mathcal{W}(Y_2) \ .$$

Note that Y_1 and Y_2 are conjugate under the outer isomorphism of the Dynkin diagram. Hence we obtain two conjugate involutions on $\mathfrak{g}$ by

$$\tau_j(X) = (\mathrm{Ad}(\exp(i2Y_j)) \circ \theta)(X)$$

for $j = 1, 2$. From Lemma 3.4, Theorem 3.5 and Theorem 3.6 we thus obtain that:

Proposition 3.22. *If $\mathfrak{g} = \mathfrak{e}_{6(-26)}$, then*

$$\partial_d \Xi \simeq E_{6(-26)}/E_{6(-26)}^{\tau_1} \amalg E_{6(-26)}/E_{6(-26)}^{\tau_2}$$

with $\mathfrak{g}^{\tau_1} \simeq \mathfrak{g}^{\tau_2} \simeq \mathfrak{f}_{4(-20)}$.

The case of $E_{7(7)}$. Let $\mathfrak{g} = \mathfrak{e}_{7(7)}$. Note that $\mathfrak{g}$ is a normal real form of $\mathfrak{g}_{\mathbb{C}}$, i.e., $\mathfrak{a}$ is a Cartan algebra of $\mathfrak{g}$. In particular, the restricted root system is of type E_7.

By Lemma 3.19 we have $\partial_e \Omega = \mathcal{W}(\frac{\pi}{2}\omega_7)$. As before, we conclude that the prescription

$$\tau(X) = (\mathrm{Ad}(\exp(i\pi\omega_7)) \circ \theta)(X)$$

defines an involution on $\mathfrak{g}$. Hence Lemma 3.4, Theorem 3.5 and the list in Remark 3.3(a) imply:

Proposition 3.23. *If $\mathfrak{g} = \mathfrak{e}_{7(7)}$, then*

$$\partial_d \Xi \simeq E_{7(7)}/E_{7(7)}^{\tau},$$

where $\mathfrak{g}^\tau \simeq \mathfrak{su}^*(8)$.

The case of E_7. Let $\mathfrak{g} = \mathfrak{e}_7$. This case is completely analogous to the $\mathfrak{e}_{7(7)}$-case, and one obtains:

Proposition 3.24. *If $\mathfrak{g} = \mathfrak{e}_7$, then*

$$\partial_d \Xi \simeq E_7/E_{7(-25)} \ .$$

Classification of distinguished boundaries. Taking all our results from the previous discussions together, we have proved the following classification result:

Theorem 3.25. *Let $\mathfrak{g}$ be a non-compactly causal Lie algebra. Asuume that G is a connected Lie group with Lie algebra $\mathfrak{g}$ sitting in a simply connected complexification $G_{\mathbb{C}}$.*

1. *If G is locally a classical group, then the situation is as follows:*
   (a) *If G is not a special orthogonal group, then*

$$\partial_d \Xi \simeq \coprod_{j=1}^{m} G/H_j,$$

where every G/H_j is a non-compactly causal symmetric space. With $\mathfrak{h}_j := \mathrm{Lie}(H_j)$ we have

$\mathfrak{g}$	$\coprod_{j=1}^{m} \mathfrak{h}_j$
$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{gl}(n,\mathbb{R})$
$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{sp}(n,\mathbb{R})$
$\mathfrak{sp}(n,n)$	$\mathfrak{sp}(n,\mathbb{C})$
$\mathfrak{su}(n,n)$	$\mathfrak{sl}(n,\mathbb{C}) \times \mathbb{R}$
$\mathfrak{so}^*(4n)$	$\mathfrak{sl}(n,\mathbb{H}) \times \mathbb{R}$
$\mathfrak{sl}(n,\mathbb{R})$	$\coprod_{q=1}^{n-1} \mathfrak{so}(q,n-q)$
$\mathfrak{sl}(n,\mathbb{C})$	$\coprod_{q=1}^{n-1} \mathfrak{su}(q,n-q)$
$\mathfrak{sl}(n,\mathbb{H})$	$\coprod_{q=1}^{n-1} \mathfrak{sp}(q,n-q)$

(b) *If G is locally $\mathrm{SO}(n,n)$ or $\mathrm{SO}(2n,\mathbb{C})$ for $n \geq 3$, then*

$$\partial_d \Xi \simeq G/H_1 \amalg G/H_2 \amalg G/H_2$$

with G/H_1 and G/H_2 non-compactly causal symmetric spaces and

$\mathfrak{g}$	$\mathfrak{h}_1$	$\mathfrak{h}_2$
$\mathfrak{so}(n,n)$	$\mathfrak{so}(n-1,1) \times \mathfrak{so}(n-1,1)$	$\mathfrak{so}(n,\mathbb{C})$
$\mathfrak{so}(2n,\mathbb{C})$	$\mathfrak{so}(2,2n-2)$	$\mathfrak{so}^*(2n)$

(c) *If G is locally $\mathrm{SO}(p,q)$ for $1 \leq p < q$ or $\mathrm{SO}(2n+1,\mathbb{C})$, then for $p, n \geq 3$ one has*

$$\partial_d \Xi \simeq G/H_1 \amalg G/H_2$$

with G/H_1 non-compactly causal and G/H_2 a homogeneous but non-symmetric space:

$\mathfrak{g}$	$\mathfrak{h}_1$	$\mathfrak{h}_2$
$\mathfrak{so}(p,q)$	$\mathfrak{so}(p-1,1) \times \mathfrak{so}(q-1,1)$	$\mathfrak{so}(p,\mathbb{C}) \times \mathfrak{so}(q-p)$
$\mathfrak{so}(2n+1,\mathbb{C})$	$\mathfrak{so}(2,2n-1)$	$\mathfrak{so}^*(2n)$

In the low-dimensional cases $p = 1, 2$ and $n = 2$ one has

$$\partial_d \Xi \simeq G/H$$

with G/H non-compactly causal and:

$\mathfrak{g}$	$\mathfrak{h}$
$\mathfrak{so}(1,q)$	$\mathfrak{so}(1,q-1)$
$\mathfrak{so}(2,q)$	$\mathfrak{so}(1,q-1) \times \mathbb{R}$
$\mathfrak{so}(5,\mathbb{C})$	$\mathfrak{so}(2,3)$

2. *For the exceptional cases with $\mathfrak{g}_\mathbb{C} \simeq \mathfrak{e}_6$ or $\mathfrak{g}_\mathbb{C} \simeq \mathfrak{e}_6 \oplus \mathfrak{e}_6$ we have $\partial_d \Xi \simeq G/H \amalg G/H$ with G/H non-compactly causal. If $\mathfrak{h} = \mathrm{Lie}(H)$, then*

$\mathfrak{g}$	$\mathfrak{h}$
$\mathfrak{e}_{6(6)}$	$\mathfrak{sp}(2,2)$
$\mathfrak{e}_{6(-26)}$	$\mathfrak{f}_{4(-20)}$
$\mathfrak{e}_6$	$\mathfrak{e}_{6(-14)}$

3. *For the exceptional cases with* $\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{e}_7$ *or* $\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{e}_7 \oplus \mathfrak{e}_7$ *the distinguished boundary* $\partial_d \Xi \simeq G/H$ *is connected. Further,* G/H *is non-compactly causal, and we have*

$\mathfrak{g}$	$\mathfrak{h}$
$\mathfrak{e}_{7(7)}$	$\mathfrak{su}^*(8)$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-26)} \times \mathbb{R}$
$\mathfrak{e}_7$	$\mathfrak{e}_{7(-25)}$

From the classification we obtain the following important result:

Theorem 3.26. *Let* $\mathfrak{g}$ *be a non-compactly causal Lie algebra and*

$$\partial_d \Xi \simeq \coprod_{j=1}^{n} G/H_j$$

the decomposition of $\partial_d \Xi$ *into* G*-orbits. Then the following assertions hold:*

1. *If* G/H *is a non-compactly causal symmetric space, then* G/H *appears locally as a component of* $\partial_d \Xi$.
2. *A boundary component* G/H_j *of* $\partial_d \Xi$ *is totally real if and only if* G/H_j *is symmetric.*

Remark 3.27. In the situation of Theorem 3.25 there is a boundary component of $\partial_d \Xi$ with complex dimensions if and only if G is locally $\mathrm{SO}(p,q)$ for $2 < p < q$ or $\mathrm{SO}(2n+1,\mathbb{C})$ for $n > 2$. In these cases we have $\partial_d \Xi = G/H_1 \amalg G/H_2$ with G/H_1 totally real and G/H_2 a component admitting complex submanifolds of $G_{\mathbb{C}}/K_{\mathbb{C}}$.

4. Relations to Jordan algebras the cases with A_n-type root systems

We first recall some standard facts of Jordan algebras and explain some results from [15].

Let V be a Euclidean Jordan algebra. For $x \in V$ we define $L(x) \in \mathrm{End}(V)$ by $L(x)y = xy$, $y \in V$. We denote by e the identity element of V. The symmetric cone $W \subseteq V$ associated to V can be defined as

$$W := \mathrm{int}\{x^2 : x \in V\},$$

where $\mathrm{int}(\cdot)$ denotes the interior of $(\cdot)$. We write $G := \mathrm{Aut}(W)_0$ for the connected component of the automorphism group of W which contains the identity. Recall that G is a reductive subgroup of $\mathrm{GL}(V)$. The isotropy group in e

$$K := G_e := \{g \in G : g(e) = e\}$$

is a maximal compact subgroup of G. The mapping

$$G/K \to W, \quad gK \mapsto g(e)$$

is a homeomorphism.

We write $V_{\mathbb{C}} = V + iV$ for the complexification of V and define the tube domain

$$S_W := V + iW \subseteq V_{\mathbb{C}} \ .$$

Let S denote the connected component of the complex automorphism group of S_W which contains the identity. Write $U := S_{ie}$ for the stabilizer of $ie \in S_W$. Then U is a maximal compact subgroup of S extending K, and the mapping

$$S/U \to S_W, \quad gU \mapsto g(ie)$$

is a biholomorphism of the Hermitian symmetric space S/U onto S_W. Note that $G/K \subseteq S/U$.

For the convenience of the reader we list here all irreducible Euclidean Jordan algebras and their associated groups G and S (cf. [4, p. 213]).

V	G	S
$\mathrm{Symm}(n,\mathbb{R})$	$\mathrm{GL}(n,\mathbb{R})_+$	$\mathrm{Sp}(n,\mathbb{R})$
$\mathrm{Herm}(n,\mathbb{C})$	$\mathrm{Sl}(n,\mathbb{C}) \times \mathbb{R}^+$	$\mathrm{SU}(n,n)$
$\mathrm{Herm}(n,\mathbb{H})$	$\mathrm{Sl}(n,\mathbb{H}) \times \mathbb{R}^+$	$\mathrm{SO}^*(4n)$
$\mathbb{R} \times \mathbb{R}^n$	$\mathrm{SO}(1,n) \times \mathbb{R}^+$	$\mathrm{SO}(2,n+1)$
$\mathrm{Herm}(3,\mathbb{O})$	$E_{6(-26)} \times \mathbb{R}^+$	$E_{7(-25)}$

Let $c_1, \ldots, c_n$ be a Jordan frame (cf. [4, p. 44]) of V and set $V^0 := \bigoplus_{j=1}^n \mathbb{R}c_j$. Recall that $K(V^0) = V$ (cf. [4, Cor. IV.2.7])

The choice $\mathfrak{a} := \bigoplus_{j=1}^n \mathbb{R}L(c_j)$ defines a maximal abelian hyperbolic subspace orthogonal to $\mathfrak{k}$. As the table shows, the root system $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ is classical and of type A_{n-1}. If we define $\varepsilon_j \in \mathfrak{a}^*$ by $\varepsilon_j(L(c_i)) = \delta_{ij}$, then we have

$$\Sigma = \{\frac{1}{2}(\varepsilon_i - \varepsilon_j) : i \neq j\}$$

(cf. [4, Prop. VI.3.3]). Thus

$$\Omega = \{x = \sum_{j=1}^n x_j L(c_j) : x_j \in \mathbb{R},\ |x_i - x_j| < \pi\}\ .$$

In particular, for

$$\Omega_0 := \bigoplus_{j=1}^n \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[L(c_j)$$

we have $\Omega_0 \subseteq \Omega$. Note that the domain

$$\Xi_0 := G\exp(i\Omega_0)K_\mathbb{C}/K_\mathbb{C} \subseteq G_\mathbb{C}/K_\mathbb{C}$$

is a G-invariant open subdomain of Ξ (cf. [15, Lemma 1.4]).

Theorem 4.1. *The mapping*

$$\Xi_0 \to S_W, \quad gK_\mathbb{C} \mapsto g(ie)$$

is a biholomorphism.

Proof. See [15, Th. 7.2.]. □

The distinguished boundary of Ξ_0. As in Section 2, we prove the following result:

Proposition 4.2. *The distinguished boundary of Ξ_0 in $G_\mathbb{C}/K_\mathbb{C}$ is given by*

$$\partial_d \Xi_0 = G\exp(i\partial_e \mathfrak{a}_1)K_\mathbb{C}/K_\mathbb{C}$$

with

$$\partial_e \Omega_0 = \{\frac{\pi}{2}(\pm L(c_1) \pm \ldots \pm L(c_n))\}\ .$$

Recall that the Weyl group $\mathcal{W}$ is isomorphic to S_n and acts as the full permutation group of $L(c_1), \ldots, L(c_n)$. Thus we immediately obtain that:

Lemma 4.3. *We have the disjoint union*

$$\partial_e \Omega_0 = \coprod_{p=0}^{n} \mathcal{W}(Y_p)$$

with

$$Y_p := \frac{\pi}{2}(L(c_1) + \ldots + L(c_p) - L(c_{p+1}) - \ldots - L(c_n)) \ .$$

Recall that $G_{\mathbb{C}}/K_{\mathbb{C}}$ sits canonically in $V_{\mathbb{C}}$ via the embedding

$$G_{\mathbb{C}}/K_{\mathbb{C}} \to V_{\mathbb{C}}, \quad gK_{\mathbb{C}} \mapsto g(ie) \ .$$

In the sequel we identify $G_{\mathbb{C}}/K_{\mathbb{C}}$ as a subset of $V_{\mathbb{C}}$.

Set $z_p := \exp(iY_p)K_{\mathbb{C}} \in V_{\mathbb{C}}$. Then we have:

Lemma 4.4. *For all* $0 \leq p \leq n$,

$$z_p = c_1 + \ldots + c_p - c_{p+1} - \ldots - c_n \ .$$

Write G_p for the stabilizer of z_p in G.

Example 4.5. We take $V = \mathrm{Symm}(n, \mathbb{R})$ with $G = \mathrm{GL}(n, \mathbb{R})_+$. The action of G is given by

$$G \times V \to V, \quad (g, X) \mapsto gXg^t \ .$$

Further, we have

$$z_p = \mathrm{diag}(1, \ldots 1, -1, \ldots, -1)$$

with $+1$ on the diagonal p times. Thus we see that

$$(\forall 0 \leq p \leq n) \qquad G_p = \mathrm{SO}(p, n-p) \ .$$

In particular, $G_0 = G_n = \mathrm{SO}(n, \mathbb{R})$ is compact and all other G_p are non-compact. Write V_p in V for the symmetric matrices with signature $(p, n-p)$, write V^{reg} for the invertible matrices in V, and set $V_p^{\mathrm{reg}} := V_p \cap V^{\mathrm{reg}}$. Then we have

$$V^{\mathrm{reg}} = \coprod_{p=0}^{n} V_p^{\mathrm{reg}}$$

and

$$V_p^{\mathrm{reg}} = G(z_p) \simeq G/G_p \ .$$

In particular, we see that $\partial_d \Xi_0$ is a Zariski open subset of V.

We are now going to generalize the results in Example 4.5 to arbitrary irreducible Euclidean Jordan algebras.

Let V be an irreducible Euclidean Jordan algebra. Write $\det(x)$ for the Jordan algebra determinant of V, and note that det is a polynomial function on V. Define the subset of regular elements of V by

$$V^{\mathrm{reg}} := \{x \in V : \det x \neq 0\}$$

and note that V^{reg} is a Zariski open subset of V.

For every $x \in V$ we define a real polynomial

$$f(\lambda, x) := \det(\lambda e - x) \ .$$

Note that for fixed x, the polynomial $f(\lambda, x)$ is completely reducible over $\mathbb{R}$. If x is regular, then it has degree n and all roots are non-zero. If $f(\lambda)$ is a polynomial, then we define its signature $\operatorname{sgn} f$ to be the number of its positive roots. For every $0 \leq p \leq n$ we now set

$$V_p := \{x \in V : \operatorname{sgn} f(\cdot, x) = p\} .$$

Note that

$$V^{\mathrm{reg}} = \coprod_{p=0}^{n} V_p^{\mathrm{reg}}$$

is a disjoint decomposition in cones. The only convex cones in this decomposition are $V_0^{\mathrm{reg}} = -W$ and $V_n^{\mathrm{reg}} = W$.

Observe that $z_p \in V_p$ for all $0 \leq p \leq n$.

Proposition 4.6. *For every $0 \leq p \leq n$ the set V_p^{reg} is G-invariant and G acts transitively on it. Hence the map*

$$G/G_p \to V_p^{\mathrm{reg}}, \quad gG_p \mapsto g(z_p)$$

is an isomorphism.

Proof. First we show that V_p^{reg} is invariant under K. For that, fix $x \in V_p^{\mathrm{reg}}$ and $k \in K$. Recall that $k(e) = e$ and that K acts on V by Jordan algebra automorphisms. We have

$$f(\lambda, k(x)) = \det(\lambda e - k(x)) = \det(k(\lambda(e) - x)) = \det(\lambda e - x),$$

where in the last equality we used the fact that $\det k(y) = \det y$ for all $y \in V$ and $k \in K$.

It is straightforward to check that

$$V_p^{\mathrm{reg}} \cap V^0 = A(z_p) .$$

Since V_p^{reg} is K-invariant and $G = KAK$, the assertions of the proposition now follow. □

Corollary 4.7. *The distinguished boundary $\partial_d \Xi_0$, realized in $V_{\mathbb{C}}$, is a Zariski open subset of V.*

Next we compute the isotropy groups G_p for $0 \leq p \leq n$.

Proposition 4.8. *For an irreducible Euclidean Jordan algebra V the isotropy groups G_p for $0 \leq p \leq n$ are given as follows:*

1. *For the classical matrix Jordan algebras one has:*

V	G	G_p
$\mathrm{Symm}(n, \mathbb{R})$	$\mathrm{GL}(n, \mathbb{R})_+$	$\mathrm{SO}(p, n-p)$
$\mathrm{Herm}(n, \mathbb{C})$	$\mathrm{Sl}(n, \mathbb{C}) \times \mathbb{R}^+$	$\mathrm{SU}(p, n-p)$
$\mathrm{Herm}(n, \mathbb{H})$	$\mathrm{Sl}(n, \mathbb{H}) \times \mathbb{R}^+$	$\mathrm{Sp}(p, n-p)$

2. *For $V = \mathbb{R} \times \mathbb{R}^n$ and $G = \mathrm{SO}(1, n) \times \mathbb{R}^+$ one has:*
   (a) *$G_p = \mathrm{SO}(n)$ for $p = 0, 2$.*
   (b) *$G_p = \mathrm{SO}(1, n-1)$ for $p = 1$.*
3. *For $V = \mathrm{Herm}(3, \mathbb{O})$ and $E_{6(-26)} \times \mathbb{R}^+$ one has:*
   (a) *$G_p = K = F_{4(-52)}$ for $p = 0, 3$.*
   (b) *$G_p \simeq F_{4(-20)}$ for $p = 1, 2$.*

Proof. 1. For all classical matrix algebras this is the same computation as in Example 4.5.

2. Straightforward calculation similar to the one in 1.

3. (1) is clear, and (2) is shown as in Proposition 3.22. □

Corollary 4.9. *The distinguished boundary $\partial_d \Xi_0$ is G-isomorphic to*

$$\partial_d \Xi_0 = \coprod_{p=0}^{n} G/G_p,$$

where

1. *G/G_p is a Riemannian symmetric space for $p = 0, n$, and*
2. *G/G_p is a non-compactly causal symmetric space for $p \neq 0, n$.*

Remark 4.10. (a) Comparing Corollary 4.9 to our results for the distinguished boundary of the bigger domain Ξ, we see that the distinguished boundary of Ξ_0 is the distinguished boundary of Ξ plus two copies of the Riemannian symmetric space. Also observe that every boundary component of $\partial_d \Xi_0$ is totally real.

(b) The observation (a) fits into the following general philosphy: Non-compactly causal symmetric Lie algebras $(\mathfrak{g}, \tau)$ are the class of symmetric Lie algebras which are closest to non-compactly Riemannian symmetric algebras $(\mathfrak{g}, \theta)$. In fact, these two classes make up the class of symmetric Lie algebras $(\mathfrak{g}, \tau)$, where $\mathfrak{q}$ admits a non-trivial open $\mathrm{Ad}(H)$-invariant hyperbolic convex set. These Lie algebras were the subject of systematic study in [13].

5. Further results on the special orthogonal groups

The examples discussed in Section 4 fit into the broader context of comparing Ξ with symmetric spaces S/U of Hermitian type which contain G/K as a totally real submanifold (cf. [15]).

In this section we will focus on the special orthogonal groups $G = \mathrm{SO}_e(p, q)$ and $G = \mathrm{SO}(n, \mathbb{C})$, which in a sense are the classical groups with the most complicated structure of Ξ (cf. Theorem 3.25).

Our choice of the maximal compact subgroup $K < G$ is as in Section 3. If $G = \mathrm{SO}_e(p, q)$, then we take $S = \mathrm{SU}(p, q)$ and $U = S(U(p) \times U(q))$, and if $G = \mathrm{SO}(n, \mathbb{C})$, then we choose $S = \mathrm{SO}^*(2n)$ and $U = U(n)$. The embedding

$$G/K \hookrightarrow S/U, \quad gK \mapsto gU \tag{5.1}$$

realizes G/K as a totally real submanifold of the symmetric space of Hermitian type S/U.

Let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal abelian subspace as in Section 3. Write $\widehat{\Sigma} := \Sigma(\mathfrak{a}, \mathfrak{s})$ for the doubly restricted root system of $\mathfrak{s}$ with respect to $\mathfrak{a}$. Notice that $\widehat{\Sigma}$ is of type C_n if $\mathfrak{g} = \mathfrak{so}(n, n)$ or $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$, and of type BC_n otherwise. In the notation of Section 3 we have

$$\widehat{\Sigma} = \begin{cases} \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i, j \leq n\} \backslash \{0\} & \text{for } \mathfrak{g} = \mathfrak{so}(n, n), \mathfrak{so}(2n, \mathbb{C}), \\ \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i, j \leq p\} \backslash \{0\} \amalg \{\pm\varepsilon_i : 1 \leq i \leq p\} & \\ \qquad \text{for } \mathfrak{g} = \mathfrak{so}(p, q), \mathfrak{so}(2p+1, \mathbb{C}), \ (p < q). & \end{cases}$$

Define $\Omega_0 := \{X \in \mathfrak{a} : (\forall \alpha \in \widehat{\Sigma}) \ |\alpha(X)| < \frac{\pi}{2}\}$. Note that $\Omega_0 \subseteq \Omega$, since $\widehat{\Sigma} \supseteq \Sigma$. Define

$$\Xi_0 := G \exp(i\Omega_0)K_{\mathbb{C}}/K_{\mathbb{C}}$$

and notice that Ξ_0 is a G-invariant open subdomain of Ξ. Finally, define the distinguished boundary of Ξ_0 by $\partial_d \Xi_0 := G \exp(i\partial_e \Omega_0)K_{\mathbb{C}}/K_{\mathbb{C}}$.

Write $\mathfrak{u}$ for the Lie algebra of U and $\mathfrak{s}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{u}_{\mathbb{C}} \oplus \mathfrak{p}^-$ for the Harish-Chandra decomposition of $\mathfrak{s}_{\mathbb{C}}$. Let $\mathcal{D} \subseteq \mathfrak{p}^+$ be the Harish-Chandra realization of S/U as a bounded symmetric domain. The embedding in (5.1) extends to an embedding

$$G_{\mathbb{C}}/K_{\mathbb{C}} \hookrightarrow S_{\mathbb{C}}/U_{\mathbb{C}}P^-, \tag{5.2}$$

and it follows from [15, Th. 7.5] that the image of Ξ_0 under (5.2) is precisely $S/U \simeq \mathcal{D}$, i.e., we have a G-equivariant biholomorphism

$$\Xi_0 \simeq S/U \ .$$

Theorem 5.1. *Let $G = \mathrm{SO}_e(p,q)$ or $G = \mathrm{SO}(n,\mathbb{C})$. Then the distinguished boundary $\partial_d \Xi_0$ is given as follows:*

1. *If $G = \mathrm{SO}_e(n,n)$ or $G = \mathrm{SO}(2n,\mathbb{C})$, then $\partial_d \Xi_0 = G/H \amalg G/H$, with $\mathfrak{h} := \mathrm{Lie}(H)$ given by*

$$\mathfrak{h} = \begin{cases} \mathfrak{so}(n,\mathbb{C}) & \text{for } \mathfrak{g} = \mathfrak{so}(n,n), \\ \mathfrak{so}^*(2n) & \text{for } \mathfrak{g} = \mathfrak{so}(2n,\mathbb{C}). \end{cases}$$

2. *If $G = \mathrm{SO}_e(p,q)$ for $p < q$ or $G = \mathrm{SO}(2n+1,\mathbb{C})$, then $\partial_d \Xi_0 = G/H$, with $\mathfrak{h} := \mathrm{Lie}(H)$ given by*

$$\mathfrak{h} = \begin{cases} \mathfrak{so}(p,\mathbb{C}) \oplus \mathfrak{so}(q-p) & \text{for } \mathfrak{g} = \mathfrak{so}(p,q), \\ \mathfrak{so}^*(2n) & \text{for } \mathfrak{g} = \mathfrak{so}(2n+1,\mathbb{C}). \end{cases}$$

Proof. 1. Keep the notation of Lemma 3.12. Then it follows from the structure of $\widehat{\Sigma}$ that

$$\partial_e \Omega_0 = \mathcal{W}(Y_2) \amalg \mathcal{W}(Y_3) \ .$$

Now the assertion follows from the computations before Proposition 3.13 and Proposition 3.15.

2. This is analogous to 1. □

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