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THE DYNAMICS OF EXPANSIVE INVERTIBLE ONESIDED CELLULAR AUTOMATA

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Abstract. Using textile systems, we prove the conjecture of Boyle and Maass that the dynamical system defined by an expansive invertible onesided cellular automaton is topologically conjugate to a topological Markov shift. We also study expansive leftmost-permutive onesided cellular automata and bipermutive endomorphisms of mixing topological Markov shifts.

1. Introduction

Let A be the alphabet of N symbols. Let $A^{\mathbf{N}}$ be endowed with a metric compatible with the product topology of the discrete topology on A. Let $k \in \mathbf{N}$. Let $f : A^{k+1} \to A$ be a mapping. Let $\tilde{\varphi}_f : A^{\mathbf{N}} \to A^{\mathbf{N}}$ be the mapping defined as follows: for $(a_i)_{i\in\mathbf{N}} \in A^{\mathbf{N}}, a_i \in A$,

$$\tilde{\varphi}_f((a_i)_{i\in\mathbf{N}}) = (f(a_i \dots a_{i+k}))_{i\in\mathbf{N}}.$$

The map $\tilde{\varphi}_f$ is the *onesided cellular automaton map* defined by a *local rule* f. In [BM], Boyle and Maass have proved the following theorem.

Theorem 1.1 (Boyle and Maass [BM]). *If the dynamical system $(A^{\mathbf{N}}, \tilde{\varphi}_f)$ is topologically conjugate to a sofic system, then it is topologically conjugate to a topological Markov shift which is shift equivalent to some full J-shift, where J and N are divisible by the same primes.*

They have also given the following conjecture.

Conjecture 1.2 (Boyle and Maass [BM]). *If $\tilde{\varphi}_f$ is an expansive homeomorphism, then the dynamical system $(A^{\mathbf{N}}, \tilde{\varphi}_f)$ is topologically conjugate to a topological Markov shift.*

In this paper, we shall establish a property of a certain class of textile systems (Proposition 3.5), and using it we shall prove

Theorem 1.3. *The conjecture of Boyle and Maass is true.*

Boyle and Maass have also given two more conjectures in [BM]; one combined with Theorem 1.3 is such that if $\tilde{\varphi}_f$ is an expansive homeomorphism, then $(A^{\mathbf{N}}, \tilde{\varphi}_f)$ is topologically conjugate to a full shift, and the other combined with Theorem 1.3,

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in particular, implies that under the same hypothesis, N is divisible by p^2 for every prime p dividing N. We cannot prove these.

A local rule $f : A^{k+1} \to A$ is said to be *leftmost-permutive* if for any distinct $a, a' \in A$ and any $w \in A^k$, $f(aw)$ and $f(a'w)$ are different.

It is clear that if the onesided cellular automaton map $\tilde{\varphi_f}$ defined by a local rule f is injective, then f must be leftmost-permutive. We see that every expansive leftmost-permutive onesided cellular automaton map is necessarily accompanied by expansive invertible onesided cellular automaton maps with the "same" dynamics as it has (Proposition 5.4), so that Theorem 1.1 together with Theorem 1.3 is generalized to expansive leftmost-permutive onesided cellular automata (Theorem 5.5). We also add the result that if a mixing topological Markov shift has a noninvertible bipermutive endomorphism, then the shift must be topologically conjugate to some full shift (Proposition 6.1), which shows that unknown examples mentioned in [N2, p. 52] actually do not exist.

In [N4], the following question was asked. If τ is a positively expansive continuous map of a compact metric space X onto itself and has the pseudo-orbit tracing property and if φ is an expansive homeomorphism with $\varphi\tau = \tau\varphi$, then does φ have the pseudo-orbit tracing property? In view of Theorem 3.12(2) of [N2] (a onesided topological Markov shift having an expansive automorphism every power of which is topologically transitive is topologically conjugate to some onesided full shift), Theorem 1.3 answers the question for the case where X is 0-dimensional in the affirmative under the condition that φ^n is topologically transitive for all $n \in \mathbf{N}$.

The reader is referred to [BM] for more information on the subject of this paper, to [Ki] or [LM] for a comprehensive introduction to symbolic dynamics, and to [AH] for information on topological dynamics.

2. Preliminaries

We follow the notation and terminology of [N2].

Let A be an alphabet (i.e., a finite nonempty set of symbols). Let $A^{\mathbf{Z}}$ be endowed with the metric d such that for $x = (a_j)_{j\in\mathbf{Z}}$ and $y = (b_j)_{j\in\mathbf{Z}}$ with $a_j, b_j \in A$, $d(x, y) = 0$ if $x = y$, and otherwise, $d(x, y) = 1/(1 + k)$, where $k = \min\{|j| \mid a_j \neq b_j\}$. The metric d is compatible with the product topology of the discrete topology on A. We also endow $A^{\mathbf{N}}$ with a similar metric.

Let $\sigma_A : A^{\mathbf{Z}} \to A^{\mathbf{Z}}$ be defined by $\sigma_A((a_j)_{j\in\mathbf{Z}}) = (a_{j+1})_{j\in\mathbf{Z}}$. The dynamical system $(A^{\mathbf{Z}}, \sigma_A)$ is called the *full shift* over A or the *full N-shift* if the cardinality of A is N. For a closed subset X of $A^{\mathbf{Z}}$ with $\sigma(X) = X$, the dynamical system (X, σ_X) is called a *subshift* over A, where σ_X is the restriction of σ_A on X.

Let $\tilde{\sigma}_A : A^{\mathbf{N}} \to A^{\mathbf{N}}$ be defined by $\tilde{\sigma}_A((a_i)_{i\in\mathbf{N}}) = (a_{i+1})_{i\in\mathbf{N}}$. The dynamical system $(A^{\mathbf{N}}, \tilde{\sigma}_A)$ is called the *onesided full shift* over A. For a closed subset $\tilde{X}$ of $A^{\mathbf{N}}$ with $\tilde{\sigma}(\tilde{X}) = \tilde{X}$, the dynamical system $(\tilde{X}, \tilde{\sigma}_{\tilde{X}})$ is called a *onesided subshift* over A, where $\tilde{\sigma}_{\tilde{X}}$ is the restriction of $\tilde{\sigma}_A$ on $\tilde{X}$.

Let G be a (directed) graph which may have multiple (directed) arcs and loops. It is represented by

$$V_G \xleftarrow{i_G} A_G \xrightarrow{t_G} V_G,$$

where A_G and V_G are the arc-set and the vertex-set, i_G and t_G are the mappings such that for each $a \in A_G$, $i_G(a)$ and $t_G(a)$ are the initial and terminal vertices of

a, respectively. Let

$$X_G = \{(a_j)_{j\in\mathbf{Z}} \mid \forall j,\, a_j \in A_G,\, t_G(a_j) = i_G(a_{j+1})\}$$

and let

$$\tilde{X}_G = \{(a_j)_{j\in\mathbf{N}} \mid \exists (a_j)_{j\in\mathbf{Z}} \in X_G\}.$$

Then the subshift (X_G, σ_G) is called a *topological Markov shift* defined by G. We call the onesided subshift $(\tilde{X}_G, \tilde{\sigma}_G) = (\tilde{X}_G, \tilde{\sigma}_{\tilde{X}_G})$ the *onesided topological Markov shift* defined by G.

Let $n \geq 1$. Let $L_n(G)$ denote the set of all *path*s of length n in G, that is,

$$L_n(G) = \{a_1 \dots a_n \mid a_j \in A_G,\ t_G(a_j) = i_G(a_{j+1}) \text{ for } j = 1, \dots, n-1\}.$$

For $n \geq 2$, let $G^{[n]}$ be the graph defined as follows: $A_{G^{[n]}} = L_n(G), V_{G^{[n]}} = L_{n-1}(G)$, and let $i_{G^{[n]}}$ and $t_{G^{[n]}}$ be the mappings such that for $w = a_1 \dots a_n \in A_{G^{[n]}}, a_j \in A_G, i_{G^{[n]}}(w) = a_1 \dots a_{n-1}$ and $t_{G^{[n]}}(w) = a_2 \dots a_n$. We define $G^{[1]} = G$. We call $G^{[n]}$ the *higher block graph of order n* of G.

A λ*-graph* $\mathcal{G}$ over an alphabet A is a pair (G, λ) of a graph G, which is called the *support* of $\mathcal{G}$, and an onto mapping $\lambda : A_G \to A$. For a λ-graph $\mathcal{G} = (G, \lambda)$, let

$$X_{\mathcal{G}} = \{(\lambda(a_j))_{j\in\mathbf{Z}} \mid (a_j)_{j\in\mathbf{Z}} \in X_G\}.$$

Then we have a subshift $(X_{\mathcal{G}}, \sigma_{\mathcal{G}})$, which is the *sofic system* defined by $\mathcal{G}$. We also have the 1-block map $\pi : X_G \to X_{\mathcal{G}}$ defined by $(a_j)_{j\in\mathbf{Z}} \mapsto (\lambda(a_j))_{j\in\mathbf{Z}}$, which is called the *sofic cover* defined by $\mathcal{G}$.

Let $n \geq 1$. For a λ-graph $\mathcal{G} = (G, \lambda)$, the *higher block λ-graph of order n* is defined as $\mathcal{G}^{[n]} = (G^{[n]}, \lambda^{[n]})$, where $\lambda^{[n]}$ is the onto mapping such that $\lambda^{[n]}(a_1 \dots a_n) = \lambda(a_1) \dots \lambda(a_n)$ for $a_1 \dots a_n \in A_{G^{[n]}}, a_j \in A_G$.

The following is a key lemma for our proof of Theorem 1.3, which is an easy consequence of well-known fundamental results in symbolic dynamics.

Lemma 2.1. *Let $\mathcal{G}_i = (G, \lambda_i), i = 1, 2$, be two λ-graphs over an alphabet A with the same irreducible support G. Let $\pi_i : X_G \to X_{\mathcal{G}_i}$ be the sofic cover defined by $\mathcal{G}_i$ for $i = 1, 2$. If both π_1 and π_2 are bounded-to-one and $\pi_1(X_G) \subset \pi_2(X_G)$, then $\pi_1(X_G) = \pi_2(X_G)$.*

Proof. Assume that $\pi_1(X_G) \subsetneq \pi_2(X_G)$. Then there is a word w over A which appears on a point (bisequence) of $\pi_2(X_G) = X_{\mathcal{G}_2}$ but does not appear on any point of $\pi_1(X_G) = X_{\mathcal{G}_1}$. Let l be the length of w . Consider the higher block λ-graphs $\mathcal{G}_i^{[l]} = (G^{[l]}, \lambda_i^{[l]})$ of order l of $\mathcal{G}_i = (G, \lambda_i)$ for $i = 1, 2$. Let $\mathcal{H} = (H, \lambda_{\mathcal{H}})$ be the λ-graph obtained from $\mathcal{G}_2^{[l]}$ by deleting all arcs having w as their $\lambda_2^{[l]}$-labels. Then we have

$$h(\sigma_H) < h(\sigma_{G^{[l]}}) = h(\sigma_G) \tag{2.1}$$

and

$$X_{\mathcal{G}_1^{[l]}} \subset X_{\mathcal{H}} \subset X_{\mathcal{G}_2^{[l]}}, \tag{2.2}$$

where h denotes topological entropy. By (2.2), we have

$$h(\sigma_{\mathcal{G}_1}) = h(\sigma_{\mathcal{G}_1^{[l]}}) \leq h(\sigma_{\mathcal{H}}) \leq h(\sigma_{\mathcal{G}_2}^{[l]}) = h(\sigma_{\mathcal{G}_2}).$$

Since π_1 and π_2 are bounded-to-one sofic covers, it follows that $h(\sigma_{\mathcal{G}_i}) = h(\sigma_G)$ for $i = 1, 2$ and $h(\sigma_{\mathcal{H}}) = h(\sigma_H)$. Hence we have $h(\sigma_G) = h(\sigma_H)$, which contradicts (2.1). □

Now we recall the notion of a textile system introduced in [N2]. A *graph-homomorphism* p of a graph

$$\Gamma : \quad V_\Gamma \xleftarrow{\ i_\Gamma\ } A_\Gamma \xrightarrow{\ t_\Gamma\ } V_\Gamma$$

into a graph

$$G : \quad V_G \xleftarrow{\ i_G\ } A_G \xrightarrow{\ t_G\ } V_G$$

is defined to be a pair of mappings $p_A : A_\Gamma \to A_G$ and $p_V : V_\Gamma \to V_G$ such that the diagram

$$\begin{array}{ccccc} V_G & \xleftarrow{\ i_G\ } & A_G & \xrightarrow{\ t_G\ } & V_G \\ \uparrow{\scriptstyle p_V} & & {\scriptstyle p_A}\uparrow & & {\scriptstyle p_V}\uparrow \\ V_\Gamma & \xleftarrow{\ i_\Gamma\ } & A_\Gamma & \xrightarrow{\ t_\Gamma\ } & V_\Gamma \end{array}$$

commutes. We write $p : \Gamma \to G$ to denote it. We will often use p to denote p_A.

A *textile system T over* a graph G is defined to be an ordered pair of graph-homomorphisms $p : \Gamma \to G$ and $q : \Gamma \to G$ such that each $\alpha \in A_\Gamma$ is uniquely determined by the quadruple $(i_\Gamma(\alpha), t_\Gamma(\alpha), p_A(\alpha), q_A(\alpha))$. We write

$$T = (p, q : \Gamma \to G).$$

We have the following commutative diagram.

$$\begin{array}{ccccc} V_G & \xleftarrow{\ i_G\ } & A_G & \xrightarrow{\ t_G\ } & V_G \\ \uparrow{\scriptstyle p_V} & & {\scriptstyle p_A}\uparrow & & {\scriptstyle p_V}\uparrow \\ V_\Gamma & \xleftarrow{\ i_\Gamma\ } & A_\Gamma & \xrightarrow{\ t_\Gamma\ } & V_\Gamma \\ \downarrow{\scriptstyle q_V} & & {\scriptstyle q_A}\downarrow & & {\scriptstyle q_V}\downarrow \\ V_G & \xleftarrow{\ i_G\ } & A_G & \xrightarrow{\ t_G\ } & V_G \end{array}$$

If we observe this diagram vertically, then we have the ordered pair of graph homomorphisms

$$\begin{array}{ccc} V_G & \xleftarrow{\ i_G\ } & A_G \\ \uparrow{\scriptstyle p_V} & & {\scriptstyle p_A}\uparrow \\ V_\Gamma & \xleftarrow{\ i_\Gamma\ } & A_\Gamma \\ \downarrow{\scriptstyle q_V} & & {\scriptstyle q_A}\downarrow \\ V_G & \xleftarrow{\ i_G\ } & A_G \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A_G & \xrightarrow{\ t_G\ } & V_G \\ {\scriptstyle p_A}\uparrow & & {\scriptstyle p_V}\uparrow \\ A_\Gamma & \xrightarrow{\ t_\Gamma\ } & V_\Gamma \\ {\scriptstyle q_A}\downarrow & & {\scriptstyle q_V}\downarrow \\ A_G & \xrightarrow{\ t_G\ } & V_G \end{array}$$

of the graph

$$\begin{array}{c} A_G \\ {\scriptstyle p_A}\big\uparrow \\ A_\Gamma \\ {\scriptstyle q_A}\big\downarrow \\ A_G \end{array}$$

into the graph

$$\begin{array}{c} V_G \\ {\scriptstyle p_V}\big\uparrow \\ V_\Gamma \\ {\scriptstyle q_V}\big\downarrow \\ V_G \end{array}$$

which gives the definition of the *dual* T^* of the textile system T.

Let $T = (p, q : \Gamma \to G)$ be a textile system. Let $\xi : X_\Gamma \to X_G$ and $\eta : X_\Gamma \to X_G$ be the *1-block maps* given by p and q, respectively, that is,

$$\xi((\alpha_j)_{j\in\mathbf{Z}}) = (p(\alpha_j))_{j\in\mathbf{Z}}, \quad \eta((\alpha_j)_{j\in\mathbf{Z}}) = (q(\alpha_j))_{j\in\mathbf{Z}}, \qquad (\alpha_j)_{j\in\mathbf{Z}} \in X_\Gamma,\ \alpha_j \in A_\Gamma.$$

We say that T is *nondegenerate* if both ξ and η are onto. We also say that T is *upwardly nondegenerate* if η is onto.

Assume that T is upwardly nondegenerate. Let X_n and $Z_n, n \geq 0$, be defined as follows:

$$\begin{aligned} X_0 &= X_G, & Z_0 &= X_\Gamma, \\ X_n &= \xi(Z_{n-1}), & Z_n &= \eta^{-1}(X_n) \qquad n = 1, 2 \dots. \end{aligned}$$

Then by induction on n, we have

$$X_n \supset X_{n+1}, \quad Z_n \supset Z_{n+1} \qquad n = 0, 1, \dots.$$

Define

$$X_T = \bigcap_{n=0}^{\infty} X_n \quad \text{and} \quad Z_T = \bigcap_{n=0}^{\infty} Z_n.$$

We call the subshift (X_T, σ_T) the *woof shift* of T. The woof shift of the dual T^* of T is called the *warp shift* of T. Let $\xi_T : Z_T \to X_T$ and $\eta_T : Z_T \to X_T$ be the restrictions of ξ and η, respectively. We say that T is *onesided 1-1* if ξ_T is 1-1. If T is onesided 1-1, then we define an endomorphism φ_T of the woof shift (X_T, σ_T) by $\varphi_T = \eta_T \xi_T^{-1}$.

Let $n \geq 1$. We call the subshift (X_n, σ_n) the *woof shift in height* n of T. Let $\xi_n : Z_{n-1} \to X_n$ be defined by the restriction of ξ. We say that T is *onesided 1-1 in height* n if ξ_n is 1-1. We say that T is *onesided bounded-to-one in height* n if ξ_n is bounded-to-one. We also say that T is *saturated in height* n if $X_n = X_T$.

Further we say that T is *finitely onesided 1-1* if there is $n \geq 1$ such that ξ_n is 1-1. We also say that T is *finitely saturated* if there is $n \geq 0$ such that $X_n = X_T$.

For $n \in \mathbf{N}$, a 2-dimensional configuration $(\alpha_{i,j})_{1\le i\le n, j\in\mathbf{Z}}$ of arcs $\alpha_{i,j} \in A_\Gamma$ is called an *obi* of width n woven by T if $(\alpha_{i,j})_{j\in\mathbf{Z}} \in X_\Gamma$ for all $1 \le i \le n$ and $\eta((\alpha_{i,j})_{j\in\mathbf{Z}}) = \xi((\alpha_{i+1,j})_{j\in\mathbf{Z}})$ for all $1 \le i \le n-1$. The definition of a *half-textile* $(\alpha_{i,j})_{i\in\mathbf{N}, j\in\mathbf{Z}}$ woven by T should be clear.

We define $\tilde{X}_T = \{(a_j)_{j\in\mathbf{N}} \mid \exists(a_j)_{j\in\mathbf{Z}} \in X_T\}$ and $\tilde{Z}_T = \{(\alpha_j)_{j\in\mathbf{N}} \mid \exists(\alpha_j)_{j\in\mathbf{Z}} \in Z_T\}$, and let $\tilde{\xi}_T : \tilde{Z}_T \to \tilde{X}_T$ and $\tilde{\eta}_T : \tilde{Z}_T \to \tilde{X}_T$ be defined by $\tilde{\xi}_T((\alpha_j)_{j\in\mathbf{N}}) = (p(\alpha_j))_{j\in\mathbf{N}}$ and $\tilde{\eta}_T((\alpha_j)_{j\in\mathbf{N}}) = (q(\alpha_j))_{j\in\mathbf{N}}$. If $\tilde{\xi}_T$ is 1-1, then we have an endomorphism $\tilde{\varphi}_T$ of the *onesided woof shift* $(\tilde{X}_T, \tilde{\sigma}_T) = (\tilde{X}_T, \tilde{\sigma}_{\tilde{X}_T})$ by $\tilde{\varphi}_T = \tilde{\eta}_T\tilde{\xi}_T^{-1}$.

3. Onesided 1-1 q-biresolving textile systems

Proposition 3.1. *Let T be an upwardly nondegenerate textile system. Then T is onesided 1-1 if and only if T is finitely onesided 1-1.*

Proof. We follow the notation of Section 2. We first note that for $z \in X_\Gamma$ and $n \ge 1$, $z \in Z_{n-1}$ if and only if z is the uppermost sub-obi $(\alpha_{1,j})_{j\in\mathbf{Z}}$ of width 1 of some obi $(\alpha_{i,j})_{1\le i\le n, j\in\mathbf{Z}}$ of width n woven by T, and $z \in Z_T$ if and only if z is the uppermost sub-obi $(\alpha_{1,j})_{j\in\mathbf{Z}}$ of width 1 of some half textile $(\alpha_{i,j})_{i\in\mathbf{N}, j\in\mathbf{Z}}$ woven by T.

By definition the "if" part holds. To prove the converse, suppose that T is not finitely onesided 1-1. Then for each $n \in \mathbf{N}$, ξ_n is not 1-1. Hence for each $n \in \mathbf{N}$, there exists a pair of obis $(\alpha^{(n)}_{i,j})_{1\le i\le n, j\in\mathbf{Z}}$ and $(\beta^{(n)}_{i,j})_{1\le i\le n, j\in\mathbf{Z}}$ of width n woven by T such that

$$\xi((\alpha^{(n)}_{1,j})_{j\in\mathbf{Z}}) = \xi((\beta^{(n)}_{1,j})_{j\in\mathbf{Z}}) \quad \text{and} \quad \alpha^{(n)}_{1,0} \ne \beta^{(n)}_{1,0}.$$

Using a standard compactness argument on the product space $(A_\Gamma^{\mathbf{Z}})^{\mathbf{N}} \times (A_\Gamma^{\mathbf{Z}})^{\mathbf{N}}$, we see that there exists a pair of half-textiles $(\alpha_{i,j})_{i\in\mathbf{N}, j\in\mathbf{Z}}$ and $(\beta_{i,j})_{i\in\mathbf{N}, j\in\mathbf{Z}}$ woven by T with $\xi((\alpha_{1,j})_{j\in\mathbf{Z}}) = \xi((\beta_{1,j})_{j\in\mathbf{Z}})$ and $\alpha_{1,0} \ne \beta_{1,0}$. This implies that ξ_T is not 1-1. □

A graph G is said to be *nondegenerate* if both i_G and t_G are onto. A graph-homomorphism $h : \Gamma \to G$ is said to be *onto* if both h_A and h_V are onto.

An onto graph-homomorphism $q : \Gamma \to G$ between nondegenerate graphs is said to be *right resolving* if for each $u \in V_\Gamma$, the restriction of q_A on $i_\Gamma^{-1}(\{u\})$ is a bijection onto $i_G^{-1}(\{q_V(u)\})$. It is said to be *left resolving* if for each $u \in V_\Gamma$, the restriction of q_A on $t_\Gamma^{-1}(\{u\})$ is a bijection onto $t_G^{-1}(\{q_V(u)\})$. It is said to be *biresolving* if it is both right resolving and left resolving. If $q : \Gamma \to G$ is a right resolving or left resolving graph-homomorphism, then the 1-block map $\eta : X_\Gamma \to X_G$ given by q is onto.

Proposition 3.2. *Let $T = (p, q : \Gamma \to G)$ be a textile system with q biresolving. If T is onesided bounded-to-one in height $n+1$ with $n \ge 0$ and the woof shift in height n of T is topologically transitive, then T is saturated in height n.*

Proof. We follow the notation of Section 2.

Since the woof shift (X_n, σ_n) in height n of T is a topologically transitive sofic system (see [N2, pp. 20–21] or the paragraph preceding Proposition 3.3 below), there exists a λ-graph $\mathcal{D} = (D, \lambda_{\mathcal{D}})$ such that D is an irreducible graph, the sofic system $(X_{\mathcal{D}}, \sigma_{\mathcal{D}})$ defined by $\mathcal{D}$ is equal to (X_n, σ_n), and the sofic cover $\pi_{\mathcal{D}}$ of $\mathcal{D}$ is bounded-to-one. In fact, we can take the right or left Fischer λ-graph of (X_n, σ_n) as $\mathcal{D}$ [F].

We define a textile system $\hat{T} = (\hat{p}, \hat{q} : \hat{\Gamma} \to G)$ using $\mathcal{D} = (D, \lambda_{\mathcal{D}})$ and $T = (p, q : \Gamma \to G)$ by

$$A_{\hat{\Gamma}} = \{(\delta, \alpha) \in A_D \times A_\Gamma \mid \lambda_{\mathcal{D}}(\delta) = q(\alpha)\}$$
$$V_{\hat{\Gamma}} = \{(i_D(\delta), i_\Gamma(\alpha)) \mid (\delta, \alpha) \in A_{\hat{\Gamma}}\}$$

and

$$i_{\hat{\Gamma}}((\delta, \alpha)) = (i_D(\delta), i_\Gamma(\alpha)) \qquad t_{\hat{\Gamma}}((\delta, \alpha)) = (t_D(\delta), t_\Gamma(\alpha))$$
$$\hat{p}((\delta, \alpha)) = p(\alpha) \qquad \hat{q}((\delta, \alpha)) = q(\alpha) \qquad \text{for } (\delta, \alpha) \in A_{\hat{\Gamma}}.$$

Claim. $\hat{\Gamma}$ is the disjoint union of irreducible graphs.

Since q is biresolving, each $b \in A_G$ induces the bijection $Q_b : q_V^{-1}(\{i_G(b)\}) \to q_V^{-1}(\{t_G(b)\})$ such that $Q_b(a) = t_\Gamma(\alpha)$ for $a \in q_V^{-1}(\{i_G(b)\})$, where α is the unique arc in Γ with $i_\Gamma(\alpha) = a$ and $q(\alpha) = b$. For each $u \in V_D$, let

$$W_u = \{(u, a) \in V_{\hat{\Gamma}} \mid a \in V_\Gamma\}.$$

Let $\delta \in A_D$. If $u = i_D(\delta), v = t_D(\delta)$ and $\lambda_{\mathcal{D}}(\delta) = b$, then we can define a bijection $J_\delta : W_u \to W_v$ by

$$J_\delta((u, a)) = (v, Q_b(a)).$$

To prove the claim, it suffices to show that if $(\delta_1, \alpha_1) \dots (\delta_l, \alpha_l)$ is a path in $\hat{\Gamma}$ with $\delta_i \in A_D$ and $\alpha_i \in A_\Gamma$, then there exists a path going from vertex $(t_D(\delta_l), t_\Gamma(\alpha_l))$ to vertex $(i_D(\delta_1), i_\Gamma(\alpha_1))$ in $\hat{\Gamma}$. Put $i_D(\delta_1) = u$ and put $t_D(\delta_l) = v$. Then $\delta_1 \dots \delta_l$ is a path going from u to v in D. Since D is irreducible, there exists a path $\delta'_1 \dots \delta'_m$ going from v to u in D. Then $\delta_1 \dots \delta_l \delta'_1 \dots \delta'_m$ is a cycle in D, i.e., a path going from u to itself. Let

$$J = J_{\delta'_m} \dots J_{\delta'_1} J_{\delta_l} \dots J_{\delta_1}.$$

Then J is a bijection of W_u onto itself. There exists $r \in \mathbf{N}$ such that J^r is the identity map of W_u. If $J' = J^{r-1} J_{\delta'_m} \dots J_{\delta'_1}$, then J' is a bijection of W_v onto W_u with $J'(t_\Gamma(\alpha_l)) = i_\Gamma(\alpha_1)$, which implies that there exists a path going from vertex $(t_D(\delta_l), t_\Gamma(\alpha_l))$ to vertex $(i_D(\delta_1), i_\Gamma(\alpha_1))$ in $\hat{\Gamma}$. Hence the claim is proved.

Let $\check{\Gamma}$ be any one of the irreducible components of $\hat{\Gamma}$. Let $\check{p}$ and $\check{q}$ be the restrictions of $\hat{p}$ and $\hat{q}$, respectively, on $A_{\check{\Gamma}}$. Let $\check{\xi} : X_{\check{\Gamma}} \to X_G$ and $\check{\eta} : X_{\check{\Gamma}} \to X_G$ be the 1-block maps given by $\check{p}$ and $\check{q}$, respectively.

Let $\delta_1 \dots \delta_l$ be any path in D with $\delta_i \in A_D$. Then there exists a path of the form $(\delta_1, \alpha_1) \dots (\delta_l, \alpha_l)$ with $\alpha_i \in A_\Gamma$ in $\check{\Gamma}$. For let $(u, a) \in V_{\check{\Gamma}}$ with $u \in V_D$ and $a \in V_\Gamma$. Since D is an irreducible graph, there exists a path $\delta'_1 \dots \delta'_m$ going from u to $i_D(\delta_1)$ in D. It follows that $q_V(a) = i_G(\lambda_{\mathcal{D}}(\delta'_1))$. Therefore, since q is right resolving, there is a path $\alpha'_1 \dots \alpha'_m \alpha_1 \dots \alpha_l$ starting from a with

$$q(\alpha'_1) \dots q(\alpha'_m) q(\alpha_1) \dots q(\alpha_l) = \lambda_{\mathcal{D}}(\delta'_1) \dots \lambda_{\mathcal{D}}(\delta'_m) \lambda_{\mathcal{D}}(\delta_1) \dots \lambda_{\mathcal{D}}(\delta_l).$$

This implies that $(\delta'_1, \alpha'_1) \dots (\delta'_m, \alpha'_m)(\delta_1, \alpha_1) \dots (\delta_l, \alpha_l)$ is a path in $\hat{\Gamma}$. Moreover, it is a path in $\check{\Gamma}$ because it starts from $(u, a) \in V_{\check{\Gamma}}$. Hence $(\delta_1, \alpha_1) \dots (\delta_l, \alpha_l)$ is a path in $\check{\Gamma}$.

Therefore, we have

$$\check{\eta}(X_{\check{\Gamma}}) = \pi_{\mathcal{D}}(X_D) = X_n. \tag{3.1}$$

Let $((\delta_j, \alpha_j))_{j\in\mathbf{Z}} \in X_{\check{\Gamma}}$ with $\delta_j \in A_D$ and $\alpha_j \in A_\Gamma$. Then $(\alpha_j)_{j\in\mathbf{Z}} \in Z_n$ because

$$\eta((\alpha_j)_{j\in\mathbf{Z}}) = \pi_{\mathcal{D}}((\delta_j)_{j\in\mathbf{Z}}) \in X_{\mathcal{D}} = X_n.$$

Hence if $\check{\pi} : X_{\check{\Gamma}} \to X_\Gamma$ is the mapping $((\delta_j, \alpha_j))_{j\in\mathbf{Z}} \mapsto (\alpha_j)_{j\in\mathbf{Z}}$, then

$$\check{\pi}(X_{\check{\Gamma}}) \subset Z_n,$$

$$\check{\xi} = \xi\check{\pi} = \xi_{n+1}\check{\pi} \quad \text{and} \quad \check{\eta} = \eta\check{\pi}.$$

Thus we have

$$\check{\xi}(X_{\check{\Gamma}}) \subset \xi(Z_n) = X_{n+1}. \tag{3.2}$$

Since $\pi_{\mathcal{D}}$ is bounded-to-one, it follows that $\check{\pi}$ is bounded-to-one. Since $\check{\pi}, \xi_{n+1}$ and η are all bounded-to-one, $\check{\xi}$ and $\check{\eta}$ are bounded-to-one. Therefore, noting (3.1) and (3.2) with $X_n \supset X_{n+1}$, we apply Lemma 2.1 to the textile system $\check{T} = (\check{p}, \check{q} : \check{\Gamma} \to G)$. Then we have $X_n = X_{n+1}$ and Proposition 3.2 is proved. □

Let $T = (p, q : \Gamma \to G)$ be a textile system with p or q right resolving or left resolving. Let $n \geq 1$. We consider the *n-th composition power* T^n of T (see [N2]). This is given as the dual of the *n-th product power* $(T^*)^{(n)} = ((p^*)^{(n)}, (q^*)^{(n)} : (\Gamma^*)^n \to (G^*)^n)$ of the dual $T^* = (p^*, q^* : \Gamma^* \to G^*)$ of T such that the graphs $(\Gamma^*)^n$ and $(G^*)^n$ and the graph-homomorphisms $(p^*)^{(n)}$ and $(q^*)^{(n)}$ are given as follows: $A_{(\Gamma^*)^n} = L_n(\Gamma^*), V_{(\Gamma^*)^n} = V_{\Gamma^*}, A_{(G^*)^n} = L_n(G^*), V_{(G^*)^n} = V_{G^*}$, and

$$i_{(\Gamma^*)^n}(\alpha_1 \dots \alpha_n) = i_{\Gamma^*}(\alpha_1) \qquad t_{(\Gamma^*)^n}(\alpha_1 \dots \alpha_n) = t_{\Gamma^*}(\alpha_n)$$
$$i_{(G^*)^n}(a_1 \dots a_n) = i_{G^*}(a_1) \qquad t_{(G^*)^n}(a_1 \dots a_n) = t_{G^*}(a_n)$$
$$(p^*)^{(n)}(\alpha_1 \dots \alpha_n) = p^*(\alpha_1) \dots p^*(\alpha_n) \quad (q^*)^{(n)}(\alpha_1 \dots \alpha_n) = q^*(\alpha_1) \dots q^*(\alpha_n),$$

where $\alpha_1 \dots \alpha_n \in L_n(\Gamma^*)$ with $\alpha_i \in A_{\Gamma^*}$ and $a_1 \dots a_n \in L_n(G^*)$ with $a_i \in A_{G^*}$. We can put

$$T^n = ((T^*)^{(n)})^* = (\bar{p}, \bar{q} : \bar{\Gamma} \to G).$$

We note that the vertex-set $V_{\bar{\Gamma}}$ is equal to $L_n(G^*)$. If q is biresolving, then $\bar{q}$ is biresolving. We observe that $X_{\bar{\Gamma}}$ is the set of all elements $((\alpha_{i,j})_{1\leq i\leq n})_{j\in\mathbf{Z}}, \alpha_{i,j} \in A_\Gamma$, such that $(\alpha_{i,j})_{1\leq i\leq n, j\in\mathbf{Z}}$ is an obi of width n woven by T. Hence $X_{\bar{\Gamma}}$ will be identified with the set of all obis of width n woven by T. Let $\bar{\xi} : X_{\bar{\Gamma}} \to X_G$ and $\bar{\eta} : X_{\bar{\Gamma}} \to X_G$ be the 1-block maps given by $\bar{p}$ and $\bar{q}$, respectively. If $\bar{z} = (\alpha_{i,j})_{1\leq i\leq n, j\in\mathbf{Z}}$ is in $X_{\bar{\Gamma}}$, then $\bar{\xi}(\bar{z})$ is its uppermost thread $\xi((\alpha_{1,j})_{j\in\mathbf{Z}})$ and $\bar{\eta}(\bar{z})$ is its lowermost thread $\eta((\alpha_{n,j})_{j\in\mathbf{Z}})$. We have $\bar{\xi}(X_{\bar{\Gamma}}) = X_n$, where (X_n, σ_n) is the woof shift in height n of T. The subshift (X_n, σ_n) is a sofic system defined by the λ-graph $(\bar{\Gamma}, \bar{p})$.

Proposition 3.3. *Let $T = (p, q : \Gamma \to G)$ be a textile system with q biresolving, G irreducible and its warp shift (X_{T^*}, σ_{T^*}) topologically mixing. If T is onesided 1-1 in height $n + 1$ with $n \geq 1$, then the woof shift (X_n, σ_n) in height n of T is topologically transitive. If in addition G is aperiodic, then (X_n, σ_n^m) is topologically transitive for all $m \in \mathbf{N}$.*

Proof. We follow the notation above and that of Section 2.

First we observe that $T^n = (\bar{p}, \bar{q} : \bar{\Gamma} \to G)$ is onesided 1-1 in height 2. Since T is onesided 1-1 in height $n + 1$, if $(\alpha_{i,j})_{1\leq i\leq n+1, j\in\mathbf{Z}}$ is an obi of width $n + 1$ woven by $T = (p, q : \Gamma \to G)$ with $\alpha_{i,j} \in A_\Gamma$, then $(\alpha_{1,j})_{j\in\mathbf{Z}}$ is uniquely determined by $\xi((\alpha_{1,j})_{j\in\mathbf{Z}})$, because $(\alpha_{1,j})_{j\in\mathbf{Z}} \in Z_n$ and $\xi_{n+1} = \xi/Z_n$ is 1-1. Assume that

$(\bar{\alpha}_{i,j})_{1\le i\le 2, j\in\mathbf{Z}}$ is an obi of width 2 woven by T^n. Then $\bar{\alpha}_{1,j}$ and $\bar{\alpha}_{2,j}$ are of the form $(\alpha_{i,j})_{1\le i\le n}$ and $(\alpha_{i,j})_{n+1\le i\le 2n}$, respectively, with $(\alpha_{i,j})_{1\le i\le 2n, j\in\mathbf{Z}}, \alpha_{i,j}\in A_\Gamma$, woven by T. Since $(\alpha_{s+i,j})_{0\le i\le n, j\in\mathbf{Z}}$ is an obi of width $n+1$ for $1\le s\le n$, $(\alpha_{s,j})_{j\in\mathbf{Z}}$ is uniquely determined by $\xi((\alpha_{s,j})_{j\in\mathbf{Z}})$ for $1\le s\le n$. Hence it follows that $(\alpha_{i,j})_{1\le i\le n, j\in\mathbf{Z}} = (\bar{\alpha}_{1,j})_{j\in\mathbf{Z}}$ is uniquely determined by $\xi((\alpha_{1,j})_{j\in\mathbf{Z}}) = \bar{\xi}((\bar{\alpha}_{1,j})_{j\in\mathbf{Z}})$. Thus T^n is onesided 1-1 in height 2.

Assume that $\bar{\Gamma}$ is not irreducible. Then, since $\bar{q}$ is biresolving and G is irreducible, $\bar{\Gamma}$ is the disjoint union of at least two irreducible components. (In fact, this follows from the claim in the proof of Proposition 3.2 if we consider $\hat{\Gamma}$ constructed by using the λ-graph (G, id_{A_G}) and $\bar{q}:\bar{\Gamma}\to G$ instead of $\mathcal{D}$ and $q:\Gamma\to G$.) Let $\breve{\Gamma}$ be one of the components.

Let $\breve{T}=(\breve{p},\breve{q}:\breve{\Gamma}\to G)$ be the textile system such that $\breve{p}$ and $\breve{q}$ are the restrictions of $\bar{p}$ and $\bar{q}$ on $\breve{\Gamma}$. Then since $\bar{q}$ is biresolving, so is $\breve{q}$. Since T^n is onesided 1-1 in height 2, so is $\breve{T}$. Since $\breve{\Gamma}$ is an irreducible graph, the woof shift in height 1 of $\breve{T}$ is topologically transitive. Therefore, if we apply Proposition 3.2 to $\breve{T}$, then we see that $\breve{T}$ is saturated in height 1.

This implies that if $(\alpha_{i,j})_{1\le i\le n, j\in\mathbf{Z}}, \alpha_{i,j}\in A_\Gamma$, is an obi of width n woven by T such that $(\alpha_{i,j})_{1\le i\le n, j\in\mathbf{Z}}\in X_{\breve{\Gamma}}$, then there exists a half-textile $(\beta_{i,j})_{i\in\mathbf{N}, j\in\mathbf{Z}}, \beta_{i,j}\in A_\Gamma$, woven by T such that

$$\begin{aligned} \xi((\beta_{1,j})_{j\in\mathbf{Z}}) &= \xi((\alpha_{1,j})_{j\in\mathbf{Z}}) \\ (\beta_{tn+i,j})_{1\le i\le n, j\in\mathbf{Z}} &\in X_{\breve{\Gamma}} \qquad \forall t\ge 0. \end{aligned}$$

Let $\bar{u}\in V_{\breve{\Gamma}}$ and let $\bar{v}\in V_{\bar{\Gamma}} - V_{\breve{\Gamma}}$. Since q is biresolving, the dual $T^* = (p^*, q^*:\Gamma^*\to G^*)$ of T is LL, that is, p^* and q^* are left resolving, so that T^* is nondegenerate. Hence $(X_{T^*}, \sigma_{T^*}) = (X_{G^*}, \sigma_{G^*})$, so that $\bar{u}\in L_n(G^*)$ appears on some point in X_{T^*} and so does $\bar{v}\in L_n(G^*)$. Since (X_{T^*}, σ_{T^*}) is topologically mixing, there exists a half-textile $(\alpha_{i,j})_{i\in\mathbf{N}, j\in\mathbf{Z}}$ woven by T such that

$$\begin{aligned} (i_\Gamma(\alpha_{i,0}))_{1\le i\le n} &= \bar{u} \\ (i_\Gamma(\alpha_{tn+i,0}))_{1\le i\le n} &= \bar{v} \qquad \exists t\ge 1. \end{aligned}$$

Since $\breve{\Gamma}$ is an irreducible graph, $(\alpha_{i,j})_{1\le i\le n, j\in\mathbf{Z}}\in X_{\breve{\Gamma}}$. Therefore, it follows from the above that there exists a half-textile $(\beta_{i,j})_{i\in\mathbf{N}, j\in\mathbf{Z}}, \beta_{i,j}\in A_\Gamma$, woven by T such that

$$\begin{aligned} \xi((\beta_{1,j})_{j\in\mathbf{Z}}) &= \xi((\alpha_{1,j})_{j\in\mathbf{Z}}) \\ (\beta_{tn+i,j})_{1\le i\le n, j\in\mathbf{Z}} &\in X_{\breve{\Gamma}} \qquad \forall t\ge 0. \end{aligned}$$

Clearly the half-textiles $(\alpha_{i,j})_{i\in\mathbf{N}, j\in\mathbf{Z}}$ and $(\beta_{i,j})_{i\in\mathbf{N}, j\in\mathbf{Z}}$ are different, but they have the same uppermost thread, which contradicts that T is onesided 1-1. Thus we have proved that $\bar{\Gamma}$ is irreducible. Therefore, (X_n, σ_n) is topologically transitive.

Assume that G is aperiodic. Let $m\in\mathbf{N}$. Let $T^{(m)} = (p^{(m)}, q^{(m)}:\Gamma^m\to G^m)$ be the m-th product power of T. Then $q^{(m)}$ is biresolving, G^m is irreducible, $(X_{(T^{(m)})^*}, \sigma_{(T^{(m)})^*}) = (X_{T^*}, \sigma_{T^*})$ and $T^{(m)}$ is onesided 1-1 in height $n+1$. Hence it follows from the above that the woof shift of $T^{(m)}$ in height n is topologically transitive. Therefore, (X_n, σ_n^m) is topologically transitive, because it is topologically conjugate to the woof shift of $T^{(m)}$ in height n. □

Proposition 3.4 (Proposition 2.1(2) of [N2])**.** *If T is an upwardly nondegenerate, onesided 1-1 and finitely saturated textile system, then (X_T, σ_T) is topologically conjugate to a topological Markov shift.*

Proposition 3.5. *If $T = (p, q : \Gamma \to G)$ is a onesided 1-1 textile system with q biresolving, G irreducible and (X_{T^*}, σ_{T^*}) topologically mixing, then (X_T, σ_T) is topologically conjugate to a topologically transitive topological Markov shift. If in addition G is aperiodic, then (X_T, σ_T) is topologically conjugate to a topologically mixing topological Markov shift.*

Proof. The proposition follows from Propositions 3.1, 3.2, 3.3, and 3.4. □

Let $T = (p, q : \Gamma \to G)$ be a textile system. Let $n \in \mathbf{N}$. The *higher block system $T^{[n]}$ of order n* of T is defined by $T^{[n]} = (p^{[n]}, q^{[n]} : \Gamma^{[n]} \to G^{[n]})$ with $p^{[n]}, q^{[n]}$ such that

$$p^{[n]}(\alpha_1 \dots \alpha_n) = p(\alpha_1) \dots p(\alpha_n), \quad q^{[n]}(\alpha_1 \dots \alpha_n) = q(\alpha_1) \dots q(\alpha_n)$$

for $\alpha_1 \dots \alpha_n \in A_{\Gamma^{[n]}} = L_n(\Gamma), \alpha_i \in A_\Gamma$.

Corollary 3.6. *Let $T = (p, q : \Gamma \to G)$ be a onesided 1-1 textile system with q biresolving, G irreducible and (X_{T^*}, σ_{T^*}) topologically mixing. Then there exist $n \in \mathbf{N}$ and a onesided 1-1, nondegenerate textile system $T' = (p', q' : \Gamma' \to G')$ with q' biresolving and G' irreducible such that T' is a subsystem of $T^{[n]}$, $(X_{G'}, \sigma_{G'}) = (X_{T^{[n]}}, \sigma_{T^{[n]}})$ and $\varphi_{T'} = \varphi_{T^{[n]}}$ (hence there exists a conjugacy $\rho : (X_T, \sigma_T) \to (X_{G'}, \sigma_{G'})$ such that $\varphi_T = \rho^{-1} \varphi_{T'} \rho$).*

Proof. By Proposition 3.5, (X_T, σ_T) is a topologically transitive subshift of finite type, and so is (Z_T, ς_T) because ξ_T is 1-1. Therefore there exists $n \in \mathbf{N}$ such that both $(Z_{T^{[n]}}, \varsigma_{T^{[n]}}) = (Z_T^{[n]}, \varsigma_T^{[n]})$ and $(X_{T^{[n]}}, \sigma_{T^{[n]}}) = (X_T^{[n]}, \sigma_T^{[n]})$ are topological Markov shifts. Let Γ' and G' be the irreducible graphs which define $(Z_{T^{[n]}}, \varsigma_{T^{[n]}})$ and $(X_{T^{[n]}}, \sigma_{T^{[n]}})$, respectively. Let $T' = (p', q' : \Gamma' \to G')$ be the subsystem of $T^{[n]} = (p^{[n]}, q^{[n]} : \Gamma^{[n]} \to G^{[n]})$, that is, p' and q' are the restrictions of $p^{[n]}$ and $q^{[n]}$, respectively, on Γ'. Then T' is nondegenerate. Since $\xi_{T'} = \xi_{T^{[n]}}$, $\eta_{T'} = \eta_{T^{[n]}}$ and ξ_T is 1-1, we see that $\xi_{T'}$ is 1-1 and $\varphi_{T'} = \varphi_{T^{[n]}}$. Since q' is a restriction of $q^{[n]}$ and $q^{[n]}$ is biresolving, it follows that q' is *weakly right resolving* (i.e., the restriction of q'_A on $i_{\Gamma'}^{-1}(\{u\})$ is injective for all $u \in V_{\Gamma'}$) and *weakly left resolving* (i.e., the restriction of q'_A on $t_{\Gamma'}^{-1}(\{u\})$ is injective for all $u \in V_{\Gamma'}$). The irreducible graphs G' and Γ' have the same spectral radius. It is well known (see [LM, Proposition 8.2.2]) that for a graph-homomorphism between irreducible graphs with the same spectral radius, "weakly right resolving" and "right resolving" are the same, and so are "weakly left resolving" and "left resolving". Thus q' is biresolving. □

4. Proof of Theorem 1.3

First we restate Theorem 1.3 in an explicit form.

Let A be an alphabet. Let $k \in \mathbf{N}$. Let $f : A^{k+1} \to A$ and $g : A^{k+1} \to A$ be mappings. Let $\tilde{\varphi}_f : A^{\mathbf{N}} \to A^{\mathbf{N}}$ and $\tilde{\varphi}_g : A^{\mathbf{N}} \to A^{\mathbf{N}}$ be defined as in Section 1. Assume that $\tilde{\varphi}_g$ is a homeomorphism, and define

$$\tilde{\varphi}_{f,g} = \tilde{\varphi}_f (\tilde{\varphi}_g)^{-1}.$$

Theorem 1.3. *If $\tilde{\varphi}_{f,g}$ is an expansive homeomorphism, then $(A^{\mathbf{N}}, \tilde{\varphi}_{f,g})$ is topologically conjugate to a topological Markov shift.*

Let $T^* = (p^*, q^* : \Gamma^* \to G_A)$ be the textile system defined as follows. The graph G_A is the 1-vertex graph with arc-set A, that is, $A_{G_A} = A$, V_{G_A} is a singleton, say $\{v_0\}$, and $i_{G_A}(a) = t_{G_A}(a) = v_0$ for all $a \in A_{G_A}$. $\Gamma^* = G_A^{[k+1]}$. The graph-homomorphisms $p^* : \Gamma^* \to G_A$ and $q^* : \Gamma^* \to G_A$ are defined by

$$p^*(w) = g(w), \quad q^*(w) = f(w), \qquad w \in A_{\Gamma^*} = A^{k+1}.$$

We note:

(1) $(X_{T^*}, \sigma_{T^*}) = (A^{\mathbf{Z}}, \sigma_A)$.

Let T be the dual of T^*. Let $B = A^k$. Then we can write $T = (p, q : \Gamma \to G_B)$, where G_B is the 1-vertex graph with arc-set B and vertex-set $\{v_0\}$. Let $\tilde{\xi}^*$ and $\tilde{\eta}^*$ be the mappings of $\tilde{X}_{\Gamma^*}$ onto $A^{\mathbf{N}}$ defined by $\tilde{\xi}^*((\alpha_i)_{i\in\mathbf{N}}) = (p^*(\alpha_i))_{i\in\mathbf{N}}$ and $\tilde{\eta}^*((\alpha_i)_{i\in\mathbf{N}}) = (q^*(\alpha_i))_{i\in\mathbf{N}}$, where $(\alpha_i)_{i\in\mathbf{N}} \in \tilde{X}_{\Gamma^*}, \alpha_i \in A_{\Gamma^*}$. Since $\tilde{\varphi}_f$ and $\tilde{\varphi}_g$ are bijections, so are $\tilde{\xi}^*$ and $\tilde{\eta}^*$, and $\tilde{\varphi}_{f,g} = \tilde{\eta}^*(\tilde{\xi}^*)^{-1}$. Since f and g are leftmost-permutive, T^* is LL, i.e., p^* and q^* are left resolving. Hence

(2) $T = (p, q : \Gamma \to G_B)$ has the property that q is biresolving.

Since $\tilde{\varphi}_{f,g}$ is expansive with $\tilde{\varphi}_{T^*} = \tilde{\varphi}_{f,g}$ and $(A^{\mathbf{N}}, \tilde{\varphi}_{f,g}) = (\tilde{X}_{T^*}, \tilde{\varphi}_{T^*})$, applying Theorem 2.11 of [N2] to T^*, we know:

(3) T is onesided 1-1 and $(A^{\mathbf{N}}, \tilde{\varphi}_{f,g})$ is topologically conjugate to (X_T, σ_T).

Theorem 1.3 is proved by (1),(2),(3) and Proposition 3.5 (or Propositions 3.1, 3.2, 3.3 and Theorem 1.1).

5. Leftmost-permutive onesided cellular automata

The following is the dual expression of a refinement of Lemma 3.11 of [N2].

Proposition 5.1. *Let $T^* = (p^*, q^* : \Gamma^* \to G^*)$ be a onesided 1-1, LL textile system with dual T. Suppose that σ_T^m is topologically transitive for all $m \in \mathbf{N}$. Then there exist an alphabet A and a onesided 1-1, LL textile system ${T_0}^* = ({p_0}^*, {q_0}^* : \Gamma^* \to G_A)$ with dual T_0 , where G_A is the one-vertex graph with $A_{G_A} = A$, such that there exists a topological conjugacy $\tilde{\psi} : (\tilde{X}_{G^*}, \tilde{\sigma}_{G^*}) \to (A^{\mathbf{N}}, \tilde{\sigma}_A)$ with $\tilde{\varphi}_{T^*} = \tilde{\psi}^{-1}\tilde{\varphi}_{T_0^*}\tilde{\psi}$, and $\xi_{T_0} = \xi_T$ and $\eta_{T_0} = \eta_T$.*

Proof. The proof of Lemma 3.10 of [N2] proves that for two left resolving graph-homomorphisms $h : \Gamma^* \to G^*$ and $h' : \Gamma^* \to G^*$ with G^* "column reduced", if h gives a conjugacy of $(X_{\Gamma^*}, \sigma_{\Gamma^*})$ onto (X_{G^*}, σ_{G^*}), then for $u, v \in V_{\Gamma^*}$, $h(u) = h(v)$ implies $h'(u) = h'(v)$. Using this, the proof of Lemma 3.11 of [N2] with a few obvious modifications proves Proposition 5.1. □

Let $\tilde{\varphi}$ be an endomorphism of a onesided topological Markov shift $(\tilde{X}_{G^*}, \tilde{\sigma}_{G^*})$. Then we call $\tilde{\varphi}$ an *LL endomorphism* of the shift if there exists a onesided 1-1, LL textile system over G^* such that $\tilde{\varphi}_{T^*} = \tilde{\varphi}$. If $\tilde{\varphi} = \tilde{\varphi}_f$ for some mapping (local rule) $f : L_{k+1}(G^*) \to A_{G^*}$ with $k \geq 0$ which induces a left resolving graph-homomorphism of $(G^*)^{[k+1]}$ onto G^*, then $\tilde{\varphi}$ is called an *leftmost-permutive endomorphism* of $(\tilde{X}_{G^*}, \tilde{\sigma}_{G^*})$, where $\tilde{\varphi}_f$ is defined by

$$\tilde{\varphi}_f((a_i)_{i\in\mathbf{N}}) = (f(a_i \dots a_{i+k}))_{i\in\mathbf{N}}, \qquad (a_i)_{i\in\mathbf{N}} \in \tilde{X}_{G^*}, \quad a_i \in A_{G^*}.$$

Remark 5.2. Let $\tilde{\varphi}$ be an endomorphism of a onesided topological Markov shift. Then $\tilde{\varphi}$ is an LL endomorphism of the shift if and only if $\tilde{\varphi}$ is a leftmost-permutive endomorphism of the shift.

Proof. Let G^* be a graph. If a local rule $f : L_{k+1}(G^*) \to A_{G^*}$ with $k \geq 0$ induces a left resolving graph-homomorphism $q^* : (G^*)^{[k+1]} \to G^*$, then we have $\tilde{\varphi}_f = \tilde{\varphi}_{T^*}$ for the onesided 1-1, LL textile system $T^* = (p^*, q^* : (G^*)^{[k+1]} \to G^*)$, where p^* is such that $p^*(a_1 \dots a_{k+1}) = a_1$ for $a_1 \dots a_{k+1} \in A_{(G^*)^{[k+1]}}, a_i \in A_{G^*}$.

Conversely let $T^* = (p^*, q^* : \Gamma^* \to G^*)$ be a onesided 1-1, LL textile system. Since $\tilde{\xi}^* : \tilde{X}_{\Gamma^*} \to \tilde{X}_{G^*}$ given by the left resolving graph-homomorphism p^* is 1-1, there exists $k \in \mathbf{N}$ such that for any $\alpha\alpha_1 \dots \alpha_k \in L_{k+1}(\Gamma^*)$ with $\alpha, \alpha_i \in A_{\Gamma^*}$, the path $p^*(\alpha)p^*(\alpha_1) \dots p^*(\alpha_k)$ in G^* uniquely determines α. Hence we can define a local rule $f : L_{k+1}(G^*) \to A_{G^*}$ by

$$f(p^*(\alpha)p^*(\alpha_1) \dots p^*(\alpha_k)) = q^*(\alpha).$$

Clearly we have $\tilde{\varphi}_{T^*} = \tilde{\varphi}_f$. Since p^* is left resolving, it follows that for any $\alpha_1 \dots \alpha_k \in L_k(\Gamma^*)$ with $\alpha_i \in A_{\Gamma^*}$, the path $p^*(\alpha_1) \dots p^*(\alpha_k)$ uniquely determines $i_{\Gamma^*}(\alpha_1)$. Hence we can define a mapping $F : L_k(G^*) \to V_{\Gamma^*}$ by $F(p^*(\alpha_1) \dots p^*(\alpha_k)) = i_{\Gamma^*}(\alpha_1)$. Let $Q^* : (G^*)^{[k+1]} \to G^*$ be the graph-homomorphism induced by f. Then $Q^*_V(a_1 \dots a_k) = q^*_V(F(a_1 \dots a_k))$ for $a_1 \dots a_k \in L_k(G^*) = V_{(G^*)^{[k+1]}}$. For $aa_1 \dots a_k \in L_{k+1}(G^*) = A_{(G^*)^{[k+1]}}$, we have $Q^*_A(aa_1 \dots a_k) = q^*_A(\alpha)$, where α is the unique arc in Γ^* such that $t_{\Gamma^*}(\alpha) = F(a_1 \dots a_k)$ and $p^*_A(\alpha) = a$. Since q^* is left resolving, it easily follows that Q^* is left resolving. □

The following is an extension of a major part of Theorem 3.12 (2) of [N2].

Corollary 5.3. *If $\tilde{\varphi}$ is a leftmost-permutive endomorphism of a onesided topological Markov shift $(\tilde{X}_{G^*}, \tilde{\sigma}_{G^*})$ such that $\tilde{\varphi}^m$ is topologically transitive for all $m \in \mathbf{N}$, then there exists a leftmost-permutive endomorphism $\tilde{\varphi}_0$ of the onesided full shift $(A^{\mathbf{N}}, \tilde{\sigma}_A)$ over an alphabet A such that there exists a topological conjugacy $\tilde{\psi} : (\tilde{X}_{G^*}, \tilde{\sigma}_{G^*}) \to (A^{\mathbf{N}}, \tilde{\sigma}_A)$ with $\tilde{\varphi} = \tilde{\psi}^{-1}\tilde{\varphi}_0\tilde{\psi}$.*

Proof. By Remark 5.2, there exists a onesided 1-1, LL textile system T^* over G^* with $\tilde{\varphi}_{T^*} = \tilde{\varphi}$. Since $\tilde{\xi}_{T^*}$ is 1-1, there exists a factor map (semi-conjugacy) of $(\tilde{X}_{T^*}, \tilde{\varphi}_{T^*})$ onto the onesided woof shift $(\tilde{X}_T, \tilde{\sigma}_T)$ of the dual T of T^*. Therefore, for each $m \in \mathbf{N}$, since $\tilde{\varphi}^m_{T^*}$ is topologically transitive, so is $\tilde{\sigma}^m_T$, so that σ^m_T is topologically transitive . Thus the result follows from Proposition 5.1 and Remark 5.2. □

The preceding corollary shows that any (not necessarily expansive) leftmost-permutive or LL endomorphism $\tilde{\varphi}$ of any onesided topological Markov shift such that $\tilde{\varphi}^m$ is topologically transitive for all $m \in \mathbf{N}$ is "topologically conjugate" to some leftmost-permutive endomorphism of some onesided full shift (some leftmost-permutive onesided cellular automaton map).

We consider generalizing Theorems 1.1 and 1.3 which concern automorphisms of onesided full shifts (invertible onesided cellular automata) to leftmost-permutive endomorphisms of onesided full shifts (leftmost-permutive onesided cellular automata). As is clear by the proof of Theorem 1.3, the generalization of Theorem 1.3 follows directly from Proposition 3.5. But it will turn out that the whole generalization can be reduced to Theorems 1.1 and 1.3, since every expansive, leftmost-permutive endomorphism of a onesided full shift is necessarily accompanied by expansive automorphisms which have the "same" dynamics as it has.

Let $\varphi : X \to X$ be an onto continuous map of a compact metric space with metric d_X. Let

$$\mathcal{O}_\varphi = \{(x_i)_{i\in\mathbf{Z}} \mid \forall i, x_i \in X \text{ and } \varphi(x_i) = x_{i+1}\}.$$

Let $\mathcal{O}_\varphi$ be endowed with the metric such that

$$d_{\mathcal{O}_\varphi}((x_i)_{i\in\mathbf{Z}}, (y_i)_{i\in\mathbf{Z}}) = \sup\{2^{-|i|} d_X(x_i, y_i) \mid i \in \mathbf{Z}\}.$$

We define a homeomorphism $\sigma_\varphi : \mathcal{O}_\varphi \to \mathcal{O}_\varphi$ by

$$\sigma_\varphi((x_i)_{i\in\mathbf{Z}}) = (x_{i+1})_{i\in\mathbf{Z}} \qquad (x_i)_{i\in\mathbf{Z}} \in \mathcal{O}_\varphi.$$

Proposition 5.4. *Let A be an alphabet. Let $k \in \mathbf{N}$. Let $f : A^{k+1} \to A$ be a leftmost-permutive local rule. Then for some $l \in \mathbf{N}$, there exists a leftmost-permutive local rule $\hat{f} : (A^{2k})^l \to A^{2k}$ satisfying the following conditions:*

1. *$\tilde{\varphi}_{\hat{f}} : (A^{2k})^{\mathbf{N}} \to (A^{2k})^{\mathbf{N}}$ is a homeomorphism;*
2. *$\tilde{\varphi}_{\hat{f}}$ is expansive if and only if $\tilde{\varphi}_f$ is expansive;*
3. *if $\tilde{\varphi}_f$ is expansive, then $(\mathcal{O}_{\tilde{\varphi}_f}, \sigma_{\tilde{\varphi}_f})$ is topologically conjugate to $((A^{2k})^{\mathbf{N}}, \tilde{\varphi}_{\hat{f}})$.*

Proof. Let $T^* = (p^*, q^* : \Gamma^* \to G_A)$ be the textile system in the proof of Theorem 1.3 (Section 4) in which g is such that $\tilde{\varphi}_g$ is the identity mapping. Then $\Gamma^* = G_A^{[k+1]}$, and $p^* : \Gamma^* \to G_A$ and $q^* : \Gamma^* \to G_A$ are defined by

$$p^*(a_1 \dots a_{k+1}) = a_1, \;\; q^*(a_1 \dots a_{k+1}) = f(a_1 \dots a_{k+1}), \quad a_1 \dots a_{k+1} \in A_{\Gamma^*} = A^{k+1}.$$

We have $\tilde{\varphi}_f = \tilde{\varphi}_{T^*}$. Let $T = (p, q : \Gamma \to G_B)$ be the dual of T^* with $B = A^k$. Since f is leftmost-permutive, T^* is LL, so that q is biresolving. Let $\bar{T} = (\bar{p}, \bar{q} : \bar{\Gamma} \to G_B)$ be the k-th composition power of T. Then $\bar{q}$ is biresolving. By construction, $V_{\bar{\Gamma}} = A^k$ and for any arc $\alpha \in A_{\bar{\Gamma}}$, $i_{\bar{\Gamma}}(\alpha)$ uniquely determines $\bar{p}(\alpha)$; in fact, $i_{\bar{\Gamma}}(\alpha)$ and $\bar{p}(\alpha)$ are the same as an element of A^k. Hence we define $F : V_{\bar{\Gamma}} \to B$ by

$$F(i_{\bar{\Gamma}}(\alpha)) = \bar{p}(\alpha), \qquad \alpha \in A_{\bar{\Gamma}}.$$

Let $\bar{\bar{T}} = (\bar{\bar{p}}, \bar{\bar{q}} : \bar{\bar{\Gamma}} \to G_B)$ be the second composition power of $\bar{T}$. Then each arc γ in $\bar{\bar{\Gamma}}$ is of the form $\genfrac{}{}{0pt}{}{\alpha}{\beta}$, where α and β are arcs in $\bar{\Gamma}$ with $\bar{q}(\alpha) = \bar{p}(\beta)$. For $\gamma = \genfrac{}{}{0pt}{}{\alpha}{\beta}$, $\bar{\bar{p}}(\gamma) = \bar{p}(\alpha)$ and $\bar{\bar{q}}(\gamma) = \bar{q}(\beta)$. Since $\bar{q}$ is biresolving, so is $\bar{\bar{q}}$. Let $\hat{T} = (\hat{p}, \hat{q} : \hat{\Gamma} \to G_B)$ be the textile system such that $\hat{\Gamma}$ and $\hat{q}$ are the same as $\bar{\bar{\Gamma}}$ and $\bar{\bar{q}}$, respectively, but $\hat{p}$ is given as follows. For $\gamma = \genfrac{}{}{0pt}{}{\alpha}{\beta}$ in $A_{\hat{\Gamma}} = A_{\bar{\bar{\Gamma}}}$,

$$\hat{p}(\gamma) = F(t_{\bar{\Gamma}}(\alpha)),$$

alternatively $\hat{p}(\gamma) = \bar{\bar{p}}(\delta)$, where δ is an arc in $\bar{\bar{\Gamma}}$ such that $\gamma\delta$ is a path of length 2 in $\bar{\bar{\Gamma}}$. Then it is observed that $\bar{\bar{T}}$ weaves a half-textile $(\gamma_{i,j})_{i\in\mathbf{N}, j\in\mathbf{Z}}$ if and only if $\hat{T}$ weaves the half-textile $(\gamma_{i,j-i})_{i\in\mathbf{N}, j\in\mathbf{Z}}$. Note that these half-textiles have the same uppermost thread, i.e., $(\bar{\bar{p}}(\gamma_{1,j}))_{j\in\mathbf{Z}} = (\hat{p}(\gamma_{1,j-1}))_{j\in\mathbf{Z}}$. Hence we have

$$X_{\hat{T}} = X_{\bar{\bar{T}}} = X_T,$$

and we see that $\xi_{\hat{T}}$ is 1-1 if and only if $\xi_{\bar{\bar{T}}}$ is 1-1, so that $\xi_{\hat{T}}$ is 1-1 if and only if ξ_T is 1-1. (If ξ_T is 1-1, then $\varphi_{\hat{T}} = \varphi_{\bar{\bar{T}}}^2 \sigma_{\bar{\bar{T}}}^{-1}$.) We see that $\hat{T}^*$ is over $G_{A^{2k}}$. Since $\hat{q}$ is biresolving, $\hat{T}^*$ is LL, so that $\hat{T}^*$ is nondegenerate. Hence

$$(\tilde{X}_{\hat{T}^*}, \tilde{\sigma}_{\hat{T}^*}) = ((A^{2k})^{\mathbf{N}}, \tilde{\sigma}_{A^{2k}}).$$

We shall prove that $\tilde{\xi}_{\hat{T}^*}$ and $\tilde{\eta}_{\hat{T}^*}$ are bijections. Let $(\hat{\gamma}_{i,j})_{i\in\mathbf{N},j\in\mathbf{Z}}$ be a half-textile woven by $\hat{T}$. Then $(\hat{\gamma}_{i,j})_{i\in\mathbf{N},j\in\mathbf{Z}} = (\gamma_{i,j-i})_{i\in\mathbf{N},j\in\mathbf{Z}} = \begin{pmatrix}\alpha_{i,j-i}\\ \beta_{i,j-i}\end{pmatrix}_{i\in\mathbf{N},j\in\mathbf{Z}}$ for a half-textile $(\gamma_{i,j})_{i\in\mathbf{N},j\in\mathbf{Z}} = \begin{pmatrix}\alpha_{i,j}\\ \beta_{i,j}\end{pmatrix}_{i\in\mathbf{N},j\in\mathbf{Z}}$ woven by $\bar{\bar{T}}$. It suffices to show that $(i_{\hat{\Gamma}}(\hat{\gamma}_{i,0}))_{i\in\mathbf{N}}$ uniquely determines $(\hat{\gamma}_{i,0})_{i\in\mathbf{N}}$ and $(\hat{\gamma}_{i,-1})_{i\in\mathbf{N}}$. To do this, it suffices to show that $\begin{matrix}i_{\hat{\Gamma}}(\hat{\gamma}_{1,0})\\ i_{\hat{\Gamma}}(\hat{\gamma}_{2,0})\end{matrix}$ uniquely determines $\hat{\gamma}_{1,0}$ and $\hat{\gamma}_{1,-1}$. Note that

$$\begin{matrix}\hat{\gamma}_{1,0}\\ \hat{\gamma}_{2,0}\end{matrix} = \begin{matrix}\alpha_{1,-1}\\ \beta_{1,-1}\\ \alpha_{2,-2}\\ \beta_{2,-2}\end{matrix}$$

where $\alpha_{i,-i}, \beta_{i,-i} \in A_{\bar{\Gamma}}, \bar{q}(\alpha_{i,-i}) = \bar{p}(\beta_{i,-i}), i = 1, 2$, and $F(t_{\bar{\Gamma}}(\alpha_{2,-2})) = \bar{q}(\beta_{1,-1})$.

Assume that

$$\begin{matrix}i_{\hat{\Gamma}}(\hat{\gamma}_{1,0})\\ i_{\hat{\Gamma}}(\hat{\gamma}_{2,0})\end{matrix} = \begin{matrix}i_{\bar{\Gamma}}(\alpha_{1,-1})\\ i_{\bar{\Gamma}}(\beta_{1,-1})\\ i_{\bar{\Gamma}}(\alpha_{2,-2})\\ i_{\bar{\Gamma}}(\beta_{2,-2})\end{matrix}$$

is given. Then $i_{\bar{\Gamma}}(\beta_{2,-2})$ uniquely determines $F(i_{\bar{\Gamma}}(\beta_{2,-2})) = \bar{p}(\beta_{2,-2}) = \bar{q}(\alpha_{2,-2})$. Since $\bar{q}$ is right resolving, $i_{\bar{\Gamma}}(\alpha_{2,-2})$ and $\bar{q}(\alpha_{2,-2})$ determine $\alpha_{2,-2}$, so that $t_{\bar{\Gamma}}(\alpha_{2,-2})$ is determined. Hence $F(t_{\bar{\Gamma}}(\alpha_{2,-2})) = \bar{q}(\beta_{1,-1})$ is determined. Together with $\bar{q}(\beta_{1,-1})$, $\begin{matrix}i_{\bar{\Gamma}}(\alpha_{1,-1})\\ i_{\bar{\Gamma}}(\beta_{1,-1})\end{matrix}$ determines $\begin{matrix}\alpha_{1,-1}\\ \beta_{1,-1}\end{matrix} = \hat{\gamma}_{1,0}$, because $\bar{q}$ is right resolving.

We also see that $i_{\bar{\Gamma}}(\alpha_{2,-2})$ determines $F(i_{\bar{\Gamma}}(\alpha_{2,-2})) = \bar{p}(\alpha_{2,-2}) = \bar{q}(\beta_{1,-2})$. Together with this, $\begin{matrix}i_{\bar{\Gamma}}(\alpha_{1,-1})\\ i_{\bar{\Gamma}}(\beta_{1,-1})\end{matrix} = \begin{matrix}t_{\bar{\Gamma}}(\alpha_{1,-2})\\ t_{\bar{\Gamma}}(\beta_{1,-2})\end{matrix}$ determines $\begin{matrix}\alpha_{1,-2}\\ \beta_{1,-2}\end{matrix} = \hat{\gamma}_{1,-1}$, because $\bar{q}$ is left resolving.

Thus we have proved that $\tilde{\xi}_{\hat{T}^*}$ and $\tilde{\eta}_{\hat{T}^*}$ are bijections. Therefore, $\tilde{\varphi}_{\hat{T}^*} = \tilde{\eta}_{\hat{T}^*}(\tilde{\xi}_{\hat{T}^*})^{-1}$ is an automorphism of $(\tilde{X}_{\hat{T}^*}, \tilde{\sigma}_{\hat{T}^*})$. Since $(\tilde{X}_{\hat{T}^*}, \tilde{\sigma}_{\hat{T}^*}) = ((A^{2k})^{\mathbf{N}}, \tilde{\sigma}_{A^{2k}})$, there exist $l \in \mathbf{N}$ and a local rule $\hat{f} : (A^{2k})^l \to A^{2k}$ such that $\tilde{\varphi}_{\hat{f}} = \tilde{\varphi}_{\hat{T}^*}$. Since $\xi_{\hat{T}}$ is 1-1 if and only if ξ_T is 1-1, $\tilde{\varphi}_{\hat{f}} = \tilde{\varphi}_{\hat{T}^*}$ is expansive if and only if $\tilde{\varphi}_f = \tilde{\varphi}_{T^*}$ is expansive (by [N2, Theorem 2.11(2)]). If $\tilde{\varphi}_f$ is expansive, then

$$(\mathcal{O}_{\tilde{\varphi}_f}, \sigma_{\tilde{\varphi}_f}) \cong (X_T, \sigma_T) = (X_{\hat{T}}, \sigma_{\hat{T}}) \cong ((A^{2k})^{\mathbf{N}}, \tilde{\varphi}_{\hat{f}}),$$

where $\cong$ denotes topological conjugacy. □

Combined with Theorem 1.3 and Proposition 5.4, the theorem of Boyle and Maass (Theorem 1.1) is improved to

Theorem 5.5. *Let A be the alphabet of N symbols. Let $f : A^{k+1} \to A$ be a leftmost-permutive local rule with $k \in \mathbf{N}$. If $\tilde{\varphi}_f$ is expansive, then $(\mathcal{O}_{\tilde{\varphi}_f}, \sigma_{\tilde{\varphi}_f})$ is topologically conjugate to a topological Markov shift which is shift equivalent to some full J-shift, where J and N are divisible by the same primes.*

There exists an expansive, noninvertible, leftmost-permutive onesided cellular automaton map which is not positively expansive. Let A_1 and A_2 be alphabets with cardinality not less than 2. Let $f_1 : A_1^{k+1} \to A_1$ be a *bipermutive* (i.e., leftmost-permutive and rightmost-permutive) local rule with $k \in \mathbf{N}$. Then $\tilde{\varphi}_{f_1}$

is noninvertible and positively expansive. Let $\tilde{\varphi}_2 : A_2^{\mathbf{N}} \to A_2^{\mathbf{N}}$ be an expansive invertible onesided cellular automaton map, which is not positively expansive by Theorem 3.9 (Schwartzman) in [BL]. Then the direct product $\tilde{\varphi}_{f_1} \times \tilde{\varphi}_2$ is an example of the above.

6. Bipermutive endomorphisms of mixing topological Markov shifts

For integers $k, l \geq 0$, an endomorphism φ of a topological Markov shift (X_G, σ_G) is called a *bipermutive endomorphism of* (k,l)*-type* if there exists a local rule $f : L_{k+l+1}(G) \to A_G$ which induces a biresolving graph-homomorphism of $G^{[k+l+1]}$ onto G and gives φ by

$$\varphi((a_j)_{j\in\mathbf{Z}}) = (f(a_{j-l} \dots a_{j+k}))_{j\in\mathbf{Z}}, \qquad (a_j)_{j\in\mathbf{Z}}, \quad a_j \in A_G.$$

A bipermutive endomorphism of an irreducible topological Markov shift is N-to-one for some $N \in \mathbf{N}$ (see [N1]). Bipermutive endomorphisms of topological Markov shifts were treated in [N2, Proposition 3.14], but no example has been known for noninvertible bipermutive endomorphisms of a mixing topological Markov shift which is not topologically conjugate to any full shift [N2, p. 52]. By virtue of Proposition 5.1, we can know that there exists no such example.

Proposition 6.1. *Let (X_G, σ_G) be a mixing topological Markov shift having an N-to-one bipermutive endomorphism φ of (k,l)-type with $N \in \mathbf{N}, k, l \geq 0, k+l \geq 1$. Then the following statements are valid.*

1. *(X_G, σ_G) is topologically conjugate to the full J-shift with $J^{k+l} = N$.*
2. *If $k > 0$ and $l > 0$, then (X_G, φ) is topologically conjugate to the onesided full N-shift.*
3. *If $k = 0$, then $(\tilde{X}_G, \tilde{\varphi})$ is topologically conjugate to the onesided full N-shift, where $\tilde{\varphi}$ is the endomorphism of the onesided topological Markov shift $(\tilde{X}_G, \tilde{\sigma}_G)$ naturally induced by φ.*

Proof. Let $f : L_{k+l+1}(G) \to A_G$ be the bipermutive local rule defining φ. Let $\Gamma = G^{[k+l+1]}$. Let $q : \Gamma \to G$ be the biresolving graph-homomorphism induced by f. Let $T = (p, q : \Gamma \to G)$ be the textile system with p defined by $p(a_1 \dots a_{k+l+1}) = a_1, a_1 \dots a_{k+l+1} \in L_{k+l+1}(G), a_j \in A_G$. Then T is a onesided 1-1 nondegenerate textile system. Clearly p is left resolving and $\tilde{\xi}_T$ is 1-1. Using the property that q is right resolving, we easily see that $\tilde{\xi}_{T^*}$ is 1-1 (see [N2, the proof of Proposition 3.14]). Since q is biresolving, T^* is LL. Hence T^* is nondegenerate, so that (X_{T^*}, σ_{T^*}) is a topological Markov shift. Since $\tilde{\xi}_{T^*}$ is 1-1, it follows from Theorem 2.12 of [N2] that $(\tilde{X}_T, \tilde{\varphi}_T) \cong (\tilde{X}_{T^*}, \tilde{\sigma}_{T^*})$. Since $\tilde{\varphi}_T$ is conjugate to a onesided topological Markov shift and commutes with the mixing onesided topological Markov shift $\tilde{\sigma}_T$, it follows from Lemma 2.9 of [BFF] or Theorem 1.6 of [N4] that $\tilde{\varphi}_T$ is topologically mixing, so that $\tilde{\sigma}_{T^*}$ is topologically mixing. Since T is onesided 1-1 and LL and $(\tilde{\sigma}_{T^*})^m$ is topologically transitive for all $m \in \mathbf{N}$, it follows from Proposition 5.1 that (X_G, σ_G) is topologically conjugate to the full J-shift for some $J \in \mathbf{N}$.

Let M_G and M_Γ be the adjacency matrices of G and Γ, respectively. Let $l_0 = (l_0(u))_{u\in V_G}$ and $r_0 = (r_0(u))_{u\in V_G}$ be left and right eigenvectors of M_G corresponding to J. Let $l_1 = (l_1(v))_{v\in V_\Gamma}$ and $r_1 = (r_1(v))_{v\in V_\Gamma}$ be the row and column vectors such that for $v = a_1 \dots a_{k+l} \in V_\Gamma = L_{k+l}(G), a_i \in A_\Gamma$,

$$l_1(v) = l_0(i_G(a_1)) \quad \text{and} \quad r_1(v) = r_0(t_G(a_{k+l})).$$

Then l_1 and r_1 are left and right eigenvectors of M_Γ corresponding to J. We have $l_1 r_1 = l_0 M_G^{k+l} r_0 = J^{k+l} l_0 r_0$. Let $l_2 = (l_2(v))_{v\in V_\Gamma}$ and $r_2 = (r_2(v))_{v\in V_\Gamma}$ be the row and column vectors such that $l_2(v) = l_0(q_V(v))$ and $r_2(v) = r_0(q_V(v))$ for all $v \in V_\Gamma$. Then, since q is a biresolving graph-homomorphism with q_V N-to-one, it follows that l_2 and r_2 are left and right eigenvectors of M_Γ corresponding to J and $l_2 r_2 = N l_0 r_0$ (see [N1, Proposition 2.3]). Since l_1 and l_2 are eigenvectors of the same irreducible nonnegative matrix corresponding to its spectral radius, we have $l_1 = c\, l_2$ for some $c > 0$. Since $\{l_1(v) \mid v \in V_\Gamma\} = \{l_2(v) \mid v \in V_\Gamma\}$, we have $c = 1$, so that $l_1 = l_2$. Similarly $r_1 = r_2$. Thus we have $N = J^{k+l}$. We have proved (1).

Let $k \geq 0$ and $l > 0$. Let $T_0 = (p_0, q : \Gamma \to G)$ be the textile system with p_0 defined by $p_0(a_1 \dots a_{k+l+1}) = a_{k+1}$, $a_1 \dots a_{k+l+1} \in L_{k+l+1}(G), a_j \in A_G$. Then T_0 is a onesided 1-1 nondegenerate textile system with $\varphi_{T_0} = \varphi$. Since $l > 0$, it follows that $\tilde{\xi}_{T_0^*}$ is injective. Hence T_0^* is a onesided 1-1, LL textile system with $\sigma_{T_0}^m$ topologically transitive for all $m \in \mathbf{N}$. Therefore, by Proposition 5.1, $(\tilde{X}_{T_0^*}, \tilde{\sigma}_{T_0^*})$ is topologically conjugate to a onesided full shift. If T_0^* is over a graph G^*, then every vertex of G^* has exactly N arcs ending in it, because q_V is N-to-one. Hence $(\tilde{X}_{T_0^*}, \tilde{\sigma}_{T_0^*}) = (\tilde{X}_{G^*}, \tilde{\sigma}_{G^*})$ is topologically conjugate to the onesided full N-shift.

If $k > 0$, then it follows that $\tilde{\eta}_{T_0^*}$ is injective, so that we have $(X_{T_0}, \varphi_{T_0}) \cong (\tilde{X}_{T_0^*}, \tilde{\sigma}_{T_0^*})$, by Theorem 2.11 of [N2]. If $k = 0$, then, by Theorem 2.12 of [N2], we have $(\tilde{X}_{T_0}, \tilde{\varphi}_{T_0}) \cong (\tilde{X}_{T_0^*}, \tilde{\sigma}_{T_0^*})$. Thus we have proved (2) and (3). ☐

Proposition 6.1 for the case that (X_G, σ_G) is a full shift (i.e., G is a one-vertex graph) was already known: (2) is found in [SA] and [N2], and (3) is contained in the result in [C] that for a rightmost-permutive endomorphism φ of $(0, l)$-type of a full shift $(A^{\mathbf{Z}}, \sigma_A)$ with $l \geq 1$, $(A^{\mathbf{N}}, \tilde{\varphi})$ is topologically conjugate to $(A^{\mathbf{N}}, \tilde{\sigma}_A^l)$.

In [BM, Theorem 9.2], the following result has been given. If an expansive invertible onesided cellular automaton map $\tilde{\varphi}_f$ with $f : A^2 \to A$ has its inverse $\tilde{\varphi}_e$ with $e : A^2 \to A$ and if for $a, b \in A$, $e(ab)af(ab)$ uniquely determines b, then $(A^{\mathbf{N}}, \tilde{\varphi}_f)$ is topologically conjugate to the full J-shift with $J^2 = N$, where N is the cardinality of A. In view of Theorem 1.3, Proposition 6.1 can be considered to be an extension of it. For let $T^* = (p^*, q^* : G_A^{[2]} \to G_A)$ be the textile system such that p^* and q^* are given by $p^*(a_1 a_2) = a_1$ and $q^*(a_1 a_2) = f(a_1 a_2)$ with $a_1, a_2 \in A$. Then the dual T of T^* is onesided 1-1, $(X_T, \sigma_T) \cong (A^{\mathbf{N}}, \tilde{\varphi}_f)$ and φ_T is an endomorphism of (1,1)-type of (X_T, σ_T) given by the local rule $e(a, b)af(a, b) \mapsto b$. By Theorem 1.3 and its proof, (X_T, σ_T) is topologically conjugate to a mixing topological Markov shift. Hence, for sufficiently large n, $(X_{T^{[n]}}, \sigma_{T^{[n]}}) = (X_T^{[n]}, \sigma_T^{[n]})$ is a mixing topological Markov shift and it is easily seen that $\varphi_{T^{[n]}}$ is a bipermutive endomorphism of (1,1)-type of the shift.

Let an endomorphism φ of a topological Markov shift (X_G, σ_G) be called a *q-biresolving endomorphism* if there exists a onesided 1-1 nondegenerate textile system $T = (p_0, q_0 : \Gamma \to G)$ with q_0 biresolving and $\varphi_T = \varphi$. A q-biresolving endomorphism of a transitive topological Markov shift is N-to-one for some $N \in \mathbf{N}$ (see [N1]). A bipermutive endomorphism of a topological Markov shift is q-biresolving. A positively expansive endomorphism φ of a transitive topological Markov shift (X_G, σ_G) is *essentially q-biresolving*, that is, there exist a q-biresolving endomorphism φ_0 of a topological Markov shift (X_{G_0}, σ_{G_0}) and a topological conjugacy $\psi : (X_G, \sigma_G) \to (X_{G_0}, \sigma_{G_0})$ with $\varphi = \psi^{-1} \varphi_0 \psi$. This follows from Theorem 3.9 of [N2] and the result of Kůrka [Ku] that a positively expansive endomorphism of a transitive topological Markov shift is topologically conjugate to a (surjective) onesided topological Markov shift. (The versions of this and the next result in which the endomorphism is additionally assumed to be onto were obtained independently by the author; see [N3, Section 8].) Combining Kůrka's result and Theorem 3.12 of [N2], we have the result that a positively expansive endomorphism of a mixing topological Markov shift is topologically conjugate to some onesided full N-shift. Therefore, the theorem of Boyle and Maass (Theorem 1.1) proves the following. If a mixing topological Markov shift has a positively expansive endomorphism which is N-to-one, then the shift is shift equivalent to some full J-shift, where J and N are divisible by the same primes. The two conjectures of Boyle and Maass mentioned in Section 1 imply that under the same hypothesis as above, the shift is topologically conjugate to a full shift and N is divisible by p^2 for every prime p dividing N. Proposition 6.1 is consistent with this.

It follows from Proposition 3.5 and Corollary 3.6 that if $T = (p, q : \Gamma \to G)$ is a onesided 1-1 textile system with q biresolving, G irreducible and aperiodic and σ_{T^*} topologically mixing, then for some $n \in \mathbf{N}$, $\varphi_{T^{[n]}}$ is a q-biresolving endomorphism of the mixing topological Markov shift $(X_T^{[n]}, \sigma_T^{[n]})$.

Question 6.2. Does there exist a onesided 1-1, nondegenerate, q-biresolving textile system T such that (X_T, σ_T) is not conjugate to any full shift, σ_T and σ_{T^*} are topologically mixing, and φ_T is noninvertible (but not necessarily positively expansive)?

Acknowledgements

In [U] Masaya Uchida made a computer program which generates all local rules $f : A^2 \to A$ with $\tilde{\varphi}_f$ bijective and $f(aa) = a$ for $a \in A$, and obtained the list of them, where A is the 4-symbol alphabet. In [Ko], Tadashi Kobayashi took away a large part of the pairs (f, g) with $\tilde{\varphi}_{f,g}$ nonexpansive from all the pairs $(f : A^2 \to A, g : A^2 \to A)$ with $\tilde{\varphi}_f, \tilde{\varphi}_g$ bijective (given by using the rules in Uchida's list) and obtained the list of the remaining of relatively small size. Examining some from this was helpful for the author to find the proof of Theorem 1.3.

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