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INVOLUTIONS FIXING $\mathbb{RP}^{\mathbf{odd}} \sqcup P(h,i)$, I

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Abstract. This paper studies the equivariant cobordism classification of all involutions fixing a disjoint union of an odd-dimensional real projective space $\mathbb{RP}^j$ with its normal bundle nonbounding and a Dold manifold $P(h,i)$ with $h>0$ and $i>0$. For odd h, the complete analysis of the equivariant cobordism classes of such involutions is given except that the upper and lower bounds on codimension of $P(h,i)$ may not be best possible; for even h, the problem may be reduced to the problem for even projective spaces.

1. Introduction

The objective of this paper is to classify up to equivariant cobordism the smooth involutions fixing the disjoint union of an odd-dimensional real projective space $\mathbb{RP}^j$ and a Dold manifold $P(h,i)$ with $h>0$ and $i>0$, where $P(h,i)$ is defined as $S^h \times \mathbb{CP}^i / -1\times$ (conjugation); see [Do]. The special cases $j=1,3$ have been considered in [Gu] and [L-L]. Here we deal with the general case.

Suppose (M^m, T) is a closed manifold with involution fixing a disjoint union of $\mathbb{RP}^j$ with normal bundle ν^{m-j} and $P(h,i)$ with normal bundle ν^k; so $m = h+2i+k$. In order to avoid the possibility that (M^m,T) is cobordant to an involution fixing only either $\mathbb{RP}^j$ or $P(h,i)$, one may assume that $(\mathbb{RP}^j, \nu^{m-j})$ is nonbounding, and thus $w(\nu^{m-j}) = (1+\alpha)^q$ with q odd where $H^*(\mathbb{RP}^j;\mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/(\alpha^{j+1}=0)$ and $\alpha \in H^1(\mathbb{RP}^j;\mathbb{Z}_2)$. In fact, since $w_1(\nu^{m-j}) = q\alpha \neq 0$, one has $m>j$. Since $(\mathbb{RP}^j, \nu^{m-j})$ is nonbounding and every involution fixing $\mathbb{RP}^j$ bounds, the component of M containing $\mathbb{RP}^j$ must contain $P(h,i)$; so $m > h+2i$ or $k>0$. Also, $(P(h,i),\nu^k)$ must be nonbounding, for if not, (M,T) is cobordant to an involution fixing $(\mathbb{RP}^j,\nu^{m-j})$. Here one uses the convention that $(\mathbb{RP}^j,\nu^{m-j})$ *is nonbounding*, and thus (M^m,T) does not bound equivariantly if (M^m,T) exists.

Letting $2^p < j < 2^{p+1}$, q is only determined modulo 2^{p+1}; so it is assumed that $q < 2^{p+1}$.

The mod 2 cohomology of the Dold manifold is given by

$$H^*(P(h,i);\mathbb{Z}_2) = \mathbb{Z}_2[c,d]/(c^{h+1} = d^{i+1} = 0)$$

where $c \in H^1(P(h,i);\mathbb{Z}_2)$ and $d \in H^2(P(h,i);\mathbb{Z}_2)$. According to the recent work of Stong [St], one may write the total Stiefel-Whitney class of ν^k in the form

$$w(\nu^k) = (1+c)^a(1+c+d)^b w(\rho)^\varepsilon$$

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where $\varepsilon = 0$ or 1 and $w(\rho) = 1+$ terms of dimension at least 4 is an exotic class ($\varepsilon = 0$ except for $h = 2, 4, 5$, or 6).

Now form the class

$$w[r] = \frac{w(\mathbb{RP}(\nu))}{(1+e)^{m-h-2i-r}}$$

where e is the characteristic class of the double cover of $\mathbb{RP}(\nu)$ by the sphere bundle of ν, so that each $w[r]_x$ is a polynomial in $w_y(\mathbb{RP}(\nu))$ and e. Then

(1.1)

$$w[r] = \begin{cases} (1+c)^h(1+c+d)^{i+1}\left\{(1+e)^r + (a+b)c(1+e)^{r-1} + \cdots\right\} & \text{on } P(h,i) \\ (1+\alpha)^{j+1}\left\{(1+e)^{h+2i+r-j} + q\alpha(1+e)^{h+2i+r-j-1} + \cdots\right\} & \text{on } \mathbb{RP}^j. \end{cases}$$

According to Conner and Floyd [C-F], $\mathbb{RP}(\nu^k)$ and $\mathbb{RP}(\nu^{m-j})$ are cobordant in $B\mathbb{Z}_2$, and thus the characteristic numbers

$$\begin{aligned} &w[r_1]_{\omega_1} \cdots w[r_s]_{\omega_s} e^{m-1-|\omega_1|-\cdots-|\omega_s|}[\mathbb{RP}(\nu^k)] \\ &\quad = w[r_1]_{\omega_1} \cdots w[r_s]_{\omega_s} e^{m-1-|\omega_1|-\cdots-|\omega_s|}[\mathbb{RP}(\nu^{m-j})] \end{aligned}$$

where each $\omega = (i_1, ..., i_t)$ is a partition of $|\omega| = i_1 + \cdots + i_t$. This provides a method of studying involutions fixing $\mathbb{RP}^j \sqcup P(h,i)$. Such a method was first used by Pergher and Stong to study involutions fixing a disjoint union of a point and a closed manifold (see [P-S]).

The argument is divided into two cases: (i) h is odd; (ii) h is even. When h is odd, we give the complete analysis of the cobordism classes for such involutions except that the lower and upper bounds on k may not be best possible. The result is stated as follows.

Theorem 1.1. *Suppose (M^m, T) is a manifold with involution fixing $\mathbb{RP}^j \sqcup P(h,i)$ with j and h odd and with the fixed component $\mathbb{RP}^j$ with its normal bundle nonbounding. Let $2^p < j < 2^{p+1}$ and write $i = 2^u(2v+1)$. Then*

(1) *$h = j$ and i is even.*

(2) *The Stiefel-Whitney class of the normal bundle of $\mathbb{RP}^j$ is of the form $(1+\alpha)^q$ with $q =$ odd, well-defined modulo 2^{p+1}.*

(3) *The Stiefel-Whitney class of the normal bundle of $P(j,i)$ is of the form $(1+c)^a(1+c+d)$, with $a =$ even, well-defined modulo 2^{p+1}, and*

(*a*) *$q \equiv a + i + 1$ modulo 2^{p+1};*

(*b*) *$a \leq 2^u$ and if $u > 1, a < 2^u$.*

(4) *Writing $m = j+2i+k$, the involutions exist for k in a range $k_{\min} \leq k \leq k_{\max}$ where*

$$2 \leq k \leq \begin{cases} 2^{u+1}+2 & \text{if } u = 1 \\ 2^{u+1}-1 & \text{if } u > 1 \end{cases}$$

and more precisely

(*a*) *for $a < j$, $k_{\min} = a+2$ and for $a > j$, $k_{\min} \leq j+1 < a+2$ is the minimum dimension of a vector bundle with Stiefel-Whitney class $(1+c)^a(1+c+d)$ over $P(j,i)$;*

(*b*) *for $u = 1$,*

$$k_{\max} = \begin{cases} 4 & \text{if } a = 0 \text{ and } j \geq 3 \\ 6 & \text{if } a = 2 \text{ or } a = 0 \text{ and } j = 1 \end{cases}$$

and for $u > 1$,

$$2a+2 \le k_{\max} \le \begin{cases} 2^u + a + 1 & \textit{if } p \ge u \\ 2^{u+1} - (j - \textit{common}\ (j,a)) & \textit{if } p < u \end{cases}$$

where common (j,a) *is the common part of the 2-adic expansions of* j *and* a.

When h is even, one will prove that

Proposition 1.2. *If* (M^m, T) *fixes* $\mathbb{RP}^j \sqcup P(h,i)$ *with* j *odd and* h *even and with the fixed component* $\mathbb{RP}^j$ *with its normal bundle nonbounding, then* $m = j + q$.

One will see that this case may be reduced to a problem about finding involutions that fix $\mathbb{RP}^{q-1} \sqcup P(h,i)$, which is the problem for *even* projective spaces.

The paper is organized as follows. In Section 2, some involutions fixing $\mathbb{RP}^j \sqcup P(j,i)$ with j odd are constructed. With the help of these examples, in Section 3 we will complete the proof of Theorem 1.1. In Section 4, we discuss the case for which h is even and give the proof of Proposition 1.2. Throughout this paper, the coefficient group is $\mathbb{Z}_2$. w denotes the total Stiefel-Whitney class and w_s denotes the s-th Stiefel-Whitney class.

In concluding this introduction, the author would like to express his gratitude to Professor R.E. Stong for his valuable suggestions.

2. Examples for which involutions exist

Now let us build some involutions fixing $\mathbb{RP}^j \sqcup P(j,i)$ with j odd.

Write $i = 2^u(2v+1)$ and let

$$k_0 = \begin{cases} 2^u + 1 & \text{if } u = 1 \\ 2^u & \text{if } u \ne 1. \end{cases}$$

From [P-S], there is an involution (N^{i+l}, T_l) with $1 \le l \le k_0$ having fixed point set $* \sqcup \mathbb{RP}^i$ with the normal bundle of $\mathbb{RP}^i$ in N^{i+l} being $\iota \oplus (l-1)\mathbb{R}$, with ι the nontrivial line bundle, where $*$ denotes a point. This is constructed by applying the operation Γ $l-1$ times to the involution $(\mathbb{RP}^{i+1}, T_1)$ defined by

$$T_1([x_0, x_1, ..., x_{i+1}]) = [-x_0, x_1, ..., x_{i+1}],$$

which fixes $\mathbb{RP}^0 \sqcup \mathbb{RP}^i$ with the normal bundle ι on $\mathbb{RP}^i$ and cobording away various bounding fixed components (see Royster [Ro]).

Consider the involution $T_{N^{i+l}}$ on

$$P(j, N^{i+l}) = \frac{S^j \times N^{i+l} \times N^{i+l}}{-1 \times \text{twist}}$$

induced by $1 \times T_l \times T_l$. The fixed point set of this involution is

(1) $\frac{S^j \times \text{point} \times \text{point}}{-1 \times \text{twist}} = \mathbb{RP}^j$ and the normal bundle is formed by $\frac{S^j \times \mathbb{R}^{i+l} \times \mathbb{R}^{i+l}}{-1 \times \text{twist}}$, so is $(i+l)\iota \oplus (i+l)\mathbb{R}$;

(2) $\frac{S^j \times ((\mathbb{RP}^i \times \text{point}) \sqcup (\text{point} \times \mathbb{RP}^i))}{-1 \times \text{twist}}$ and the twist exchanges the two copies of $\mathbb{RP}^i$; so the quotient is $S^j \times (\mathbb{RP}^i \times \text{ point})$ with normal bundle $S^j \times$(normal bundle of $\mathbb{RP}^i \times$ point). Since S^j bounds, this component bounds away.

(3) $\frac{S^j \times \mathbb{RP}^i \times \mathbb{RP}^i}{-1 \times \text{twist}}$ with the normal bundle $\frac{S^j \times ((\iota \oplus (l-1)\mathbb{R}) \times (\iota \oplus (l-1)\mathbb{R}))}{-1 \times \text{twist}}$ and this is cobordant to $\frac{S^j \times \mathbb{CP}^i}{-1 \times \text{conjugation}} = P(j,i)$ with the normal bundle $\eta \oplus (l-1)\xi \oplus (l-1)\mathbb{R}$,

where ξ induced by ι is a 1-plane bundle over $P(j,i)$, and η is a 2-plane bundle over $P(j,i)$. Note that $w(\xi)=1+c$ and $w(\eta)=1+c+d$ (see [Do], [Uc]).

This produces an involution $(P(j,N^{i+l}),T_{N^{i+l}})$ fixing $\mathbb{RP}^j$ with the normal bundle ν^{2i+2l} having $w(\nu^{2i+2l})=(1+\alpha)^{i+l}$ and $P(j,i)$ with the normal bundle ν^{2l} having $w(\nu^{2l})=(1+c)^{l-1}(1+c+d)$.

Now let us look at $P(j,N^{i+l})$. One has

Lemma 2.1. *For $1\le l<k_0$, $P(j,N^{i+l})$ bounds.*

Proof. When $1\le l<k_0$, N^{i+l} bounds. Furthermore, one has that $(N^{i+l}\times N^{i+l}$, twist) fixing N^{i+l} with the normal bundle τ, the tangent bundle of N^{i+l}, bounds equivariantly, and thus the bundle $(N^{i+l},\tau\oplus s\mathbb{R})$ bounds for any $s\ge 0$. So, $\mathbb{RP}(\tau\oplus(s+1)\mathbb{R})$ bounds. On the other hand, consider the involution on $P(j,N^{i+l})$ induced by $T'\times 1\times 1$ on $S^j\times N^{i+l}\times N^{i+l}$ where

$$T'(x_0,x_1,\cdots,x_j)=(-x_0,x_1,\cdots,x_j).$$

It is easy to see that the fixed data is $(N^{i+l},\tau\oplus j\mathbb{R})\sqcup(P(j-1,N^{i+l}),\xi^1)$ where ξ^1 is a real line bundle over $P(j-1,N^{i+l})$. Therefore, by [C-F] one obtains that the cobordism class $\{P(j,N^{i+l})\}=\{\mathbb{RP}(\tau\oplus(j+1)\mathbb{R})\}+\{\mathbb{RP}(\xi^1\oplus\mathbb{R})\}=0$. $\square$

Note that if $l=k_0$, then N^{i+k_0} does not bound. It will be proved later that $P(j,N^{i+k_0})$ must be nonbounding. So $\Gamma(P(j,N^{i+k_0}),T_{N^{i+k_0}})$ does not have the same fixed information as $(P(j,N^{i+k_0}),T_{N^{i+k_0}})$.

If $l<k_0$, by Lemma 2.1 and applying the operation Γ to $(P(j,N^{i+l}),T_{N^{i+l}})$, then the resulting involutions $\Gamma^x(P(j,N^{i+l}),T_{N^{i+l}})$ denoted by $(M^{j+2i+2l+x},T)$ have the following properties:

(i) There is an integer X_0 such that for $x<X_0$, $M^{j+2i+2l+x}$ bounds, but $M^{j+2i+2l+X_0}$ does not bound.

(ii) For $x\le X_0$, $(M^{j+2i+2l+x},T)$ has the same fixed information as $(P(j,N^{i+l}),T_{N^{i+l}})$.

If $l-1\le j$, then the normal bundle to the fixed point set of $(P(j,N^{i+l}),T_{N^{i+l}})$ has only $l-1$ sections. Thus, there exist involutions (M^{j+2i+k},T) with $l+1\le k\le 2l$, each of which has the same fixed information as $(P(j,N^{i+l}),T_{N^{i+l}})$ such that M^{j+2i+k} bounds for $k<2l$. Furthermore, by applying the inverse operation Γ^{-1} $l-1$ times to $(P(j,N^{i+l}),T_{N^{i+l}})$, one has that (M^{j+2i+k},T) is cobordant to $\Gamma^{k-2l}(P(j,N^{i+l}),T_{N^{i+l}})$ for $l+1\le k\le 2l$.

Generally, stability says that every vector bundle over $\mathbb{RP}^j$ is realizable by a j-plane bundle. Hence $(l-1)\xi$, which is induced from a bundle $(l-1)\iota$ over $\mathbb{RP}^j$, can be realized by a j-plane bundle. If $l-1$ is even and $l-1>j$, then $(l-1)\iota$ over $\mathbb{RP}^j$ is a complex vector bundle $(=\frac{l-1}{2}(\iota\otimes\mathbb{C}))$. One can then find a section (nonvanishing) of the real bundle and use it to split off a trivial complex line bundle. Thus $(l-1)\iota$ is stably equivalent to a complex vector bundle μ of dimension $\frac{j-1}{2}$ (or real dimension $j-1$). So, $\eta\oplus(l-1)\xi$ over $P(j,i)$ is stably equivalent to $\eta\oplus\mu'$ and is realized by a $(j+1)$-plane bundle, where μ' is induced by μ. In this case, the normal bundle to the fixed point set of $(P(j,N^{i+l}),T_{N^{i+l}})$ has at least $2l-X_1$ sections with $X_1\le j+1$. Therefore, one can apply the inverse operation Γ^{-1} $2l-X_1$ times to $(P(j,N^{i+l}),T_{N^{i+l}})$, so that there exist the involutions (M^{j+2i+k},T) with $X_1\le k\le 2l$ such that (M^{j+2l+k},T) is cobordant to $\Gamma^{k-2l}(P(j,N^{i+l}),T_{N^{i+l}})$.

Combining the above discussions, one has

Proposition 2.1. *Let $l < k_0$. There exist involutions (M^{j+2i+k}, T) fixing $\mathbb{RP}^j \sqcup P(j,i)$ with $l+1 \le k \le 2l + X_0$ if $l-1 \le j$, and with $X_1 \le k \le 2l + X_0$ if $l-1$ is even and $l-1 > j$, such that*

(i) (M^{j+2i+k}, T) is cobordant to $\Gamma^{k-2l}(P(j, N^{i+l}), T_{N^{i+l}})$ for each k;

(ii) M^{j+2i+k} bounds for $k < 2l + X_0$, but not for $k = 2l + X_0$.

3. The case in which h is odd

Following the notation of section 1, we discuss the case in which h is odd. Our task is to prove Theorem 1.1.

From (1.1) one then has

$$w[0]_1 = \begin{cases} (h+i+1+a+b)c & \text{on } P(h,i) \\ \alpha & \text{on } \mathbb{RP}^j. \end{cases}$$

So

$$w[0]_1^j e^{m-1-j}[\mathbb{RP}(\nu^{m-j})] = \alpha^j e^{m-1-j}[\mathbb{RP}(\nu^{m-j})] = \alpha^j[\mathbb{RP}^j] \neq 0$$

and

$$0 \neq w[0]_1^j e^{m-1-j}[\mathbb{RP}(\nu^k)] = (h+i+1+a+b)c^j e^{m-1-j}[\mathbb{RP}(\nu^k)],$$

which implies that $h+i+1+a+b \not\equiv 0 \bmod 2$ and $c^j \neq 0$, and so $h \ge j$.

Now, there are certain operations in the bordism of $B\mathbb{Z}_2$. For $x = e, w_1$, or $w_1 + e$, one may dualize any power of x, giving homomorphisms

$$(\text{dual } x^t) : \mathfrak{N}_n(B\mathbb{Z}_2) \longrightarrow \mathfrak{N}_{n-t}(B\mathbb{Z}_2).$$

Dualizing e is the Smith homomorphism of Conner and Floyd [C-F]. Dualizing w_1 and w_1^2 was used by C.T.C. Wall [Wa] in studying oriented bordism.

Consider the operation

$$(\text{dual } w[0]_1^2) = (\text{dual } (w_1 + (m-h-2i)e)^2) : \mathfrak{N}_{m-1}(B\mathbb{Z}_2) \longrightarrow \mathfrak{N}_{m-3}(B\mathbb{Z}_2).$$

When applied to $\mathbb{RP}(\nu^{m-j})$, $w[0]_1 = \alpha$ and the dual is $\mathbb{RP}(\nu^{m-j}|_{\mathbb{RP}^{j-2}})$, which is the projective space bundle of ν^{m-j} with $w(\nu^{m-j}) = (1+\alpha)^q$ over $\mathbb{RP}^{j-2}$. When applied to $\mathbb{RP}(\nu^k)$, $w[0]_1 = c$, and the dual is $\mathbb{RP}(\nu^k|_{P(h-2,i)})$, which is the projective space bundle of $\nu^k|_{P(h-2,i)}$ over $P(h-2,i)$. Since $\mathbb{RP}(\nu^{m-j})$ is cobordant to $\mathbb{RP}(\nu^k)$, the duals will be cobordant in $B\mathbb{Z}_2$, and one has

Proposition 3.1. *If (M^m, T) fixing $\mathbb{RP}^j \sqcup P(h,i)$ with h odd exists, then there is an involution (M^{m-2}, T) fixing $\mathbb{RP}^{j-2}$ with $w(\nu^{m-j}) = (1+\alpha)^q$ and $P(h-2,i)$ with normal bundle $\nu^k|_{P(h-2,i)}$.*

Note. When restricted to $P(0,i) = \mathbb{CP}^i$, $w(\nu^k)$ becomes $(1+d)^b$ and b does not change under restriction since i is unchanged. The values of a and q may reduce to smaller equivalent values.

By iterating this procedure, one may reduce j to 1 and quote results of Guo [Gu] ($j = 1$). Since Guo assumes $w(\nu^k) = (1+c)^a(1+c+d)^b$, which is not valid, we will not use her results.

So, by iteration one may consider the case $j = 1$ with h odd (so $h \ge 1$ obviously).

Proposition 3.2. *Suppose (M^{h+2i+k}, T) fixes $\mathbb{RP}^1 \sqcup P(h,i)$ with h odd. Then*

(1) $q = h = b = 1, a = \varepsilon = 0$, and i is even.

(2) *Letting* $i = 2^u(2v+1)$ *with* $u > 0$,

$$2 \le k \le \begin{cases} 2^{u+1}+2 & \text{if } u = 1 \\ 2^{u+1}-1 & \text{if } u > 1. \end{cases}$$

Furthermore, (M^{1+2i+k}, T) *fixing* $\mathbb{RP}^1 \sqcup P(1,i)$ *is cobordant to* $\Gamma^{k-2}(P(1,N^{i+1}), T_{N^{i+1}})$.

Proof. Obviously, $q = 1$ holds since $j = 1$. Now one computes the values of $w[1]_2$. On $P(h,i)$,

$$\begin{aligned} w[1] &= \left\{1 + (h+i+1)c + (\binom{h+i+1}{2}c^2 + (i+1)d) + \cdots \right\} \\ &\quad \times \left\{1 + e + (a+b)c + (\binom{a+b}{2}c^2 + bd)(1+e)^{-1} + \cdots \right\}; \end{aligned}$$

so

$$\begin{aligned} w[1]_2 &= \left\{\binom{h+i+1}{2}c^2 + (i+1)d\right\} + (h+i+1)c\{e + (a+b)c\} \\ &\quad + \binom{a+b}{2}c^2 + bd \\ &= (h+i+1)ce + (i+1+b)d + \binom{h+i+1+a+b}{2}c^2. \end{aligned}$$

On $\mathbb{RP}^1$,

$$w[1] = (1+e)^{h+2i} + \alpha(1+e)^{h+2i-1};$$

so

$$w[1]_2 = \binom{h+2i}{2}e^2.$$

Form the class

$$\begin{aligned} \hat{w}_2 &= w[1]_2 + (h+i+1)w[0]_1 e + \binom{h+i+1+a+b}{2}w[0]_1^2 \\ &= \begin{cases} (i+1+b)d & \text{on } P(h,i) \\ \binom{h+2i}{2}e^2 + (h+i+1)\alpha e & \text{on } \mathbb{RP}^1. \end{cases} \end{aligned}$$

If i is odd, then $P(h,i)$ bounds, for

$$w(P(h,i)) = (1+c)^h(1+c+d)^{i+1}$$

has only even powers of d. Since $(P(h,i), \nu^k)$ is nonbounding,

$$w(\nu^k) = (1+c)^a(1+c+d)^b w(\rho)^\varepsilon$$

must have some term with an odd power of d. Since h is odd, the only exotic class occurs for $h = 5$ and then by [St],

$$w(\rho) = 1 + \frac{c^4d^2}{(1+d)^2},$$

which has no odd powers of d. Thus b is odd. This gives

$$\hat{w}_2 = \begin{cases} d & \text{on } P(h,i) \\ \binom{h+2i}{2}e^2 + \alpha e & \text{on } \mathbb{RP}^1. \end{cases}$$

Then

$$w[0]_1^h \hat{w}_2^i e^{k-1}[\mathbb{RP}(\nu^k)] = c^h d^i e^{k-1}[\mathbb{RP}(\nu^k)] \neq 0$$

and so

$$\begin{aligned} w[0]_1^h \hat{w}_2^i e^{k-1} &= \alpha^h \left\{ \binom{h+2i}{2} e^2 + \alpha e \right\}^i e^{k-1} \\ &= \alpha^h \binom{h+2i}{2} e^{k+2i-1} \end{aligned}$$

must be nonzero on $\mathbb{RP}^1$. This implies $h = 1$ and then $\binom{h+2i}{2} = \binom{2i+1}{2} = 1$.

Since $w_2(\nu^k) = bd + \binom{a+b}{2} c^2 \neq 0$, $k \geq 2$ and $2(i+1) < 1+2i+2 \leq h+2i+k = m$. Furthermore, one has that

$$\hat{w}_2^{i+1} e^{m-1-2(i+1)}[\mathbb{RP}(\nu^k)] = d^{i+1} e^{m-1-2(i+1)}[\mathbb{RP}(\nu^k)] = 0$$

but

$$\begin{aligned} \hat{w}_2^{i+1} e^{m-1-2(i+1)}[\mathbb{RP}(\nu^{m-1})] &= (e^2 + \alpha e)^{i+1} e^{m-1-2(i+1)}[\mathbb{RP}(\nu^{m-1})] \\ &= e^{m-1}[\mathbb{RP}(\nu^{m-1})] \\ &= \text{coefficient of } \alpha \text{ in } \frac{1}{(1+\alpha)^q} \\ &\neq 0, \end{aligned}$$

which is a contradiction.

Thus, i *is even.*

If b is even, then

$$\hat{w}_2 = \begin{cases} d & \text{on } P(h,i) \\ \binom{h+2i}{2} e^2 & \text{on } \mathbb{RP}^1 \end{cases}$$

and $w[0]_1^h \hat{w}_2^i e^{k-1} = c^h d e^{k-1} \neq 0$ for $P(h,i)$, and for $\mathbb{RP}^1$ this is $\binom{h+2i}{2} \alpha e^{2i+k-1}$. Since this is nonzero, one must have $h = 1$, but then $\binom{h+2i}{2} = \binom{2i+1}{2} = 0$ since $2i + 1 \equiv 1 \bmod 4$.

Thus, b *is odd.* Moreover, a *is even* since $h + i + 1 + a + b \not\equiv 0 \bmod 2$.

For b odd,

$$\hat{w}_2 = \begin{cases} 0 & \text{on } P(h,i) \\ \binom{h+2i}{2} e^2 & \text{on } \mathbb{RP}^1 \end{cases}$$

gives $\hat{w}_2 e^{m-3} = 0$ on $P(h,i)$, but on $\mathbb{RP}^1$ this is $\binom{h+2i}{2} e^{m-1}$ with the value of e^{m-1} on $\mathbb{RP}(\nu^{m-1})$ being the coefficient of α in $\frac{1}{(1+\alpha)^q} = \frac{1}{1+\alpha}$, which is 1. Thus $\binom{h+2i}{2} = 0$, which says $h \equiv 1 \bmod 4$.

If $h > 1$, dualizing $w[0]_1^h$ gives an involution (M^{2i+k}, T) fixing $P(0,i) = \mathbb{CP}^i$ with $k > 0$ and with normal bundle $\nu^k|_{\mathbb{CP}^i}$. The involutions fixing $\mathbb{CP}^i$ are well known and one has $k = 2i$ and $b = i + 1$ with (M^{2i+k}, T) being cobordant to $(\mathbb{CP}^i \times \mathbb{CP}^i, \text{twist})$.

Now, let us find $w[2]_4$. One has that on $P(h,i)$,

$$w[2] = (1+w_1+w_2+\cdots)\{(1+e)^2 + u_1(1+e) + u_2 + u_3(1+e)^{-1} + u_4(1+e)^{-2} + \cdots\}$$

where $w_s = w_s(P(h,i))$ and $u_t = w_t(\nu^k)$ from which

$$w[2]_4 = w_2 e^2 + x_3 e + x_4 \quad (\dim x_3 = 3, \dim x_4 = 4)$$

and

$$w_2(P(h,i)) = (i+1)d + \binom{h+i+1}{2}c^2 = d + \binom{h+i+1}{2}c^2;$$

so

$$w[2]_4 = de^2 + \binom{h+i+1}{2}c^2e^2 + x_3e + x_4$$

and on $\mathbb{RP}^1$

$$w[2] = (1+e)^{h+2i+1} + \alpha(1+e)^{h+2i}.$$

So

$$w[2]_4 = \binom{h+2i+1}{4}e^4 + \alpha\binom{h+2i}{3}e^3 = \binom{h+2i+1}{4}e^4$$

since $h \equiv 1 \bmod 4$. Then

$$\hat{w}_4 = w[2]_4 + \binom{h+i+1}{2}w[0]_1^2e^2 = \begin{cases} de^2 + x_3e + x_4 & \text{on } P(h,i) \\ \binom{h+2i+1}{4}e^4 & \text{on } \mathbb{RP}^1 \end{cases}$$

and so

$$\begin{aligned} w[0]_1^{h-1}\hat{w}_4^ie^{k-2i} &= c^{h-1}(de^2 + x_3e + x_4)^ie^{k-2i} \\ &= c^{h-1}d^ie^{2i}e^{k-2i} \\ &= c^{h-1}d^ie^k \end{aligned}$$

on $P(h,i)$ for since i is even, all other terms have dimension more than $h+2i$ in c and d. The value of this on $\mathbb{RP}(\nu^k)$ is

$$\begin{aligned} c^{h-1}d^i\bar{w}_1(\nu^k)[P(h,i)] &= c^{h-1}d^iw_1(\nu^k)[P(h,i)] \\ &= c^{h-1}d^i(a+b)c[P(h,i)] \\ &= a+b \end{aligned}$$

and $a+b$ is odd. Thus, this is nonzero. However,

$$w[0]_1^{h-1}\hat{w}_4^ie^{k-2i} = \alpha^{h-1}\binom{h+2i+1}{4}e^{k+2i} = 0$$

on $\mathbb{RP}^1$ since $h-1$ is even and positive.

Thus, $h = 1$. Moreover, $a = 0$ and the exotic class cannot occur; so $w(\nu^k) = (1+c+d)^b$.

Now dualizing $w[0]_1$ gives an involution (M^{2i+k}, T) fixing a point $= \mathbb{RP}^0$ and $P(0,i) = \mathbb{CP}^i$ with the normal bundle of $\mathbb{CP}^i$ being $\nu^k|_{\mathbb{CP}^i}$ with $w(\nu^k) = (1+d)^b$. Royster's argument for involutions fixing (point) $\sqcup$ $\mathbb{RP}^{\text{even}}$ also works for fixing (point) $\sqcup$ $\mathbb{CP}^{\text{even}}$ to give

$$b = 1.$$

Furthermore, one knows the possible values of k (see [P-S]). Writing $i = 2^u(2v+1)$ with $u > 0$ one has

$$2 \le k \le \begin{cases} 2^{u+1}+2 & \text{if } u = 1 \\ 2^{u+1} & \text{if } u > 1. \end{cases}$$

Next, it suffices only to show that $k = 2^{u+1}$ with $u \neq 1$ is impossible.

If $k = 2^{u+1}$ with $u > 1$, then $m = 1 + 2^{u+1}(2v+1) + 2^{u+1} = 1 + 2^{u+2}(v+1)$, and by direct computations, one has that on $P(1, 2^u(2v+1))$,

$$w[1]_4 = cde + de^2 + d^2$$

and on $\mathbb{RP}^1$,

$$w[1]_4 = 0.$$

Then

$$w[1]_4^{2^u(v+1)}[\mathbb{RP}(\nu^{m-1})] = 0$$

but

$$\begin{aligned} w[1]_4^{2^u(v+1)}[\mathbb{RP}(\nu^{2^{u+1}})] &= (cde + de^2 + d^2)^{2^u(v+1)}[\mathbb{RP}(\nu^{2^{u+1}})] \\ &= \frac{(cd + d + d^2)^{2^u(v+1)}}{w(\nu^{2^{u+1}})}[P(1, 2^u(2v+1))] \\ &= \frac{(cd + d + d^2)^{2^u(v+1)}}{1 + c + d}[P(1, 2^u(2v+1))] \\ &= d^{2^u(v+1)}(1 + c + d)^{2^u(v+1)-1}[P(1, 2^u(2v+1))] \\ &= \binom{2^u(v+1) - 1}{2^u v}\binom{2^u - 1}{1} cd^{2^u(2v+1)}[P(1, 2^u(2v+1))] \\ &= \binom{2^u v + 2^u - 1}{2^u v} \\ &= 1, \end{aligned}$$

which is a contradiction.

Finally, let us observe the involution $(P(1, N^{i+l}), T_{N^{i+l}})$ with $i = 2^u(2v+1)$ even. Taking $l = 1$, one sees that $(P(1, N^{i+1}), T_{N^{i+1}})$ has the fixed data $\mathbb{RP}^1$ with $w(\nu^{2i+2}) = 1 + \alpha$ and $P(1, i)$ with $w(\nu^2) = 1 + c + d$. If $u = 1$, choosing $l = 2^u + 1 = 3$ one has that $(P(1, N^{i+3}), T_{N^{i+3}})$ also fixes $\mathbb{RP}^1$ with $w(\nu^{2i+6}) = 1 + \alpha$ and $P(1, i)$ with $w(\nu^6) = 1 + c + d$. Hence, for $2 \le k \le 2^{u+1} + 2$ with $u = 1$, (M^{1+2i+k}, T) fixing $\mathbb{RP}^1 \sqcup P(1, i)$ exists and then (M^{1+2i+k}, T) is cobordant to $\Gamma^{k-2}(P(1, N^{i+1}), T_{N^{i+1}})$. If $u > 1$, taking $l = 2^u - 1$, it is easy to see that $(P(1, N^{i+2^u-1}), T_{N^{i+2^u-1}})$ has the same fixed information as $(P(1, N^{i+1}), T_{N^{i+1}})$. Since $l = 2^u - 1 < 2^u$, $P(1, N^{i+2^u-1})$ bounds, and one may apply the operation Γ one time to $(P(1, N^{i+2^u-1}), T_{N^{i+2^u-1}})$, so that $\Gamma(P(1, N^{i+2^u-1}), T_{N^{i+2^u-1}})$ has the same fixed information as $(P(1, N^{i+1}), T_{N^{i+1}})$. Thus, for $2 \le k \le 2^{u+1} - 1$ with $u > 1$, (M^{1+2i+k}, T) fixing $\mathbb{RP}^1 \sqcup P(1, i)$ exists, and so is cobordant to $\Gamma^{k-2}(P(1, N^{i+1}), T_{N^{i+1}})$. □

Note. In her paper [Gu], Guo showed that when $u = 1$, there exists an involution with $k = 7$. This is false.

Returning to the general case of j, one has

Lemma 3.1. *Suppose that (M^{h+2i+k}, T) fixes $\mathbb{RP}^j \sqcup P(h, i)$ with h odd. Then*

(1) $h = j$.

(2) *i is even.*

(3) *$b = 1$, and a is even.*

(4) *For $i = 2^u(2v+1)$, one has*

$$2 \le k \le \begin{cases} 2^{u+1} + 2 & \text{if } u = 1 \\ 2^{u+1} - 1 & \text{if } u > 1. \end{cases}$$

(5) *Exotic characteristic classes do not occur in the bundle ν^k. Thus $w(\nu^k) = (1+c)^a(1+c+d)$.*

Proof. (1)–(4) follow by applying Propositions 3.1 and 3.2. It suffices only to show that (5) holds. For this, one need only consider the case $j = 5$ and suppose $w(\nu^k) = (1+c)^a(1+c+d)(1+\frac{c^4d^2}{(1+d)^2})$. Then

$$\begin{aligned} w(\nu^k) &= (1+c)^a(1+c+d)(1+\frac{c^4d^2}{(1+d)^2}) \\ &= (1+c)^{a-4}(1+c+d)(1+\frac{c^4}{(1+d)^2}) \\ &= (1+c)^{a-4}(1+c+d)^{-1}(1+c^2+d^2+c^4) \\ &= (1+c)^{a-4}(1+c+d) + \frac{c^4(1+c)^{a-4}}{1+c+d} \\ &= (1+c)^{a-4}(1+c+d) + \frac{c^4}{1+c+d} \end{aligned}$$

since a is even. Now

$$\frac{1}{1+c+d} = \frac{1}{1+c}\cdot\frac{1}{1+\frac{d}{1+c}} = \frac{1}{1+c}\left\{1+\frac{d}{1+c}+\frac{d^2}{(1+c)^2}+\cdots+\frac{d^i}{(1+c)^i}\right\}$$

and so

$$\frac{c^4}{1+c+d} = c^4\left\{\frac{1}{1+c}+\frac{d}{(1+c)^2}+\frac{d^2}{(1+c)^3}+\cdots+\frac{d^i}{(1+c)^{i+1}}\right\}$$

and

$$\frac{c^4}{(1+c)^{2s}} = c^4, \quad \frac{c^4}{(1+c)^{2s+1}} = c^4+c^5.$$

Thus

$$w(\nu^k) = \{(1+c)^{a-3}+c^4+c^5\}+d\{(1+c)^{a-4}+c^4\}+d^2(c^4+c^5)+d^3c^4+\cdots+d^i(c^4+c^5).$$

Furthermore, it follows that $w_{2i+5}(\nu^k) \neq 0$ and so $k \geq 2i+5$. However, k never exceeds $2i+2$ since

$$2 \leq k \leq \begin{cases} 2^{u+1}+2 & \text{if } u = 1 \\ 2^{u+1}-1 & \text{if } u > 1. \end{cases}$$

Hence the exotic class cannot occur. □

Note. (1) From the results for $j = 3$ [L-L], one finds that there exist examples with

$$\begin{aligned} u = 1 \text{ for } & q = 1, a = 2 \text{ and } 4 \leq k \leq 6 \\ & q = 3, a = 0 \text{ and } 2 \leq k \leq 4 \end{aligned}$$

and

$$\begin{aligned} u \neq 1 \text{ for } & q = 1, a = 0 \text{ and } 2 \leq k \leq 2^{u+1}-3 \\ & q = 3, a = 2 \text{ and } 4 \leq k \leq 2^{u+1}-1. \end{aligned}$$

One sees that given a pair (q, a), there are $k_{\min}$ and $k_{\max}$ with

$$2 \leq k_{\min}, k_{\max} \leq \begin{cases} 2^{u+1}+2 & \text{if } u = 1 \\ 2^{u+1}-1 & \text{if } u > 1 \end{cases}$$

such that $k_{\min} \le k \le k_{\max}$, but $k_{\min}$ may not be equal to 2, and $k_{\max}$ may not be equal to

$$\begin{cases} 2^{u+1}+2 & \text{if } u=1 \\ 2^{u+1}-1 & \text{if } u>1. \end{cases}$$

However, for $j=1$, $k_{\min}=2$ and

$$k_{\max} = \begin{cases} 2^{u+1}+2 & \text{if } u=1 \\ 2^{u+1}-1 & \text{if } u>1. \end{cases}$$

This is because (q,a) has only one choice, i.e., $(q,a)=(1,0)$ for $j=1$, but not for $j \ge 3$. Thus, Lemma 3.1 does not provide complete information for the general case of j, and the argument is not yet finished.

(2) It is known that $P(j, N^{i+l})$ bounds if $l < k_0$. Let $l = k_0$. One claims that $P(j, N^{i+k_0})$ does not bound. If $P(j, N^{i+k_0})$ bounds, then one may apply the operation Γ one time to $(P(j, N^{i+k_0}), T_{N^{i+k_0}})$, so that the resulting involution $\Gamma(P(j, N^{i+k_0}), T_{N^{i+k_0}})$ fixes $\mathbb{RP}^j$ with $w(\nu^{2i+2k_0+1}) = (1+\alpha)^{i+k_0}$ and $P(j, N^{i+k_0})$ with $w(\nu^{2k_0+1}) = (1+c)^{k_0-1}(1+c+d)$ and has dimension $j+2i+2k_0+1$. However,

$$2k_0+1 > \begin{cases} 2^{u+1}+2 & \text{if } u=1 \\ 2^{u+1}-1 & \text{if } u>1 \end{cases}$$

gives a contradiction.

Recall that $2^p < j < 2^{p+1}$ and $q < 2^{p+1}$. Since $j=h$, a is only determined modulo 2^{p+1} too and it is assumed that $a < 2^{p+1}$. Throughout the following discussions, (M^m, T) fixing $\mathbb{RP}^j \sqcup P(h,i)$ is always assumed to satisfy (1)–(5) stated in Lemma 3.1.

Lemma 3.2. *Suppose (M^m, T) fixes $\mathbb{RP}^j \sqcup P(j,i)$. Then $q \equiv a+i+1 \mod 2^{p+1}$.*

Proof. One first claims that $m > q$. If $q \le j$, then $w_q(\nu^{m-j}) = \binom{q}{q}\alpha^q = \alpha^q \ne 0$; so $m \ge j+q > q$. If $2^p < j < q < 2^{p+1}$, then $w_{2^p}(\nu^{m-j}) = \binom{q}{2^p}\alpha^{2^p} \ne 0$; so $m \ge j + 2^p > 2^{p+1} > q$.

Now let $x \equiv a+i+1 \mod 2^{p+1}$. One claims again that $m > x$. If $i \ge 2^p$, then $m = j+2i+k > 2i \ge 2^{p+1} > x$. If $i < 2^p$ and $a \ge 2^p$, then $w_{2^p+2}(\nu^k) = \binom{a+1}{2^p+2}c^{2^p+2} + c^{2^p}d \ne 0$; so $k \ge 2^p+2$ and $m > j+k > j+2^p > 2^{p+1} > x$. If $i < 2^p$ and $a < 2^p$, then $x = a+i+1$ and $w_{a+2}(\nu^k) = \binom{a}{a}c^a d = c^a d \ne 0$; so $k \ge a+2$ and $m > i+k > i+a+1 = x$.

From (1.1) one has that

$$w[1]_1 = \begin{cases} e+c & \text{on } P(j,i) \\ e+\alpha & \text{on } \mathbb{RP}^j. \end{cases}$$

The argument proceeds as follows.

(i) If $x > q$, then $x-(q+1) \geq 0$. When $0 \leq x-(q+1) < j$, one has

$$\begin{aligned}
w[1]_1^{x-1}e^{m-x}[\mathbb{RP}(\nu^k)] &= (e+c)^{x-1}e^{m-x}[\mathbb{RP}(\nu^k)] \\
&= \frac{(1+c)^{x-1}}{(1+c)^a(1+c+d)}[P(j,i)] \\
&= \frac{(1+c)^{x-1}}{(1+c)^{a+1}} \cdot \frac{1}{1+\frac{d}{1+c}}[P(j,i)] \\
&= \frac{(1+c)^{x-1}}{(1+c)^{a+1}} \left\{ 1 + \frac{d}{1+c} + \cdots + \frac{d^i}{(1+c)^i} \right\} [P(j,i)] \\
&= \frac{d^i}{1+c}[P(j,i)] \\
&= c^j d^i [P(j,i)] \\
&= 1
\end{aligned}$$

but

$$\begin{aligned}
w[1]_1^{x-1}e^{m-x}[\mathbb{RP}(\nu^{m-j})] &= (e+\alpha)^{x-1}e^{m-x}[\mathbb{RP}(\nu^{m-j})] \\
&= \frac{(1+\alpha)^{x-1}}{(1+\alpha)^q}[\mathbb{RP}^j] \\
&= (1+\alpha)^{x-q-1}[\mathbb{RP}^j] \\
&= 0
\end{aligned}$$

since $x-q-1 < j$, which leads to a contradiction. When $j \leq x-(q+1) < 2^{p+1}$, one has

$$\begin{aligned}
w[1]_1^{q-1}e^{m-q}[\mathbb{RP}(\nu^k)] &= (e+c)^{q-1}e^{m-q}[\mathbb{RP}(\nu^k)] \\
&= \frac{(1+c)^{q-1}}{(1+c)^a(1+c+d)}[P(j,i)] \\
&= \frac{1}{(1+c)^{x+1-q}} d^i [P(j,i)] \\
&= (1+c)^{2^{p+1}-1-x+q} d^i [P(j,i)] \\
&= \binom{2^{p+1}-1-x+q}{j} \\
&= 0
\end{aligned}$$

since $2^{p+1}-1-x+q = 2^{p+1}-2-(x-q-1) \leq 2^{p+1}-2-j < j$, but

$$\begin{aligned}
w[1]_1^{q-1}e^{m-q}[\mathbb{RP}(\nu^{m-j})] &= (e+\alpha)^{q-1}e^{m-q}[\mathbb{RP}(\nu^{m-j})] \\
&= \frac{(1+\alpha)^{q-1}}{(1+\alpha)^q}[\mathbb{RP}^j] \\
&= \frac{1}{1+\alpha}[\mathbb{RP}^j] \\
&= 1.
\end{aligned}$$

Thus, $x > q$ is impossible.

(ii) If $x < q$, in a similar way to (i), one may obtain that this is also impossible.

Combining (i) and (ii), x must be equal to q. □

Since the case $j = 1$ is understood well (see Proposition 3.2), one always assumes $j \geq 3$ in the following discussions. Now one divides the argument into two cases: (I) $u = 1$; (II) $u > 1$.

Case (I). $u = 1$**.** For $u = 1$ one has $i = 4v + 2$. Suppose $(M^{j+8v+4+k}, T)$ fixes $\mathbb{RP}^j \sqcup P(j, 4v+2)$. The argument proceeds as follows.

First, one cannot have $a > 6$. For $a \geq 8$, one must have $j \geq 8$ (else a is taken mod 8) and a must have a power of 2 that is at least 8 and less than j in its 2-adic expansion. Then there is at least a nonzero term $w_s(\nu^k)$ with $s > 6$ in $w(\nu^k)$, and ν^k cannot be realized by a bundle of dimension less than or equal to 6.

For $a = 6$, one cannot have $j \geq 7$, for then $\binom{6}{6}c^6 d \neq 0$ making $k \geq 8$. Thus $a = 6$ can occur only for $j = 5$, and one must have $k = 6$ and $q \equiv 4v + 1 \mod 8$. In particular, $q = 1$ if v is even, and $q = 5$ if v is odd.

Claim. $a = 6$ *is impossible.*

Proof. One computes the values of $w[1]_4$ and $w[1]_{8v+6}$. On $P(5, 4v+2)$, one has

$$\begin{aligned} w[1] \;&=\; (1+c)^5(1+c+d)^{4v+3}\{1+e+c+(c^2+d)(1+e)^{-1}+c^3(1+e)^{-2} \\ &\quad + (c^4+c^2d)(1+e)^{-3}+c^5(1+e)^{-4}+c^4d(1+e)^{-5}\} \end{aligned}$$

and so

$$w[1]_4 = cde + c^2e^2 + de^2 + c^4 + \binom{v}{1}c^4 = \begin{cases} cde + c^2e^2 + de^2 + c^4 & \text{if } v \text{ is even} \\ cde + c^2e^2 + de^2 & \text{if } v \text{ is odd} \end{cases}$$

and

$$w[1]_{8v+6} = d^{4v+2}(ce + e^2) + \text{terms of degree less than } 4v+2 \text{ in } d.$$

On $\mathbb{RP}^5$,

$$\begin{aligned} w[1] &= (1+\alpha)^6\left\{(1+e)^{8v+5} + \alpha(1+e)^{8v+4} + \binom{q}{4}\alpha^4(1+e)^{8v+1} + \binom{q}{5}\alpha^5(1+e)^{8v}\right\} \\ &= (1+\alpha)^6\left\{(1+e)^5 + \alpha(1+e)^4 + \binom{q}{4}\alpha^4(1+e) + \binom{q}{5}\alpha^5\right\}(1+e)^{8v} \end{aligned}$$

and so

$$w[1]_4 = \begin{cases} \alpha^4 + e^4 & \text{if } q = 1 \\ e^4 & \text{if } q = 5 \end{cases}$$

and $w[1]_{8v+6} = \alpha^2 e^{8v+4}$ for $q = 1$ or $q = 5$.

If v is even, then

$$w[1]_4 + w[1]_1^4 = \begin{cases} cde + c^2e^2 + de^2 + e^4 & \text{on } P(5, 4v+2) \\ 0 & \text{on } \mathbb{RP}^5 \end{cases}$$

with $w[1]_1$ and $w[1]_{8v+6}$ together giving

$$w[1]_1^3(w[1]_4 + w[1]_1^4)w[1]_{8v+6}e[\mathbb{RP}(\nu^{8v+10})] = 0,$$

but

$$
\begin{aligned}
&w[1]_1^3(w[1]_4+w[1]_1^4)w[1]_{8v+6}e[\mathbb{RP}(\nu^6)]\\
&\quad=\{(e+c)^3(cde+c^2e^2+de^2+e^4)\\
&\qquad\times(cd^{4v+2}e+d^{4v+2}e^2+\text{terms of degree}<4v+2\text{ in }d)\}e[\mathbb{RP}(\nu^6)]\\
&\quad=\frac{(1+c)^3(1+cd+c^2+d)(cd^{4v+2}+d^{4v+2}+\text{terms of degree}<4v+2\text{ in }d)}{(1+c)^6(1+c+d)}\\
&\qquad\times[P(5,4v+2)]\\
&\quad=\frac{(1+c+d+c+c^2+cd)\{d^{4v+2}(1+c)+\text{terms of degree}<4v+2\text{ in }d\}}{(1+c)^3(1+c+d)}\\
&\qquad\times[P(5,4v+2)]\\
&\quad=\frac{(1+c)(1+c+d)(d^{4v+2}(1+c)+\text{terms of degree}<4v+2\text{ in }d)}{(1+c)^3(1+c+d)}[P(5,4v+2)]\\
&\quad=\frac{d^{4v+2}(1+c)+\text{terms of degree}<4v+2\text{ in }d}{(1+c)^2}[P(5,4v+2)]\\
&\quad=\frac{d^{4v+2}}{1+c}[P(5,4v+2)]+\frac{\text{terms of degree}<4v+2\text{ in }d}{(1+c)^2}[P(5,4v+2)]\\
&\quad=1+0\\
&\quad=1.
\end{aligned}
$$

If v is odd, in a similar way to the above, then

$$
w[1]_4+w[1]_1^4+w[0]_1^4=\begin{cases}cde+c^2e^2+de^2+e^4 & \text{on } P(5,4v+2)\\ 0 & \text{on } \mathbb{RP}^5\end{cases}
$$

with $w[1]_1$ and $w[1]_{8v+6}$ together giving

$$
w[1]_1^3(w[1]_4+w[1]_1^4+w[0]_1^4)w[1]_{8v+6}e[\mathbb{RP}(\nu^{8v+10})]=0,
$$

but

$$
w[1]_1^3(w[1]_4+w[1]_1^4+w[0]_1^4)w[1]_{8v+6}e[\mathbb{RP}(\nu^6)]=1.
$$

Therefore, $a=6$ is impossible. □

For $a=4$, $\nu^6=4\xi\oplus\eta$ provides a suitable ν^k and, of course, $k=6$ is the only possibility. However, dualizing $w[0]_1^{j-3}$ changes this case into the case $j=3$ with $a=0$, and the range of the values of k must lie in $2\le k\le 4$. Therefore, $a=4$ is impossible.

For $a=2$, one has $q\equiv 4v+5 \mod 2^{p+1}$. Dualizing $w[0]_1^{j-3}$ changes this case into the case $j=3$ with $q=1$. Thus one has that $4\le k\le 6$. Taking $l=3$ in the involution $(P(j,N^{4v+2+l}),T_{N^{4v+2+l}})$, then for each $4\le k\le 6$, $\Gamma^{k-6}(P(j,N^{4v+5}),T_{N^{4v+5}})$ fixes $\mathbb{RP}^j$ with $w(\nu^{8v+4+k})=(1+\alpha)^q$ and $P(j,4v+2)$ with $w(\nu^k)=(1+c)^2(1+c+d)$. Hence, $(M^{j+8v+4+k},T)$ is cobordant to $\Gamma^{k-6}(P(j,N^{4v+5}),T_{N^{4v+5}})$ for $4\le k\le 6$.

For $a=0$, one has $q\equiv 4v+3 \mod 2^{p+1}$. Now dualizing $w[0]_1^{j-3}$ changes the general case j into the case $j=3$ with $q=3$. Then one knows that $2\le k\le 4$. Proposition 2.1 provides the examples of the involutions of this type. $\Gamma^{k-2}(P(j,N^{4v+3}),T_{N^{4v+3}})$ fixing $\mathbb{RP}^j$ with $w(\nu^{8v+4+k})=(1+\alpha)^{4v+3}$ and $P(j,4v+2)$ with $w(\nu^k)=1+c+d$ belongs to the involution of this type for $2\le k\le 2+X_0$, and so X_0 must be less than or equal to 2, and $X_0\ge 1$ since

$P(j, N^{4v+3})$ bounds. Although one knows that $X_0 = 2$ for $j = 3$, this does not ensure that $X_0 = 2$ for $j > 3$.

Claim. *For $a = 0$, $X_0 = 2$.*

Proof. It suffices to prove that $\Gamma(P(j, N^{4v+3}))$ bounds. According to Conner and Floyd [C-F], this is equivalent to showing that $\mathbb{RP}(\nu^{8v+6} \oplus 2\mathbb{R})$ is cobordant to $\mathbb{RP}(\nu^2 \oplus 2\mathbb{R})$, where the disjoint union of $(\mathbb{RP}^j, \nu^{8v+6})$ with $w(\nu^{8v+6}) = (1+\alpha)^{4v+3}$ and $(P(j, 4v+2), \nu^2)$ with $w(\nu^2) = 1+c+d$ is the fixed data of $(P(j, N^{4v+3}), T_{N^{4v+3}})$.

First, let us look at the total Stiefel-Whitney classes of $\mathbb{RP}(\nu^{8v+6} \oplus 2\mathbb{R})$ and $\mathbb{RP}(\nu^2 \oplus 2\mathbb{R})$:

$$\begin{aligned} w(\mathbb{RP}(\nu^{8v+6} \oplus 2\mathbb{R})) &= (1+\alpha)^{j+1}\{(1+e)^{8v+8} + \alpha(1+e)^{8v+7} + \cdots \\ &\quad + \binom{4v+3}{z}\alpha^z(1+e)^{8v+8-z} + \cdots + \alpha^{4v+3}(1+e)^{4v+5}\} \\ &= (1+\alpha)^{j+1}(1+e)^{4v+5}(1+e+\alpha)^{4v+3} \\ &= (1+\alpha)^{j+1}(1+\alpha+\alpha e+e^2)^{4v+3}(1+e^2) \end{aligned}$$

and

$$\begin{aligned} w(\mathbb{RP}(\nu^2 \oplus 2\mathbb{R})) &= (1+c)^j(1+c+d)^{4v+3}\{(1+e)^4 + c(1+e)^3 + d(1+e)^2\} \\ &= (1+c)^j(1+c+d)^{4v+3}(1+e)^2(1+c+e^2+ce+d). \end{aligned}$$

According to Borel and Hirzebruch [B-H] (see also [C-F]), one knows that on $\mathbb{RP}(\nu^2 \oplus 2\mathbb{R})$,

$$e^4 = ce^3 + de^2.$$

Let $\sigma = e^2 + ce + d$. Then $\sigma e^2 = 0$. Replacing d by $ce + e^2 + \sigma$ in $w(\mathbb{RP}(\nu^2 \oplus 2\mathbb{R}))$, one has

$$\begin{aligned} &w(\mathbb{RP}(\nu^2 \oplus 2\mathbb{R})) \\ &\quad = (1+c)^j(1+c+d)^{4v+3}(1+e)^2(1+c+e^2+ce+d) \\ &\quad = (1+c)^j(1+c+ce+e^2+\sigma)^{4v+3}(1+e^2)(1+c+\sigma) \\ &\quad = (1+c)^{j+1}(1+c+ce+e^2+\sigma)^{4v+3}(1+e^2) + (1+c)^j(1+c+ce+\sigma)^{4v+3}\sigma \\ &\quad = (1+c)^{j+1}(1+e^2)\{(1+c+ce+e^2)^{4v+3} + (1+c+ce+e^2)^{4v+2}\sigma + \cdots \\ &\qquad + \binom{4v+3}{z}(1+c+ce+e^2)^{4v+3-z}\sigma^z + \cdots + \sigma^{4v+3}\} \\ &\qquad + (1+c)^j(1+c+ce+\sigma)^{4v+3}\sigma \\ &\quad = (1+c)^{j+1}(1+c+ce+e^2)^{4v+3}(1+e^2) + (1+c)^{j+1}\{(1+c+ce)^{4v+2}\sigma + \cdots \\ &\qquad + \binom{4v+3}{z}(1+c+ce)^{4v+3-z}\sigma^z + \cdots + \sigma^{4v+3}\} \\ &\qquad + (1+c)^j(1+c+ce+\sigma)^{4v+3}\sigma \\ &\quad = (1+c)^{j+1}(1+c+ce+e^2)^{4v+3}(1+e^2) + \phi(c, ce, \sigma). \end{aligned}$$

One sees that if one writes the ℓ-th Stiefel-Whitney class w_ℓ of $\mathbb{RP}(\nu^{8v+6} \oplus 2\mathbb{R})$ in a polynomial

$$p_\ell(\alpha, e^2, \alpha e + e^2),$$

then the ℓ-th Stiefel-Whitney class w_ℓ of $\mathbb{RP}(\nu^2 \oplus 2\mathbb{R})$ is of the form

$$p_\ell(c, e^2, ce + e^2) + \phi_\ell(c, ce, \sigma)$$

where $\phi_\ell(c, ce, \sigma)$ is the sum of all terms of degree ℓ in $\phi(c, ce, \sigma)$. Thus, for any characteristic class

$$w_{\ell_1} \cdots w_{\ell_s} \text{ with } \ell_1 + \cdots + \ell_s = j + 8v + 7,$$

one has that

$$\begin{aligned} & w_{\ell_1} \cdots w_{\ell_s}([\mathbb{RP}(\nu^{8v+6} \oplus 2\mathbb{R})] + [\mathbb{RP}(\nu^2 \oplus 2\mathbb{R})]) \\ &\quad = (p_{\ell_1} \cdots p_{\ell_s})(\alpha, e^2, \alpha e + e^2)[\mathbb{RP}(\nu^{8v+6} \oplus 2\mathbb{R})] \\ &\qquad + (p_{\ell_1} \cdots p_{\ell_s})(c, e^2, ce + e^2)[\mathbb{RP}(\nu^2 \oplus 2\mathbb{R})] \\ &\qquad + (\phi_{\ell_1} \cdots \phi_{\ell_s})(c, ce, \sigma)[\mathbb{RP}(\nu^2 \oplus 2\mathbb{R})]. \end{aligned}$$

Since

$$\begin{aligned} & (p_{\ell_1} \cdots p_{\ell_s})(\alpha, e^2, \alpha e + e^2)[\mathbb{RP}(\nu^{8v+6} \oplus 2\mathbb{R})] \\ &\quad + (p_{\ell_1} \cdots p_{\ell_s})(c, e^2, ce + e^2)[\mathbb{RP}(\nu^2 \oplus 2\mathbb{R})] \\ &= \frac{(p_{\ell_1} \cdots p_{\ell_s})(\alpha, 1, 1+\alpha)}{w(\nu^{8v+6} \oplus 2\mathbb{R})}[\mathbb{RP}^j] + \frac{(p_{\ell_1} \cdots p_{\ell_s})(c, 1, 1+c)}{w(\nu^2 \oplus 2\mathbb{R})}[P(j, 4v+2)] \\ &= \frac{(p_{\ell_1} \cdots p_{\ell_s})(\alpha, 1, 1+\alpha)}{(1+\alpha)^{4v+3}}[\mathbb{RP}^j] + \frac{(p_{\ell_1} \cdots p_{\ell_s})(c, 1, 1+c)}{1+c+d}[P(j, 4v+2)] \\ &= \frac{(p_{\ell_1} \cdots p_{\ell_s})(\alpha, 1, 1+\alpha)}{(1+\alpha)^{4v+3}}[\mathbb{RP}^j] + \frac{(p_{\ell_1} \cdots p_{\ell_s})(c, 1, 1+c)}{(1+c)(1+\frac{d}{1+c})}[P(j, 4v+2)] \\ &= \frac{(p_{\ell_1} \cdots p_{\ell_s})(\alpha, 1, 1+\alpha)}{(1+\alpha)^{4v+3}}[\mathbb{RP}^j] + (p_{\ell_1} \cdots p_{\ell_s})(c, 1, 1+c)\{\frac{1}{1+c} + \frac{d}{(1+c)^2} \\ &\quad + \cdots + \frac{d^{4v+2}}{(1+c)^{4v+3}}\}[P(j, 4v+2)] \\ &= \frac{(p_{\ell_1} \cdots p_{\ell_s})(\alpha, 1, 1+\alpha)}{(1+\alpha)^{4v+3}}[\mathbb{RP}^j] + \frac{(p_{\ell_1} \cdots p_{\ell_s})(c, 1, 1+c)}{(1+c)^{4v+3}}d^{4v+2}[P(j, 4v+2)] \\ &= 0, \end{aligned}$$

one has that

$$w_{\ell_1} \cdots w_{\ell_s}([\mathbb{RP}(\nu^{8v+6} \oplus 2\mathbb{R})] + [\mathbb{RP}(\nu^2 \oplus 2\mathbb{R})]) = (\phi_{\ell_1} \cdots \phi_{\ell_s})(c, ce, \sigma)[\mathbb{RP}(\nu^2 \oplus 2\mathbb{R})].$$

If

$$(\phi_{\ell_1} \cdots \phi_{\ell_s})(c, ce, \sigma)[\mathbb{RP}(\nu^2 \oplus 2\mathbb{R})] = 0,$$

then

$$w_{\ell_1} \cdots w_{\ell_s}([\mathbb{RP}(\nu^{8v+6} \oplus 2\mathbb{R})] + [\mathbb{RP}(\nu^2 \oplus 2\mathbb{R})]) = 0;$$

so $\mathbb{RP}(\nu^{8v+6} \oplus 2\mathbb{R})$ is cobordant to $\mathbb{RP}(\nu^2 \oplus 2\mathbb{R})$. Thus, to complete the proof, it suffices merely to show that

$$(\phi_{\ell_1} \cdots \phi_{\ell_s})(c, ce, \sigma)[\mathbb{RP}(\nu^2 \oplus 2\mathbb{R})] = 0.$$

The argument proceeds as follows.

For each monomial $c^{h_1}e^{h_2}\sigma^{h_3}$ of $(\phi_{\ell_1} \cdots \phi_{\ell_s})(c, ce, \sigma)$ in c, e and σ (note that $h_3 > 0$ always holds), if $h_2 \geq 2$, then

$$\begin{aligned} c^{h_1}e^{h_2}\sigma^{h_3}[\mathbb{RP}(\nu^2 \oplus 2\mathbb{R})] &= \frac{c^{h_1}(1+c+d)^{h_3}}{1+c+d}[P(j, 4v+2)] \\ &= c^{h_1}(1+c+d)^{h_3-1}[P(j, 4v+2)] = 0 \end{aligned}$$

since $h_1+2(h_3-1) = h_1+h_2+2h_3-(h_2+2) = j+8v+7-(h_2+2) < j+8v+4$. Thus, the possibility for $c^{h_1}e^{h_2}\sigma^{h_3}[\mathbb{RP}(\nu^2 \oplus 2\mathbb{R})] \neq 0$ is that $h_2 = 0$ or 1. Furthermore, it is easy to see that $c^{h_1}e^{h_2}\sigma^{h_3}[\mathbb{RP}(\nu^2 \oplus 2\mathbb{R})]$ is nonzero if and only if $c^{h_1}e^{h_2}\sigma^{h_3}$ is either $c^j e\sigma^{4v+3}$ or $c^{j-1}\sigma^{4v+4}$.

Next, one further analyses $\phi(c, ce, \sigma)$. Write $1+c = 1+c+ce+ce$. Then $(1+c)^{j+1} = (1+c)^j(1+c+ce) + (1+c)^j ce$, and so

$$
\begin{aligned}
&(1+c)^{j+1}\{(1+c+ce)^{4v+2}\sigma + \cdots + \binom{4v+3}{z}(1+c+ce)^{4v+3-z}\sigma^z + \cdots + \sigma^{4v+3}\} \\
&= (1+c)^j\{(1+c+ce)^{4v+3}\sigma + \cdots + \binom{4v+3}{z}(1+c+ce)^{4v+4-z}\sigma^z \\
&\quad + \cdots + (1+c+ce)\sigma^{4v+3}\} + (1+c)^j ce\{(1+c+ce)^{4v+2}\sigma + \cdots \\
&\quad + \binom{4v+3}{z}(1+c+ce)^{4v+3-z}\sigma^z + \cdots + \sigma^{4v+3}\} \\
&= (1+c)^j\{(1+c+ce)^{4v+3}\sigma + \cdots + \binom{4v+3}{z}(1+c+ce)^{4v+4-z}\sigma^z \\
&\quad + \cdots + (1+c+ce)\sigma^{4v+3}\} + (1+c)^j ce\{(1+c)^{4v+2}\sigma + \cdots \\
&\quad + \binom{4v+3}{z}(1+c)^{4v+3-z}\sigma^z + \cdots + \sigma^{4v+3}\} \\
&= (1+c)^j\{(1+c+ce)^{4v+3}\sigma + \cdots + \binom{4v+3}{z}(1+c+ce)^{4v+4-z}\sigma^z \\
&\quad + \cdots + (1+c+ce)\sigma^{4v+3}\} + \sum_{0\le x\le v}\{\binom{4v+3}{4x+1}(1+c)^{j+4v-4x+2}ce\sigma^{4x+1} \\
&\quad + \binom{4v+3}{4x+2}(1+c)^{j+4v-4x+1}ce\sigma^{4x+2} + \binom{4v+3}{4x+3}(1+c)^{j+4v-4x}ce\sigma^{4x+3} \\
&\quad + \binom{4v+3}{4x+4}(1+c)^{j+4v-4x-1}ce\sigma^{4x+4}\}
\end{aligned}
$$

since $\sigma e^2 = 0$. 384 Note that $\binom{4v+3}{4x+4} = 0$ when $x = v$. Since

$$
\begin{aligned}
&(1+c)^j(1+c+ce+\sigma)^{4v+3}\sigma \\
&= (1+c)^j\left\{\binom{4v+3}{0}(1+c+ce)^{4v+3}\sigma + \cdots + \binom{4v+3}{z-1}(1+c+ce)^{4v+4-z}\sigma^z\right. \\
&\qquad\qquad \left.\binom{4v+3}{z-1} + \cdots + \sigma^{4v+4}\right\}
\end{aligned}
$$

one has that

$$
\begin{aligned}
&(1+c)^j\{(1+c+ce)^{4v+3}\sigma + \cdots + \binom{4v+3}{z}(1+c+ce)^{4v+4-z}\sigma^z \\
&\quad + \cdots + (1+c+ce)\sigma^{4v+3}\} + (1+c)^j(1+c+ce+\sigma)^{4v+3}\sigma \\
&= (1+c)^j\sum_{0\le x\le v}\binom{4v+4}{4x+4}(1+c+ce)^{4v-4x}\sigma^{4x+4} \\
&= \sum_{0\le x\le v}\binom{4v+4}{4x+4}(1+c)^{j+4v-4x}\sigma^{4x+4}.
\end{aligned}
$$

Thus, $\phi(c, ce, \sigma)$ can be expressed as follows.

$$\begin{aligned}
&\phi(c, ce, \sigma)\\
&= \sum_{0\le x\le v}\Big\{\binom{4v+3}{4x+1}(1+c)^{j+4v-4x+2}ce\sigma^{4x+1}\\
&\quad + \binom{4v+3}{4x+2}(1+c)^{j+4v-4x+1}ce\sigma^{4x+2}\\
&\quad + (1+c)^{j+4v-4x}[\binom{4v+3}{4x+3}ce\sigma^{4x+3} + \binom{4v+4}{4x+4}\sigma^{4x+4}]\\
&\quad + \binom{4v+3}{4x+4}(1+c)^{j+4v-4x-1}ce\sigma^{4x+4}\Big\}.
\end{aligned}$$

With the above understood, let $\varphi(x) = \binom{4v+3}{4x+3}ce\sigma^{4x+3} + \binom{4v+4}{4x+4}\sigma^{4x+4}$. Then

$$(\phi_{\ell_1}\cdots\phi_{\ell_s})(c, ce, \sigma)$$

can also be expressed as a sum of those monomials generated by the factors of the following forms:

$$c,\ ce\sigma^{4x+1},\ ce\sigma^{4x+2},\ ce\sigma^{4x+4},\ \varphi(x).$$

Obviously, if such a monomial of $(\phi_{\ell_1}\cdots\phi_{\ell_s})(c, ce, \sigma)$ contains the factor of the form $ce\sigma^{4x+1}$ or $ce\sigma^{4x+2}$ or $ce\sigma^{4x+4}$, then its expression in c, e and σ has no terms $c^j e\sigma^{4v+3}$ and $c^{j-1}\sigma^{4v+4}$; so its value on $[\mathbb{RP}(\nu^2 \oplus 2\mathbb{R})]$ is zero. Now one considers the monomials of the form

$$c^t\varphi(x_1)\cdots\varphi(x_\gamma)$$

of $(\phi_{\ell_1}\cdots\phi_{\ell_s})(c, ce, \sigma)$ where $t + 8x_1 + 8 + \cdots + 8x_\gamma + 8 = j + 8v + 7$. First, let us look at

$$\varphi(x) = \binom{4v+3}{4x+3}ce\sigma^{4x+3} + \binom{4v+4}{4x+4}\sigma^{4x+4}.$$

It is easy to see that when v is even, $\binom{4v+4}{4x+4} = 1$ if and only if $\binom{4v+3}{4x+3} = 1$; when v is odd, if $\binom{4v+4}{4x+4} = 1$, then $\binom{4v+3}{4x+3} = 1$, but conversely, it may not be true. For example, take $v = 3$ and $x = 1$. Then $\binom{4\times3+3}{4+3} = 1$ but $\binom{4\times3+4}{4+4} = 0$.

When v is odd, let $x_0 = \min\{x | \binom{4v+4}{4x+4} = 1\}$. Then one easily sees that

(1) $\binom{v}{x_0} = 1$ and $x_0 + 1$ is of the form 2^t.

(2) For x with $\text{common}(x, x_0) < x_0$ or $\text{common}(x, x_0)$ empty, if $\binom{4v+3}{4x+3} = 1$, then $\binom{4v+4}{4x+4} = 0$. In particular, if $x < x_0$, then $\binom{4v+3}{4x+3} = 1$ but $\binom{4v+4}{4x+4} = 0$.

(3) For x with $\text{common}(x, x_0) = x_0$, $\binom{4v+4}{4x+4} = 1$ if and only if $\binom{4v+3}{4x+3} = 1$.

Let x' be such that $\binom{4v+3}{4x'+3} = 1$ and $\text{common}(x', x_0)$ is either less than x_0 or empty. Then $\binom{4v+4}{4x'+4} = 0$. If $c^t\varphi(x_1)\cdots\varphi(x_\gamma)$ contains the factor $ce\sigma^{4x'+3} = \varphi(x')$, one claims that

$$c^t\varphi(x_1)\cdots\varphi(x_\gamma)[\mathbb{RP}(\nu^2 \oplus 2\mathbb{R})] = 0.$$

When the 2-adic expansion of v has no gap, one has $x_0 = v$; so for any $x < v$, $\binom{4v+3}{4x+3} = 1$ but $\binom{4v+4}{4x+4} = 0$, and for $x = v$, $\binom{4v+3}{4x+3} = \binom{4v+4}{4x+4} = 1$. In this case, since

$x' \neq v$, obviously one has that

$$c^t\varphi(x_1)\cdots\varphi(x_\gamma)[\mathbb{RP}(\nu^2\oplus 2\mathbb{R})] = 0.$$

When the 2-adic expansion of v has at least one gap, one has $x_0 < v$. If

$$c^t\varphi(x_1)\cdots\varphi(x_\gamma)[\mathbb{RP}(\nu^2\oplus 2\mathbb{R})] = 1,$$

then $ce\sigma^{4x'+3}$ must appear only one time in $c^t\varphi(x_1)\cdots\varphi(x_\gamma)$, and since $x' \neq v$, one has $\gamma > 1$ and each $\varphi(x_y)(\neq ce\sigma^{4x'+3})$ of $c^t\varphi(x_1)\cdots\varphi(x_\gamma)$ must contain the factor $ce\sigma^{4x_0+3}+\sigma^{4x_0+4}$ by the above (3), so that each term of $c^t\varphi(x_1)\cdots\varphi(x_\gamma)$ in c, e and σ must be of two forms: (1) $e^{a_1}c^{a_2}\sigma^{a_3}$ with $a_1 > 1$; (2) $ec^{l_1}\sigma^{4l_2+3}$ with the property that either $\text{common}(4l_2, 4x_0 - \text{common}(4x', 4x_0))$ is empty (this corresponds to the case $\text{common}(x', x_0) < x_0$) or $\text{common}(4l_2, 4x_0)$ is empty (this corresponds to the case $\text{common}(x', x_0)$ is empty). Since $\text{common}(4v, 4x_0) = 4x_0 = 4(2^t-1)$, one has that the expression of $c^t\varphi(x_1)\cdots\varphi(x_\gamma)$ in c, e and σ has no terms $c^je\sigma^{4v+3}$ and $c^{j-1}\sigma^{4v+4}$; so the value of $c^t\varphi(x_1)\cdots\varphi(x_\gamma)$ on $\mathbb{RP}(\nu^2\oplus 2\mathbb{R})$ must be zero. This is a contradiction.

If each factor $\varphi(x_y)$ of $\varphi(x_1)\cdots\varphi(x_\gamma)$ is of the form $ce\sigma^{4x+3}+\sigma^{4x+4}$, then it is easy to see that if the expression of $c^t\varphi(x_1)\cdots\varphi(x_\gamma)$ in c, e and σ contains one of both $c^je\sigma^{4v+3}$ and $c^{j-1}\sigma^{4v+4}$, then it must contain the other one, too. Thus one concludes that

$$c^t\varphi(x_1)\cdots\varphi(x_\gamma)[\mathbb{RP}(\nu^2\oplus 2\mathbb{R})] = 0.$$

Together with the above arguments, one has

$$(\phi_{\ell_1}\cdots\phi_{\ell_s})(c, ce, \sigma)[\mathbb{RP}(\nu^2\oplus 2\mathbb{R})] = 0.$$

This completes the proof. □

Combining the above arguments, one has

Proposition 3.3. *Suppose that* $(M^{j+8v+4+k}, T)$ *fixes* $\mathbb{RP}^j \sqcup P(j, 4v+2)$ *with* $j \geq 3$. *Then either*

(1) $a = 0$, $q \equiv 4v+3 \mod 2^{p+1}$ *and* $2 \leq k \leq 4$. *Furthermore,* $(M^{j+8v+4+k}, T)$ *is cobordant to* $\Gamma^{k-2}(P(j, N^{4v+3}), T_{N^{4v+3}})$*; or*

(2) $a = 2$, $q \equiv 4v+5 \mod 2^{p+1}$ *and* $4 \leq k \leq 6$. *Furthermore,* $(M^{j+8v+4+k}, T)$ *is cobordant to* $\Gamma^{k-6}(P(j, N^{4v+5}), T_{N^{4v+5}})$.

Case (II). $u > 1$. From the case $u = 1$, one sees that $a \leq 2^u = 2$, so that the involutions may correspond to those examples constructed in section 2. Also, for the special cases $j = 1, 3$ with $u > 1$, one has $a < 2^u$. Now one considers the general cases with $u > 1$.

Lemma 3.3. *If* $u > 1$, *then* $a < 2^u$.

Proof. First, one computes the values of $w[1]_4$. On $P(j,i)$,

$$\begin{aligned}
w[1] &= (1+c)^j(1+c+d)^{i+1} \\
&\quad \times \{1+e+c+\frac{\binom{a+1}{2}c^2+d}{1+e}+\frac{\binom{a+1}{3}c^3}{(1+e)^2}+\frac{\binom{a+1}{4}c^4+\binom{a}{2}c^2d}{(1+e)^3}+\cdots\} \\
&= \{(1+c)^{i+j+1}+(1+c)^{i+j}d+\binom{i+1}{2}(1+c)^{i+j-1}d^2+\cdots\} \\
&\quad \times \{1+e+c+\binom{a+1}{2}c^2+d+\binom{a+1}{2}c^2e+de+\binom{a+1}{2}c^2e^2 \\
&\quad + de^2+\binom{a+1}{3}c^3+\binom{a+1}{4}c^4+\binom{a}{2}c^2d+\cdots\} \\
&= \{1+\binom{i+j+1}{2}c^2+\binom{i+j+1}{4}c^4+d+cd+\binom{i+j}{2}c^2d+\cdots\} \\
&\quad \times \{1+e+c+\binom{a+1}{2}c^2+d+\binom{a+1}{2}c^2e+de+\binom{a+1}{2}c^2e^2 \\
&\quad + de^2+\binom{a+1}{3}c^3+\binom{a+1}{4}c^4+\binom{a}{2}c^2d+\cdots\}
\end{aligned}$$

(note that $i+j+1$ is even and $\binom{i+1}{2}=0$); so

$$\begin{aligned}
w[1]_4 &= \binom{a+1}{2}c^2e^2+de^2+\binom{a+1}{4}c^4+\binom{a}{2}c^2d+\binom{i+j+1}{2}\binom{a+1}{2}c^4 \\
&\quad + \binom{i+j+1}{2}c^2d+\binom{a+1}{2}c^2d+d^2+c^2d+cde \\
&\quad + \binom{i+j+1}{4}c^4+\binom{i+j}{2}c^2d \\
&= \binom{a+1}{2}c^2e^2+\left\{\binom{a+1}{4}+\binom{i+j+1}{2}\binom{a+1}{2}+\binom{i+j+1}{4}\right\}c^4 \\
&\quad + de^2+d^2+cde.
\end{aligned}$$

Note that $\binom{a}{2}+\binom{a+1}{2}=0$ and $\binom{i+j+1}{2}+\binom{i+j}{2}=1$. On $\mathbb{RP}^j$, one has

$$\begin{aligned}
w[1] &= (1+\alpha)^{j+1}\{(1+e)^{2i+1}+\alpha(1+e)^{2i}+\binom{q}{2}\alpha^2(1+e)^{2i-1} \\
&\quad +\binom{q}{3}\alpha^3(1+e)^{2i-2}+\binom{q}{4}\alpha^4(1+e)^{2i-3}+\cdots\} \\
&= \{1+\binom{j+1}{2}\alpha^2+\binom{j+1}{4}\alpha^4+\cdots\} \\
&\quad \times\{1+e+\alpha+\binom{q}{2}\alpha^2+\binom{q}{2}\alpha^2e+\binom{q}{2}\alpha^2e^2+\binom{q}{2}\alpha^3+\binom{q}{4}\alpha^4+\cdots\}
\end{aligned}$$

and so

$$w[1]_4=\binom{q}{2}\alpha^2e^2+\binom{q}{4}\alpha^4+\binom{j+1}{2}\binom{q}{2}\alpha^4+\binom{j+1}{4}\alpha^4.$$

Form the class

$$\begin{aligned}\hat{w}_4 &= w[1]_4 + \binom{q}{2} w[0]_1^2 (w[0]_1 + w[1]_1)^2 + \left\{ \binom{q}{4} + \binom{j+1}{2}\binom{q}{2} + \binom{j+1}{4} \right\} w[0]_1^4 \\ &= \begin{cases} de^2 + cde + d^2 & \text{on } P(j,i) \\ 0 & \text{on } \mathbb{RP}^j \end{cases}\end{aligned}$$

since $\binom{q}{2} + \binom{a+1}{2} = \binom{j+1}{2} + \binom{i+j+1}{2} = 0$ and

$$\binom{q}{4} + \binom{j+1}{4} = \binom{a+1}{4} + \binom{i+j+1}{4}$$

by Lemma 3.2.

Now suppose that $a \geq 2^u$. Then $2^{p+1} > a \geq 2^u$ and so $p \geq u$ (this happens in the case $j \geq 5$). If $a < j$, then one has that $k \geq a + 2$; so

$$a - 2^u + 2^{u+2}(v+1) = a - 2^u + 2i + 2^{u+1} = a + 2i + 2^u < j + 2i + (a+2) \leq m.$$

Furthermore, one has that

$$\begin{aligned} & w[1]_1^{a-2^u} \hat{w}_4^{2^u(v+1)} e^{m-1-a-2i-2^u}[\mathbb{RP}(\nu^k)] \\ &= (e+c)^{a-2^u}(de^2 + cde + d^2)^{2^u(v+1)} e^{m-1-a-2i-2^u}[\mathbb{RP}(\nu^k)] \\ &= \frac{(1+c)^{a-2^u}(d + cd + d^2)^{2^u(v+1)}}{(1+c)^a(1+c+d)}[P(j,i)] \\ &= \frac{d^{2^u(v+1)}(1+c+d)^{2^u(v+1)-1}}{(1+c)^{2^u}}[P(j,i)] \\ &= \binom{2^u(v+1)-1}{2^u v} \frac{d^{2^u(2v+1)}}{1+c}[P(j,i)] \\ &= c^j d^i [P(j,i)] \\ &= 1, \end{aligned}$$

but

$$w[1]_1^{a-2^u} \hat{w}_4^{2^u(v+1)} e^{m-1-a-2i-2^u}[\mathbb{RP}(\nu^{m-j})] = 0,$$

which is a contradiction.

If $a > j$, then $a > 2^p$ and so $w_{2^p+2}(\nu^k) = \binom{a+1}{2^p+2} c^{2^p+2} + c^{2^p} d \neq 0$. Thus $2^p + 2 \leq k \leq 2^{u+1} - 1$. This implies that $u \geq p$, and so $u = p$. By Lemma 3.2, $q = a - 2^u + 1 < 2^u$. Since $\mathrm{common}(j, 2^{u+1} - q) \geq 2^u$, one has $j - \mathrm{common}(j, 2^{u+1} - q) \leq j - 2^u$. Let $j_0 = \min\{j - \mathrm{common}(j, 2^{u+1} - q), q - 1\}$. Then

$$j_0 + 2^{u+2}(v+1) \leq j - 2^u + 2^{u+2}(v+1) = j + 2i + 2^u < j + 2i + k = m$$

and

$$\binom{j_0 + 2^{u+1} - q}{j} = 1.$$

Thus, one has

$$\begin{aligned}
&w[1]_1^{j_0}\hat{w}_4^{2^u(v+1)}e^{m-1-j_0-2^{u+2}(v+1)}[\mathbb{RP}(\nu^k)]\\
&\quad= (e+c)^{j_0}(de^2+cde+d^2)^{2^u(v+1)}e^{m-1-j_0-2^{u+2}(v+1)}[\mathbb{RP}(\nu^k)]\\
&\quad= \frac{(1+c)^{j_0}(d+cd+d^2)^{2^u(v+1)}}{(1+c)^a(1+c+d)}[P(j,i)]\\
&\quad= \frac{d^{2^u(v+1)}(1+c+d)^{2^u(v+1)-1}}{(1+c)^{a-j_0}}[P(j,i)]\\
&\quad= \binom{2^u(v+1)-1}{2^uv}\frac{d^{2^u(2v+1)}}{(1+c)^{a-2^u+1-j_0}}[P(j,i)]\\
&\quad= (1+c)^{j_0+2^{u+1}-q}d^i[P(j,i)]\\
&\quad= \binom{j_0+2^{u+1}-q}{j}c^jd^i[P(j,i)]\\
&\quad= 1,
\end{aligned}$$

but

$$w[1]_1^{j_0}\hat{w}_4^{2^u(v+1)}e^{m-1-j_0-2^{u+2}(v+1)}[\mathbb{RP}(\nu^{m-j})] = 0.$$

This is a contradiction.

Therefore, $a \geq 2^u$ is impossible. □

Now, by Lemmas 3.1, 3.2, 3.3 and Proposition 2.1, one has

Proposition 3.4. *If (M^{j+2i+k}, T) fixes $\mathbb{RP}^j \sqcup P(j,i)$ with $u > 1$, then (M^{j+2i+k}, T) exists when k is restricted to a range $k_{\min} \leq k \leq k_{\max}$ and is cobordant to*

$$\Gamma^{k-2a-2}(P(j, N^{i+a+1}), T_{N^{i+a+1}})$$

where $k_{\min} = a+2$ for $a < j$, and $k_{\min} \leq j+1 < a+2$ for $a > j$, and $k_{\max} = 2a+2+X_0$.

However, X_0 is only an unknown number. We wish to know the value of X_0. This is equivalent to determining the upper bound of k.

Now let us estimate the maximum k value for realizing the Stiefel-Whitney class $w(\nu^k) = (1+c)^a(1+c+d)$.

When $p \geq u$, one has $j > 2^p \geq 2^u > a$ (this only happens in the case $j \geq 5$). If $k > 2^u + a + 1$, then

$$j - 2^u + a + 1 + 2^{u+2}(v+1) = j + 2i + 2^u + a + 1 < j + 2i + k = m.$$

Using the class $\hat{w}_4$ in the proof of Lemma 3.3, one has

$$\begin{aligned}
0 &= w[1]_1^{j-2^u+a+1}\hat{w}_4^{2^u(v+1)}e^{m-j-2i-2^u-a-2}([\mathbb{RP}(\nu^k)] + [\mathbb{RP}(\nu^{m-j})])\\
&= (e+c)^{j-2^u+a+1}(de^2+cde+d^2)^{2^u(v+1)}e^{m-j-2i-2^u-a-2}[\mathbb{RP}(\nu^k)] + 0\\
&= (1+c)^{j-2^u+1}d^{2^u(v+1)}(1+c+d)^{2^u(v+1)-1}[P(j,i)]\\
&= \binom{2^u(v+1)-1}{2^uv}d^{2^u(2v+1)}(1+c)^j[P(j,i)]\\
&= c^jd^i[P(j,i)]\\
&= 1,
\end{aligned}$$

which is impossible. Thus k must be less than or equal to $2^u + a + 1$.

When $p < u$, one has $q = a+1$ by Lemma 3.2. Let $a_0 = \text{common}(j,a)$. It is easy to see that a_0 is even, and $a_0 < j$ and $a_0 \le a$; in particular, $\binom{2^u-1-a+a_0}{j} = 1$. If $k > 2^{u+1} - j + a_0$, then

$$a_0 + 2^{u+2}(v+1) = a_0 + 2i + 2^{u+1} < j + 2i + k = m.$$

Using the class $\hat{w}_4$ in the proof of Lemma 3.3, one has

$$\begin{aligned} 0 &= w[1]_1^{a_0}\hat{a}w_4^{2^u(v+1)}e^{m-1-a_0-2^{u+2}(v+1)}([\mathbb{RP}(\nu^k)] + [\mathbb{RP}(\nu^{m-j})]) \\ &= (e+c)^{a_0}(de^2+cde+d^2)^{2^u(v+1)}e^{m-1-a_0-2^{u+2}(v+1)}[\mathbb{RP}(\nu^k)] + 0 \\ &= (1+c)^{a_0-a}d^{2^u(v+1)}(1+c+d)^{2^u(v+1)-1}[P(j,i)] \\ &= d^{2^u(2v+1)}(1+c)^{2^u-1-a+a_0}[P(j,i)] \\ &= \binom{2^u-1-a+a_0}{j}c^jd^i[P(j,i)] \\ &= 1, \end{aligned}$$

which is a contradiction. Thus k must be less than or equal to $2^{u+1} - j + a_0 = 2^{u+1} - (j - \text{common}(j,a))$.

Combining the discussions of this section, one completes the proof of Theorem 1.1.

Observation. For $u > 1$, the upper bound of k estimated as above is attainable in some special cases. For example, when $a = 2^u - 2$, the above arguments show that if $p \ge u$, then the upper bound of k should be $2^u + a + 1 = 2^{u+1} - 1$, and if $u \ge p+1$, then $a_0 = j-1$ and so the upper bound of k should be $2^{u+1} - j + a_0 = 2^{u+1} - 1$. The examples in section 2 make sure that $2^{u+1} - 1$ with $a = 2^u - 2$ can become the upper bound of k. In fact, if $u > 1$, then $P(j, N^{i+2^u-1})$ bounds, and thus one may apply the operation Γ just one time to $(P(j, N^{i+2^u-1}), T_{N^{i+2^u-1}})$ such that the resulting involution $\Gamma(P(j, N^{i+2^u-1}), T_{N^{i+2^u-1}})$ has the same fixed information as $(P(j, N^{i+2^u-1}), T_{N^{i+2^u-1}})$ and has dimension $j + 2i + 2^{u+1} - 1$. Also, if $u > 1$ and $j = 1$, or 3, then $u \ge p+1$ must be satisfied. It is easy to see that $a_0 = a$ when $j = 1, 3$; so the upper bound of k should be $2^{u+1} - j + a$. This just corresponds to those results showed in Proposition 3.2 and [L-L, Theorem 5.1]. For the general case, the proof for which the upper bound of k estimated as above is attainable seems to be a difficult thing.

It is extremely tempting to conjecture that the upper bound of k is $2^u + a + 1$ if $p \ge u$, and $2^{u+1} - j + a_0$ if $u \ge p+1$. In other words, X_0 should be $2^u - a - 1$ if $p \ge u$, and $2^{u+1} - j - 2a - 2 + a_0$ if $u \ge p+1$.

4. The case in which h is even

In this section, one considers the involution (M^m, T) fixing $\mathbb{RP}^j \sqcup P(h,i)$ with h even. First, let us prove some lemmas.

Lemma 4.1. *If h is even, then $h \ge q-1$ and $j+1 \ge 2i+k$.*

Proof. From (1.1) one then has

$$w[0]_1 = \begin{cases} (i+1+a+b)c & \text{on } P(h,i) \\ e+\alpha & \text{on } \mathbb{RP}^j. \end{cases}$$

Since $m > q$ (see the proof of Lemma 3.2), one may form the characteristic number for

$$w[0]_1^{q-1}e^{m-1-(q-1)} = (e+\alpha)^{q-1}e^{m-q},$$

which has value on $\mathbb{RP}(\nu^{m-j})$ equal to the coefficient of α^j in $\frac{(1+\alpha)^{q-1}}{(1+\alpha)^q} = \frac{1}{1+\alpha}$ and that coefficient is nonzero. On $\mathbb{RP}(\nu^k)$,

$$w[0]_1^{q-1}e^{m-1-(q-1)} = (i+1+a+b)c^{q-1}e^{m-q}$$

and the value of this on $\mathbb{RP}(\nu^k)$ must be nonzero. Thus, one has that if $q > 1$, then $i+1+a+b$ must be odd and $h \geq q-1$. If $q = 1$, it is obvious that $h \geq q-1$.

Now for $h < t \leq m-1$, one has that on $\mathbb{RP}(\nu^k)$,

$$w[0]_1^t e^{m-1-t} = (i+1+a+b)c^t e^{m-1-t} = 0$$

and the coefficient of α^j in $\frac{(1+\alpha)^t}{(1+\alpha)^q}$ is zero. If one writes

$$\frac{(1+\alpha)^{h+1}}{(1+\alpha)^q} = 1 + \cdots + \alpha^{s_0}$$

where s_0 is the degree of the highest term, $0 \leq s_0 \leq j$, and s_0 is even since $h+1$ and q are odd, then $s_0 < j$. Furthermore,

$$\frac{(1+\alpha)^{h+1+(j-s_0)}}{(1+\alpha)^q} = (1+\alpha)^{j-s_0}(1+\cdots+\alpha^{s_0})$$

has the coefficient of α^j being 1. Since $h+1+j-s_0 > h$, this makes $h+1+j-s_0 \geq m = h+2i+k$ and so $j+1 \geq s_0+2i+k \geq 2i+k$. This completes the proof. □

Lemma 4.2. *If $m \neq j+q$, then*
(1) *$i+1+a+b$ is odd,*
(2) *$h \geq 2i+k$.*

Proof. If $m < j+q$, then q must be more than j; so $q-1 \geq j+1$ and $q > 1$. By Lemma 4.1, one has

$$h \geq q-1 \geq j+1 \geq s_0+2i+k \geq 2i+k$$

and $i+1+a+b$ is odd. If $m > j+q$, then the characteristic number for

$$w[0]_1^{j+q}e^{m-1-j-q} = (e+\alpha)^{j+q}e^{m-1-j-q}$$

has value on $\mathbb{RP}^j$ equal to the coefficient of α^j in $\frac{(1+\alpha)^{j+q}}{(1+\alpha)^q} = (1+\alpha)^j$, which is nonzero. On $P(h,i)$,

$$w[0]_1^{j+q}e^{m-1-j-q} = (i+1+a+b)c^{j+q}e^{m-1-j-q}$$

and the value of this on $P(h,i)$ must be nonzero. Thus, $i+1+a+b$ is odd and

$$h \geq j+q.$$

Furthermore, by Lemma 4.1 one has

$$h \geq j+q = (j+1)+(q-1) \geq 2i+k+(q-1) \geq 2i+k.$$

This completes the proof. □

Lemma 4.3. *If $m \neq j+q$, then the exotic classes cannot occur in $w(\nu^k)$.*

Proof. Since $k>0$, by Lemma 4.2(2) one has that $h \geq 2i+k \geq 3$ and so $h \geq 4$ for h even.

Claim. If $k<4$, then the exotic classes cannot occur in $w(\nu^k)$.

If $k<4$, then $w(\nu^k) = 1+\lambda c + (\beta d + \gamma c^2) + (\delta cd + \varepsilon c^3)$ and

$$w_3(\nu^k) = Sq^1 w_2(\nu^k) + w_1(\nu^k) w_2(\nu^k) = \beta cd + \lambda c(\beta d + \gamma c^2) = \beta(\lambda+1)cd + \lambda\gamma c^3.$$

If $\beta = 0$, then $w(\nu^k) = 1+\lambda c+\gamma c^2+\lambda\gamma c^3 = (1+c)^{\lambda+2\gamma}$ is standard (here $w(\nu^k)$ is standard if $w(\nu^k)$ can be expressed as $(1+c)^a(1+c+d)^b$); so one may suppose $\beta=1$.

For $\lambda=0$, $w(\nu^k) = 1+(d+\gamma c^2)+cd$ gives $w_2(\nu^k)w_3(\nu^k) = Sq^2 w_3(\nu^k)$ or $cd^2+\gamma c^3 d = c^2 cd + cd^2$ and $(\gamma+1)c^3 d = 0$. Thus $\gamma = 1$ (i.e., $w(\nu^k) = (1+c)(1+c+d)$ is standard) or $c^3=0$ (so $h \leq 2$ but this is impossible).

For $\lambda = 1$, $w(\nu^k) = 1+c+(d+\gamma c^2)+\gamma c^3$. If $\gamma=0$, this is $w(\nu^k) = 1+c+d$, which is standard. If $\gamma = 1$, $w(\nu^k) = 1+c+(d+c^2)+c^3$ and $w_2(\nu^k)w_3(\nu^k) = Sq^2 w_3(\nu^k)$. Furthermore, $c^3 d + c^5 = Sq^2 c^3 = c^5$; so $c^3 d = 0$ and $c^3 = 0$. Thus $h \leq 2$, but this is impossible.

If $k \geq 4$, then $h \geq 2i+k \geq 6$, and the only possibility for which the exotic classes may occur is that $h=6, k=4, i=1$, and $j=5$. If the exotic class occurs when $h=6$, then $w(\rho) = 1+c^6 d$ by [St] and $w(\nu^k) = (1+c)^a(1+c+d)^b(1+c^6 d) = \{(1+c)^{a+b} + \binom{b}{1}(1+c)^{a+b-1}d\}(1+c^6 d)$. If b is even, then $w(\nu^k)$ has nonzero term $c^6 d$ and so $k \geq 8$. This is a contradiction. Thus b is odd, and so $w(\nu^k) = (1+c)^{a+b} + (1+c)^{a+b-1}d + c^6 d$. Since $k=4$, each term of degree more than 4 in $w(\nu^k)$ must be zero. This forces $\binom{a+b-1}{6}$ to be 1; so $a+b-1$ must have terms 2 and 2^2 in its 2-adic expansion, and thus $\binom{a+b-1}{4} = 1$. This means that there is a nonzero term $c^4 d$ in $w(\nu^k)$ and so $k \geq 6$, which leads to a contradiction. Thus, the exotic classes cannot occur in $w(\nu^k)$ if $k \geq 4$. □

Letting $2^A \leq h < 2^{A+1}$ and $2^B \leq i < 2^{B+1}$, one may assume that $a < 2^{A+1}$ and $b < 2^{B+1}$ since a (resp. b) is only determined modulo 2^{A+1} (resp. 2^{B+1}). Let $C = \max\{A+1, B+1\}$.

Lemma 4.4. *If $m \neq j+q$, then*

(1) *$b \leq 2^B \leq i$ and furthermore, $k \geq 2b$,*

(2) *$\frac{(1+\alpha)^h}{(1+\alpha)^q}[\mathbb{RP}^j] = 1$,*

(3) *$k > 2i+4b$ for $a \geq h$.*

Proof. By Lemma 4.3, one can write $w(\nu^k) = (1+c)^a(1+c+d)^b$.

(1) Since

$$w[0]_1^{q-1} e^{m-q}[\mathbb{RP}(\nu^{m-j})] = \frac{1}{1+\alpha}[\mathbb{RP}^j] = 1,$$

one has that

$$\begin{aligned} w[0]_1^{q-1} e^{m-q}[\mathbb{RP}(\nu^k)] &= \frac{c^{q-1}}{(1+c)^a(1+c+d)^b}[P(h,i)] \\ &= \binom{2^C-b}{i} d^i c^{q-1}(1+c)^{2^C+2^{A+1}-a-b-i}[P(h,i)] \\ &= \binom{2^C-b}{i}\binom{2^C+2^{A+1}-a-b-i}{h-q+1} \end{aligned}$$

is nonzero, and so

$$\binom{2^C - b}{i} = 1. \tag{4.1}$$

Since $\binom{2^C-b}{i} = \binom{2^{B+1}-b}{i} = 1$, one has that $b \le 2^B \le i$, for if not, $2^{B+1} - b$ is less than 2^B; so $\binom{2^{B+1}-b}{i} = 0$, but this is a contradiction. Furthermore, it follows that $k \ge 2b$ since there exists the nonzero term d^b in $w(\nu^k) = (1+c)^a(1+c+d)^b$.

(2) The characteristic number for

$$w[0]_1^h e^{m-1-h} = c^h e^{m-1-h}$$

has value on $\mathbb{RP}(\nu^k)$ equal to $\binom{2^C-b}{i}$, which is 1 by (4.1). Thus on $\mathbb{RP}(\nu^{m-j})$,

$$w[0]_1^h e^{m-1-h}[\mathbb{RP}(\nu^{m-j})] = \frac{(1+\alpha)^h}{(1+\alpha)^q}[\mathbb{RP}^j]$$

must be nonzero.

(3) If $a \ge h$, then $a \ge 2^A$. So the coefficient of the term $c^{2^A} d^b$ is nonzero in $w(\nu^k) = (1+c)^a(1+c+d)^b$. Thus

$$k \ge 2^A + 2b. \tag{4.2}$$

On the other hand, by Lemma 4.2(2), one has that $2^{A+1} > h \ge 2i + k$ and so

$$2^A > \frac{2i+k}{2}. \tag{4.3}$$

From (4.2) and (4.3), it follows that

$$k \ge 2^A + 2b > \frac{2i+k}{2} + 2b$$

and thus

$$k > 2i + 4b.$$

This completes the proof. □

Now one begins with the proof of Proposition 1.2.

Proof of Proposition 1.2. Suppose that $m \ne j + q$. By Lemma 4.3, since the exotic classes cannot occur in $w(\nu^k)$, it is easy to see from (1.1) that on $P(h, i)$,

$$w[0] = (1+c)^h(1+c+d)^{i+1}\frac{(1+c+e)^a(1+c+e^2+ce+d)^b}{(1+e)^{a+2b}}.$$

By the proof of Lemma 4.1 and Lemma 4.2(1), one has $w[0]_1 = c$ on $P(h,i)$. When multiplied by $w[0]_1^h = c^h$ on $P(h,i)$,

$$\begin{aligned} w[0] &\sim (1+d)^{i+1}\frac{(1+e)^a(1+e^2+d)^b}{(1+e)^{a+2b}} \\ &\sim (1+d)^{i+1}(1+\frac{d}{1+e^2})^b \\ &\sim \{1+(i+1)d+\binom{i+1}{2}d^2+\cdots\}\{1+b\frac{d}{1+e^2}+\binom{b}{2}\frac{d^2}{(1+e^2)^2}+\cdots\} \\ &\sim 1+(i+1+b)d+\{\binom{i+1}{2}d^2+(i+1)bd^2+\binom{b}{2}d^2+bde^2\}+\cdots \\ &\sim 1+(i+1+b)d+\{bde^2+\binom{i+1+b}{2}d^2\}+\cdots . \end{aligned}$$

So

$$w[0]_2 \sim (i+1+b)d$$

and

$$w[0]_4 \sim bde^2+\binom{i+1+b}{2}d^2.$$

Now on $\mathbb{RP}^j$, $w[0]_1 = e+\alpha$ and $\alpha = w[0]_1 + e$ and

$$w[0]_2 = p_2(e,\alpha), \quad w[0]_4 = p_4(e,\alpha)$$

are polynomials in e and α. So one can form classes

$$x_2 = w[0]_2 + p_2(e, w[0]_1+e),$$
$$x_4 = w[0]_4 + p_4(e, w[0]_1+e),$$

obtaining characteristic classes that have

$$x_2 = 0 \text{ and } x_4 = 0$$

on $\mathbb{RP}^j$.

On $P(h,i)$, when multiplied by $w[0]_1^h = c^h$, these become

$$\begin{aligned} x_2 &= w[0]_2 + p_2(e,e+c) \\ &\sim (i+1+b)d + p_2(e,e) \\ &\sim (i+1+b)d+\lambda e^2 \end{aligned}$$

and

$$\begin{aligned} x_4 &= w[0]_4 + p_4(e,e+c) \\ &\sim bde^2+\binom{i+1+b}{2}d^2+p_4(e,e) \\ &\sim bde^2+\binom{i+1+b}{2}d^2+\mu e^4. \end{aligned}$$

One can even determine the values of λ and μ, if desired, because

$$\begin{aligned} w[r] &= (1+\alpha)^{j+1}\{(1+e)^{h+2i+r-j}+q\alpha(1+e)^{h+2i+r-j-1}+\cdots\} \\ &= (1+\alpha)^{j+1}(1+e+\alpha)^q(1+e)^{h+2i+r-j-q} \end{aligned}$$

and so

$$w[0] = (1+\alpha)^{j+1}(1+e+\alpha)^q(1+e)^{h+2i-j-q}.$$

Replacing α by $w[0]_1 + e$ and letting $w[0]_1 = c$, which becomes 0,

$$\begin{aligned} w[0] &\sim (1+e)^{j+1}(1+e+e)^q(1+e)^{h+2i-j-q} \\ &\sim (1+e)^{h+2i+1-q} \end{aligned}$$

and so

$$\lambda = \binom{h+2i+1-q}{2} \text{ and } \mu = \binom{h+2i+1-q}{4}.$$

The argument proceeds as follows.

(I) The case in which $i+1+b$ is odd.

If $i+1+b$ is odd, then a is even by Lemma 4.2(1), and on $P(h,i)$, $w[0]_1^h x_2$ is either $c^h d$ or $c^h(e^2+d)$.

When $w[0]_1^h x_2 = c^h d$ on $P(h,i)$, one has that

$$w[0]_1^h x_2^i e^{m-1-h-2i} = \begin{cases} c^h d^i e^{k-1} & \text{on } P(h,i) \\ 0 & \text{on } \mathbb{RP}^j \end{cases}$$

gives a nonzero value on $\mathbb{RP}(\nu^k)$, but the value of this on $\mathbb{RP}(\nu^{m-j})$ is zero. This is a contradiction.

When $w[0]_1^h x_2 = c^h(e^2+d)$ on $P(h,i)$, if $k > 2b$, then one has that

$$w[0]_1^h x_2^{i+b} e^{m-1-h-2(i+b)}[\mathbb{RP}(\nu^{m-j})] = 0,$$

but

$$\begin{aligned} w[0]_1^h x_2^{i+b} e^{m-1-h-2(i+b)}[\mathbb{RP}(\nu^k)] &= c^h(e^2+d)^{i+b} e^{m-1-h-2(i+b)}[\mathbb{RP}(\nu^k)] \\ &= \frac{c^h(1+d)^{i+b}}{(1+c)^a(1+c+d)^b}[P(h,i)] \\ &= c^h(1+d)^i[P(h,i)] \\ &= 1, \end{aligned}$$

which leads to a contradiction (note that $m-1-h-2(i+b) = k-1-2b \geq 0$). Thus $k = 2b$ by Lemma 4.4(1); so $a = 0$ and $w(\nu^k) = (1+c+d)^b$. If $b > 1$, one has that the value of $w[0]_1^h x_2^{b-1} e^{m-1-h-2(b-1)}$ on $\mathbb{RP}(\nu^{m-j})$ is zero, but

$$\begin{aligned} w[0]_1^h x_2^{b-1} e^{m-1-h-2(b-1)}[\mathbb{RP}(\nu^k)] &= c^h(e^2+d)^{b-1} e^{m-h-2b+1}[\mathbb{RP}(\nu^k)] \\ &= \frac{c^h(1+d)^{b-1}}{(1+c+d)^b}[P(h,i)] \\ &= \frac{c^h}{1+d}[P(h,i)] \\ &= 1. \end{aligned}$$

This is impossible. So, $b = 1$ and $w(\nu^k) = 1+c+d$ since $b = 0$ is obviously impossible. By direct computations, one has that

$$w[1]_1 = \begin{cases} \alpha & \text{on } \mathbb{RP}^j \\ e+c & \text{on } P(h,i) \end{cases}$$

and so

$$w[1]_1^j e^{m-1-j}[\mathbb{RP}(\nu^{m-j})] = \frac{\alpha^j}{(1+\alpha)^q}[\mathbb{RP}^j] = 1,$$

but

$$\begin{aligned} w[1]_1^j e^{m-1-j}[\mathbb{RP}(\nu^k)] &= (e+c)^j e^{m-1-j}[\mathbb{RP}(\nu^k)] \\ &= \frac{(1+c)^j}{1+c+d}[P(h,i)] \\ &= (1+c)^j\{1+\frac{d}{1+c}+\cdots+\frac{d^i}{(1+c)^i}\}[P(h,i)] \\ &= (1+c)^{j-i}d^i[P(h,i)] \\ &= 0 \end{aligned}$$

since $j-i<j+1\leq h$ by the proof of Lemma 4.2.

Thus, the case of odd $i+1+b$ does not happen.

(II) The case in which $i+1+b$ is even.

Let $i+1+b$ be even. Then a is odd by Lemma 4.2(1). If i is odd, then b is odd since $\binom{2^C-b}{i}=1$ by (4.1), and furthermore $i+1+b$ is odd. This is impossible.

Thus, i must be even and b must be odd.

Since i is even, one has that

$$\chi(M^m)=\chi(\mathbb{RP}^j)+\chi(P(h,i))=0+\chi(P(h,i))=(h+1)(i+1)$$

is nonzero modulo 2 where $\chi(\cdot)$ denotes the Euler characteristic number, and thus m must be even since the Euler characteristic number of any odd-dimensional manifold is always zero. Furthermore, k is also even. Since b is odd, by Lemma 4.4(1), one has that $b<i$ and so $2i+k>4b$. With these understood, now the argument is divided into the following two cases.

(i) The case $\binom{i+1+b}{2}=0$.

If $\binom{i+1+b}{2}=0$, then on $P(h,i)$,

$$w[0]_1^h x_4 = c^h de^2 \text{ or } c^h(e^4+de^2).$$

When $w[0]_1^h x_4=c^h de^2$ on $P(h,i)$, one has that

$$\begin{aligned} w[0]_1^h(x_4+e^4)^b e^{m-1-h-4b}[\mathbb{RP}(\nu^k)] &= c^h(de^2+e^4)^b e^{m-1-h-4b}[\mathbb{RP}(\nu^k)] \\ &= \frac{c^h(1+d)^b}{(1+c)^a(1+c+d)^b}[P(h,i)] \\ &= c^h[P(h,i)] \\ &= 0, \end{aligned}$$

but

$$\begin{aligned} w[0]_1^h(x_4+e^4)^b e^{m-1-h-4b}[\mathbb{RP}(\nu^{m-j})] &= (e+\alpha)^h e^{m-1-h}[\mathbb{RP}(\nu^{m-j})] \\ &= \frac{(1+\alpha)^h}{(1+\alpha)^q}[\mathbb{RP}^j] \\ &= 1 \end{aligned}$$

by Lemma 4.4(2). This is impossible.

When $w[0]_1^h x_4=c^h(e^4+de^2)$ on $P(h,i)$, if $b>1$, then one has that

$$\begin{aligned} w[0]_1^h x_4^{b-1}e^{m-1-h-4(b-1)}[\mathbb{RP}(\nu^k)] &= \frac{c^h(1+d)^{b-1}}{(1+c)^a(1+c+d)^b}[P(h,i)] \\ &= \frac{c^h}{1+d}[P(h,i)]=1 \end{aligned}$$

but

$$w[0]_1^h x_4^{b-1} e^{m-1-h-4(b-1)}[\mathbb{RP}(\nu^{m-j})] = 0.$$

If $b = 1$ and $a < h$, then the top nonzero Stiefel-Whitney class in $w(\nu^k) = (1+c)^a(1+c+d)$ is $c^a d$ and so $k > a+2$ (note that a is odd and k is even). Thus, one has that

$$w[1]_1^j e^{m-1-j}[\mathbb{RP}(\nu^{m-j})] = \frac{\alpha^j}{(1+\alpha)^q}[\mathbb{RP}^j] = 1,$$

but

$$\begin{aligned} w[1]_1^j e^{m-1-j}[\mathbb{RP}(\nu^k)] &= (e+c)^j e^{m-1-j}[\mathbb{RP}(\nu^k)] \\ &= \frac{(1+c)^j}{(1+c)^a(1+c+d)}[P(h,i)] \\ &= \frac{(1+c)^j}{(1+c)^{a+1}} \cdot \frac{1}{1+\frac{d}{1+c}}[P(h,i)] \\ &= (1+c)^{j-a-1}\{1 + \frac{d}{1+c} + \cdots + \frac{d^i}{(1+c)^i}\}[P(h,i)] \\ &= (1+c)^{j-a-1-i} d^i [P(h,i)] \\ &= 0 \end{aligned}$$

since $a+1+i < 2i+k \leq j+1 \leq h$ by the proof of Lemma 4.2. If $b = 1$ and $a \geq h$, by Lemma 4.4(3) one knows that $k > 2i+4$. So $m-1 = h+2i+k-1 \geq h+4i+5$ for k even. Now

$$w[0]_1^h x_4^{i+1} e^{m-1-h-4i-4} = \begin{cases} c^h(e^4+de^2)^{i+1} e^{m-1-h-4i-4} & \text{on } P(h,i) \\ 0 & \text{on } \mathbb{RP}^j \end{cases}$$

has a nonzero value on $\mathbb{RP}(\nu^k)$, but the value of this on $\mathbb{RP}(\nu^{m-j})$ is zero, which gives a contradiction.

(ii) The case $\binom{i+1+b}{2} = 1$.

If $\binom{i+1+b}{2} = 1$, then on $P(h,i)$,

$$w[0]_1^h x_4 = c^h(de^2+d^2) \text{ or } c^h(e^4+de^2+d^2).$$

When $w[0]_1^h x_4 = c^h(de^2+d^2)$ on $P(h,i)$, if $b > 1$, then

$$w[0]_1^h x_4^{b-1} e^{m-1-h-4(b-1)} = \begin{cases} c^h(de^2+d^2)^{b-1} e^{m-1-h-4(b-1)} & \text{on } P(h,i) \\ 0 & \text{on } \mathbb{RP}^j \end{cases}$$

gives a nonzero value on $\mathbb{RP}(\nu^k)$ but not on $\mathbb{RP}(\nu^{m-j})$, which leads to a contradiction. As in case (i), one may conclude that $b = 1$ is impossible.

When $w[0]_1^h x_4 = c^h(e^4+de^2+d^2)$ on $P(h,i)$, one has that

$$\begin{aligned} w[0]_1^h (x_4+e^4)^b e^{m-1-h-4b}[\mathbb{RP}(\nu^k)] &= \frac{c^h d^b (1+d)^b}{(1+c)^a(1+c+d)^b}[P(h,i)] \\ &= c^h d^b [P(h,i)] = 0 \end{aligned}$$

but

$$\begin{aligned} w[0]_1^h (x_4+e^4)^b e^{m-1-h-4b}[\mathbb{RP}(\nu^{m-j})] &= (e+\alpha)^h e^{m-1-h}[\mathbb{RP}(\nu^{m-j})] \\ &= \frac{(1+\alpha)^h}{(1+\alpha)^q}[\mathbb{RP}^j] \\ &= 1 \end{aligned}$$

by Lemma 4.4(2).

Thus, the case of even $i+1+b$ does not happen.

Combining the above arguments, one completes the proof. □

For the case $m = j+q$, consider the involution T_q on $\mathbb{RP}^{j+q}$ defined by

$$T_q([x_0,\cdots,x_j,x_{j+1},\cdots,x_{j+q}]) = [x_0,\cdots,x_j,-x_{j+1},\cdots,-x_{j+q}]$$

fixing $\mathbb{RP}^j$ with normal bundle $\nu^q = q\iota$ having $w(\nu^q) = (1+\alpha)^q$ and $\mathbb{RP}^{q-1}$ with normal bundle $\nu^{j+1} = (j+1)\iota$ having $w(\nu^{j+1}) = (1+\alpha)^{j+1}$. Forming the union $(M^m,T)\sqcup(\mathbb{RP}^{j+q},T_q)$ one obtains an involution $(\bar{M}^{j+q},\bar{T})$ fixing $\mathbb{RP}^{q-1}$ with $w(\nu^{j+1}) = (1+\alpha)^{j+1}$ and $P(h,i)$ with normal bundle ν^k, with $h \geq q-1$.

Observation. Finding involutions fixing $\mathbb{RP}^j$ and $P(h,i)$ with h even reduces to a problem about finding involutions that fix $\mathbb{RP}^{q-1}$ and $P(h,i)$, which is the problem for *even* projective spaces. Studying the case of even j is beyond what one wants to consider at this point.

Finally, one points out that there exist examples for the case $m = j+q$. For $h = q-1$, there is an obvious way to get an involution fixing $\mathbb{RP}^{q-1}$ and $P(h,i)$, which is to begin with the involution on $P(q-1,i+1)$ induced by $T_1([z_0,\cdots,z_i,z_{i+1}]) = [z_0,\cdots,z_i,-z_{i+1}]$. This fixes $P(q-1,i)$ with normal bundle η and $P(q-1,0) = \mathbb{RP}^{q-1}$ with normal bundle $(i+1)\eta = (i+1)\iota \oplus (i+1)\mathbb{R}$. In order that the normal bundle of $\mathbb{RP}^{q-1}$ has dimension $j+1$, one needs $2(i+1) = j+1$ or $i = \frac{j+1}{2}-1 = \frac{j-1}{2}$. The normal bundle of $\mathbb{RP}^{q-1}$ has $w(\nu^{2(i+1)}) = (1+\alpha)^{i+1} = (1+\alpha)^{\frac{j+1}{2}}$ and one wants it to have $w(\nu^{j+1}) = (1+\alpha)^{j+1}$. This occurs only for $(1+\alpha)^{\frac{j+1}{2}} = 1$, which means $\frac{j+1}{2} = 2^u(2v+1)$ with $2^u > q-1$. Thus $j = 2^{u+1}(2v+1)-1$ and $2^u \geq q$, and $i = \frac{j+1}{2}-1 = 2^u(2v+1)-1$. Thus one has

Proposition 4.1. *For $j = 2^{u+1}(2v+1)-1$ and $q \leq 2^u$, there is an involution (M^{j+q},T) fixing $\mathbb{RP}^j$ with $w(\nu^q) = (1+\alpha)^q$ and $P(q-1,\frac{j-1}{2})$ with normal bundle η where $w(\eta) = 1+c+d$.*

Note. For $j = 3 = 2^2-1$ this gives $q \leq 2$; so $q = 1$ and $P(q-1,\frac{j-1}{2}) = P(0,1)$, which was excluded since $h = 0$. Thus, this involution does not *occur* for $j = 3$. For $q > 1$, this is a valid involution.

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