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A MEASURE-VALUED ANALOGUE OF WIENER MEASURE AND THE MEASURE-VALUED FEYNMAN-KAC FORMULA

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Dedicated to Professor Kun Soo Chang on his sixtieth birthday

ABSTRACT. In this article, we consider a complex-valued and a measure-valued measure on $C[0,t]$, the space of all real-valued continuous functions on $[0,t]$. Using these concepts, we establish the measure-valued Feynman-Kac formula and we prove that this formula satisfies a Volterra integral equation. The work here is patterned to some extent on earlier works by Kluvanek in 1983 and by Lapidus in 1987, but the present setting requires a number of new concepts and results.

1. INTRODUCTION

In 1923, Wiener showed that one can define a reasonable measure on the space $C_0[0,t]$ of all continuous functions on a closed interval $[0,t]$ that vanish at the origin 0, the so-called Wiener space [20]. Since then, the theory of this measure was investigated extensively and applied to various subjects by many mathematicians and many mathematical physicists. In 1968, Cameron and Storvick presented the definitions and theories for the operator-valued function space integral on $C_0[0,t]$ [2]. In 1986, Johnson and Lapidus obtained a perturbation expansion for the operator-valued function space integral under quite general conditions. Further, in 1987, Lapidus showed that for an exponential functional, the result in their paper satisfies a Volterra integral equation [9], [12], [13], [14], thereby establishing in this context the Feynman-Kac formula with a (complex-valued) measure. In 1983, Kluvanek introduced a measure μ_φ on the space $C_0[0,t]$ of all continuous functions on a closed interval by a method similar to the Wiener case and introduced an operator-valued measure M_t on $C_0[0,t]$ associated with a measure μ_φ [10]. Moreover, he found an operator-valued Feynman-Kac formula with respect to an operator-valued measure M_t and proved that his results satisfy a Volterra integral equation.

In this article, we introduce a complex-valued analogue of Wiener measure ω_φ on $C[0,t]$, associated with a complex-valued measure φ on $\mathbb{R}$. Indeed, if φ is the Dirac measure δ_0 at the origin in $\mathbb{R}$, ω_φ is the concrete Wiener measure. Using these concepts and the concepts of the conditional Wiener integral [21], we derive the measure-valued measure V_φ on $C[0,t]$ and establish the measure-valued Feynman-Kac formula. Moreover, we prove that our results—the measure-valued Feynman-Kac formula—satisfy a Volterra integral equation.

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This article consists of six sections. In section 2, we introduce some notation, some definitions and some basic facts which are needed to understand the contents of the subsequent sections. In section 3, we define a complex-valued analogue of Wiener measure ω_φ on $C[0,t]$, associated with a complex Borel measure φ on $\mathbb{R}$ and investigate the basic properties and some examples of it. In section 4, using the concepts of the conditional Wiener integral [21], we define a measure-valued measure V_φ on $C[0,t]$, associated with a measure ω_φ and investigate its basic properties. In section 5, we establish the Feynman-Kac formula with respect to V_φ. In the last section, we prove that our result in section 5 satisfies a suitable Volterra integral equation.

2. Preliminaries

In this section, we introduce some notation, definitions and facts which are needed to understand the subsequent sections.

(A) Let $\mathbb{R}$ be the real number system and let $\mathbb{C}$ be the complex number system. For a natural number n, let $\mathbb{R}^n$ be the n-times product space of $\mathbb{R}$. Let $\mathcal{B}(\mathbb{R})$ be the set of all Borel measurable subsets of $\mathbb{R}$ and let m_L be the Lebesgue measure on the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let $\alpha_1 = 1$, $\alpha_2 = -1$, $\alpha_3 = i$ and $\alpha_4 = -i$.

(B) For a positive real number t, let $C[0,t]$ be the space of all real-valued continuous functions on a closed bounded interval $[0,t]$ with the supremum norm $\|\cdot\|_\infty$. By the Stone-Weierstrass theorem,

$$(C[0,t], \|\cdot\|_\infty) \text{ is a real separable Banach space.} \tag{2.1}$$

Let $\mathcal{M}(\mathbb{R})$ be the space of all finite complex-valued countably additive measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. For p in $\mathbb{R}$, let δ_p be the Dirac measure concentrated at p with total mass one. For μ in $\mathcal{M}(\mathbb{R})$ and for E in $\mathcal{B}(\mathbb{R})$, the total variation $|\mu|(E)$ on E is defined by

$$|\mu|(E) = \sup \sum_{i=1}^{n} |\mu(E_i)| \tag{2.2}$$

where the supremum is taken over all finite sequences $\langle E_i \rangle$ of disjoint sets in $\mathcal{B}(\mathbb{R})$. Then $|\mu|$ is in $\mathcal{M}(\mathbb{R})$ and, by the Jordan decomposition theorem [7, (19.13) Theorem, p. 307], there are unique nonnegative measures μ_j $(j = 1,2,3,4)$ in $\mathcal{M}(\mathbb{R})$ such that

$$\mu = \sum_{j=1}^{4} \alpha_j \mu_j \ . \tag{2.3}$$

By [3, Theorem 4.1.7, p. 128] $(\mathcal{M}(\mathbb{R}), |\cdot|(\mathbb{R}))$ is a complex Banach space. Let $\mathcal{RM}(\mathbb{R})$ be the space of all finite complex-valued measures μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ that are absolutely continuous with respect to m_L, that is, the Radon-Nikodym derivative $\frac{d|\mu|}{dm_L}$ exists.

(C) Let $(X, \mathcal{B}, \mu)$ be a measure space. For a positive real number p, let $\mathcal{L}^p(X,\mu)$ be the space of complex-valued μ-measurable functions f on X such that $|f|^p$ is $|\mu|$-integrable. Let $\mathcal{L}^\infty(X,\mu)$ be the space of complex-valued μ-measurable functions f on X that are $|\mu|$-essentially bounded. The elements of $\mathcal{L}^p(X,\mu)$ and $\mathcal{L}^\infty(X,\mu)$ are equivalence classes of functions in $\mathcal{L}^p(X,\mu)$ and $\mathcal{L}^\infty(X,\mu)$, respectively, with f_1 and f_2 said to be equivalent if they are equal $|\mu|$-a.e. Since $\mathcal{RM}(\mathbb{R})$ is isomorphic to $L^1(\mathbb{R}, m_L)$, $\mathcal{RM}(\mathbb{R})$ is a Banach space and the dual space $\mathcal{RM}(\mathbb{R})^*$ of $\mathcal{RM}(\mathbb{R})$ is isomorphic to $L^\infty(\mathbb{R}, m_L)$. For x^* in $\mathcal{RM}(\mathbb{R})^*$, there is a function θ in $L^\infty(\mathbb{R}, m_L)$ such that $x^*(\mu) = \int_{\mathbb{R}} \theta(s)\, d\mu(s)$ for μ in $\mathcal{RM}(\mathbb{R})$.

Let $\mathbb{B}$ be a complex Banach space and let $\mathbb{B}^*$ be the dual space of $\mathbb{B}$. For a $\mathbb{B}$-valued countably additive measure ν on $(X, \mathcal{B})$ and for E in $\mathcal{B}$, the semivariation $\| \nu \| (E)$ of ν on E is given by

$$\| \nu \| (E) = \sup\{|x^*\nu|(E) \mid x^* \text{ is in } \mathbb{B}^* \text{ and } \| x^* \|_{\mathbb{B}^*} \le 1\} \tag{2.4}$$

where $|x^*\nu|(E)$ is the total variation on E of the complex-valued measure $x^*\nu$.

(D) Let $\mathbb{B}$ be a complex Banach space and let $(X, \mathcal{B}, \mu)$ be a complex measure space. A function $f : X \to \mathbb{B}$ is called μ-measurable if there exists a sequence $\langle f_n \rangle$ of $\mathbb{B}$-valued simple functions with

$$\lim_{n\to\infty} \| f_n - f \|_{\mathbb{B}} = 0 \quad |\mu|\text{-a.e.} \tag{2.5}$$

A function f is called μ-weakly measurable if $x^* f$ is μ-measurable for each x^* in $\mathbb{B}^*$, the dual space of $\mathbb{B}$. By Pettis' measurability theorem [4, Theorem 2, p. 42],

$$\begin{aligned} &f \text{ is } \mu\text{-measurable if and only if } f \text{ is } |\mu|\text{-essentially,} \\ &\quad \text{separably valued and } f \text{ is } \mu\text{-weakly measurable.} \end{aligned} \tag{2.6}$$

We say that f is μ-Bochner integrable if there exists a sequence $\langle f_n \rangle$ of $\mathbb{B}$-valued simple functions such that $\langle f_n \rangle$ converges to f in the norm sense in $\mathbb{B}$ for $|\mu|$-a.e. and $\lim_{n\to\infty} \int_X \| f(t) - f_n(t) \|_{\mathbb{B}} \; d|\mu|(t) = 0$. In this case, $(Bo) - \int_X f(t)\, d\mu(t)$ is defined by

$$(Bo) - \int_X f(t)\, d\mu(t) \;=\; \lim_{n\to\infty} \int_X f_n(t)\, d\mu(t) \tag{2.7}$$

where the limit means the limit in the norm sense. By [4, Theorem 2, p. 45],

$$f \text{ is } \mu\text{-Bochner integrable if and only if } \int_X \| f(t) \|_{\mathbb{B}} \; d|\mu|(t) \text{ is finite.} \tag{2.8}$$

By [23, Corollary 2, p. 134],

$$\begin{aligned} &\text{if } U \text{ is a bounded linear operator on } \mathbb{B} \text{ into a Banach} \\ &\text{space } \mathbb{B}_1 \text{ and } f \text{ is a } \mathbb{B}\text{-valued } \mu\text{-Bochner integrable function,} \\ &\text{then } Uf \text{ is a } \mathbb{B}_1\text{-valued } \mu\text{-Bochner integrable function, and} \\ &(Bo) - \int_X (Uf)(t)\, d\mu(t) \;=\; U\left((Bo) - \int_X f(t)\, d\mu(t)\right). \end{aligned} \tag{2.9}$$

Theorem 2.1. *Let $(X, \mathcal{B}, \mu)$ be a complex measure space and let $f : X \to \mathcal{M}(\mathbb{R})$ be a μ-Bochner integrable function. Then for E in $\mathcal{B}(\mathbb{R})$, $[f(t)](E)$ is a complex-valued μ-integrable function of t and*

$$\left[(Bo) - \int_X f(t)\, d\mu(t)\right](E) = \int_X [f(t)](E)\, d\mu(t). \tag{2.10}$$

Proof. For B in $\mathcal{B}$ and for m in $\mathcal{M}(\mathbb{R})$ we let $f(t) = \chi_B(t) m$ where $\chi_B(t)$ is a characteristic function associated with B. Then, for E in $\mathcal{B}(\mathbb{R})$, trivially, $[f(t)](E) =$

$\chi_B(t)m(E)$ is μ-integrable in t and

$$\begin{aligned}\left[(Bo)-\int_X f(t)\ d\mu(t)\right](E) \\ = \ [\mu(B)m](E) \\ = \mu(B)m(E) \\ = \int_X [f(t)](E)\ d\mu(t);\end{aligned} \tag{2.11}$$

so this theorem holds whenever $f(t)=\chi_B(t)m$ and B is in $\mathcal{B}$. By the basic properties of the Bochner integral and the Lebesgue integral, clearly this theorem holds whenever f is a simple function. Now, we assume that f is μ-Bochner integrable. Then there is a sequence $\langle f_n\rangle$ of simple functions such that $\lim_{n\to\infty}|f_n(t)-f(t)|(\mathbb{R})=0$ $|\mu|$-a.e. t and $\lim_{n\to\infty}\int_X |f_n(t)-f(t)|(\mathbb{R})\ d|\mu|(t)=0$. Let E be in $\mathcal{B}(\mathbb{R})$. Since $[f_n(t)](E)$ is simple for all natural numbers n and $|[f_n(t)](E)-[f(t)](E)|\le |f_n(t)-f(t)|(\mathbb{R})\to 0$ as $n\to+\infty$, $|\mu|$-a.e. t, $[f(t)](E)$ is a complex-valued μ-measurable function. By (2.8), $\int_X |[f(t)](E)|\ d|\mu|(t)\le\int_X |f(t)|(\mathbb{R})\ d|\mu|(t)<+\infty$; so $[f(t)](E)$ is $|\mu|$-integrable. By [7, (19.16) Theorem, p. 311], $[f(t)](E)$ is μ-integrable. Moreover,

$$\begin{aligned}&\left|\left[(Bo)-\int_X f(t)\ d\mu(t)\right](E)-\int_X [f(t)](E)\ d\mu(t)\right| \\ &\overset{(1)}{=}\lim_{n\to\infty}\left|\left[(Bo)-\int_X f_n(t)\ d\mu(t)\right](E)-\int_X [f(t)](E)\ d\mu(t)\right| \\ &\overset{(2)}{=}\lim_{n\to\infty}\left|\int_X [f_n(t)](E)\ d\mu(t)-\int_X [f(t)](E)\ d\mu(t)\right| \\ &\overset{(3)}{\le}\lim_{n\to\infty}\int_X |[f_n(t)](E)-[f(t)](E)|\ d|\mu|(t) \\ &\overset{(4)}{\le}\lim_{n\to\infty}\int_X |f_n(t)-f(t)|(\mathbb{R})\ d|\mu|(t) \\ &\overset{(5)}{=}0\ .\end{aligned} \tag{2.12}$$

By (2.7) and the continuity of the absolute value, we have Step (1). We proved already that this theorem holds for $\mathcal{M}(\mathbb{R})$-valued simple functions in (2.11); so Step (2) is true. Step (3) results from the basic properties of the Lebesgue integral. By the definition (2.2) of the total variation for a measure μ on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$, we obtain Step (4). Step (5) follows from the assumptions for a sequence $\langle f_n\rangle$. Thus, we have $[(Bo)-\int_X f(t)\ d\mu(t)](E)=\int_X [f(t)](E)\ d\mu(t)$. □

Remark 2.2. Consider a function H on $[0,1]\times[0,1]$ defined by $H(x,y)=\chi_{[0,x]}(y)$. Then H is $(m_L\times m_L)$-integrable on $[0,1]\times[0,1]$; so by the Fubini theorem, $H(x,y)$ is an m_L-integrable function of x for all y and $H(x,\cdot)$ is in $L^\infty([0,1],m_L)$ for all x in $[0,1]$. But $H(x,\cdot)$ has no essentially separable range; so $H(x,\cdot)$ is not m_L-Bochner integrable. Hence, in general, the equality (2.10) is not true in the theory of the Bochner integral.

(E) Let $\mathbb{B}$ be a complex Banach space and let $(Y,\mathcal{C},\nu)$ be a $\mathbb{B}$-valued measure space. Let g be a complex-valued $\|\ \nu\ \|$-measurable function on Y, that is, there exists a sequence $\langle g_n \rangle$ of complex-valued simple functions with $\lim_{n\to\infty} |g_n - g| = 0$ $\| \nu \|$-a.e. We say that g is ν-Bartle integrable if there exists a sequence $\langle g_n \rangle$ of simple functions such that $\langle g_n \rangle$ converges to g $\| \nu \|$-a.e. and the sequence $\langle \int_Y g_n(s) d\nu(s) \rangle$ is Cauchy in the norm sense. In this case, $(Ba) - \int_Y g(s) d\nu(s)$ is defined by

$$(Ba) - \int_Y g(s)\, d\nu(s) = \lim_{n\to\infty} \int_Y g_n(s)\, d\nu(s) \tag{2.13}$$

where the limit means the limit in the norm sense. By [5, Theorem 8, p. 323],

(2.14) if f is a ν-measurable function that is $\| \nu \|$-essentially bounded, then f is ν-Bartle integrable and

$$\left\| (Ba) - \int_Y f(s) d\nu(s) \right\|_{\mathbb{B}} \leq (\| \nu \| - \operatorname{ess\,sup} |f(s)|) \| \nu \| (Y) .$$

By [15, Theorem 2.4, p. 162],

(2.15) g is ν-Bartle integrable if and only if for each x^* in $\mathbb{B}^*$, g is $x^*\nu$-integrable and for each E in $\mathcal{C}$, there is an element $(Ba) - \int_E g(s)\, d\nu(s)$ in $\mathbb{B}$ such that $x^* \left[(Ba) - \int_E g(s)\, d\nu(s) \right]$ $= \int_E g(s)\, dx^*\nu(s)$ for x^* in $\mathbb{B}^*$.

By [5, Theorem 8, p. 324],

(2.16) if U is a bounded linear operator from $\mathbb{B}$ into a Banach space $\mathbb{B}_1$ and g is ν-Bartle integrable, then g is $U\nu$-Bartle integrable. In this case,

$$U \left[(Ba) - \int_Y g(s)\, d\nu(s) \right] = (Ba) - \int_Y g(s)\, dU\nu(s) .$$

By [5, Theorem 10, p. 328],

(2.17) if $\langle f_n \rangle$ is a sequence of ν-Bartle integrable functions that converges $\| \nu \|$-a.e. to f and if g is a ν-Bartle integrable function such that $|f_n(s)| \leq g(s)$ $\| \nu \|$-a.e. s for all natural numbers n, then f is ν-Bartle integrable and for E in $\mathcal{C}$, $(Ba) - \int_E f(s)\, d\nu(s) = \lim_{n\to\infty} (Ba) - \int_E f_n(s)\, d\nu(s)$.

(F) Let $\mathbb{B}$ be a complex Banach space. Let $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ be two measurable spaces and let $\mathcal{B} \otimes \mathcal{C}$ be the σ-algebra of sets in the space $X \times Y$ generated by the family of rectangles $E \times F$ for all E in $\mathcal{B}$ and F in $\mathcal{C}$. Let μ be a complex-valued measure on $(X, \mathcal{B})$ and let ν be a $\mathbb{B}$-valued measure on $(Y, \mathcal{C})$. For G in $\mathcal{B} \otimes \mathcal{C}$, we let

$$(\mu \times \nu)(G) = (Ba) - \int_Y \left[\int_X \chi_G(u, v)\, d\mu(u) \right] d\nu(v) . \tag{2.18}$$

By [10, Proposition 2, p. 169], using the dominated convergence theorem in [11, Theorem 2, p. 30], Kluvanek proved that $\mu \times \nu$ is a $\mathbb{B}$-valued measure on $\mathcal{B} \otimes \mathcal{C}$ and for G in $\mathcal{B} \otimes \mathcal{C}$, the following holds:

$$
\begin{aligned}
&(\mu \times \nu)(G) \\
&= (Ba) - \int_Y \left[\int_X \chi_G(u,v)\ d\mu(u) \right] d\nu(v) \\
&= (Bo) - \int_X \left[(Ba) - \int_Y \chi_G(u,v)\ d\nu(v) \right] d\mu(u).
\end{aligned} \tag{2.19}
$$

Moreover, in [10, Proposition 3, p. 170], he showed that

$$
x^*(\mu \times \nu) \ = \ \mu \times (x^* \nu) \tag{2.20}
$$

for all x^* in $\mathbb{B}^*$.

When both measures μ and ν are complex-valued, a sufficient condition for validity of the Fubini theorem is the integrability of the function with respect to $\mu \times \nu$. But, if ν is a vector measure, then the integrability of the function with respect to $\mu \times \nu$ is no longer a sufficient condition for the validity of the Fubini theorem. Indeed, we can find a counterexample for this fact in [10, Example, p. 170].

Theorem 2.3. *Let $\mathbb{B}$ be a separable complex Banach space, $(X, \mathcal{B}, \mu)$ a complex-valued measure space and let $(Y, \mathcal{C}, \nu)$ be a $\mathbb{B}$-valued measure space. Let $f : X \times Y \to \mathbb{C}$ be $(\mathcal{B} \otimes \mathcal{C})$-measurable and $(\mu \times \nu)$-Bartle integrable. Then*

$$
\text{for } \| \nu \| \text{-a.e. } v,\ f(u,v) \text{ is a } \mu\text{-integrable function of } u, \tag{2.21}
$$

$$
\int_X f(u,v)\ d\mu(u) \text{ is } \nu\text{-Bartle integrable, and} \tag{2.22}
$$

$$
\begin{aligned}
&(Ba) - \int_{X \times Y} f(u,v)\ d\mu \times \nu(u,v) \\
&\quad = (Ba) - \int_Y \left[\int_X f(u,v)\ d\mu(u) \right]\ d\nu(v)\ .
\end{aligned} \tag{2.23}
$$

Moreover, if for $|\mu|$-a.e. u, $f(u,v)$ is a ν-Bartle integrable function of v and $(Ba)-\int_Y f(u,v)\ d\nu(v)$ is μ-Bochner integrable, then

$$
\begin{aligned}
&(Ba) - \int_{X \times Y} f(u,v)\ d\mu \times \nu(u,v) \\
&\quad = (Bo) - \int_X \left[(Ba) - \int_Y f(u,v)\ d\nu(v) \right]\ d\mu(u) \\
&\quad = (Ba) - \int_Y \left[\int_X f(u,v)\ d\mu(u) \right]\ d\nu(v)\ .
\end{aligned} \tag{2.24}
$$

Proof. In [10, Proposition 4, p. 171], one can find that the facts (2.21), (2.22) and (2.23) hold. By the classical Fubini theorem, for x^* in $\mathbb{B}^*$,

$$
x^* \left[(Ba) - \int_Y f(u,v)\ d\nu(v) \right] \ = \ \int_Y f(u,v)\ dx^*\nu(v)
$$

is a $|\mu|$-measurable function of u. Since $\mathbb{B}$ is separable, by condition (2.6), $(Ba)-\int_Y f(u,v)\, d\nu(v)$ is a $|\mu|$-measurable function of u. For x^* in $\mathbb{B}^*$,

$$\begin{aligned}
x^*&\left[(Ba)-\int_{X\times Y} f(u,v)\, d\mu\times\nu(u,v)\right] \\
&\overset{(1)}{=} \int_{X\times Y} f(u,v)\, d[x^*(\mu\times\nu)](u,v) \\
&\overset{(2)}{=} \int_{X\times Y} f(u,v)\, d[\mu\times(x^*\nu)](u,v) \\
&\overset{(3)}{=} \int_X\left[\int_Y f(u,v)\, d(x^*\nu)(v)\right]\, d\mu(u) \\
&\overset{(4)}{=} \int_X x^*\left[(Ba)-\int_Y f(u,v)\, d\nu(v)\right]\, d\mu(u) \\
&\overset{(5)}{=} x^*\left[(Bo)-\int_X\left\{(Ba)-\int_Y f(u,v)\, d\nu(v)\right\}\, d\mu(u)\right] .
\end{aligned} \tag{2.25}$$

Step (1) results from (2.15). Step (2) follows from (2.20). By the classical Fubini theorem, we have Step (3). From (2.16), we obtain Step (4). By (2.9), we have Step (5).

Using the Hahn-Banach theorem, one can show

$$\begin{aligned}
&(Ba)-\int_{X\times Y} f(u,v)\, d\mu\times\nu(u,v) \\
&\quad = (Bo)-\int_X\left[(Ba)-\int_Y f(u,v)\, d\nu(v)\right]\, d\mu(u) ,
\end{aligned} \tag{2.26}$$

as desired. □

(G) Let φ be in $\mathcal{M}(\mathbb{R})$ and η be a complex-valued Borel measure on $[0,t]$. A complex-valued Borel measurable function θ on $[0,t]\times\mathbb{R}$ is said to belong to $L_{\varphi;\infty,1;\eta}$ (or $L^t_{\varphi;\infty,1;\eta}$) if

$$\parallel \theta \parallel_{\varphi;\infty,1;\eta} = \int_{[0,t]} \parallel \theta(s,\cdot) \parallel_{\varphi;\infty}\ d|\eta|(s) \tag{2.27}$$

is finite where $\parallel \theta(0,\cdot) \parallel_{\varphi;\infty}$ is $\inf\{\lambda>0 \mid |\varphi|(\{\xi \text{ in } \mathbb{R} \mid |\theta(0,\xi)|>\lambda\})=0\}$ and $\parallel \theta(s,\cdot) \parallel_{\varphi;\infty}$ is $\inf\{\lambda>0 \mid m_L(\{\xi \text{ in } \mathbb{R} \mid |\theta(s,\xi)|>\lambda\})=0\}$ for $0<s\le t$. If θ is bounded Borel measurable, then θ is in $L_{\varphi;\infty,1;\eta}$.

(H) For θ in $L^\infty(\mathbb{R},m_L)$, we consider an operator M_θ from $\mathcal{RM}(\mathbb{R})$ into itself such that

$$[M_\theta(\mu)](E) = \int_E \frac{d\mu}{dm_L}(\xi)\theta(\xi)\, dm_L(\xi) \tag{2.28}$$

for E in $\mathcal{B}(\mathbb{R})$ and for μ in $\mathcal{RM}(\mathbb{R})$. Then

$$\frac{dM_\theta(\mu)}{dm_L}(\xi) = \frac{d\mu}{dm_L}(\xi)\theta(\xi); \tag{2.29}$$

so M_θ is well-defined. Since $|M_\theta(\mu)|(\mathbb{R}) \le \int_{\mathbb{R}} |\frac{d\mu}{dm_L}(\xi)|\ |\theta(\xi)|\ dm_L(\xi) \le \parallel \theta \parallel_\infty |\mu|(\mathbb{R})$, M_θ is a bounded linear operator.

For $s > 0$, we let

$$P_s(E) \;=\; \int_E \frac{1}{\sqrt{2\pi s}} \exp\left\{-\frac{u^2}{2s}\right\} \, dm_L(u) \tag{2.30}$$

for E in $\mathcal{B}(\mathbb{R})$.

For $s > 0$, we consider an operator S_s from $\mathcal{RM}(\mathbb{R})$ into itself such that

$$\begin{aligned}[S_s(\mu)](E) \;&=\; (\mu * P_s)(E) \\ &=\; \frac{1}{\sqrt{2\pi s}} \int_{\mathbb{R}} \left[\int_E \exp\left\{-\frac{(u-v)^2}{2s}\right\} \, dm_L(u)\right] \, d\mu(v) \; .\end{aligned} \tag{2.31}$$

Then $\frac{dS_s(\mu)}{dm_L}(\xi) \;=\; \frac{1}{\sqrt{2\pi s}} \int_{\mathbb{R}} \exp\{-\frac{(\xi-v)^2}{2s}\} \, d\mu(v)$; so S_s is well-defined. It is not hard to show that S_s is a bounded linear operator and the operator norm $\| S_s \|$ of S_s is less than or equal to one.

Let s_1 and s_2 be two positive real numbers. Then by the Chapman-Kolmogorov equation in [8, Proposition 3.2.3, p. 37] and the classical Fubini theorem, we have

$$S_{s_1} \circ S_{s_2} \;=\; S_{s_1+s_2} \; . \tag{2.32}$$

For $s > 0$, for φ in $\mathcal{M}(\mathbb{R})$, for a Borel measurable $|\varphi|$-essentially bounded function θ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and for E in $\mathcal{B}(\mathbb{R})$, we let

$$[T(s,\varphi,\theta)](E) = \frac{1}{\sqrt{2\pi s}} \int_{\mathbb{R}} \left[\int_E \theta(v) \exp\left\{-\frac{(u-v)^2}{2s}\right\} \, dm_L(u)\right] \, d\varphi(v) \; . \tag{2.33}$$

Then $T(s,\varphi,\theta)$ is in $\mathcal{RM}(\mathbb{R})$ and

$$\frac{dT(s,\varphi,\theta)}{dm_L}(u) = \frac{1}{\sqrt{2\pi s}} \int_{\mathbb{R}} \theta(v) \exp\left\{-\frac{(u-v)^2}{2s}\right\} \, d\varphi(v) \; . \tag{2.34}$$

3. The complex-valued analogue of the Wiener measure ω_φ

In this section, we will introduce a complex-valued analogue of the Wiener measure ω_φ on $C[0,t]$ and we will give some examples of it.

Let t be a positive real number and n a nonnegative integer. For $\vec{t} = (t_0, t_1, \cdots, t_n)$ with $0 = t_0 < t_1 < \cdots < t_n \le t$, let $J_{\vec{t}} : C[0,t] \to \mathbb{R}^{n+1}$ be a function with

$$J_{\vec{t}}(x) \;=\; (x(t_0), x(t_1), \cdots, x(t_n)) \; . \tag{3.1}$$

For B_j $(j = 0, 1, 2, \cdots, n)$ in $\mathcal{B}(\mathbb{R})$, the subset $J_{\vec{t}}^{-1}(\prod_{j=0}^{n} B_j)$ of $C[0,t]$ is called an interval. Let $\mathcal{I}$ be the set of all intervals. For a nonnegative finite Borel measure φ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, we let

(3.2)

$$\begin{aligned} &m_\varphi(J_{\vec{t}}^{-1}(\prod_{j=0=0}^{n} B_j)) \\ &\quad= \int_{B_0} \left[\int_{\prod_{j=1}^{n} B_j} W(n+1; \vec{t}; u_0, u_1, \cdots, u_n) \; d\prod_{j=1}^{n} m_L(u_1, \cdots, u_n)\right] \; d\varphi(u_0)\end{aligned}$$

where

$$W(n+1;\vec{t};u_0,u_1,\cdots,u_n) = \left(\prod_{j=1}^{n}\frac{1}{\sqrt{2\pi(t_j-t_{j-1})}}\right)\exp\left\{-\frac{1}{2}\sum_{j=1}^{n}\frac{(u_j-u_{j-1})^2}{t_j-t_{j-1}}\right\}.$$

By [18, Theorem 5.1, p. 144] and [18, Theorem 2.1, p. 212], $\mathcal{B}(C[0,t])$, the set of all Borel subsets in $C[0,t]$, coincides with the smallest σ-algebra generated by $\mathcal{I}$ and there exists a unique positive measure ω_φ on $(C[0,t],\mathcal{B}(C[0,t]))$ such that $\omega_\varphi(I)=m_\varphi(I)$ for all I in $\mathcal{I}$.

For φ in $\mathcal{M}(\mathbb{R})$ with the Jordan decomposition $\varphi=\sum_{j=1}^{4}\alpha_j\varphi_j$, let $\omega_\varphi=\sum_{j=1}^{4}\alpha_j\omega_{\varphi_j}$. We say that ω_φ is the complex-valued analogue of the Wiener measure on $(C[0,t],\mathcal{B}(C[0,t]))$, associated with φ. If φ is a Dirac measure δ_0 at the origin in $\mathbb{R}$, then ω_φ is the classical Wiener measure.

By the change of variables formula, we can easily prove the following theorem.

Theorem 3.1 (The Wiener integration formula)**.** *If $f:\mathbb{R}^{n+1}\to\mathbb{C}$ is a Borel measurable function, then the following equality holds:*

$$\int_{C[0,t]} f(x(t_0),x(t_1),\cdots,x(t_n))\,d\omega_\varphi(x) \stackrel{*}{=} \int_{\mathbb{R}^{n+1}} f(u_0,u_1,\cdots,u_n)W(n+1;\vec{t};u_0,u_1,\cdots,u_n)\, d(\prod_{j=1}^{n} m_L\times\varphi)((u_1,u_2,\cdots,u_n),u_0) \tag{3.3}$$

where $\stackrel{*}{=}$ means that if one side exists, then both sides exist and the two values are equal.

Remark 3.2. Let φ be in $\mathcal{M}(\mathbb{R})$.

(1) It is not hard to show that ω_φ has no atoms.

(2) $\omega_\varphi(C[0,t]) = \varphi(\mathbb{R})$.

(3) Let $J_t: C[0,t]\to\mathbb{C}$ be a function with $J_t(x)=x(t)$. Then for E in $\mathcal{B}(\mathbb{R})$, $\omega_\varphi(J_t^{-1}(E)) = [S_t(\varphi)](E)$.

Example 3.3. Let φ be in $\mathcal{M}(\mathbb{R})$.

(1) Let $I = \{x \text{ in } C[0,t] \mid x(0) \text{ is in } B\}$ where B is in $\mathcal{B}(\mathbb{R})$. Then $\omega_\varphi(I) = \varphi(B)$.

(2) Suppose that $f(u)=u$ is φ-integrable. Then for $0\le s\le t$,

$$\int_{C[0,t]} x(s)\ d\omega_\varphi(x) \ = \ \int_{\mathbb{R}} u\ d\varphi(u)\ .$$

If $\varphi=\delta_p$, then $\int_{C[0,t]} x(s)\ d\omega_\varphi(x) = p$; and if φ has a normal distribution with mean α and variation σ^2, then $\int_{C[0,t]} x(s)\ d\omega_\varphi(x)=\alpha$.

(3) Suppose that $g(u)=u^2$ is φ-integrable. Then for $0\le s\le t$,

$$\int_{C[0,t]} x(s)^2\ d\omega_\varphi(x) \ = \ \int_{\mathbb{R}} u^2\ d\varphi(u)+s\varphi(\mathbb{R})\ .$$

If $\varphi = \delta_p$, then $\int_{C[0,t]} x(s)^2 \, d\omega_\varphi(x) = p^2 + s$ and if φ has a normal distribution with mean α and variance σ^2, then

$$\int_{C[0,t]} x(s)^2 \, d\omega_\varphi(x) \;=\; \alpha^2 + \sigma^2 + s \; .$$

(4) Let $\mathcal{F}(\varphi)$ be the Fourier transform of a measure φ, that is, $[\mathcal{F}(\varphi)](\xi) = \int_{\mathbb{R}} \exp\{i\xi u\} \, d\varphi(u)$. Then for $0 \le s \le t$,

$$\int_{C[0,t]} \exp\{i\xi x(s)\} \, d\omega_\varphi(x) \;=\; \exp\left\{-\frac{s\xi^2}{2}\right\} [\mathcal{F}(\varphi)](\xi) \; .$$

If $\varphi = \delta_p$, then $\int_{C[0,t]} \exp\{i\xi x(s)\} \, d\omega_\varphi(x) \;=\; \exp\{-\frac{s\xi^2}{2} + ip\xi\}$ and if φ has a normal distribution with mean α and variance σ^2 then

$$\int_{C[0,t]} \exp\{i\xi x(s)\} \, d\omega_\varphi(x) \;=\; \exp\left\{-\frac{(s+\sigma^2)\xi^2}{2} + i\alpha\xi\right\} \; .$$

Let $0 < s \le t$ be given and let $J_s : C[0,t] \to \mathbb{R}$ be a function with $J_s(x) = x(s)$. We assume that $\langle \varphi_n \rangle$ converges to φ weakly. By a similar calculation as in this example, since $\langle \mathcal{F}(\varphi_n) \rangle$ converges to $\mathcal{F}(\varphi)$ pointwise, $\langle \mathcal{F}(\omega_{\varphi_n}(J_s^{-1}(\cdot))) \rangle$ converges to $\mathcal{F}(\omega_\varphi(J_s^{-1}(\cdot)))$ pointwise; so by the continuity theorem in [1, Theorem 12-5A, p. 273], $\langle \omega_{\varphi_n}(J_s^{-1}(\cdot)) \rangle$ converges to $\omega_\varphi(J_s^{-1}(\cdot))$ weakly.

(5) We assume that $k(u) = u^2$ is φ-integrable. For $0 \le s_1, s_2 \le t$,

$$\int_{C[0,t]} x(s_1)x(s_2) \, d\omega_\varphi(x) \;=\; (\min\{s_1, s_2\})\varphi(\mathbb{R}) + \int_{\mathbb{R}} u^2 \, d\varphi(u) \; .$$

If $\varphi = \delta_p$, then $\int_{C[0,t]} x(s_1)x(s_2) \, d\omega_\varphi(x) \;=\; \min\{s_1, s_2\} + p^2$ and if φ has a normal distribution with mean α and variance σ^2, then

$$\int_{C[0,t]} x(s_1)x(s_2) \, d\omega_\varphi(x) \;=\; \min\{s_1, s_2\} + \alpha^2 + \sigma^2 \; .$$

(6) For $0 \le s_1 < s_2 \le s_3 < s_4 \le t$ and for α, β in $\mathbb{R}$, using the change of variable formula, we have

$$\begin{aligned} &\omega_\varphi(\{x \text{ in } C[0,t] \mid x(s_2) - x(s_1) \le \alpha \text{ and } x(s_4) - x(s_3) \le \beta\}) \\ &\quad = \varphi(\mathbb{R})\omega_\varphi(\{x \text{ in } C[0,t] \mid x(s_2) - x(s_1) \le \alpha\}) \\ &\quad\quad \cdot \omega_\varphi(\{x \text{ in } C[0,t] \mid x(s_4) - x(s_3) \le \beta\}). \end{aligned}$$

Hence, if φ is a probability measure, then $x(s_2) - x(s_1)$ and $x(s_4) - x(s_3)$ are independent.

Theorem 3.4. *For φ in $\mathcal{M}(\mathbb{R})$, $|\omega_\varphi| = \omega_{|\varphi|}$ on $(C[0,t], \mathcal{B}(C[0,t]))$.*

Proof. By [19, Theorem 6.12, p. 124], there exists a measurable function h such that $|h(u)| = 1$ for all u in $\mathbb{R}$ and $d\varphi = hd|\varphi|$. For $\vec{t} = (t_0, t_1, \cdots, t_n)$ with $0 = t_0 < t_1, \cdots, t_n \le t$ and for B_j $(j = 0, 1, \cdots, n)$ in $\mathcal{B}(\mathbb{R})$, by [19, Theorem 6.13,

p. 125] and (3.3),

(3.4)

$$\begin{aligned}\omega_\varphi(J_{\vec{t}}^{-1}(\prod_{j=0}^{n} B_j)) \\ &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}^n} W(n+1;\vec{t};u_0,u_1,\cdots,u_n) d(\prod_{j=1}^{n} m_L)(u_1,\cdots,u_n) \right] d\varphi(u_0) \\ &= \int_{\mathbb{R}} h(u_0) \left[\int_{\mathbb{R}^n} W(n+1;\vec{t};u_0,u_1,\cdots,u_n)\ d(\prod_{j=1}^{n} m_L)(u_1,\cdots,u_n) \right] d|\varphi|(u_0)\end{aligned}$$

and so

(3.5)

$$\begin{aligned}|\omega_\varphi|(J_{\vec{t}}^{-1}(\prod_{j=0}^{n} B_j)) \\ &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}^n} W(n+1;\vec{t};u_0,u_1,\cdots,u_n)\ d(\prod_{j=1}^{n} m_L)(u_1,\cdots,u_n) \right] d|\varphi|(u_0) \\ &= \omega_{|\varphi|}(J_{\vec{t}}^{-1}(\prod_{j=0}^{n} B_j)) .\end{aligned}$$

□

We consider a set $\mathcal{A} = \{E \text{ in } \mathcal{B}(C[0,t]) \mid |\omega_\varphi|(E) = \omega_{|\varphi|}(E)\}$. Then, by the equality (3.5), we have $\mathcal{I} \subset \mathcal{A}$. Since $|\omega_\varphi|$ and $\omega_{|\varphi|}$ are both measures on $(C[0,t], \mathcal{B}(C[0,t]))$, by [18, Theorem 2.1, p. 212], $|\omega_\varphi| = \omega_{|\varphi|}$ on $\mathcal{B}(C[0,t])$.

4. The measure-valued measure V_φ on $(C[0,t], \mathcal{B}(C[0,t]))$

In this section, we introduce a measure-valued measure V_φ on $(C[0,t], \mathcal{B}(C[0,t]))$ from the concepts of an analogue of the Wiener measure ω_φ and investigate its basic properties.

Let φ be in $\mathcal{M}(\mathbb{R})$ with the Jordan decomposition $\varphi = \sum_{j=1}^{4} \alpha_j \varphi_j$ and let $J_t : C[0,t] \to \mathbb{R}$ be a function defined by $J_t(x) = x(t)$.

For a measure μ in $\mathcal{M}(\mathbb{R})$, for B in $\mathcal{B}(C[0,t])$ and for E in $\mathcal{B}(\mathbb{R})$, let

$$P_\mu(E) = \omega_\mu(J_t^{-1}(E)) \tag{4.1}$$

and

$$Q_{B,\mu}(E) = \omega_\mu(J_t^{-1}(E) \cap B) . \tag{4.2}$$

Then both P_μ and $Q_{B,\mu}$ are measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. By Theorem 3.4, we have

$$P_\varphi = \sum_{j=1}^{4} \alpha_j P_{\varphi_j} , \tag{4.3}$$

$$Q_{B,\varphi} = \sum_{j=1}^{4} \alpha_j Q_{B,\varphi_j}, \tag{4.4}$$

and

$$|P_\varphi| \;=\; P_{|\varphi|}\ . \tag{4.5}$$

For $j = 1,2,3,4$, Q_{B,φ_j} is absolutely continuous with respect to P_{φ_j}, and P_{φ_j} is absolutely continuous with respect to $|P_\varphi| \;=\; P_{|\varphi|}$. Hence, for $j = 1,2,3,4$, by the Radon-Nikodym theorem, there exists a measurable function $\widetilde{\psi}_{B,\varphi_j}$ such that $0 \le \widetilde{\psi}_{B,\varphi_j} \le 1$ and $Q_{B,\varphi_j}(E) \;=\; \int_E \widetilde{\psi}_{B,\varphi_j}(\xi)\ dP_{|\varphi|}(\xi)$ for E in $\mathcal{B}(\mathbb{R})$. By [19, Theorem 6.12, p. 124], there is a measurable function h such that $|h(\xi)| = 1$ for all ξ in $\mathbb{R}$ and $dP_\varphi \;=\; h\ dP_{|\varphi|}$; so $dP_{|\varphi|} \;=\; \bar{h}\ dP_\varphi$ where $\bar{h}$ is conjugate of h. Hence, we can write

$$Q_{B,\varphi_j}(E) \;=\; \int_E \widetilde{\psi}_{B,\varphi_j}(\xi)\ \bar{h}(\xi)\ dP_\varphi(\xi) \tag{4.6}$$

for E in $\mathcal{B}(\mathbb{R})$ and for $j = 1,2,3,4$. Thus,

$$Q_{B,\varphi}(E) \;=\; \int_E \sum_{j=1}^{4} \alpha_j \widetilde{\psi}_{B,\varphi_j}(\xi)\ \bar{h}(\xi)\ dP_\varphi(\xi)\ . \tag{4.7}$$

For B in $\mathcal{B}(C[0,t])$, we let

$$\psi_{B,\varphi} \;=\; \left(\sum_{j=1}^{4} \alpha_j \widetilde{\psi}_{B,\varphi_j} \right)\ \bar{h}\ . \tag{4.8}$$

By Theorem 3.1, the Radon-Nikodym derivative $\frac{dP_\varphi}{dm_L}$ exists and

$$\frac{dP_\varphi}{dm_L}(\xi) \;=\; \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} \exp\left\{ -\frac{(\xi-u)^2}{2t} \right\}\ d\varphi(u); \tag{4.9}$$

so for E in $\mathcal{B}(\mathbb{R})$,

$$Q_{B,\varphi}(E) = \int_{\mathbb{R}} \left[\int_E \frac{1}{\sqrt{2\pi t}} \psi_{B,\varphi}(\xi) \exp\left\{ -\frac{(\xi-u)^2}{2t} \right\}\ dm_L(\xi) \right]\ d\varphi(u)\ . \tag{4.10}$$

For B in $\mathcal{B}(C[0,t])$, we let

$$[V_\varphi(B)](E) \;=\; Q_{B,\varphi}(E) \tag{4.11}$$

for E in $\mathcal{B}(\mathbb{R})$.

Remark 4.1. Let φ be in $\mathcal{M}(\mathbb{R})$.

(1) From (4.2), for E in $\mathcal{B}(\mathbb{R})$, $Q_{\emptyset,\varphi}(E) = 0$, which implies that $[V_\varphi(\emptyset)](E) = 0$ for E in $\mathcal{B}(\mathbb{R})$.

(2) $P_\varphi \;=\; Q_{C[0,t],\varphi}$, and so, by (4.7), $\psi_{C[0,t],\varphi} \;=\; 1\ \ P_{|\varphi|}$-a.e. Hence, by (2.31), we have

$$[V_\varphi(C[0,t])](E) \;=\; [S_t(\varphi)](E)$$

for E in $\mathcal{B}(\mathbb{R})$.

(3) Let φ be in $\mathcal{M}(\mathbb{R})$ with the Jordan decomposition $\varphi \;=\; \sum_{j=1}^{4} \alpha_j \varphi_j$. Then from (4.4) and (4.11), we have $V_\varphi \;=\; \sum_{j=1}^{4} \alpha_j V_{\varphi_j}$.

(4) By Theorem 3.1 and equation (4.11), we know that $[V_{\alpha\varphi_1+\beta\varphi_2}(B)](E) = \alpha[V_{\varphi_1}(B)](E) + \beta[V_{\varphi_2}(B)](E)$ for any α, β in $\mathbb{C}$, for any φ_1, φ_2 in $\mathcal{M}(\mathbb{R})$, for B in $\mathcal{B}(C[0,t])$ and for E in $\mathcal{B}(\mathbb{R})$. For B in $\mathcal{B}(C[0,t])$, by (C) in section 2,

$$\begin{aligned}
&\| V_\varphi(B) \| \\
&\quad \le \sup_{\substack{\|\theta\|_\infty \le 1 \\ \theta \in L^\infty(\mathbb{R}, m_L)}} \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |\psi_{B,\varphi}(\xi)| \exp\left\{ -\frac{(\xi-u)^2}{2t} \right\} |\theta(\xi)| \, dm_L(\xi) \right] d|\varphi|(u) \\
&\quad \le 4|\varphi|(\mathbb{R}) .
\end{aligned} \tag{4.12}$$

Hence, letting $V(B) : \mathcal{M}(\mathbb{R}) \to \mathcal{M}(\mathbb{R})$ with $[V(B)](\varphi) = V_\varphi(B)$, $V(B)$ is a bounded linear operator on $\mathcal{M}(\mathbb{R})$.

Theorem 4.2. *For φ in $\mathcal{M}(\mathbb{R})$, V_φ is a measure-valued measure on $(C[0,t]$, $\mathcal{B}(C[0,t]))$ in the total variation norm sense.*

Proof. By Remark 4.1 (1), $V_\varphi(\emptyset)$ is a zero measure and the finite additivity of V_φ is clear. Hence, it suffices to show that V_φ is countably additive in the total variation norm sense. Suppose φ is a positive measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. In this case, $dP_\varphi = dP_{|\varphi|}$. Let $\langle B_n \rangle$ be a sequence of disjoint sets in $\mathcal{B}(C[0,t])$ with $B = \bigcup_{j=1}^\infty B_j$. Then for E in $\mathcal{B}(\mathbb{R})$,

$$\begin{aligned}
&\int_E \widetilde{\psi}_{B,\varphi}(\xi) \, dP_\varphi(\xi) \\
&\quad \overset{(1)}{=} \int_{J_t^{-1}(E)} \chi_B(x) \, d\omega_\varphi(x) \\
&\quad \overset{(2)}{=} \sum_{j=1}^\infty \int_{J_t^{-1}(E)} \chi_{B_j}(x) \, d\omega_\varphi(x) \\
&\quad \overset{(3)}{=} \sum_{j=1}^\infty \int_E \widetilde{\psi}_{B_j,\varphi}(\xi) \, dP_\varphi(\xi) \\
&\quad \overset{(4)}{=} \int_E \sum_{j=1}^\infty \widetilde{\psi}_{B_j,\varphi}(\xi) \, dP_\varphi(\xi) .
\end{aligned} \tag{4.13}$$

Steps (1) and (3) follow from equality (4.10). Steps (2) and (4) result from the monotone convergence theorem.

Since E is an arbitrary Borel set in (4.13), by [22, Theorem 8.1, p. 76],

$$\widetilde{\psi}_{B,\varphi}(\xi) = \sum_{j=1}^\infty \widetilde{\psi}_{B_j,\varphi}(\xi)$$

for P_φ-a.e. ξ.

Since $\widetilde{\psi}_{B_j,\varphi} \geq 0$ for all natural numbers j, by [16], $\sum_{j=1}^{\infty} \widetilde{\psi}_{B_j,\varphi}$ converges to $\widetilde{\psi}_{B,\varphi}$ in the $L^1(\mathbb{R}, P_\varphi)$-norm sense. Hence

$$
\begin{aligned}
&|\sum_{j=k}^{\infty} V_\varphi(B_j)|(\mathbb{R}) \\
&= \sum_{j=k}^{\infty} \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} \widetilde{\psi}_{B_j,\varphi}(\xi) \exp\left\{ -\frac{(\xi-u)^2}{2t} \right\} \, dm_L(\xi) \right] \, d\varphi(u) \\
&= \sum_{j=k}^{\infty} \| \widetilde{\psi}_{B_j,\varphi} \|_{L^1(\mathbb{R},P_\varphi)} \\
&\to 0
\end{aligned} \tag{4.14}
$$

as $k \to +\infty$.

Hence, V_φ is a measure-valued measure in the total variation norm sense if φ is a positive measure.

Now, let φ be in $\mathcal{M}(\mathbb{R})$ with the Jordan decomposition $\varphi = \sum_{j=1}^{4} \alpha_j \varphi_j$ and let $\langle B_j \rangle$ be a sequence of disjoint sets in $\mathcal{B}(C[0,t])$ with $B = \bigcup_{j=1}^{\infty} B_j$. Then

$$
\begin{aligned}
&|\sum_{j=k}^{\infty} V_\varphi(B_j)|(\mathbb{R}) \\
&\leq \sum_{j=k}^{\infty} \left[(\sum_{n=1}^{2} \| \widetilde{\psi}_{B_j,\varphi_n} \|_{L^1(\mathbb{R},P_\varphi)})^2 + (\sum_{n=1}^{2} \| \widetilde{\psi}_{B_j,\varphi_{n+2}} \|_{L^1(\mathbb{R},P_\varphi)})^2 \right]^{\frac{1}{2}} \\
&\leq \sum_{j=k}^{\infty} \sum_{n=1}^{4} \| \widetilde{\psi}_{B_j,\varphi_n} \|_{L^1(\mathbb{R},P_\varphi)} \\
&\to 0
\end{aligned} \tag{4.15}
$$

as $k \to +\infty$, as desired. □

Theorem 4.3. *Let φ be in $\mathcal{M}(\mathbb{R})$ and let B be in $\mathcal{B}(C[0,t])$ with $|\omega_\varphi|(B) = 0$. Then $V_\varphi(B)$ is a zero measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.*

Proof. Let $\varphi = \sum_{j=1}^{4} \alpha_j \varphi_j$ be the Jordan decomposition. Then $\omega_{\varphi_j}(B) = 0$ for $j = 1,2,3,4$; so for $j = 1,2,3,4$, $\widetilde{\psi}_{B,\varphi_j} = 0$ P_{φ_j}-a.e., which implies that for $j = 1,2,3,4$, $\widetilde{\psi}_{B,\varphi_j} = 0$ m_L-a.e. Hence $\psi_{B,\varphi} = 0$, m_L-a.e. By (4.10) and (4.11), for E in $\mathcal{B}(\mathbb{R})$, $[V_\varphi(B)](E) = 0$, as desired. □

Theorem 4.4. *Let φ be in $\mathcal{M}(\mathbb{R})$ and let $\vec{t} = (t_0, t_1, \cdots, t_n)$ be a vector in $\mathbb{R}^{n+1}$ with $0 = t_0 < \cdots < t_n = t$. Let $f : \mathbb{R}^{n+1} \to \mathbb{C}$ be a Borel measurable function such that $f(u_0, u_1, \cdots, u_n) W(n+1; \vec{t}; u_0, \cdots, u_n)$ is $|\varphi| \times \prod_{j=1}^{n} m_L$-integrable. Let $F : C[0,t] \to \mathbb{C}$ be a function with $F(x) = (f \circ J_{\vec{t}})(x) = f(x(t_0), x(t_1), \cdots, x(t_n))$.*

Then F is V_φ-Bartle integrable on $C[0,t]$ and for E in $\mathcal{B}(\mathbb{R})$,

$$\begin{aligned}(4.16)\qquad &\left[(Ba)-\int_{C[0,t]} F(x)\ dV_\varphi(x)\right](E)\\ &= \int_E \Bigg\{ \int_{\mathbb{R}^{n-1}} \Big(\int_{\mathbb{R}} f(u_0,u_1,\cdots,u_n) W(n+1;\vec{t};u_0,\cdots,u_n)\\ &\qquad d\varphi(u_0)\Big)\ d\Big(\prod_{j=1}^{n-1} m_L\Big)(u_1,\cdots,u_{n-1})\Bigg\}\ dm_L(u_n).\end{aligned}$$

Proof. Let $f=\chi_B$ where B is a Borel subset of $\mathbb{R}^{n+1}$. Then $F(x)=(\chi_B\circ J_{\vec{t}})(x)$ $=\chi_{J_{\vec{t}}^{-1}(B)}(x)$ is V_φ-Bartle integrable and for E in $\mathcal{B}(\mathbb{R})$,

$$\begin{aligned}(4.17)\qquad &\left[(Ba)-\int_{C[0,t]} F(x)\ dV_\varphi(x)\right](E)\\ &= [V_\varphi(J_{\vec{t}}^{-1}(B))](E)\\ &= \omega_\varphi(J_{\vec{t}}^{-1}(B)\cap J_t^{-1}(E))\\ &= \int_E \Bigg\{ \int_{\mathbb{R}^{n-1}} \Big(\int_{\mathbb{R}} \chi_B(u_0,u_1,\cdots,u_n) W(n+1;\vec{t};u_0,\cdots,u_n)\\ &\qquad d\varphi(u_0)\Big)\ d\Big(\prod_{j=1}^{n-1} m_L\Big)(u_1,u_2,\cdots,u_{n-1})\Bigg\}\ dm_L(u_n)\ .\end{aligned}$$

If f is a simple function, then by the basic properties of the Lebesgue integral and the Bartle integral, it is not hard to show that F is V_φ-Bartle integrable and the equality (4.16) holds for f. We assume that φ is a positive measure in $\mathcal{M}(\mathbb{R})$ and f is a nonnegative real-valued Borel measurable function on $\mathbb{R}^{n+1}$ such that $f(u_0,u_1,\cdots,u_n)W(n+1;\vec{t};u_0,\cdots,u_n)$ is $|\varphi|\times\prod_{j=1}^n m_L$-integrable. Then there is a sequence $\langle f_m\rangle$ of nonnegative real-valued simple functions such that $\langle f_m(u_0,u_1,\cdots,u_n)\rangle$ is increasing for $(u_0,u_1,\cdots,u_n)$ in $\mathbb{R}^{n+1}$, $\langle f_m\rangle$ converges to f for $|\varphi|\times\prod_{j=1}^n m_L$-a.e. and

$$\langle \int_{\mathbb{R}^{n+1}} f_m(u_0,u_1,\cdots,u_n)W(n+1;\vec{t};u_0,\cdots,u_n)\ d(\varphi\times\prod_{j=1}^n m_L)(u_0,(u_1,\cdots,u_n))\rangle$$

converges to

$$\int_{\mathbb{R}^{n+1}} f(u_0,u_1,\cdots,u_n)W(n+1;\vec{t};u_0,\cdots,u_n)\ d(\varphi\times\prod_{j=1}^n m_L)(u_0,(u_1,\cdots,u_n)).$$

Let $N = \{(u_0, u_1, \cdots, u_n)$ in $\mathbb{R}^{n+1}$ | either the limit $\lim_{m\to\infty} f_m(u_0, u_1, \cdots, u_n)$ does not exist or the limit $\lim_{m\to\infty} f_m(u_0, u_1, \cdots, u_n)$ exists but is not equal to $f(u_0, u_1, \cdots, u_n)\}$. Then N is $|\varphi| \times \prod_{j=1}^{n} m_L$-null. By Theorem 3.1, $J_{\vec{t}}^{-1}(N)$ is $\omega_{|\varphi|}$-null. For a natural number m and for E in $\mathcal{B}(\mathbb{R})$, let

$$F_m = f_m \circ J_{\vec{t}} \tag{4.18}$$

and

$$\begin{aligned}\mu_m(E) = \int_E \Bigg\{ \int_{\mathbb{R}^{n-1}} \Big(\int_{\mathbb{R}} f_m(u_0, u_1, \cdots, u_n) W(n+1; \vec{t}; u_0, \cdots, u_n)\ d\varphi(u_0)\Big) \\ d\Big(\prod_{j=1}^{n-1} m_L\Big)(u_1, \cdots, u_{n-1}) \Bigg\}\ dm_L(u_n)\ .\end{aligned} \tag{4.19}$$

Then $\langle F_m \rangle$ converges to F $\omega_{|\varphi|}$-a.e.; so by Theorem 4.3, $\langle F_m \rangle$ converges to F $\| V_\varphi \|$-a.e. By the monotone convergence theorem, for E in $\mathcal{B}(\mathbb{R})$, $\langle \mu_m(E) \rangle$ converges; here, we denote $\mu(E) = \lim_{m\to\infty} \mu_m(E)$. By the Vitali-Hahn-Saks theorem, μ is a countably additive measure on $\mathcal{B}(\mathbb{R})$ and for E in $\mathcal{B}(\mathbb{R})$,

$$\begin{aligned}\mu(E) = \int_E \Bigg\{ \int_{\mathbb{R}^{n-1}} \Big(\int_{\mathbb{R}} f(u_0, u_1, \cdots, u_n) W(n+1; \vec{t}; u_0, u_1, \cdots, u_n)\ d\varphi(u_0)\Big) \\ d\Big(\prod_{j=1}^{n-1} m_L\Big)(u_1, \cdots, u_{n-1}) \Bigg\}\ dm_L(u_n)\ .\end{aligned} \tag{4.20}$$

Since for a natural number m, $|f_m - f| \leq 2|f|$, by the dominated convergence theorem,

$$\begin{aligned}&\overline{\lim_{m\to\infty}} |\mu_m - \mu|(\mathbb{R}) \\ &\quad \leq \lim_{m\to\infty} \int_{\mathbb{R}^{n+1}} |f_m(u_0, u_1, \cdots, u_n) - f(u_0, u_1, \cdots, u_n)| \\ &\qquad W(n+1; \vec{t}; u_0, u_1, \cdots, u_n)\ d\Big(|\varphi| \times \prod_{j=1}^{n} m_L\Big)(u_0, u_1, \cdots, u_n) \\ &\quad = 0\ .\end{aligned} \tag{4.21}$$

Hence $\langle \mu_m \rangle$ converges to μ in the total variation norm sense; so F is V_φ-Bartle integrable. Moreover, for E in $\mathcal{B}(\mathbb{R})$,

$$\begin{aligned}
&\left[(Ba) - \int_{C[0,t]} F(x)\, dV_\varphi(x)\right](E) \\
&\quad = \lim_{m\to\infty} \left[(Ba) - \int_{C[0,t]} F_m(x)\, dV_\varphi(x)\right](E) \\
&\quad = \lim_{m\to\infty} \int_E \Big\{ \int_{\mathbb{R}^{n-1}} \Big(\int_{\mathbb{R}} f_m(u_0, u_1, \cdots, u_n) W(n+1; \vec{t}; u_0, \cdots, u_n) \\
&\qquad d\varphi(u_0)\Big)\, d(\prod_{j=1}^{n-1} m_L)(u_1, u_2, \cdots, u_{n-1})\Big\}\, dm_L(u_n) \\
&\quad = \int_E \Big\{ \int_{\mathbb{R}^{n-1}} \Big(\int_{\mathbb{R}} f(u_0, u_1, \cdots, u_n) W(n+1; \vec{t}; u_0, \cdots, u_n) \\
&\qquad d\varphi(u_0)\Big)\, d(\prod_{j=1}^{n-1} m_L)(u_1, u_2, \cdots, u_{n-1})\Big\}\, dm_L(u_n)\ .
\end{aligned} \tag{4.22}$$

Now, it remains to show that this theorem holds in the general case. Let φ in $\mathcal{M}(\mathbb{R})$ with the Jordan decomposition $\varphi = \sum_{j=1}^4 \alpha_j \varphi_j$ and let $f : \mathbb{R}^{n+1} \to \mathbb{C}$ be a Borel measurable function such that $f(u_0, u_1, \cdots, u_n) W(n+1; \vec{t}; u_0, u_1, \cdots, u_n)$ is $|\varphi| \times \prod_{j=1}^n m_L$-integrable. Then we can write $f = \sum_{j=1}^4 \alpha_j f_j$ where f_j is a nonnegative Borel measurable function for $j = 1, 2, 3, 4$. Then $f_j(u_0, u_1, \cdots, u_n) W(n+1; \vec{t}; u_0, u_1, \cdots, u_n)$ is $\varphi_k \times \prod_{j=1}^n m_L$-integrable for $j, k = 1, 2, 3, 4$, and we proved the following fact already. Letting $F_j = f_j \circ J_{\vec{t}}$ for $j = 1, 2, 3, 4$, F_j is V_{φ_k}-Bartle integrable for $j, k = 1, 2, 3, 4$; and for $j, k = 1, 2, 3, 4$ and for E in $\mathcal{B}(\mathbb{R})$, we then have

$$\begin{aligned}
&\left[(Ba) - \int_{C[0,t]} F_j(x)\, dV_{\varphi_k}(x)\right](E) \\
&\quad = \int_E \Big\{ \int_{\mathbb{R}^{n-1}} \Big(\int_{\mathbb{R}} f_j(u_0, u_1, \cdots, u_n) W(n+1; \vec{t}; u_0, u_1, \cdots, u_n) \\
&\qquad d\varphi_k(u_0)\Big)\, d(\prod_{p=1}^{n-1} m_L)(u_1, u_2, \cdots, u_{n-1})\Big\}\, dm_L(u_n)\ .
\end{aligned} \tag{4.23}$$

By (3) in Remark 4.1 and the basic properties of the Lebesgue integral and the Bartle integral, for E in $\mathcal{B}(\mathbb{R})$,

(4.24)

$$\begin{aligned}
&\left[(Ba)-\int_{C[0,t]} F(x)\ dV_{\varphi}(x)\right](E) \\
&\quad = \sum_{j,k=1}^{4} \alpha_j\alpha_k \left[(Ba)-\int_{C[0,t]} F_j(x)\ dV_{\varphi_k}(x)\right](E) \\
&\quad = \sum_{j,k=1}^{4} \alpha_j\alpha_k \int_E \Big\{ \int_{\mathbb{R}^{n-1}} \Big(\int_{\mathbb{R}} f_j(u_0,u_1,\cdots,u_n) W(n+1;\vec{t};u_0,\cdots,u_n) \\
&\qquad d\varphi_k(u_0)\Big)\ d(\prod_{p=1}^{n-1} m_L)(u_1,\cdots,u_{n-1})\Big\}\ dm_L(u_n) \\
&\quad = \int_E \Big\{ \int_{\mathbb{R}^{n-1}} \Big(\int_{\mathbb{R}} f(u_0,u_1,\cdots,u_n) W(n+1;\vec{t};u_0,\cdots,u_n) \\
&\qquad d\varphi(u_0)\Big)\ d(\prod_{p=1}^{n-1} m_L)(u_1,\cdots,u_{n-1})\Big\}\ dm_L(u_n)\ ,
\end{aligned}$$

as desired. □

5. A measure-valued Feynman-Kac formula

In this section, we will achieve the measure-valued Feynman-Kac formula for the integral with respect to a measure-valued measure V_φ of a suitable functional.

Theorem 5.1. *Let φ be in $\mathcal{M}(\mathbb{R})$, let η be a complex-valued Borel measure on $[0,t]$ and let θ be in $L_{\varphi;\infty,1;\eta}$. Then*

$$|\theta(s,x(s))| \leq \| \theta(s,\cdot) \|_{\varphi;\infty} \tag{5.1}$$

for $|\eta|\times\omega_{|\varphi|}$-a.e. (s,x) in $[0,t]\times C[0,t]$.

Proof. We consider a propositional function $P : [0,t]\times\mathbb{R} \to \{0,1\}$ given by $P(s,v)=1$ if either $\theta(0,v)$ fails to be defined $|\varphi|$-a.e. and $\theta(s,v)$ fails to be defined m_L-a.e. for $0<s\leq t$ or $\theta(s,v)$ is defined but $|\theta(s,v)| >\| \theta(s,\cdot) \|_{\varphi;\infty}$ and $P(s,v)=0$ otherwise. Let $N=\{(s,v) \text{ in } [0,t]\times\mathbb{R} \mid P(s,v)=1\}$, $N_1=\{(0,v) \text{ in } [0,t]\times\mathbb{R} \mid P(0,v)=1\}$ and $N_2=\{(s,v) \text{ in } [0,t]\times\mathbb{R} \mid P(s,v)=1 \text{ and } 0<s\leq t\ \}$. Then both N_1 and N_2 are disjoint Borel subsets and $N=N_1\cup N_2$. Let $H:[0,t]\times C[0,t]\to[0,t]\times\mathbb{R}$ be a function with $H(s,x)=(s,x(s))$. Then H is continuous and $\theta(s,x(s))=(\theta\circ H)(s,x)$ is Borel measurable with respect to (s,x). Since $[H^{-1}(N)]^{(0)}=[H^{-1}(N_1)]^{(0)}$ and $[H^{-1}(N)]^{(s)}=[H^{-1}(N_2)]^{(s)}$ for $0<s\leq t$, by Theorem 3.1, $\omega_{|\varphi|}([H^{-1}(N)]^{(0)})=|\varphi|([N_1]^{(0)})=0$ and $\omega_{|\varphi|}([H^{-1}(N)]^{(s)})=\int_{[N_2]^{(s)}}\frac{1}{\sqrt{2\pi s}}(\int_{\mathbb{R}}\exp\{-\frac{(u-u_0)^2}{2s}\}\ d|\varphi|(u_0))\ dm_L(u)=0$ for $0<s\leq t$; that is, $\omega_{|\varphi|}([H^{-1}(N)]^{(s)})=0$ for all s in $[0,t]$ where $[A]^{(s)}$ is the s-cross section of A.

By the Fubini theorem,

$$
\begin{aligned}
&|\eta| \times \omega_{|\varphi|}(H^{-1}(N)) \\
&\quad = \int_{[0,t]\times C[0,t]} \chi_{H^{-1}(N)}(s,x)\ d(|\eta| \times \omega_{|\varphi|})(s,x) \\
&\quad = \int_{[0,t]} \omega_{|\varphi|}([H^{-1}(N)]^{(s)})\ d|\eta|(s) \\
&\quad = 0\ ,
\end{aligned} \tag{5.2}
$$

as desired. □

Throughout this section, let $\eta = \mu + \nu$ be a complex-valued Borel measure on $[0,t]$ such that μ is the continuous part of η and $\nu = \sum_{p=0}^{n} c_p \delta_{\tau_p}$ where $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_n = t$ and c_p $(p = 0, 1, \cdots, n)$ are complex numbers, let φ be in $\mathcal{M}(\mathbb{R})$ and let θ be in $L_{\varphi;\infty,1;\eta}$. For nonnegative integers q and $j_1, \cdots, j_n$ with $q = j_1 + j_2 + \cdots + j_n$, let

$$
\begin{aligned}
&\triangle_{q;j_1,j_2,\cdots,j_n} \\
&\quad = \{(s_{1,1}, s_{1,2}, \cdots, s_{1,j_1}, s_{2,1}, \cdots, s_{n-1,j_{n-1}}, s_{n,1}, \cdots, s_{n,j_n}) \mid \tau_0 = 0 < s_{1,1} < \\
&\qquad \cdots < s_{1,j_1} < \tau_1 < s_{2,1} < \cdots < \tau_{n-1} < s_{n,1} < \cdots < s_{n,j_n} < \tau_n = t\}\ .
\end{aligned} \tag{5.3}
$$

For convenience, we let $M_{\theta(s,\cdot)} \equiv M_{\theta(s)}$ for $0 \le s \le t$ and $\tau_0 = s_{0,0}$, $\tau_n = t = s_{n,j_n+1}$ and $\tau_k = s_{k+1,0} = s_{k,j_k+1}$ for $k = 1, 2, \cdots, n-1$. For nonnegative integers $m, q_0, \cdots, q_{n+1}, j_1, \cdots, j_n$ with $m = q_0 + q_1 + \cdots + q_{n+1}$ and $q_{n+1} = j_1 + j_2 + \cdots + j_n$, let $K(m,n,q,j) : \triangle_{q_{n+1};j_1,j_2,\cdots,j_n} \times C[0,t] \to \mathbb{C}$ be a function with

$$
\begin{aligned}
&K(m,n,q,j)((s_{1,1}, \cdots, s_{n,j_n}), x) \\
&\quad = \left[\prod_{i=0}^{n} \theta(\tau_i, x(\tau_i))^{q_i}\right] \left[\prod_{i=1}^{n}\prod_{j=1}^{j_i} \theta(s_{i,j}, x(s_{i,j}))\right]
\end{aligned} \tag{5.4}
$$

and let $D(m,n,q,j) : \triangle_{q_{n+1};j_1,j_2,\cdots,j_n} \to \mathbb{R}$ be a function with

$$
\begin{aligned}
&D(m,n,q,j)(s_{1,1}, \cdots, s_{n,j_n}) \\
&\quad = \left[\prod_{i=0}^{n} \| \theta(\tau_i, \cdot) \|_{\varphi;\infty}^{q_i}\right] \left[\prod_{i=1}^{n}\prod_{j=1}^{j_i} \| \theta(s_{i,j}, \cdot) \|_{\varphi;\infty}\right] .
\end{aligned} \tag{5.5}
$$

Lemma 5.2. (1) $|K(m,n,q,j)| \le D(m,n,q,j)$ $|\mu| \times \omega_{|\varphi|}$*-a.e.,*

(2) $|\int_{\triangle_{q_{n+1};j_1,\cdots,j_n}} D(m,n,q,j)(s_{1,1}, \cdots, s_{n,j_n})\ d(\prod_{i=1}^{n}\prod_{j=1}^{j_i} \mu)(s_{1,1}, \cdots, s_{n,j_n})| \le \frac{1}{q_{n+1}!}(\prod_{i=0}^{n} \| \theta(\tau_i, \cdot) \|_{\varphi;\infty}^{q_i})(\| \theta \|_{\varphi;\infty,1;\mu})^{q_{n+1}}$ *and*

(3) $D(m,n,q,j)$ *is* $(\prod_{i=1}^{n}\prod_{j=1}^{j_i} \mu) \times V_\varphi$*-Bartle integrable on* $\triangle_{q_{n+1};j_1,\cdots,j_n} \times C[0,t]$.

Proof. By Theorem 5.1, the statement (1) in this lemma is clear. Obviously, $D(m,n,q,j)$ is Borel measurable. Letting

$$\triangle_{q_{n+1}} = \{(s_1, s_2, \cdots, s_{q_{n+1}}) \text{ in } [0,t]^{q_{n+1}} \mid 0 = s_0 < s_1 < s_2 < \cdots < s_{q_{n+1}} < t\},$$

since $\triangle_{q_{n+1}}$ is Borel measurable and $\triangle_{q_{n+1};j_1,\cdots,j_n} \subset \triangle_{q_n}$,

(5.6)
$$\begin{aligned}
&\left|\int_{\triangle_{q_{n+1};j_1,\cdots,j_n}} D(m,n,q,j)(s_{1,1},\cdots,s_{n,j_n})\ d(\prod_{i=1}^{n}\prod_{j=1}^{j_i}\mu)(s_{1,1},\cdots,s_{n,j_n})\right| \\
&\quad\le\ (\prod_{i=0}^{n}\parallel\theta(\tau_i,\cdot)\parallel_{\varphi;\infty}^{q_i})\int_{\triangle_{q_{n+1}}}\prod_{i=1}^{q_{n+1}}\parallel\theta(s_i,\cdot)\parallel_{\varphi;\infty}\ d(\prod_{i=1}^{q_{n+1}}|\mu|)(s_1,s_2,\cdots,s_{q_{n+1}}) \\
&\quad=\ \frac{1}{q_{n+1}!}\left[\int_{[0,t]}\parallel\theta(s,\cdot)\parallel_{\varphi;\infty}\ d|\mu|(s)\right]^{q_{n+1}}\left[\prod_{i=0}^{n}\parallel\theta(\tau_i,\cdot)\parallel_{\varphi;\infty}^{q_i}\right] \\
&\quad=\ \frac{1}{q_{n+1}!}(\parallel\theta\parallel_{\varphi;\infty,1;\mu})^{q_{n+1}}(\prod_{i=0}^{n}\parallel\theta(\tau_i,\cdot)\parallel_{\varphi;\infty}^{q_i})\ .
\end{aligned}$$

Hence, we proved statement (2) in this lemma. Since $D(m,n,q,j)$ is $\prod_{i=1}^{n}\prod_{j=1}^{j_i}|\mu|$-integrable, there is a sequence $\langle g_k\rangle$ of simple functions such that $\langle g_k\rangle$ converges to $D(m,n,q,j)$ $\prod_{i=1}^{n}\prod_{j=1}^{j_i}|\mu|$-a.e. and a sequence $\langle\int_{\triangle_{q_{n+1};j_1,\cdots,j_n}} g_k(s_{1,1},\cdots,s_{n,j_n})$ $d(\prod_{i=1}^{n}\prod_{j=1}^{j_i}\mu)(s_{1,1},\cdots,s_{n,j_n})\rangle$ is Cauchy. Then $\langle g_k\rangle$ converges to $D(m,n,q,j)$ $\prod_{i=1}^{n}\prod_{j=1}^{j_i}|\mu|\times\parallel V_\varphi\parallel$-a.e. and for two natural numbers k and l,

(5.7)
$$\begin{aligned}
&\Big|(Ba)-\int_{\triangle_{q_{n+1};j_1,\cdots,j_n}}(g_l-g_k)(s_{1,1},\cdots,s_{n,j_n})\ d[(\prod_{i=1}^{n}\prod_{j=1}^{j_i}\mu)\times V_\varphi] \\
&\qquad((s_{1,1},\cdots,s_{n,j_n}),x)\Big| \\
&\quad\le\ 4\Big|\varphi|(\mathbb{R})|\int_{\triangle_{q_{n+1};j_1,\cdots,j_n}}(g_l-g_k)(s_{1,1},\cdots,s_{n,j_n}) \\
&\qquad d(\prod_{i=1}^{n}\prod_{j=1}^{j_i}\mu)(s_{1,1},\cdots,s_{n,j_n})\Big|\ .
\end{aligned}$$

So

$$\langle(Ba)-\int_{\triangle_{q_{n+1};j_1,\cdots,j_n}} g_k(s_{1,1},\cdots,s_{n,j_n})\ d[(\prod_{i=1}^{n}\prod_{j=1}^{j_i}\mu)\times V_\varphi]((s_{1,1},\cdots,s_{n,j_n}),x)\rangle$$

is Cauchy. Therefore, $D(m,n,q,j)$ is $(\prod_{i=1}^{n}\prod_{j=1}^{j_i}\mu)\times V_\varphi$-Bartle integrable. □

Lemma 5.3. *$\theta(s,x(s))$ is $\mu\times V_\varphi$-Bartle integrable on $[0,t]\times C[0,t]$.*

Proof. Since θ is Borel measurable on $[0,t]\times\mathbb{R}$, there is a sequence $\langle\psi_n\rangle$ of simple functions on $[0,t]\times\mathbb{R}$ such that $\langle\psi_n\rangle$ converges to θ $|\mu|\times m_L$-a.e. Let $N=\{(s,u)$ in $[0,t]\times\mathbb{R}\mid$ either $\theta(s,u)$ is not defined or $\langle\psi_n(s,u)\rangle$ does not converge to $\theta(s,u)\}$ and let $H:[0,t]\times C[0,t]\to[0,t]\times\mathbb{R}$ be a function with $H(s,x)=(s,x(s))$. Then H is continuous, and so the sequence $\langle\psi_n\circ H\rangle$ is a sequence of simple functions such that $\langle\psi_n\circ H\rangle$ converges to $\theta\circ H$ on $[0,t]\times C[0,t]\setminus H^{-1}(N)$. Moreover, by the Fubini theorem and Theorem 4.4,

(5.8)
$$\begin{aligned}
&|\mu| \times \omega_{|\varphi|}(H^{-1}(N)) \\
&\quad = \int_{[0,t]} \int_{C[0,t]} \chi_{H^{-1}(N)}(s,x)\ d\omega_{|\varphi|}(x) d|\mu|(s) \\
&\quad = \int_{[0,t]} \int_{C[0,t]} \chi_{[N]^{(s)}}(x(s))\ d\omega_{|\varphi|}(x) d|\mu|(s) \\
&\quad = \int_{(0,t]} \int_{C[0,t]} \chi_{[N]^{(s)}}(x(s))\ d\omega_{|\varphi|}(x) d|\mu|(s) \\
&\quad = \int_{(0,t]} \frac{1}{\sqrt{2\pi s}} \int_{\mathbb{R}} \chi_{[N]^{(s)}}(u_1) \exp\left\{-\frac{(u_1-u_0)^2}{2s}\right\}\ dm_L(u_1) d\varphi(u_0) d|\mu|(s) \\
&\quad = 0;
\end{aligned}$$

so $|\mu| \times \omega_{|\varphi|}(H^{-1}(N)) = 0$. If $|\mu| \times \omega_{|\varphi|}(H^{-1}(N)) = 0$, then $|\mu|$-a.e. s, $[H^{-1}(N)]^{(s)}$ is $\omega_{|\varphi|}$-null, then, by Theorem 4.3, for $|\mu|$-a.e. s, $[H^{-1}(N)]^{(s)}$ is $\| V_\varphi \|$-null, which implies that for x^* in $\mathcal{RM}^*(\mathbb{R})$ with $\| x^* \|_\infty \leq 1$, $[H^{-1}(N)]^{(s)}$ is $|x^* V_\varphi|$-null for $|\mu|$-a.e. s. Thus, we have

$$\begin{aligned}
&\| \mu \times V_\varphi \| (H^{-1}(N)) \\
&\quad = \sup\{|x^*(\mu \times V_\varphi)|(H^{-1}(N)) \mid x^* \text{ is in } L^\infty(\mathbb{R}, m_L) \text{ and } \| x^* \|_\infty \leq 1\} \\
&\quad = \sup\{|\mu \times (x^* V_\varphi)|(H^{-1}(N)) \mid x^* \text{ is in } L^\infty(\mathbb{R}, m_L) \text{ and } \| x^* \|_\infty \leq 1\} \\
&\quad = \sup\{|\mu| \times |x^* V_\varphi|(H^{-1}(N)) \mid x^* \text{ is in } L^\infty(\mathbb{R}, m_L) \text{ and } \| x^* \|_\infty \leq 1\} \\
&\quad = 0\ .
\end{aligned} \tag{5.9}$$

So, the sequence $\langle \psi_n \circ H \rangle$ of simple functions on $[0,t] \times C[0,t]$ converges to $\theta \circ H$ $\| \mu \times V_\varphi \|$-a.e., which implies that $\theta(s, x(s))$ is $\mu \times V_\varphi$-measurable. Now, by (4.12),

$$\begin{aligned}
&\int_{[0,t]\times C[0,t]} |\theta(s,x(s))|\ d|x^*(\mu \times V_\varphi)|(s,x) \\
&\quad \leq \int_{[0,t]\times C[0,t]} |\theta(s,x(s))|\ d|\mu| \times \| V_\varphi \| (s,x) \\
&\quad \leq \int_{C[0,t]} \int_{[0,t]} \| \theta(s,\cdot) \|_{\varphi;\infty}\ d|\mu|(s) d \| V_\varphi \| (x) \\
&\quad = \| \theta \|_{\varphi;\infty,1;\mu} \| V_\varphi \| (C[0,t]) \\
&\quad \leq 4|\varphi|(\mathbb{R}) \| \theta \|_{\varphi;\infty,1;\mu} \\
&\quad < +\infty\ ,
\end{aligned} \tag{5.10}$$

that is, $\theta(s, x(s))$ is $x^*(\mu \times V_\varphi)$-integrable. Now let $F : \mathcal{B}(\mathbb{R}) \to \mathbb{C}$ be a measure given by

$$\begin{aligned}
F(E) = \int_E \int_{[0,t]} \int_{\mathbb{R}} \int_{\mathbb{R}} \theta(s,u_1) \frac{1}{\sqrt{(2\pi)^2 s(t-s)}}\ \exp\left\{-\frac{(u_1-u_0)^2}{2s} - \frac{(u_2-u_1)^2}{2(t-s)}\right\} \\
d\varphi(u_0)\ dm_L(u_1)\ d\mu(s)\ dm_L(u_2)\ ,
\end{aligned}$$

for E in $\mathcal{B}(\mathbb{R})$. Then by the Tonelli theorem, we know that for E in $\mathcal{B}(\mathbb{R})$, $|F(E)| \leq \| \theta \|_{\varphi;\infty,1;\mu}$, that is, F is well-defined. So F is in $\mathcal{RM}(\mathbb{R})$. For x^* in $\mathcal{RM}^*(\mathbb{R})$, there exists a function f in $L^\infty(\mathbb{R}, m_L)$ such that $x^*(\mu) = \int_{\mathbb{R}} f(\xi)\ d\mu(\xi)$,

(5.11)

$$\begin{aligned}
&x^*(F)\\
&\overset{(1)}{=} \int_{\mathbb{R}} f(u_2)\ dF(u_2)\\
&\overset{(2)}{=} \int_{\mathbb{R}} f(u_2)\frac{dF}{dm_L}(u_2)\ dm_L(u_2)\\
&\overset{(3)}{=} \int_{\mathbb{R}} f(u_1)\Big[\int_{[0,t]}\int_{\mathbb{R}}\int_{\mathbb{R}} \theta(s,u_1)\frac{1}{\sqrt{(2\pi)^2 s(t-s)}}\ \exp\Big\{-\frac{(u_1-u_0)^2}{2s}\\
&\quad -\frac{(u_2-u_1)^2}{2(t-s)}\Big\}d\varphi(u_0)dm_L(u_1)d\mu(s)\Big]dm_L(u_2)\\
&\overset{(4)}{=} \int_{[0,t]}\Big[\int_{\mathbb{R}} f(u_1)\Big(\int_{\mathbb{R}}\int_{\mathbb{R}} \theta(s,u_1)\frac{1}{\sqrt{(2\pi)^2 s(t-s)}}\ \exp\Big\{-\frac{(u_1-u_0)^2}{2s}\\
&\quad -\frac{(u_2-u_1)^2}{2(t-s)}\Big\}d\varphi(u_0)dm_L(u_1)\Big)dm_L(u_2)\Big]d\mu(s)\\
&\overset{(5)}{=} \int_{[0,t]}\int_{C[0,t]} \theta(s,x(s))\ dx^*(V_\varphi x)d\mu(s)\\
&\overset{(6)}{=} \int_{[0,t]\times C[0,t]} \theta(s,x(s))\ d(\mu\times x^*V_\varphi)(s,x)\\
&\overset{(7)}{=} \int_{[0,t]\times C[0,t]} \theta(s,x(s))\ dx^*(\mu\times V_\varphi)(s,x)\ .
\end{aligned}$$

Step (1) follows from (C) in section 2. By the Radon-Nikodym theorem, we have Step (2). From the definition of a measure F, we obtain Step (3). Steps (4) and (6) result from the Fubini theorem. By Theorem 4.4, $\theta(s,x(s))$ is V_φ-Bartle integrable $|\mu|$-a.e. s. For x^* in $\mathcal{RM}^*(\mathbb{R})$, corresponding to f in $L^\infty(\mathbb{R}, m_L)$ and for $|\mu|$-a.e. s, by (2.15) and Theorem 4.4,

(5.12)

$$\begin{aligned}
&\int_{C[0,t]} \theta(s,x(s))\ dx^*V_\varphi(x)\\
&\quad = x^*((Ba)-\int_{C[0,t]} \theta(s,x(s))\ dV_\varphi(x))\\
&\quad = x^*\Big(\int_{\cdot}\int_{\mathbb{R}}\int_{\mathbb{R}} \theta(s,u_1)\frac{1}{\sqrt{(2\pi)^2 s(t-s)}}\exp\Big\{-\frac{(u_1-u_0)^2}{2s}-\frac{(u_2-u_1)^2}{2(t-s)}\Big\}\\
&\quad\quad d\varphi(u_0)dm_L(u_1)dm_L(u_2)\Big)\\
&\quad = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u_1)\theta(s,u_1)\frac{1}{\sqrt{(2\pi)^2 s(t-s)}}\exp\Big\{-\frac{(u_1-u_0)^2}{2s}-\frac{(u_2-u_1)^2}{2(t-s)}\Big\}\\
&\quad\quad d\varphi(u_0)dm_L(u_1)dm_L(u_2)\ .
\end{aligned}$$

So, we obtain Step (5). By (2.20), we know that Step (7) holds. Hence by [15, Definition 2.1, p. 159], $\theta(s,x(s))$ is $\mu\times V_\varphi$-Bartle integrable. □

Theorem 5.4. (1) $K(m,n,q,j)$ *is* $(\prod_{i=1}^{n}\prod_{j=1}^{j_i}\mu)\times V_\varphi$*-Bartle integrable,*

(2) *for* $\prod_{i=1}^{n}\prod_{j=1}^{j_i}|\mu|$*-a.e.* $(s_{1,1},\cdots,s_{n,j_n})$, $K(m,n,q,j)((s_{1,1},\cdots,s_{n,j_n}),\cdot)$ *is* V_φ*-Bartle integrable, and*

(3) $(Ba)-\int_{C[0,t]}K(m,n,q,j)((s_{1,1},\cdots,s_{n,j_n}),x)\;dV_\varphi(x)$ *is* $\prod_{i=1}^{n}\prod_{j=1}^{j_i}\mu$*-Bochner integrable.*

Proof. By Lemma 5.3, the statement (1) in this theorem holds. From Lemma 5.2, for $\prod_{i=1}^{n}\prod_{j=1}^{j_i}|\mu|$-a.e. $(s_{1,1},\cdots,s_{n,j_n})$,

$$
\begin{aligned}
&|K(m,n,q,j)((s_{1,1},\cdots,s_{n,j_n}),x)| \\
&\quad\leq\; D(m,n,q,j)(s_{1,1},\cdots,s_{n,j_n})
\end{aligned}
\tag{5.13}
$$

holds for $\|\,V_\varphi\,\|$-a.e. x and $D(m,n,q,j)$ is finite; that is, for $\prod_{i=1}^{n}\prod_{j=1}^{j_i}|\mu|$-a.e. $(s_{1,1},\cdots,s_{n,j_n})$, $K(m,n,q,j)((s_{1,1},\cdots,s_{n,j_n}),x)$ is $\|\,V_\varphi\,\|$-essentially bounded with respect to x. Hence, from (2.14), for $\prod_{i=1}^{n}\prod_{j=1}^{j_i}|\mu|$-a.e. $(s_{1,1},\cdots,s_{n,j_n})$, $K(m,n,q,j)((s_{1,1},\cdots,s_{n,j_n}),x)$ is a V_φ-Bartle integrable function of x.

By the classical Fubini theorem, for x^* in $\mathcal{RM}(\mathbb{R})^*$,

$$
\begin{aligned}
&x^*\left[(Ba)-\int_{C[0,t]}K(m,n,q,j)((s_{1,1},\cdots,s_{n,j_n}),x)\;dV_\varphi(x)\right] \\
&\quad=\;\int_{C[0,t]}K(m,n,q,j)((s_{1,1},\cdots,s_{n,j_n}),x)\;d(x^*V_\varphi)(x)
\end{aligned}
\tag{5.14}
$$

is $\prod_{i=1}^{n}\prod_{j=1}^{j_i}\mu$-measurable with respect to $(s_{1,1},\cdots,s_{n,j_n})$. That is, $(Ba)-\int_{C[0,t]}K(m,n,q,j)((s_{1,1},\cdots,s_{n,j_n}),x)\;dV_\varphi(x)$ is $\prod_{i=1}^{n}\prod_{j=1}^{j_i}\mu$-weakly measurable with respect to $(s_{1,1},\cdots,s_{n,j_n})$. Since $\mathcal{RM}(\mathbb{R})$ is separable, by (2.6),

$$
(Ba)-\int_{C[0,t]}K(m,n,q,j)((s_{1,1},\cdots,s_{n,j_n}),x)\;dV_\varphi(x)
$$

is $\prod_{i=1}^{n}\prod_{j=1}^{j_i}\mu$-measurable with respect to $(s_{1,1},\cdots,s_{n,j_n})$. By Lemma 5.2,

$$
\begin{aligned}
&\int_{\triangle_{q_{n+1};j_1,\cdots,j_n}}\Bigg|(Ba)-\int_{C[0,t]}K(m,n,q,j)((s_{1,1},\cdots,s_{n,j_n}),x) \\
&\qquad dV_\varphi(x)\Bigg|(\mathbb{R})\;d(\prod_{i=1}^{n}\prod_{j=1}^{j_i}|\mu|)(s_{1,1},\cdots,s_{n,j_n}) \\
&\quad\leq\;\frac{4}{q_{n+1}!}|\varphi|(\mathbb{R})(\prod_{i=0}^{n}\|\,\theta(\tau_i,\cdot)\,\|_{\varphi;\infty}^{q_i})(\|\,\theta\,\|_{\varphi;\infty,1;\mu})^{q_{n+1}} \\
&\quad<\;+\infty\;.
\end{aligned}
\tag{5.15}
$$

Hence, by (2.8), $\int_{C[0,t]}K(m,n,q,j)((s_{1,1},\cdots,s_{n,j_n}),x)\;dV_\varphi(x)$ is $\prod_{i=1}^{n}\prod_{j=1}^{j_i}\mu$-Bochner integrable. □

At this point, we establish one of our main theorems in this article. The proof of the following theorem is patterned to some extent on earlier work by Johnson and Lapidus in [9], but the present setting requires a number of new concepts and results in the previous parts of this article.

Theorem 5.5 (A measure-valued Feynman-Kac formula). *$\exp\{\int_{[0,t]}\theta(s,x(s))d\eta(s)\}$ is V_φ-Bartle integrable on $C[0,t]$ and for E in $\mathcal{B}(\mathbb{R})$,*

(5.16)
$$\left[(Ba)-\int_{C[0,t]}\exp\left\{\int_{[0,t]}\theta(s,x(s))\ d\eta(s)\right\}\ dV_\varphi(x)\right](E)$$
$$= \sum_{m=0}^{\infty}\sum_{q_0+\cdots+q_{n+1}=m}\frac{\prod_{p=0}^{n}c_p^{q_p}}{\prod_{p=0}^{n}q_p!}\sum_{j_1+\cdots+j_n=q_{n+1}}\int_{\triangle_{q_{n+1};j_1,\cdots,j_n}}$$
$$[(L_n\circ L_{n-1}\circ\cdots\circ L_1)(T(s_{1,1},\varphi,\theta(0,\cdot)^{q_0}))](E)\ d(\prod_{i=1}^{n}\prod_{j=1}^{j_i}\mu)(s_{1,1},\cdots,s_{n,j_n})\ .$$

Moreover,

$$\left|(Ba)-\int_{C[0,t]}\exp\left\{\int_{[0,t]}\theta(s,x(s))\ d\eta(s)\right\}\ dV_\varphi(x)\right|(\mathbb{R}) \le\ 4|\varphi|(\mathbb{R})[\exp\{\|\ \theta\ \|_{\varphi;\infty,1;\eta}\}]\ . \tag{5.17}$$

Here, for $k=2,3,\cdots,n$,

(5.18)
$$L_k = M_{\theta(\tau_k)^{q_k}}\circ S_{\tau_k-s_{k,j_k}}\circ M_{\theta(s_{k,j_k})}\circ S_{s_{k,j_k}-s_{k,j_k-1}}\circ\cdots\circ M_{\theta(s_{k,1})}\circ S_{s_{k,1}-s_{k,0}}$$

and

$$L_1 = M_{\theta(\tau_1)^{q_1}}\circ S_{\tau_1-s_{1,j_1}}\circ M_{\theta(s_{1,j_1})}\circ S_{s_{1,j_1}-s_{1,j_1-1}}\circ\cdots\circ M_{\theta(s_{1,1})}\ . \tag{5.19}$$

Proof. By Theorem 5.1, we have

$$\begin{aligned}&\left|\exp\left\{\int_{[0,t]}\theta(s,x(s))\ d\eta(s)\right\}\right|\\ &\le \exp\left\{\int_{[0,t]}\|\ \theta(s,\cdot)\ \|_{\varphi;\infty}\ \ d|\eta|(s)\right\}\\ &= \exp\{\|\ \theta\ \|_{\varphi;\infty,1;\eta}\}\end{aligned} \tag{5.20}$$

for $\|\ V_\varphi\ \|$-a.e. x.

Hence, by (2.14), Lemma 5.3 and the Fubini theorem, $\exp\{\int_{[0,t]}\theta(s,x(s))\ d\eta(s)\}$ is V_φ-Bartle integrable on $C[0,t]$. Moreover, by (2.14) and (4.12), we have

$$\left|(Ba)-\int_{C[0,t]}\exp\left\{\int_{[0,t]}\theta(s,x(s))\ d\eta(s)\right\}\ dV_\varphi(x)\right|(\mathbb{R}) \le\ 4|\varphi|(\mathbb{R})\exp\{\|\ \theta\ \|_{\varphi;\infty,1;\eta}\}; \tag{5.21}$$

so, we obtain the inequality (5.17). Since for any nonnegative integer n,

$$\left|(\int_{[0,t]}\theta(s,x(s))\ d\eta(s))^n\right| \le\ (\|\ \theta\ \|_{\varphi;\infty,1;\eta})^n \tag{5.22}$$

for $\|\ V_\varphi\ \|$-a.e. x, by (2.17), $(\int_{[0,t]}\theta(s,x(s))\ d\eta(s))^n$ is V_φ-Bartle integrable.

Now, we prove the equality (5.16). For E in $\mathcal{B}(\mathbb{R})$,

(5.23)
$$\begin{aligned}
&\Big[(Ba) - \int_{C[0,t]} \exp\{\int_{[0,t]} \theta(s, x(s))\ d\eta(s)\}\ dV_\varphi(x)\Big](E) \\
&\overset{(1)}{=} \Big[(Ba) - \int_{C[0,t]} \sum_{m=0}^{\infty} \frac{1}{m!}\Big(\int_{[0,t]} \theta(s, x(s))\ d\eta(s)\Big)^m\ dV_\varphi(x)\Big](E) \\
&\overset{(2)}{=} \sum_{m=0}^{\infty} \frac{1}{m!}\Big[(Ba) - \int_{C[0,t]} \Big(\int_{[0,t]} \theta(s, x(s))\ d\eta(s)\Big)^m\ dV_\varphi(x)\Big](E) \\
&\overset{(3)}{=} \sum_{m=0}^{\infty} \frac{1}{m!}\Big[(Ba) - \int_{C[0,t]} \Big(\sum_{p=0}^{n} c_p \theta(\tau_p, x(\tau_p)) \\
&\quad + \int_{[0,t]} \theta(s, x(s))\ d\mu(s)\Big)^m\ dV_\varphi(x)\Big](E) \\
&\overset{(4)}{=} \sum_{m=0}^{\infty} \sum_{q_0+\cdots+q_{n+1}=m} \frac{\prod_{j=0}^{n} c_j^{q_j}}{\prod_{j=0}^{n+1} q_j!}\Big[(Ba) - \int_{C[0,t]} (\prod_{j=0}^{n} \theta(\tau_j, x(\tau_j))^{q_j}) \\
&\quad \Big(\int_{[0,t]} \theta(s, x(s))\ d\mu(s)\Big)^{q_{n+1}}\ dV_\varphi(x)\Big](E) \\
&\overset{(5)}{=} \sum_{m=0}^{\infty} \sum_{q_0+\cdots+q_{n+1}=m} \frac{\prod_{j=0}^{n} c_j^{q_j}}{\prod_{j=0}^{n+1} q_j!}\Big[(Ba) - \int_{C[0,t]} (\prod_{j=0}^{n} \theta(\tau_j, x(\tau_j))^{q_j})(q_{n+1})! \\
&\quad \Big(\int_{\triangle_{q_{n+1}}} \prod_{j=1}^{q_{n+1}} \theta(s_j, x(s_j))\ d(\prod_{j=1}^{q_{n+1}} \mu)(s_1, s_2, \cdots, s_{q_{n+1}})\Big)\ dV_\varphi(x)\Big](E) \\
&\overset{(6)}{=} \sum_{m=0}^{\infty} \sum_{q_0+\cdots+q_{n+1}=m} \frac{\prod_{j=0}^{n} c_j^{q_j}}{\prod_{j=0}^{n} q_j!} \sum_{j_1+\cdots+j_n=q_{n+1}} \\
&\quad \Big[(Ba) - \int_{C[0,t]} \int_{\triangle_{q_{n+1};j_1,\cdots,j_n}} \\
&\quad \theta(\tau_0, x(\tau_0))^{q_0}\{\prod_{i=1}^{n}(\prod_{j=1}^{j_i} \theta(s_{i,j}, x(s_{i,j})))\theta(s_{i,j_i+1}, x(s_{i,j_i+1}))^{q_i}\} \\
&\quad d(\prod_{i=1}^{n}\prod_{j=1}^{j_i} \mu)(s_{1,1}, \cdots, s_{n,j_n})\ dV_\varphi(x)\Big](E) \\
&\overset{(7)}{=} \sum_{m=0}^{\infty} \sum_{q_0+\cdots+q_{n+1}=m} \frac{\prod_{j=0}^{n} c_j^{q_j}}{\prod_{j=0}^{n} q_j!} \sum_{j_1+\cdots+j_n=q_{n+1}} \int_{\triangle_{q_{n+1};j_1,\cdots,j_n}} \Big[(Ba) \\
&\quad - \int_{C[0,t]} \theta(\tau_0, x(\tau_0))^{q_0}\{\prod_{i=1}^{n}(\prod_{j=1}^{j_i} \theta(s_{i,j}, x(s_{i,j})))\theta(s_{i,j_i+1}, x(s_{i,j_i+1}))^{q_i}\} \\
&\quad dV_\varphi(x)\Big](E)\ d(\prod_{i=1}^{n}\prod_{j=1}^{j_i} \mu)(s_{1,1}, \cdots, s_{n,j_n})
\end{aligned}$$

$$\overset{(8)}{=} \sum_{m=0}^{\infty} \sum_{q_0+\cdots+q_{n+1}=m} \frac{\prod_{j=0}^{n} c_j^{q_j}}{\prod_{j=0}^{n} q_j!} \sum_{j_1+\cdots+j_n=q_{n+1}} \int_{\triangle_{q_{n+1};j_1,\cdots,j_n}} (2\pi)^{-\frac{1}{2}(q_{n+1}+n)}$$
$$(\prod_{i=1}^{n}\prod_{j=1}^{j_i+1}(s_{i,j}-s_{i,j-1}))^{-\frac{1}{2}} \int_{\mathbb{R}}\int_{\mathbb{R}^{q_{n+1}+n}} \theta(0,u_{0,0})^{q_0}(\prod_{i=1}^{n}(\prod_{j=1}^{j_i}\theta(s_{i,j},u_{i,j}))$$
$$\theta(s_{i,j_i+1},u_{i,j_i+1})^{q_i})\chi_E(u_{n,j_n+1}) \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{j_i+1}\frac{(u_{i,j}-u_{i,j-1})^2}{s_{i,j}-s_{i,j-1}}\right\}$$
$$d(\prod_{i=1}^{n}\prod_{j=1}^{j_i+1} m_L)(u_{1,1},\cdots,u_{n,j_n+1})\ d\varphi(u_{0,0})\ d(\prod_{i=1}^{n}\prod_{j=1}^{j_i}\mu)(s_{1,1},\cdots,s_{n,j_n+1})$$
$$\overset{(9)}{=} \sum_{m=0}^{\infty} \sum_{q_0+\cdots+q_{n+1}=m} \frac{\prod_{j=1}^{n} c_j^{q_j}}{\prod_{j=1}^{n} q_j!} \sum_{j_1+\cdots+j_n=q_{n+1}} \int_{\triangle_{q_{n+1};j_1,\cdots,j_n}} [(L_n\circ L_{n-1}\circ\cdots\circ L_1)$$
$$(T(s_{1,1};\varphi;\theta(0,\cdot)^{q_0}))](E)d(\prod_{i=1}^{n}\prod_{j=1}^{j_i}\mu)(s_{1,1},\cdots,s_{n,j_n})\ .$$

By the Taylor expansion, we have Step (1). Letting

$$f_n(x) = \sum_{m=0}^{n}\frac{1}{m!}\left(\int_{[0,t]}\theta(s,x(s))d\eta(s)\right)^m$$

on $C[0,t]$ for any nonnegative integer n, $\langle f_n\rangle$ converges to

$$\exp\left\{\int_{[0,t]}\theta(s,x(s))\ d\eta(s)\right\}$$

for $\parallel V_\varphi \parallel$-a.e. x and $\exp\{\int_{[0,t]}\theta(s,x(s))\ d\eta(s)\}$ is V_φ-Bartle integrable, by (2.17), we obtain Step (2). Step (3) follows from the elementary computation of an integral. From the multinomial expansion,

$$(a_0+a_1+\cdots+a_n)^m = \sum_{q_0+\cdots+q_n=m}\frac{m!}{q_0!\cdots q_n!}a_0^{q_0}\cdots a_n^{q_n},$$

we have Step (4). Let $\triangle_{q_{n+1}} = \{(s_1,\cdots,s_{q_{n+1}})$ in $[0,t]^{q_{n+1}}$ $\mid 0 < s_1 < s_2 < \cdots < s_{q_{n+1}} < t\ \}$, let $\mathcal{P}$ be the set of all permutations on $\{1,2,\cdots,q_{n+1}\}$, for σ in $\mathcal{P}$, let $\triangle_{q_{n+1};\sigma} = \{(s_{\sigma(1)},\cdots,s_{\sigma(q_{n+1})})$ in $[0,t]^{q_{n+1}} \mid (s_1,\cdots,s_{q_{n+1}})$ is in $\triangle_{q_{n+1}}\}$, and let $N = \{(s_1,\cdots,s_{q_{n+1}})$ in $[0,t]^{q_{n+1}} \mid$ either for some $1\le i,j\le q_{n+1}$with $i\ne j, s_i = s_j$ or for some $1\le i\le q_{n+1}, s_i = 0$ or $s_i = t\ \}$. Then for $\sigma_1\ne\sigma_2$ in $\mathcal{P}$, $\triangle_{q_{n+1};\sigma_1}\cap\triangle_{q_{n+1};\sigma_2}=\emptyset$, for σ in $\mathcal{P}$, $\triangle_{q_{n+1};\sigma}$ is Borel measurable and $[0,t]^{q_{n+1}} = N\cup(\bigcup_{\sigma\in\mathcal{P}}\triangle_{q_{n+1};\sigma})$. Since μ has no atoms, by the Fubini theorem, $(\prod_{i=1}^{q_{n+1}}\mu)(N) = 0$ and for σ in $\mathcal{P}$,

$$\int_{\triangle_{q_{n+1};\sigma}}\prod_{i=1}^{q_{n+1}}\theta(s_i,x(s_i))\ d(\prod_{i=1}^{q_{n+1}}\mu)(s_1,\cdots,s_{q_{n+1}})$$
$$=\int_{\triangle_{q_{n+1}}}\prod_{i=1}^{q_{n+1}}\theta(s_i,x(s_i))\ d(\prod_{i=1}^{q_{n+1}}\mu)(s_1,\cdots,s_{q_{n+1}}).$$

Hence, Step (5) is true. Let $N^* = \{(s_1, s_2, \cdots, s_{q_{n+1}}) \mid 0 < s_1 < s_2 < \cdots < s_{j_1} \leq \tau_1 \leq s_{j_1+1} < s_{j_1+2} < \cdots < s_{j_1+j_2} \leq \tau_2 \leq s_{j_1+j_2+1} < \cdots < s_{j_1+\cdots+j_n} \leq \tau_n = t$ and for some $k = 1, 2, \cdots, n$ and for some $j = 1, 2, \cdots, q_{n+1}, \tau_k = s_j\}$. Then for $(j_1, \cdots, j_n) \neq (j'_1, \cdots, j'_n)$, $\triangle_{q_{n+1};j_1,\cdots,j_n} \cap \triangle_{q_{n+1};j'_1,\cdots,j'_n} = \emptyset$, for all $j_1, j_2, \cdots, j_n$ with $j_1 + \cdots + j_n = q_{n+1}$, $\triangle_{q_{n+1};j_1,\cdots,j_n}$ and N^* are Borel measurable, $(\prod_{i=1}^{q_{n+1}} \mu)(N^*) = 0$ and $\triangle_{q_{n+1}} = N^* \cup (\bigcup_{j_1+\cdots+j_n=q_{n+1}} \triangle_{q_{n+1};j_1,\cdots,j_n})$. Therefore, by the relabelling $s_{j_1+\cdots+j_{k-1}+l} = s_{k-1,l}$ for $k = 0, 1, \cdots, n$ and $l = 0, 1, \cdots, j_k$, we deduce that Step (6) holds. By Theorem 2.1, Theorem 2.3 and Theorem 5.4, we obtain Step (7). For $i = 1, 2, \cdots, n$, let $u_{i,j_i+1} = u_{i+1,0}$. By the relabelling as follows: for $k = 0, 1, \cdots, n$, $\tau_k = s_{k,j_k+1} = s_{k+1,0}$ and by Theorem 4.4, Step (8) holds. By the definitions of S, M and T in (H) of section 2, we have Step (9). Thus, the proof of this theorem is finished. □

From Theorem 5.5, directly we deduce the following corollaries.

Corollary 5.6. *In Theorem 5.5, we assume that $\eta = \mu$, an arbitrary continuous measure on $[0, t]$. Then for E in $\mathcal{B}(\mathbb{R})$,*

(5.24)
$$\left[(Ba) - \int_{C[0,t]} \exp\left\{\int_{[0,t]} \theta(s, x(s))\ d\eta(s)\right\}\ dV_\varphi(x)\right](E)$$
$$= \sum_{m=0}^{\infty} \int_{\triangle_m} [(S_{t-s_m} \circ M_{\theta(s_m)} \circ \cdots \circ S_{s_2-s_1} \circ M_{\theta(s_1)})(T(s_1, \varphi, \theta^0 \equiv 1))]$$
$$(E)\ d(\prod_{i=1}^{m} \mu)(s_1, s_2, \cdots, s_m)$$

where $\triangle_m = \{(s_1, s_2, \cdots, s_m)$ in $[0, t]^m \mid 0 < s_1 < s_2 < \cdots < s_m < t\}$.

Corollary 5.7. *In Theorem 5.5, we assume that $\eta = \nu = \sum_{p=0}^{n} c_p \delta_{\tau_p}$, a discrete measure on $[0, t]$ with finite support. Then for E in $\mathcal{B}(\mathbb{R})$,*

(5.25)
$$\left[(Ba) - \int_{C[0,t]} \exp\left\{\int_{[0,t]} \theta(s, x(s))\ d\eta(s)\right\}\ dV_\varphi(x)\right](E)$$
$$= \sum_{m=0}^{\infty} \sum_{q_0+\cdots+q_n=m} \frac{\prod_{p=0}^{n} c_p^{q_p}}{\prod_{p=0}^{n} q_p!} [(M_{\theta(\tau_n)^{q_n}} \circ S_{\tau_n-\tau_{n-1}} \circ \cdots \circ S_{\tau_2-\tau_1} \circ M_{\theta(\tau_1)^{q_1}})$$
$$(T(\tau_1, \varphi, \theta(0, \cdot)^{q_0}))](E)\ .$$

Corollary 5.8. *In Theorem 5.5, we assume that $c_n = 0$. Then for E in $\mathcal{B}(\mathbb{R})$,*

$$\left[(Ba) - \int_{C[0,t]} \exp\left\{\int_{[0,t]} \theta(s,x(s))\ d\eta(s)\right\}\ dV_\varphi(x)\right](E)$$

$$(5.26)\qquad = \sum_{m=0}^{\infty} \sum_{q_0+\cdots+q_n=m} \frac{\prod_{p=0}^{n-1} c_p^{q_p}}{\prod_{p=0}^{n-1} q_p!} \sum_{j_1+\cdots+j_n=q_n} \int_{\triangle_{q_n;j_1,\cdots,j_n}} [((S_{t-s_{n,j_n}} \circ M_{\theta(s_{n,j_n})} \circ \cdots \circ S_{s_{n,1}-\tau_{n-1}}) \circ L_{n-1} \circ \cdots \circ L_1) (T(s_{1,1},\varphi,\theta(0,\cdot)^{q_0}))](E)\ d(\prod_{i=1}^{n}\prod_{j=1}^{j_i}\mu)(s_{1,1},\cdots,s_{n,j_n})\ .$$

6. A Volterra integral equation for the measure-valued Feynman-Kac formula

In this section, we prove that the equality (5.16) in Theorem 5.5, satisfies a suitable Volterra integral equation (see [10], [12], [13], [14]).

Throughout this section, let $0 = \tau_0 < \tau_1 < \cdots < \tau_n = t < \widetilde{t}$ and let η be a Borel measure on $[0,\widetilde{t}\ ]$ such that $\eta = \mu + \nu$ where μ is the continuous part of η and $\nu = \sum_{p=0}^{n} c_p \delta_{\tau_p}$; furthermore, let θ be in $L^{\widetilde{t}}_{\varphi;\infty,1;\eta}$. Let

$$(6.1)\qquad u(t') = (Ba) - \int_{C[0,t']} \exp\left\{\int_{[0,t']} \theta(s,x(s))\ d\eta(s)\right\}\ dV_\varphi(x)$$

for $t < t' \le \widetilde{t}$.

The following theorem is the counterpart for the measure-valued measure V_φ of the integral equation for the Feynman-Kac formula with Lebesgue-Stieltjes measure, obtained by Lapidus in [12], [13], [14] and for the Feynman-Kac formula with an operator-valued measure, obtained by Kluvanek in [10].

Theorem 6.1 (The measure-valued Feynman-Kac formula)**.** *For $t < t' \le \widetilde{t}$, $u(t')$ satisfies a Volterra integral equation, that is,*

$$(6.2)\qquad u(t') = S_{t'-t}(u(t)) + (Bo) - \int_{(t,t']} (S_{t'-s} \circ M_{\theta(s)}) u(s)\ d\mu(s)\ .$$

Proof. By Corollary 5.8, for $t < s \le \widetilde{t}$,

$$(6.3)\qquad u(s) = \sum_{m=0}^{\infty} \sum_{q_0+\cdots+q_{n+1}=m} \frac{\prod_{p=0}^{n} c_p^{q_p}}{\prod_{p=0}^{n} q_p!} \sum_{j_1+\cdots+j_{n+1}=q_{n+1}} (Bo) - \int_{\triangle^{(s)}_{q_{n+1};j_1,\cdots,j_{n+1}}} [S_{s-s_{n+1,j_{n+1}}} \circ M_{\theta(s_{n+1,j_{n+1}})} \circ \cdots \circ S_{s_{n+1,1}-s_{n+1,0}} \circ L_n \circ \cdots \circ L_1] (T(s_{1,1},\varphi,\theta(0,\cdot)^{q_0}))\ d(\prod_{i=1}^{n+1}\prod_{j=1}^{j_i}\mu)(s_{1,1},\cdots,s_{n+1,j_{n+1}}),$$

where $\triangle^{(s)}_{q_{n+1};j_1,\cdots,j_{n+1}} = \{(s_{1,1},\cdots,s_{n+1,j_{n+1}}) \mid 0 = s_{0,0} = \tau_0 < s_{1,1} < \cdots < s_{1,j_1} < \tau_1 < s_{2,1} < \cdots < \tau_n < s_{n+1,1} < \cdots < s_{n+1,j_{n+1}} < s\ \}$ and for $k = 1, 2, \cdots, n$, L_k is given in Theorem 5.5. For $t < s \le \widetilde{t}$, let

(6.4)

$$\begin{aligned} &Y(s; q_0, \cdots, q_{n+1}; j_1, \cdots, j_{n+1}) \\ &\quad = (Bo) - \int_{\triangle^{(s)}_{q_{n+1};j_1,\cdots,j_{n+1}}} [S_{s-s_{n+1,j_{n+1}}} \circ M_{\theta(s_{n+1,j_{n+1}})} \circ \cdots \circ S_{s_{n+1,1}-s_{n+1,0}} \\ &\qquad \circ L_n \circ \cdots \circ L_1](T(s_{1,1}, \varphi, \theta(0,\cdot)^{q_0}))\ d(\prod_{i=1}^{n+1}\prod_{j=1}^{j_i}\mu)(s_{1,1},\cdots,s_{n+1,j_{n+1}})\ . \end{aligned}$$

For $t < t' \le \widetilde{t}$, let

$$u_1(t') = \sum_{m=1}^{\infty} \sum_{q_0+\cdots+q_n=m} \frac{\prod_{p=0}^{n} c_p^{q_p}}{\prod_{p=0}^{n} q_p!} Y(t'; q_0, \cdots, q_n, 0; 0, 0, \cdots, 0),$$

$$\begin{aligned} u_2(t') = \sum_{m=1}^{\infty} \sum_{\substack{q_0+\cdots+q_{n+1}=m \\ q_{n+1}\ge 1}} \frac{\prod_{p=0}^{n} c_p^{q_p}}{\prod_{p=0}^{n} q_p!} \sum_{j_1+\cdots+j_n=q_{n+1}} \\ Y(t'; q_0, \cdots, q_n, q_{n+1}; j_1, \cdots, j_n, 0), \end{aligned}$$

$$\begin{aligned} u_3(t') = \sum_{m=1}^{\infty} \sum_{\substack{q_0+\cdots+q_{n+1}=m \\ q_{n+1}\ge 1}} \frac{\prod_{p=0}^{n} c_p^{q_p}}{\prod_{p=0}^{n} q_p!} \sum_{\substack{j_1+\cdots+j_{n+1}=q_{n+1} \\ j_{n+1}\ge 1}} \\ Y(t'; q_0, \cdots, q_{n+1}; j_1, \cdots, j_{n+1}). \end{aligned}$$

Since $Y(t'; q_0, \cdots, q_n, q_{n+1}; j_1, \cdots, j_n, 0) = S_{t'-t}(Y(t'; q_0, \cdots, q_n, q_{n+1}; j_1, \cdots, j_n))$, by (2.32),

$$\begin{aligned} &S_{t'}(\varphi) + u_1(t') + u_2(t') \\ &\quad = S_{t'-t}(S_t(\varphi) + \sum_{m=1}^{\infty} \sum_{q_0+\cdots+q_{n+1}=m} \frac{\prod_{p=0}^{n} c_p^{q_p}}{\prod_{p=0}^{n} q_p!} \sum_{j_1+\cdots+j_n=q_{n+1}} \\ &\qquad Y(t'; q_0, \cdots, q_n, q_{n+1}; j_1, \cdots, j_n)) \\ &\quad = S_{t'-t}(u(t))\ . \end{aligned} \tag{6.5}$$

On the other hand,

$$
\begin{aligned}
&(Bo) - \int_{(t,t']} (S_{t'-s} \circ M_{\theta(s)})(u(s)) \, d\mu(s) \\
&\overset{(1)}{=} \sum_{m=0}^{\infty} \sum_{q_0+\cdots+q_{n+1}=m} \frac{\prod_{p=0}^{n} c_p^{q_p}}{\prod_{p=0}^{n} q_p!} \sum_{j_1+\cdots+j_{n+1}=q_{n+1}} (Bo) \\
&\quad - \int_{(t,t']} (S_{t'-s} \circ M_{\theta(s)})(Y(s; q_0, \cdots, q_{n+1}; j_1, \cdots, j_{n+1})) \, d\mu(s) \\
&\overset{(2)}{=} \sum_{m=0}^{\infty} \sum_{q_0+\cdots+q_{n+1}=m} \frac{\prod_{p=0}^{n} c_p^{q_p}}{\prod_{p=0}^{n} q_p!} \sum_{j_1+\cdots+j_{n+1}=q_{n+1}} \\
&\quad Y(t'; q_0, \cdots, q_{n+1}+1; j_1, \cdots, j_{n+1}+1) \\
&\overset{(3)}{=} \sum_{m'=1}^{\infty} \sum_{q'_0+q'_1+\cdots+q'_{n+1}=m'} \frac{\prod_{p=0}^{n} c_p^{q'_p}}{\prod_{p=0}^{n} q'_p!} \sum_{j'_1+\cdots+j'_{n+1}=q'_{n+1}} \\
&\quad Y(t'; q'_0, \cdots, q'_{n+1}; j'_1, \cdots, j'_{n+1}) \\
&= u_3(t') \ .
\end{aligned} \tag{6.6}
$$

Step (1) follows from (2.17). By an elementary calculation, we have Step (2). If $q_{n+1} \geq 1$, then the condition "$m = 0$" has no meaning; so by making the substitution $j_k = j'_k$ $(1 \leq k \leq n)$, $j_{n+1}+1 = j'_{n+1}$, $q_k = q'_k$ $(1 \leq k \leq n)$, $q_{n+1}+1 = q'_{n+1}$ and $m+1 = m'$, we obtain Step (3).

Hence, for $t < t' \leq \widetilde{t}$,

$$
\begin{aligned}
u(t') &= (S_{t'}(\varphi) + u_1(t') + u_2(t')) + u_3(t') \\
&= S_{t'-t}(u(t)) + (Bo) - \int_{(t,t']} (S_{t'-s} \circ M_{\theta(s)})(u(s)) \, d\mu(s) \ ,
\end{aligned} \tag{6.7}
$$

as desired. □

Corollary 6.2. *Under the assumptions in Corollary 5.6, for* $0 < t' \leq \widetilde{t}$, $u(t')$ *satisfies a Volterra integral equation, that is,*

$$
u(t') = S_{t'}(\varphi) + (Bo) - \int_{(0,t']} (S_{t'-s} \circ M_{\theta(s)})(u(s)) \, d\mu(s) \ . \tag{6.8}
$$

Corollary 6.3. *Under the assumptions in Corollary 5.7, for* $0 < t' \leq \widetilde{t}$,

$$
\begin{aligned}
u(t') = \sum_{m=0}^{\infty} \sum_{q_0+\cdots+q_n=m} \frac{\prod_{p=0}^{n} c_p^{q_p}}{\prod_{p=0}^{n} q_p!} &[S_{t'-t} \circ M_{\theta(\tau_n)^{q_n}} \circ S_{\tau_n-\tau_{n-1}} \circ \cdots \\
&\circ S_{\tau_2-\tau_1} \circ M_{\theta(\tau_1)^{q_1}}](T(\tau_1, \varphi, \theta(0,\cdot)^{q_0})),
\end{aligned} \tag{6.9}
$$

$$
u(t') = S_{t'-t}(u(t)) \tag{6.10}
$$

and

$$
\begin{aligned}
&(Bo) - \int_{(t,t']} (S_{t'-s} \circ M_{\theta(s)})(u(s)) \, d\mu(s) \\
&= 0, \ \text{a zero operator} \ .
\end{aligned} \tag{6.11}
$$

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