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NONISOTROPIC STRONGLY SINGULAR INTEGRAL OPERATORS

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ABSTRACT. We consider a class of strongly singular integral operators which include those studied by Wainger, and Fefferman and Stein, and extend the results concerning the L^p boundedness of these operators to the nonisotropic setting. We also describe a geometric property of the underlying space which helps us show that our results are sharp.

1. INTRODUCTION

Let $0 < a_1 \leq a_2$, $\nu = a_1 + a_2$, and consider the one-parameter group $\{\delta_t\}_{t>0}$ of nonisotropic dilations on $\mathbb{R}^2$ given by $\delta_t : (x_1, x_2) \longmapsto (t^{a_1}x_1, t^{a_2}x_2)$. Following Stein and Wainger [9], we define a function $\rho : \mathbb{R}^2 \to [0, \infty)$ as follows. If $x \neq 0$, $|\delta_{\frac{1}{t}} x|$ as a function of t is strictly decreasing and is therefore equal to 1 for a unique value of t. Define $\rho(x)$ to be this unique t. If $x = 0$, set $\rho(x) = 0$. Then ρ is continuous, $\rho(x + y) \leq C(\rho(x) + \rho(y))$ for some $C > 0$, and $\rho(\delta_t x) = t\rho(x)$ for every $t > 0$. This function ρ is often called a δ_t–homogeneous distance function. The purpose of this paper is to study the L^p boundedness of the singular integral operator defined on the space $C_0^\infty(\mathbb{R}^2)$ of infinitely differentiable functions of compact support by

$$T\varphi(x) = \lim_{\epsilon \to 0} \int_{1 \geq \rho(y) \geq \epsilon} \frac{e^{i/\rho(y)^\beta}}{\rho(y)^\alpha} \varphi(x - y)dy, \tag{1}$$

where $\alpha, \beta > 0$. Using the generalized system of polar coordinates that one has in this setting, it is easy to see that the function $1/\rho(y)^\alpha$ is integrable near the origin if $\alpha < \nu$. So we assume $\alpha \geq \nu$. Then a straightforward argument of integration by parts shows us that the limit in (1) exists if $\beta > \alpha - \nu$.

In the special case $\rho(y) = |y|$ ($a_1 = a_2 = 1$), and in the setting of $\mathbb{R}^n$, it was shown in Wainger [10] that T extends to a bounded operator on $L^p(\mathbb{R}^n)$ for $|1/p - 1/2| < ((n/2)\beta - \alpha + n)/n\beta$, and that T is not bounded on $L^p(\mathbb{R}^n)$ if $|1/p - 1/2| > ((n/2)\beta - \alpha + n)/n\beta$. This was obtained by fully describing the asymptotic behavior near ∞ of the Fourier transform of the kernel of T. The question of whether or not T remains bounded on $L^p(\mathbb{R}^n)$ when $|1/p - 1/2| = ((n/2)\beta - \alpha + n)/n\beta$ ($\alpha > n$) was answered positively in Fefferman and Stein [3] using complex interpolation on Hardy spaces after proving the following theorem:

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Theorem A. *Let L be an integrable function on $\mathbb{R}^n$ with $L(x) = 0$ for $|x| > 1$. Assume there exists $\theta \in (0,1)$ such that*

$$\int_{|x|>2|y|^{1-\theta}} |L(x-y) - L(x)|\, dx \le B,$$

for $|y| < 1$, and

$$\left|\widehat{L}(\xi)\right| \le \frac{B}{(1+|\xi|)^{n\theta/2}}.$$

Then the transformation $S(f) = L * f$ is bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ with a bound that depends on θ and B but not on the L^1 norm of L.

The function defined by $L_\epsilon(x) = e^{i/|x|^\beta}/|x|^n$ for $\epsilon \le |x| \le 1$, and $L_\epsilon(x) = 0$ otherwise, satisfies the hypothesis of Theorem A with $\theta = \beta/(\beta+1)$ and B independent of ϵ (see [2], [3], and [10]). For further results in the radial case, we refer the reader to [4], [5], and [6].

We are going to extend the above results to the nonisotropic setting. To extend Theorem A, we introduce another distance function ρ_β which will better describe the smoothness of the kernel of a nonisotropic strongly singular integral operator and the decay of its Fourier transform. It will turn out that the balls associated to ρ, and those associated to ρ_β, are related by a geometric property which will play an important role in studying the operator T. Our main results on the L^p boundedness of T are stated in the following theorem.

Theorem 1. *Suppose $\beta > \alpha - \nu \ge 0$. For $\varphi \in C_0^\infty$, define*

$$T\varphi(x) = \lim_{\epsilon\to 0} \int_{1\ge\rho(y)\ge\epsilon} \frac{e^{i/\rho(y)^\beta}}{\rho(y)^\alpha} \varphi(x-y)dy.$$

Then:

(i) *If $\alpha > \nu$, then T extends to a bounded linear operator on $L^p(\mathbb{R}^2)$ for*

$$\left|\frac{1}{p} - \frac{1}{2}\right| \le \frac{\beta - \alpha + \nu}{2\beta}.$$

If $\alpha = \nu$, then T is bounded on $L^p(\mathbb{R}^2)$ for $1 < p < \infty$. On the other hand,

(ii) *if*

$$\left|\frac{1}{p} - \frac{1}{2}\right| > \frac{\beta - \alpha + \nu}{2\beta},$$

then T is not bounded on $L^p(\mathbb{R}^2)$.

If $x_0 \in \mathbb{R}^2$, and $r \ge 0$, we define a ρ–ball by $B(x_0,r) = \{x \in \mathbb{R}^2 : \rho(x - x_0) \le r\}$. A 1–atom is a function $a \in L^\infty(\mathbb{R}^2)$ supported in a ρ–ball $B(x_0,r)$ such that

(i) $\|a\|_{L^\infty} \le r^{-\nu}$, and
(ii) $\int a(x)dx = 0$.

Following Coifman and Weiss [1], we define $H^1_\rho(\mathbb{R}^2)$ as the set of all $f \in S'$ that can be represented in the form $f = \sum_{i=0}^\infty \mu_i a_i$, where each a_i is a 1–atom and $\sum_{i=0}^\infty |\mu_i| < \infty$. Also, for $f \in H^1_\rho(\mathbb{R}^2)$ we have $\|f\|_{H^1_\rho} = \inf\{\sum|\mu_i| : f = \sum \mu_i a_i\}$. Throughout this paper a constant is a positive real number that depends only on α, β, a_1, and a_2. c will always denote a constant which does not necessarily have the same value every time it appears.

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3. The L^p Inequality

We start this section by stating some further properties of the function ρ.

Proposition 1. (i) *$\rho(x)$ is infinitely differentiable in $\mathbb{R}^2 - 0$. Also, for $x \neq 0$,*

$$\left|\frac{\partial \rho}{\partial x_1}(x)\right| \leq C\rho(x)^{1-a_1} \text{ and } \left|\frac{\partial \rho}{\partial x_2}(x)\right| \leq C\rho(x)^{1-a_2}$$

for some $C > 0$.

(ii) *If $|x| \geq 1$, then $\rho(x)^{a_1} \leq |x| \leq \rho(x)^{a_2}$.*

(iii) *If $|x| \leq 1$, then $\rho(x)^{a_1} \geq |x| \geq \rho(x)^{a_2}$.*

(iv) *If $f \in L^1(\mathbb{R}^2)$ or $f \geq 0$, then*

$$\int_{\mathbb{R}^2} f(x)dx = \int_0^{2\pi} \Omega(\theta) \left[\int_0^\infty f(\delta_r(\cos\theta, \sin\theta)) r^{\nu-1} dr\right] d\theta$$

where $\Omega(\theta) = a_1 + (a_2 - a_1)\sin^2\theta$.

Part (iv) describes the generalized polar coordinates mentioned above. For a proof of Proposition 1, see [9].

For $\beta > 0$ we associate to ρ a function ρ_β as follows. For $t > 0$ and $x \in \mathbb{R}^2$, define

$$\gamma_t(x) = t^\beta \delta_t(y) = (t^{a_1+\beta} x_1, t^{a_2+\beta} x_2),$$

and let ρ_β be the distance function corresponding to the group $\{\gamma_t\}_{t>0}$. The geometric property, mentioned before, that relates ρ_β–balls to ρ–balls will be described in detail in the next section. For now let us note that

$$\rho(x) \leq \rho_\beta(x), \text{ if } \rho(x) \leq 1. \tag{2}$$

We start by proving the following generalization of Theorem A.

Theorem 2. *Let $K_0 \in L^1(\mathbb{R}^2)$ with $K_0(x) = 0$ for $\rho(x) > 1$. Assume there exist $\beta > 0$ and a constant C such that*

$$\int_{\rho(x) > C\rho_\beta(y)} |K_0(x-y) - K_0(x)|\, dx \leq B_0$$

for $\rho(y) < 1$, and

$$\left|\widehat{K_0}(\xi)\right| \leq \frac{B_0}{(1+\rho_\beta(\xi))^\beta}.$$

Then the transformation $T_0(f) = K_0 * f$ is bounded from $H^1_{\rho_\beta}(\mathbb{R}^2)$ to $L^1(\mathbb{R}^2)$ with a bound that depends on β, B_0 and C but not on the L^1 norm of K_0.

Proof. It suffices to show that $\|T_0(a)\|_{L^1} \leq c$ for each 1–atom a, with c independent of the L^1 norm of K_0 and the choice of a. Let a be a 1–atom supported in a ρ_β–ball $B = B(x_0, r)$. Since T_0 is translation invariant, we can take $x_0 = 0$. Then $T_0(a)$ is supported in a ρ_β–ball $B(0, c(1+r))$. By part (iv) of Proposition 1, the Lebesgue measure $|B(0, c(1+r))|$ of $B(0, c(1+r))$ is $\leq c(1+r)^{2\beta+\nu}$. So if $r \geq 1$,

$$\begin{aligned}
\|T_0(a)\|_{L^1} &\leq c(1+r)^{\beta+\nu/2}\|T_0(a)\|_{L^2} \\
&\leq c(1+r)^{\beta+\nu/2}\|a\|_{L^2} \\
&\leq c(1+r)^{\beta+\nu/2}\frac{1}{r^{\beta+\nu/2}} \\
&\leq c.
\end{aligned}$$

Suppose $r < 1$ and consider the ρ–ball $B^* = B(0, Cr)$. Then

$$\begin{aligned}
\|T_0(a)\|_{L^1(\mathbb{R}^2-B^*)} &= \int_{\mathbb{R}^2-B^*}\left|\int K_0(x-y)a(y)dy\right|dx \\
&= \int_{\mathbb{R}^2-B^*}\left|\int \left(K_0(x-y)-K_0(x)\right)a(y)dy\right|dx \\
&\leq \int |a(y)| \int_{\mathbb{R}^2-B^*} |K_0(x-y)-K_0(x)|\,dxdy \\
&\leq \int |a(y)| \int_{\rho(x)>C\rho_\beta(y)} |K_0(x-y)-K_0(x)|\,dxdy \\
&\leq B_0\|a\|_{L^1} \\
&\leq c\,,
\end{aligned}$$

and

$$\begin{aligned}
\|T_0(a)\|^2_{L^1(B^*)} &\leq |B^*|\,\|T_0(a)\|^2_{L^2} \\
&\leq c\,r^\nu\|\widehat{T_0(a)}\|^2_{L^2} \\
&= c\,r^\nu\int |\widehat{K_0}(\xi)|^2|\widehat{a}(\xi)|^2d\xi \\
&= c\,r^\nu\int_{\rho_\beta(\xi)\geq 1/r} |\widehat{K_0}(\xi)|^2|\widehat{a}(\xi)|^2d\xi \\
&\quad + c\,r^\nu\int_{\rho_\beta(\xi)\leq 1/r} |\widehat{K_0}(\xi)|^2|\widehat{a}(\xi)|^2d\xi \\
&\leq c\,r^\nu\int_{\rho_\beta(\xi)\geq 1/r} \rho_\beta(\xi)^{-2\beta}|\widehat{a}(\xi)|^2d\xi \\
&\quad + c\,r^\nu\|\widehat{a}\|^2_{L^\infty}\int_{\rho_\beta(\xi)\leq 1/r} \rho_\beta(\xi)^{-2\beta}d\xi \\
&\leq c\,r^{\nu+2\beta}\int_{\rho_\beta(\xi)\geq 1/r} |\widehat{a}(\xi)|^2d\xi \\
&\quad + c\,r^\nu\|a\|^2_{L^1}\int_0^{1/r} s^{-2\beta}s^{\nu+2\beta-1}ds \\
&\leq c\,r^{\nu+2\beta}\|a\|^2_{L^2} + c\,r^\nu\int_0^{1/r} s^{\nu-1}ds \\
&\leq c\,.
\end{aligned}$$

Hence $\|T_0(a)\|_{L^1} = \|T_0(a)\|_{L^1(B^*)} + \|T_0(a)\|_{L^1(\mathbb{R}^2-B^*)} \leq c$. This completes the proof. □

For $y \neq 0$ define $K(y) = e^{i/\rho(y)^\beta}/\rho(y)^\alpha$, and set

$$K_\epsilon(y) = \begin{cases} K(y) & \text{if } \epsilon \leq \rho(y) \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

$(0 < \epsilon \leq 1)$. Now for $f \in L^p(\mathbb{R}^2)$, $1 \leq p \leq \infty$, define $T_\epsilon f = K_\epsilon * f$. Then if $\beta > \alpha - \nu \geq 0$ and $\varphi \in C_0^\infty(\mathbb{R}^2)$, it follows that $T\varphi(x) = \lim_{\epsilon\to 0} T_\epsilon\varphi(x)$ for every $x \in \mathbb{R}^2$.

Theorem 3. *Suppose $\beta > 0$ and $\beta \geq \alpha - \nu \geq 0$. If $|1/p - 1/2| \leq (\beta - \alpha + \nu)/(2\beta)$ $(\alpha > \nu)$ or $1 < p < \infty$ $(\alpha = \nu)$, we have*

$$\|T_\epsilon f\|_{L^p} \leq A_p \|f\|_{L^p}$$

for every $f \in L^p$. The constant A_p is independent of ϵ.

A standard limiting argument shows that part (i) of Theorem 1 is an immediate consequence of Theorem 3. Part (i) of Proposition 1 tells us that

$$\left|\frac{\partial K_\epsilon}{\partial x_1}(x)\right| \leq c\rho(x)^{-\alpha-\beta-a_1} \text{ and } \left|\frac{\partial K_\epsilon}{\partial x_2}(x)\right| \leq c\rho(x)^{-\alpha-\beta-a_2}.$$

So, if $\alpha = \nu$, it can be easily checked that

$$\int_{\rho(x)>C\rho_\beta(y)} |K_\epsilon(x-y) - K_\epsilon(x)|\, dx \leq B_0,$$

uniformly in ϵ. In the next theorem, we estimate the Fourier transform of K_ϵ, and it will turn out that if $\alpha = \nu$, then $\left|\widehat{K_\epsilon}(\xi)\right| \leq B_0 \left(1 + \rho_\beta(\xi)\right)^{-\beta}$. Theorem 2 then tells us that T_ϵ is bounded from $H^1_{\rho_\beta}(\mathbb{R}^2)$ to $L^1(\mathbb{R}^2)$ with a bound that is independent of ϵ. So our next task is to estimate $\widehat{K_\epsilon}$, and for this we need the following lemma of van der Corput, which can be found in [8, pages 332–334].

Proposition 2. *Suppose ϕ is real-valued and smooth in (a,b), and that $|\phi^{(k)}(x)| \geq \lambda > 0$ for all $x \in (a,b)$. Then*

$$\left|\int_a^b e^{i\phi(x)} dx\right| \leq c_k \lambda^{-1/k} \tag{3}$$

holds when:

(i) *$k \geq 2$, or*

(ii) *$k = 1$ and $\phi''(x)$ has at most one zero.*

Also, $c_k = 5(2^k) - 4$.

Now if $0 < a < b$, ϕ and ψ are real-valued and smooth in (a,b), and $|\phi^{(k)}(x)| \geq \lambda/x^s$ $(s \geq 0)$ (when $k = 1$ we also assume that $\phi''(x)$ has at most one zero), then

$$\int_a^b e^{i\phi(x)}\psi(x)dx = \int_a^b \psi(x)F'(x)dx,$$

where $F(x) = \int_a^x e^{i\phi(t)}dt$. By Proposition 2, $|F(x)| \leq c_k\lambda^{-1/k}x^{s/k}$ for $x \in [a,b]$, and on integrating the above integral by parts it follows that

$$\left|\int_a^b e^{i\phi(x)}\psi(x)dx\right| \leq c_k\lambda^{-1/k}\left[b^{s/k}|\psi(b)| + \int_a^b x^{s/k}|\psi'(x)|dx\right]. \tag{4}$$

In particular, if $s = 0$, then

$$\left| \int_a^b e^{i\phi(x)} \psi(x) dx \right| \leq c_k \lambda^{-1/k} \left[|\psi(b)| + \int_a^b |\psi'(x)| dx \right]. \tag{5}$$

Theorem 4. *Suppose $\beta > 0$ and $\beta \geq \alpha - \nu \geq 0$. Then*

$$\left| \widehat{\left(\frac{K_\epsilon}{\rho(.)^{iv}} \right)}(\xi) \right| \leq B \frac{1 + |v|}{(1 + \rho_\beta(\xi))^{\beta - \alpha + \nu}},$$

$-\infty < v < +\infty$. The constant B is independent of ϵ.

Proof. If ρ' is the distance function corresponding to the group $\{\delta'_t\}_{t>0}$, where $\delta'_t x = (tx_1, t^{a_2/a_1} x_2)$, then it is not hard to see that $\rho(y) = \rho'(y)^{1/a_1}$ and $\rho_\beta(y) = \rho'_{\beta/a_1}(y)^{1/a_1}$ for every $y \in \mathbb{R}^2$. Therefore, we can assume $a_1 = 1$ (then $\nu = 1 + a_2 \geq 2$). If $\rho_\beta(\xi)$ is small, an easy argument of integration by parts shows that the Fourier transform of $K_\epsilon/\rho(.)^{iv}$ is bounded. So it suffices to prove the theorem for large values of $\rho_\beta(\xi)$. Furthermore, since $\rho(x_1, x_2) = \rho(-x_1, x_2) = \rho(-x_1, -x_2)$, it is enough to look at $\xi = (\xi_1, \xi_2)$ with $\xi_1, \xi_2 \geq 0$. Write

$$\widehat{\left(\frac{K_\epsilon}{\rho(.)^{iv}} \right)}(\xi) = I_1 + I_2,$$

where

$$I_1 = \int_{\rho_\beta(x) \leq C_0 \lambda(\xi)} \frac{K_\epsilon(x)}{\rho(x)^{iv}} e^{i\xi \cdot x} dx$$

and

$$I_2 = \int_{\rho_\beta(x) \geq C_0 \lambda(\xi)} \frac{K_\epsilon(x)}{\rho(x)^{iv}} e^{i\xi \cdot x} dx.$$

C_0 and $\lambda(\xi)$ are going to be chosen. For $r > 0$, set $f(r) = \frac{d}{dr} |\delta_r \xi|$. Then $f'(r) > 0$, and it follows that the equation $\beta r^{-\beta-1} = f(r)$ has a unique solution in $(0, \infty)$. Define $\lambda(\xi)$ to be this unique solution. An easy computation then shows that $\lambda(\gamma_t \xi) = (1/t)\lambda(\xi)$ for $t > 0$, and that there exist constants C_1 and C_2 such that $0 < C_1 \leq \lambda(\xi) \leq C_2$ whenever $|\xi| = 1$. So, writing $\xi = \gamma_{\rho_\beta(\xi)} \xi'$ with $|\xi'| = 1$, we conclude that

$$\frac{C_1}{\rho_\beta(\xi)} \leq \lambda(\xi) \leq \frac{C_2}{\rho_\beta(\xi)}. \tag{6}$$

In generalized polar coordinates,

$$I_1 = \int_0^{2\pi} \Omega(\theta) \left[\int_\epsilon^{C_0 \lambda(\xi)} \frac{e^{-iv \ln r}}{r^{\alpha - \nu + 1}} e^{i/r^\beta} e^{i\xi \cdot \delta_r (\cos\theta, \sin\theta)} dr \right] d\theta.$$

Writing $e^{i/r^\beta} = \frac{i}{\beta}(e^{i/r^\beta})' r^{\beta+1}$ and integrating the inner integral by parts, it follows that

$$\begin{aligned} |I_1| \quad \leq \quad & c\,\lambda(\xi)^{\beta-\alpha+\nu} + c\,(1+|v|) \int_0^{2\pi} \left| \int_\epsilon^{C_0\lambda(\xi)} e^{-iv\ln r} r^{\beta-\alpha+\nu-1} r^{i\Phi_\theta(r)} dr \right| d\theta \\ & + c\,|\xi_1| \int_0^{2\pi} \left| \int_\epsilon^{C_0\lambda(\xi)} e^{-iv\ln r} r^{\beta-\alpha+\nu} r^{i\Phi_\theta(r)} dr \right| d\theta \\ & + c\,|\xi_2| \int_0^{2\pi} \left| \int_\epsilon^{C_0\lambda(\xi)} e^{-iv\ln r} r^{\beta-\alpha+2\nu-2} r^{i\Phi_\theta(r)} dr \right| d\theta \end{aligned}$$

where $\Phi_\theta(r) = r^{-\beta} + r\xi_1\cos\theta + r^{\nu-1}\xi_2\sin\theta$. Since $|\xi_1| \leq \rho_\beta(\xi)^{\beta+1}$ and $|\xi_2| \leq \rho_\beta(\xi)^{\beta+\nu-1}$, it follows by (6) that we can find a constant C_0 small enough that $|\Phi_\theta'(r)| \geq \beta/2r^{\beta+1}$ for $r \in (0, C_0\lambda(\xi)]$ (uniformly in θ). Applying (4) to each of the integrals on the right-hand side of the above inequality, we get

$$|I_1| \leq c\,(1+|v|)\,\lambda(\xi)^{\beta-\alpha+\nu}. \tag{7}$$

Estimating I_2 takes more work. As we did for I_1, we start by expressing the integral in polar coordinates:

$$I_2 = \int_{C_0\lambda(\xi)}^1 \frac{e^{-iv\ln r}}{r^{\alpha-\nu+1}} e^{i/r^\beta} \left[\int_0^{2\pi} \Omega(\theta) e^{i\xi\cdot\delta_r(\cos\theta,\sin\theta)} d\theta \right] dr.$$

Now using the observation that $\xi\cdot\delta_r(\cos\theta,\sin\theta) = |\delta_r\xi|\cos(\theta - h(r))$, where $h(r) = \arctan(r^{\nu-2}\xi_2/\xi_1)$, we get

$$I_2 = \int_{C_0\lambda(\xi)}^1 \frac{e^{-iv\ln r}}{r^{\alpha-\nu+1}} e^{i/r^\beta} \left[\int_0^{2\pi} \Omega(\theta + h(r)) e^{i|\delta_r\xi|\cos\theta} d\theta \right] dr.$$

Note that $h'(r) \leq c/r$. By the method of stationary phase (as stated in [8, page 334]),

$$\begin{aligned} & \int_0^{2\pi} \Omega(\theta + h(r)) e^{i|\delta_r\xi|\cos\theta} d\theta \\ & \qquad = \omega_1 \frac{\Omega(h(r))}{|\delta_r\xi|^{1/2}} e^{i|\delta_r\xi|} + \ \omega_2 \frac{\Omega(h(r))}{|\delta_r\xi|^{1/2}} e^{-i|\delta_r\xi|} + O(|\delta_r\xi|^{-3/2}) \end{aligned}$$

($\omega_1 = \sqrt{2\pi}\,e^{-i\pi/4}$ and $\omega_2 = \sqrt{2\pi}\,e^{i\pi/4}$). The bounds occurring in the error term in the above equation are independent of r because all derivatives of $\Omega(\theta + h(r))$ with respect to θ are bounded uniformly in r. Let $\psi(r) = e^{-iv\ln r}\Omega(h(r))/|\delta_r\xi|^{1/2} r^{\alpha-\nu+1}$ and $\phi_\theta(r) = r^{-\beta} + |\delta_r\xi|\cos\theta$. Then

$$I_2 = \omega_1 \int_{C_0\lambda(\xi)}^1 \psi(r) e^{i\phi_0(r)} dr + \omega_2 \int_{C_0\lambda(\xi)}^1 \psi(r) e^{i\phi_\pi(r)} dr + E,$$

with $|E| \leq c\int_{C_0\lambda(\xi)}^1 |\delta_r\xi|^{-3/2} r^{-\alpha+\nu-1} dr$. Now, using the definition of $\lambda(\xi)$, one can easily see that

$$\frac{1}{|\delta_r\xi|} \leq c\,\frac{\lambda(\xi)^{\beta+1}}{r} \tag{8}$$

for $C_0\lambda(\xi) \leq r \leq 1$. Therefore,

$$|E| \leq c\,\lambda(\xi)^{\frac{3}{2}\beta-\alpha+\nu}. \tag{9}$$

It remains to estimate

$$I_3 = \int_{C_0\lambda(\xi)}^{1} \psi(r) e^{i\phi_0(r)} dr$$

and

$$I_4 = \int_{C_0\lambda(\xi)}^{1} \psi(r) e^{i\phi_\pi(r)} dr.$$

But first let us notice that (8) tells us that if $C_0\lambda(\xi) \leq r \leq 1$, then

$$|\psi(r)| \leq c \, \frac{\lambda(\xi)^{\frac{\beta}{2}+\frac{1}{2}}}{r^{\alpha-\nu+3/2}}$$

and

$$|\psi'(r)| \leq c \; (1+|v|) \, \frac{\lambda(\xi)^{\frac{\beta}{2}+\frac{1}{2}}}{r^{\alpha-\nu+5/2}}.$$

Now $\phi_\pi'(r) = -\beta r^{-\beta-1} - f(r)$, and since $f(r) > 0$, it follows that $|\phi_\pi'(r)| \geq c/\lambda(\xi)^{\beta+1}$ for $r \in [C_0\lambda(\xi), 3\lambda(\xi)/2]$. Also, for $3\lambda(\xi)/2 \leq r \leq 1$,

$$|\phi_\pi'(r)| = \beta r^{-\beta-1} + f(r) \geq f(r) \geq f(\lambda(\xi)) = \beta\lambda(\xi)^{-\beta-1}.$$

Thus $|\phi_\pi'(r)| \geq c/\lambda(\xi)^{\beta+1}$ on $[C_0\lambda(\xi), 1]$, and (5) then tells us that

$$\begin{aligned} |I_4| &\leq c\,\lambda(\xi)^{\beta+1} \left[|\psi(1)| + \int_{C_0\lambda(\xi)}^{1} |\psi'(r)|\, dr \right] \\ &\leq c \; (1+|v|) \; \lambda(\xi)^{\frac{3}{2}\beta-\alpha+\nu}. \end{aligned} \tag{10}$$

For I_3, we have

$$I_3 = \int_{C_0\lambda(\xi)}^{3\lambda(\xi)/2} \psi(r) e^{i\phi_0(r)} dr + \int_{3\lambda(\xi)/2}^{1} \psi(r) e^{i\phi_0(r)} dr = I_5 + I_6. \tag{11}$$

On $[3\lambda(\xi)/2, 1]$,

$$\begin{aligned} \phi_0'(r) &= -\beta r^{-\beta-1} + f(r) \\ &\geq -(2/3)^{\beta+1}\beta\lambda(\xi)^{-\beta-1} + f(\lambda(\xi)) \\ &= -(2/3)^{\beta+1}\beta\lambda(\xi)^{-\beta-1} + \beta\lambda(\xi)^{-\beta-1} \\ &\geq c\,\lambda(\xi)^{-\beta-1}, \end{aligned}$$

and, as before, (5) tells us that

$$|I_6| \leq c \; (1+|v|) \; \lambda(\xi)^{\frac{3}{2}\beta-\alpha+\nu}. \tag{12}$$

For $C_0\lambda(\xi) \leq r \leq 3\lambda(\xi)/2$ we have

$$\phi_0''(r) = \beta(\beta+1) r^{-\beta-2} + f'(r) \geq \beta(\beta+1) r^{-\beta-2} \geq c/\lambda(\xi)^{\beta+2},$$

and applying (5) one more time, we get

$$\begin{aligned} |I_5| &\leq c\,\lambda(\xi)^{\frac{\beta}{2}+1} \left[|\psi(\lambda(\xi)/2)| + \int_{C_0\lambda(\xi)}^{3\lambda(\xi)/2} |\psi'(r)|\, dr \right] \\ &\leq c \; (1+|v|) \; \lambda(\xi)^{\beta-\alpha+\nu}. \end{aligned} \tag{13}$$

Combining (7), (9), (10), (12), and (13), we have

$$\left|\widehat{\left(\frac{K_\epsilon}{\rho(.)^{iv}}\right)}(\xi)\right| \leq c\ (1+|v|)\ \lambda(\xi)^{\beta-\alpha+\nu},$$

and by (6),

$$\left|\widehat{\left(\frac{K_\epsilon}{\rho(.)^{iv}}\right)}(\xi)\right| \leq c\ (1+|v|)\ \rho_\beta(\xi)^{-\beta+\alpha-\nu}.$$

This completes the proof. □

We are now ready to prove Theorem 3. We use interpolation of analytic families of operators on parabolic Hardy spaces (see [1]).

Proof of Theorem 3. As we mentioned before, if $\alpha = \nu$, then K_ϵ satisfies the hypothesis of Theorem 2 with bounds independent of ϵ, and it follows that T extends to a bounded linear operator on $L^p(\mathbb{R}^2)$ for $1 < p < \infty$. Assume $\alpha > \nu$. For $z = u + iv \in \mathbb{C}$, set

$$M_z(y) = \begin{cases} \rho(y)^{\beta z-\beta-\nu} e^{i/\rho(y)^\beta} & \text{if } \epsilon \leq \rho(y) \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

We consider the family $\{R_z\}_{0\leq u\leq 1}$ of analytic operators defined on the domain of simple functions by

$$R_z f = M_z * f.$$

Clearly, $R_{\frac{\beta-\alpha+\nu}{\beta}} = T_\epsilon$.

If $u = 1$, then $\mathrm{Re}\,[-\beta z + \beta + \nu] = \nu$, and M_{1+iv} satisfies the hypothesis of Theorem 2 with $B_0 = (1+|v|)B_1$ and B_1 independent of ϵ. Thus

$$\|R_{1+iv} f\|_{L^1} \leq (1+|v|)A' \|f\|_{H^1_{\rho_\beta}}, \tag{14}$$

and the constant A' is independent of ϵ. On the other hand, Theorem 4 tells us that

$$\left|\widehat{M_{iv}}(\xi)\right| \leq B(1+|v|),$$

and it follows that

$$\|R_{iv} f\|_{L^2} \leq (1+|v|)A'' \|f\|_{L^2}. \tag{15}$$

Now we interpolate between the inequalities in (14) and (15) to conclude that

$$\|R_u f\|_{L^p} \leq A(u,p)\|f\|_{L^p}$$

whenever $0 \leq u < 1$ and $\frac{1}{p} = \frac{1-u}{2} + u$. In particular,

$$\|T_\epsilon f\|_{L^p} = \|R_{\frac{\beta-\alpha+\nu}{\beta}} f\|_{L^p} \leq A_p \|f\|_{L^p}$$

for $\frac{1}{p} - \frac{1}{2} = \frac{\beta-\alpha+\nu}{2\beta}$. It follows that

$$\|T_\epsilon f\|_{L^p} \leq A_p \|f\|_{L^p}$$

for $0 \leq \frac{1}{p} - \frac{1}{2} \leq \frac{\beta-\alpha+\nu}{2\beta}$. Finally, a duality argument shows the corresponding result for $2 < p < \infty$.

This establishes Theorem 3 and consequently part (i) of Theorem 1. □

4. The Sharp Result

In the last section we showed that, if $\alpha > \nu$, T extends to a bounded linear operator on L^p for $|1/p - 1/2| \leq (\beta - \alpha + \nu)/2\beta$. In this section we prove that this result is sharp. This was the assertion of part (ii) of Theorem 1, and for convenience, we restate it here as:

Theorem 5. *Suppose T extends to a bounded linear operator on L^p, $1 \leq p < \infty$. Then*

$$\left|\frac{1}{p} - \frac{1}{2}\right| \leq \frac{\beta - \alpha + \nu}{2\beta}. \tag{16}$$

At this point, outlining the argument that is going to be used in the proof of Theorem 5 will help in understanding some of the details that will follow. We are going to consider an appropriate $\varphi \in C_0^\infty(\mathbb{R}^2)$ supported in a small neighborhood U of the origin. The goal is, of course, to find a lower bound for $\|T\varphi\|_{L^p}$. To achieve this, we examine $|T\varphi(x)|$ at those x's such that $e^{i/\rho(y)^\beta}$ does not oscillate rapidly for y near x. For example, suppose that $e^{i/\rho(y)^\beta}$ does not oscillate rapidly for $y \in B(0,b) - B(0,a)$, where $0 < a < b \leq 1$ ($B(0,a)$ and $B(0,b)$ are ρ–balls). For $x \in E \subset B(0,b) - B(0,a)$ let $U_x = \{y \in \mathbb{R}^2 : x - y \in U\} =$ support of φ translated by x. To gain the best possible lower bound for $|T\varphi(x)|$, U_x should lie entirely in $B(0,b) - B(0,a)$. Moreover, to gain a satisfactory lower bound for $\|T\varphi\|_{L^p}$, U_x should cover most of $B(0,b) - B(0,a)$ as x varies in E. For all of this to occur, $\rho_\beta(y - x)$, rather than $\rho(y - x)$, should be small for $y \in U_x$. This geometric property is the subject of the next lemma.

Lemma 1. *Let $0 < \epsilon \leq a < b$ and $2\epsilon^{a_1+\beta} < b^{a_1} - a^{a_1}$. Suppose*

$$\left(a^{a_1} + \epsilon^{a_1+\beta}\right)^{1/a_1} \leq \rho(x) \leq \left(b^{a_1} - \epsilon^{a_1+\beta}\right)^{1/a_1}$$

and $\rho_\beta(x - y) \leq \epsilon$. Then $a \leq \rho(y) \leq b$.

Proof. Since $\rho_\beta(x-y) \leq \epsilon$, we have $|\gamma_{\frac{1}{\epsilon}}(x-y)| \leq 1$. It follows that $|\delta_{\frac{1}{\epsilon}}(x-y)| \leq \epsilon^\beta$, and since $a/\epsilon \geq 1$, we get

$$\epsilon^\beta \geq \left|\delta_{\frac{1}{\epsilon}}(x-y)\right| = \left|\delta_{\frac{1}{a}\frac{a}{\epsilon}}(x-y)\right| \geq \left(\frac{a}{\epsilon}\right)^{a_1} \left|\delta_{\frac{1}{a}}(x-y)\right|,$$

or

$$\left|\delta_{\frac{1}{a}}(x-y)\right| \leq \frac{\epsilon^{a_1+\beta}}{a^{a_1}}. \tag{17}$$

Similarly,

$$\left|\delta_{\frac{1}{b}}(x-y)\right| \leq \frac{\epsilon^{a_1+\beta}}{b^{a_1}}. \tag{18}$$

Now, since $\left(a^{a_1} + \epsilon^{a_1+\beta}\right)^{1/a_1} \leq \rho(x) \leq \left(b^{a_1} - \epsilon^{a_1+\beta}\right)^{1/a_1}$, we have

$$\left|\delta_{\frac{1}{\left(b^{a_1} - \epsilon^{a_1+\beta}\right)^{1/a_1}}}\, x\right| \leq 1 \leq \left|\delta_{\frac{1}{\left(a^{a_1} + \epsilon^{a_1+\beta}\right)^{1/a_1}}}\, x\right|. \tag{19}$$

The second inequality in (19) tells us that

$$1 \leq \left|\delta_{\frac{1}{a\left(1+\frac{\epsilon^{a_1+\beta}}{a^{a_1}}\right)^{1/a_1}}}\, x\right| \leq \frac{1}{1 + \frac{\epsilon^{a_1+\beta}}{a^{a_1}}} \left|\delta_{\frac{1}{a}}\, x\right|.$$

Therefore,

$$\left|\delta_{\frac{1}{a}}\, x\right| \geq 1 + \frac{\epsilon^{a_1+\beta}}{a^{a_1}}. \tag{20}$$

Similarly,

$$1 \geq \left|\delta_{\frac{1}{\left(b^{a_1}-\epsilon^{a_1+\beta}\right)^{1/a_1}}}\, x\right| \geq \frac{1}{1-\frac{\epsilon^{a_1+\beta}}{b^{a_1}}}\ \left|\delta_{\frac{1}{b}}\, x\right|,$$

so that

$$\left|\delta_{\frac{1}{b}}\, x\right| \leq 1 - \frac{\epsilon^{a_1+\beta}}{b^{a_1}}. \tag{21}$$

Now (17) and (20) tell us that

$$\left|\delta_{\frac{1}{a}}\, y\right| = \left|\delta_{\frac{1}{a}}\, x - \delta_{\frac{1}{a}}(x-y)\right| \geq 1 + \frac{\epsilon^{a_1+\beta}}{a^{a_1}} - \frac{\epsilon^{a_1+\beta}}{a^{a_1}} = 1.$$

Also, by (18) and (21),

$$\left|\delta_{\frac{1}{b}}\, y\right| = \left|\delta_{\frac{1}{b}}(y-x) + \delta_{\frac{1}{b}}\, x\right| \leq \frac{\epsilon^{a_1+\beta}}{b^{a_1}} + 1 - \frac{\epsilon^{a_1+\beta}}{b^{a_1}} = 1.$$

Hence $a \leq \rho(y) \leq b$. □

Next we construct subintervals I_k of $(0,1]$ such that $e^{i/\rho(y)^\beta}$ does not oscillate rapidly when $\rho(y)^{a_1} \in I_k$.

Lemma 2. *There exist two positive numbers A_0 and B_0, with $B_0 < A_0^{1/\beta} < 1$, such that whenever $0 < \epsilon < B_0$ and $1 \leq k \leq A_0\epsilon^{-\beta}$ (k an integer), the following hold.*

(i) $4\epsilon^{a_1+\beta} < \frac{1}{(2\pi k-\pi/3)^{a_1/\beta}} - \frac{1}{(2\pi k+\pi/3)^{a_1/\beta}}$ *and* $\epsilon \leq \frac{1}{(2\pi k+\pi/3)^{1/\beta}}$.

(ii) *Let*

$$I_k = \left[\frac{1}{(2\pi k+\pi/3)^{a_1/\beta}} + \epsilon^{a_1+\beta}, \frac{1}{(2\pi k-\pi/3)^{a_1/\beta}} - \epsilon^{a_1+\beta}\right]$$

and

$$J_k = \left[\frac{1}{(2\pi(k+1)-\pi/3)^{a_1/\beta}} - \epsilon^{a_1+\beta}, \frac{1}{(2\pi k+\pi/3)^{a_1/\beta}} + \epsilon^{a_1+\beta}\right].$$

Also, let k' be "the k" such that $k' \leq A_0\epsilon^{-\beta} < k'+1$. Then $2A_0^{-a_1/\beta}\epsilon^{a_1} < 7^{-a_1/\beta}$ and

$$I_{k'} \cup \left[\bigcup_{k=1}^{k'-1}(I_k \cup J_k)\right] \supset [A_0^{-a_1/\beta}\epsilon^{a_1}, 7^{-a_1/\beta}].$$

(iii) $|J_k| \leq C|I_{k+1}|$ *for some constant C that only depends on a_1 and β.*

Proof. Set

$$A_0 = \text{ Min}\left[\frac{1}{4\pi}\, , \left(\frac{a_1\pi}{6\beta(3\pi)^{\frac{a_1+\beta}{\beta}}}\right)^{\frac{\beta}{a_1+\beta}}\right]$$

and

$$B_0 = \text{ Min}\left[(\tfrac{1}{2})^{1/a_1}(\tfrac{A_0}{7})^{1/\beta}, \left((\tfrac{3}{5\pi})^{a_1/\beta} - (\tfrac{1}{7})^{a_1/\beta}\right)^{\frac{1}{a_1+\beta}}\right].$$

(i) Let $f(x) = x^{-a_1/\beta}$ $(x > 0)$. Then $f'(x) = -(a_1/\beta)x^{-\frac{a_1+\beta}{\beta}}$. For $k \geq 1$,

$$\begin{aligned} \frac{1}{(2\pi k - \pi/3)^{a_1/\beta}} - \frac{1}{(2\pi k + \pi/3)^{a_1/\beta}} &= f(2\pi k - \pi/3) - f(2\pi k + \pi/3) \\ &= (-2\pi/3)f'(t) \\ &= \frac{2\pi a_1}{3\beta}\frac{1}{t^{\frac{a_1+\beta}{\beta}}}, \end{aligned}$$

where $2\pi k - \pi/3 < t < 2\pi k + \pi/3 < 3\pi k$. Thus,

$$\begin{aligned} \frac{1}{(2\pi k - \pi/3)^{a_1/\beta}} - \frac{1}{(2\pi k + \pi/3)^{a_1/\beta}} &> \frac{2\pi a_1}{3\beta(3\pi)^{\frac{a_1+\beta}{\beta}}}\frac{1}{k^{\frac{a_1+\beta}{\beta}}} \\ &\geq 4A_0^{\frac{a_1+\beta}{\beta}}\frac{1}{k^{\frac{a_1+\beta}{\beta}}}. \end{aligned}$$

So for $1 \leq k \leq A_0\epsilon^{-\beta}$, we have

$$4\epsilon^{a_1+\beta} < \frac{1}{(2\pi k - \pi/3)^{a_1/\beta}} - \frac{1}{(2\pi k + \pi/3)^{a_1/\beta}}.$$

Also, since $A_0 \leq 1/(4\pi)$,

$$\epsilon \leq \frac{A_0^{1/\beta}}{k^{1/\beta}} \leq \frac{1}{(4\pi)^{1/\beta}}\frac{1}{k^{1/\beta}} \leq \frac{1}{(2\pi k + \pi/3)^{1/\beta}}.$$

(ii) By our choice of A_0 and B_0, we have

$$2^{1/a_1}A_0^{-1/\beta}\epsilon < 7^{-1/\beta} \text{ and } \tfrac{1}{(2\pi-\pi/3)^{a_1/\beta}} - \epsilon^{a_1+\beta} > 7^{-a_1/\beta}. \tag{22}$$

The second inequality in (22) tells us that $7^{-a_1/\beta} \in I_{k'} \cup [\bigcup_{k=1}^{k'-1}(I_k \cup J_k)]$. Now

$$2\pi k' - \frac{\pi}{3} > 4k' > k' + 1 > A_0\epsilon^{-\beta},$$

so that

$$A_0^{-a_1/\beta}\epsilon^{a_1} > \frac{1}{(2\pi k' - \pi/3)^{a_1/\beta}} > \frac{1}{(2\pi k' - \pi/3)^{a_1/\beta}} - \epsilon^{a_1+\beta}.$$

Thus,

$$I_{k'} \cup \left[\bigcup_{k=1}^{k'-1}(I_k \cup J_k)\right] \supset [A_0^{-a_1/\beta}\epsilon^{a_1}, 7^{-a_1/\beta}].$$

(iii) Let $a = 2\pi k + \pi/3$ and $d = 2\pi/3$. Then

$$|J_k| + |I_{k+1}| = f(a) - f(a+3d) = (-3d)f'(s_1),$$

where $a < s_1 < a + 3d$. On the other hand,

$$|I_{k+1}| + 2\epsilon^{a_1+\beta} = f(a+2d) - f(a+3d) = (-d)f'(s_2)$$

with $a + 2d < s_2 < a + 3d < 2a < 2s_1$. Then

$$\begin{aligned} |I_{k+1}| + |J_k| &= \frac{3da_1}{\beta}(\frac{1}{s_1})^{\frac{a_1+\beta}{\beta}} \\ &\leq \frac{3da_1}{\beta}(\frac{2}{s_2})^{\frac{a_1+\beta}{\beta}} \\ &= 3(2^{\frac{a_1+\beta}{\beta}})\frac{da_1}{\beta}(\frac{1}{s_2})^{\frac{a_1+\beta}{\beta}} \\ &= 3(2^{\frac{a_1+\beta}{\beta}})(|I_{k+1}| + 2\epsilon^{a_1+\beta}) \\ &\leq 6(2^{\frac{a_1+\beta}{\beta}})|I_{k+1}|. \end{aligned}$$

Hence

$$|J_k| \leq C|I_{k+1}|.$$

This completes the proof. □

Proof of Theorem 5. If $\alpha = \nu$, the right-hand side of (16) is $1/2$ and there is nothing to prove. So we may assume $\alpha > \nu$. Moreover, since T is translation invariant, it is enough to prove the theorem for $1 \leq p \leq 2$. Let A_0, B_0, I_k, J_k, k, k', and ϵ be as in Lemma 2. Fix $\varphi \in C_0^\infty$ such that $0 \leq \varphi \leq 1$, $\varphi(x) = 1$ for $\rho_\beta(x) \leq 1/2$, and $\varphi(x) = 0$ for $\rho_\beta(x) \geq 1$. Define

$$\varphi_\epsilon(x) = \varphi(\gamma_{1/\epsilon}x).$$

Then

$$\int |\varphi_\epsilon(x)|^p dx = A_p\epsilon^{2\beta+\nu} \tag{23}$$

for some $A_p > 0$.

Suppose $\rho(x)^{a_1} \in I_k$ and $\rho_\beta(x-y) \leq \epsilon$. Then Lemma 1, together with part (i) of Lemma 2, tell us that

$$\frac{1}{(2\pi k + \pi/3)^{1/\beta}} \leq \rho(y) \leq \frac{1}{(2\pi k - \pi/3)^{1/\beta}}\,,$$

or

$$2\pi k - \pi/3 \leq \frac{1}{\rho(y)^\beta} \leq 2\pi k + \pi/3. \tag{24}$$

Now by (2), $\rho(x-y) \leq \epsilon$. Also by part (i) of Lemma 2, $\epsilon \leq \rho(x)$. Thus,

$$\rho(y) \leq C\left(\rho(x-y) + \rho(x)\right) \leq (\epsilon + \rho(x)) \leq 2C\rho(x). \tag{25}$$

Choose ϵ' such that $0 < \epsilon' < \epsilon$. (24) and (25) tell us that if $\rho(x)^{a_1} \in I_k$, then

$$\begin{aligned}
\left| \int_{1 \geq \rho(y) \geq \epsilon'} \frac{e^{i/\rho(y)^\beta}}{\rho(y)^\alpha} \varphi_\epsilon(x-y) dy \right| &\geq \left| \int_{1 \geq \rho(y) \geq \epsilon'} \frac{\cos(1/\rho(y)^\beta)}{\rho(y)^\alpha} \varphi_\epsilon(x-y) dy \right| \\
&\geq \frac{1}{2} \int_{\rho_\beta(x-y) \leq \epsilon} \frac{1}{\rho(y)^\alpha} \varphi_\epsilon(x-y) dy \\
&\geq \frac{c}{\rho(x)^\alpha} \int_{\rho_\beta(x-y) \leq \epsilon} \varphi_\epsilon(x-y) dy \\
&= \frac{c}{\rho(x)^\alpha} \int \varphi_\epsilon(y) dy \\
&= \frac{c}{\rho(x)^\alpha} A_1 \epsilon^{2\beta+\nu}.
\end{aligned}$$

Hence, if $\rho(x)^{a_1} \in I_k$,

$$|T\varphi_\epsilon(x)| = \lim_{\epsilon' \to 0} \left| \int_{1 \geq \rho(y) \geq \epsilon'} \frac{e^{i/\rho(y)^\beta}}{\rho(y)^\alpha} \varphi_\epsilon(x-y) dy \right| \geq c\, \epsilon^{2\beta+\nu} \frac{1}{\rho(x)^\alpha} \,.$$

Then

$$\begin{aligned}
\int |T\varphi_\epsilon(x)|^p dx &\geq \sum_k \int_{\rho(x)^{a_1} \in I_k} |T\varphi_\epsilon(x)|^p dx \\
&\geq c\, \epsilon^{p(2\beta+\nu)} \sum_k \int_{\rho(x)^{a_1} \in I_k} \frac{dx}{\rho(x)^{\alpha p}} \,.
\end{aligned}$$

Changing $\int_{\rho(x)^{a_1} \in I_k} (1/\rho(x)^{\alpha p})\, dx$ into polar coordinates, and making a simple change of variables, we get

$$\int_{\rho(x)^{a_1} \in I_k} \frac{dx}{\rho(x)^{\alpha p}} \geq c \int_{I_k} \frac{dr}{r^{(\alpha p - a_2)/a_1}}.$$

Now, using the fact that $|J_k| \leq C|I_{k+1}|$ (part (iii) of Lemma 2), we have

$$\begin{aligned}
\int |T\varphi_\epsilon(x)|^p dx &\geq c\, \epsilon^{p(2\beta+\nu)} \sum_k \int_{I_k} \frac{dr}{r^{(\alpha p - a_2)/a_1}} \\
&\geq c\, \epsilon^{p(2\beta+\nu)} \left(\sum_{k=1}^{k'} \int_{I_k} \frac{dr}{r^{(\alpha p - a_2)/a_1}} + \sum_{k=1}^{k'-1} \int_{J_k} \frac{dr}{r^{(\alpha p - a_2)/a_1}} \right) \\
&\geq c\, \epsilon^{p(2\beta+\nu)} \int_{I_{k'} \cup [\bigcup_{k=1}^{k'-1} (I_k \cup J_k)]} \frac{dr}{r^{(\alpha p - a_2)/a_1}} \,.
\end{aligned}$$

Using part (ii) of Lemma 2, we get

$$\int |T\varphi_\epsilon(x)|^p dx \geq c\, \epsilon^{p(2\beta+\nu)} \int_{A_0^{-a_1/\beta} \epsilon^{a_1}}^{7^{-a_1/\beta}} \frac{dr}{r^{(\alpha p - a_2)/a_1}} \,.$$

By the assumptions made on α and p at the beginning of the proof, $\alpha p - \nu + 1 > 1$. Hence

$$\int |T\varphi_\epsilon(x)|^p dx \geq c\, \epsilon^{p(2\beta+\nu)} \epsilon^{\nu - \alpha p}.$$

Now, since T is bounded on L^p,

$$A_p \epsilon^{2\beta+\nu} = \|\varphi_\epsilon\|_{L^p}^p \geq c\, \|T\varphi_\epsilon\|_{L^p}^p = c\, \epsilon^{p(2\beta+\nu)} \epsilon^{\nu - \alpha p}.$$

Letting $\epsilon \to 0$, it follows that

$$p(2\beta + \nu) + \nu - \alpha p \geq 2\beta + \nu,$$

or

$$p(2\beta - \alpha + \nu) \geq 2\beta.$$

Therefore,

$$\frac{1}{p} - \frac{1}{2} \leq \frac{\beta - \alpha + \nu}{2\beta}.$$

This completes the proof of the theorem. □

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