

EQUILIBRIUM EXISTENCE AND TOPOLOGY IN SOME REPEATED GAMES WITH INCOMPLETE INFORMATION

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ABSTRACT. This article proves the existence of an equilibrium in any infinitely repeated, un-discounted two-person game of incomplete information on one side where the uninformed player must base his behavior strategy on state-dependent information generated stochastically by the moves of the players and the informed player is capable of sending nonrevealing signals.

This extends our earlier result stating that an equilibrium exists if additionally the information is standard. The proof depends on applying new topological properties of set-valued mappings. Given a set-valued mapping F on a compact convex set $P \subset \mathbb{R}^n$, we give further conditions which imply that every point $p_0 \in P$ belongs to the convex hull of a finite subset P_0 of the domain of F satisfying $\bigcap_{x \in P_0} F(x) \neq \emptyset$.

1. INTRODUCTION

The motivation for considering results of this paper comes from game theory, and specifically from two-person infinitely repeated games of incomplete information on one side. With such a game, there is an infinite sequence of playing stages and the relationship between the moves and the payoffs, determined by nature at the start of the game but not necessarily known by both players, does not change in the course of the game. There are finitely many different possible relationships between the moves and the payoffs and they are called “states of nature”. “Incomplete information on one side” means that one player, the “informed player”, knows which state of nature is being played and both players know the probability of the given state to occur; moreover, the informed player knows everything his opponent knows.

In [SST] the existence of equilibria was established in such games with standard information. (For only two states of nature, the result was proven by Sorin [So].)

Received by the editors August 14, 2000 and, in revised form, May 10, 2002.

2000 *Mathematics Subject Classification.* Primary 55M20, 91A20; Secondary 54C60, 52A20, 91A05, 91A10.

The first named author wishes to thank Christof Wehrsig of the Sociology Department of the University of Bielefeld for introducing him to game theory. The research of this author was supported by the German Science Foundation (Deutsche Forschungsgemeinschaft), the Institute of Mathematical Economics (Bielefeld), the Institute of Mathematical Stochastics (Goettingen), the Center for Rationality and Interactive Decision Theory (Jerusalem), the Department of Mathematics of the Hebrew University (Jerusalem), and the Edmund Landau Center for Research in Mathematical Analysis (Jerusalem), sponsored by the Minerva Foundation (Germany). The Stefan Banach International Center at the Institute of Mathematics of the Polish Academy of Sciences enabled for meetings of the three authors while the paper was in preparation.

By standard information we mean that there is perfect monitoring (both players have perfect knowledge of all the past moves) and that the uninformed player's perception of these moves is independent of the state of nature. Due to perfect monitoring, the uninformed player may of course draw inferences concerning the state of nature from the sequence of moves of his opponent.

The purpose of this paper is to show that the state independence of information and the perfect monitoring can be discarded, provided the informed player can send nonrevealing signals (distinctly perceivable messages, the perception of which is independent of the state of nature). With other assumptions on the form of the game left intact, we show it still has an equilibrium.

In fact, we establish the existence of equilibria for games that are slightly more general than those of incomplete information on one side, namely when the player uninformed about the state of nature can obtain information only from perception that is determined stochastically by the moves of the players and the state. (We keep the assumption that the player who knows the state is able to send non-revealing signals.) Here one must be careful, because there may be no equilibrium if the player knowing the state has no influence on or knowledge of some information received by the other player. Such is the case with zero-sum games of incomplete information on "one and a half sides". There, Player One knows the state of nature and the initial distribution on the states of nature, but does not know some additional revelation from Nature to Player Two of a posterior probability (see Sorin and Zamir [SZ]).

There are a few steps in the proof, related to different ideas, and they can be separated as follows.

1. First, we show that the game can be split into several games which have state-independent information and correspond to certain subsets L of the set K of states of nature. The desired result follows once we are able to find equilibria for the arising sub-games that can be pasted together into an equilibrium of the original game. Moreover, for each of the subgames, there exists a class of strategy pairs, which we call "joint plans"; they are a revision, due to J. Renault, of a concept introduced by R. Aumann, M. Maschler, and R. Stearns, see [AM], with regard to the games with standard information. Out of these joint plans, we define a class of strategy pairs for the original game.

2. This step goes back to an idea from our previous paper [SST]. To each of our strategies for the game with probability distribution on K equal to p we associate a set $F(p)$ devised so that its nonemptiness implies that we reached an equilibrium. (See also 4 below.) $F(p)$ is a subset of the cube I^K and is defined in terms of some conditions on the payoffs of the game. Identifying each probability distribution on K with a point in the simplex $\Delta(K)$ formally spanned by K , we know in turn from [SST] that $F(q) \neq \emptyset$ for all $q \in \Delta(K)$ provided the correspondence $F : p \mapsto F(p)$, when considered as a subset of $\Delta(K) \times I^K$, has a certain (co-)homological property, named property \mathcal{S} in the present paper. A similar situation exists with respect to the correspondences $F_L \subset \Delta(L) \times I^L$ associated with the subgames, with L running over an appropriate family of subsets of K . By extending one of the main results from [SST] in an essential way, we show that the F_L 's do have the property \mathcal{S} .

3. The next step is to relate the correspondence F to the F_L 's. The correspondences F_L and F share the essential feature of being "saturated", meaning (for F) that whenever $y \in F(p)$ and $p_i = 0$ for some $i \in K$, then $z \in F(p)$ for all $z \in I^K$ that differ from y in the i -th coordinate only and satisfy $z_i \geq y_i$. We show that for

a relevant class of saturated correspondences $\Delta(K) \rightarrow I^K$, property \mathcal{S} is equivalent to the requirement that its image separates the cube between a certain vertex and the union of faces disjoint from it. This allows us to use the basic properties of separating sets to show that the property \mathcal{S} of the correspondence F follows from that of the F_L 's.

4. The final step is to show that, given the nonemptiness of the set $F(p)$, the joint plans defining it do indeed generate equilibria. Here we depend largely on work of Renault [Re], who proved equilibrium existence when the uninformed player does have state-independent perception. In fact, we refer to his paper concerning the complex arguments involving the statistics of such games, and we make only a minimal revision of Renault's construction of behavior strategies from "joint plans".

The paper is not organized according to this outline. We cumulate our topological results, given in an abstract framework of saturated correspondences, in Section 2. There, steps 2 and 3 are covered. In Section 3 games of incomplete information are discussed, and we deal with steps 1 and 4.

Although the main result belongs to game theory, the topological analysis plays a very essential role in the paper. Expressed in terms of correspondences, a version of one of the two results needed says the following. Assume for a certain cover \mathcal{L} of a finite set K that we are given a family $\{F_L : \Delta(L) \rightarrow [0, 1]^L\}_{L \in \mathcal{L}}$ of saturated correspondences with property \mathcal{S} , such that the image of F_L lies in a closed and convex subset $C_L \subset [0, 1]^L$ containing $(1, 1, \dots, 1)$. Let $C := \bigcap_{L \in \mathcal{L}} C_L \times [0, 1]^{K \setminus L} \subset [0, 1]^K$. Then, for every point $p \in \Delta(K)$, there exist finitely many sets $L_1, \dots, L_s \in \mathcal{L}$ and points $p_i \in \Delta(L_i) \subset \Delta(K)$, $i = 1, \dots, s$, such that $p \in \text{conv}\{p_i\}_{i=1}^s$ and, with $\pi_i : I^K \rightarrow [0, 1]^{L_i}$ denoting the projection, we have $\pi_i(y) \in F_{L_i}(p_i)$ for some $y \in C$ and $i = 1, \dots, s$. (See Corollary 2 in Section 2.) As indicated earlier, the proof is a manifestation of the fact that certain saturated correspondences have property \mathcal{S} if and only if their image disconnects the target cube in a certain way. In addition to showing this equivalence, we formalize the property \mathcal{S} (which in [SST] was used only implicitly) and establish other results that allow for its verification. We hope that these results concerning the topological aspects of infinitely repeated games and their associated correspondences will find more applications.

Basic notation (common to both sections). Spaces are assumed metric and we write $\text{cl } C$ and \overline{C} for the closure of a set C , $\text{int } C$ for the interior of C and δC for the boundary of the interior of C . If K is a finite set, then by \mathbb{R}^K we denote the Euclidean space of all functions $K \rightarrow \mathbb{R}$, equipped with the standard inner product, and by $\Delta(K)$ the simplex in \mathbb{R}^K :

$$\Delta(K) := \{y \in \mathbb{R}^K : y_k \geq 0 \text{ for all } k \in K \text{ and } \sum_k y_k = 1\}.$$

If $y \in \mathbb{R}^K$ and $L \subset K$, then y^L denotes the projection of y to \mathbb{R}^L ; we may also (and often shall) identify $\Delta(L)$ with $\{y \in \Delta(K) : y^{K \setminus L} = 0\}$ and hence consider $\Delta(L)$ as a subset of $\Delta(K)$.

The k -th coordinate of a point $y \in \mathbb{R}^K$ is denoted both y_k (mainly in Section 2) and y^k (in Section 3, where additional indices are needed).

Given $a : \Delta(K) \rightarrow \mathbb{R}$ we denote by $\text{cav}(a) : \Delta(K) \rightarrow \mathbb{R}$ the concave envelope of a , defined by the property that the set $\{(x, t) \in \Delta(K) \times \mathbb{R} : t \leq \text{cav}(a)(x)\}$ is the convex hull of $\{(x, t) \in \Delta(K) \times \mathbb{R} : t \leq a(x)\}$.

2. TOPOLOGICAL PROPERTIES OF CORRESPONDENCES

2.1. Correspondences and their levelwise convexification. By a **correspondence** $F : X \rightarrow Y$ we mean any subset of $X \times Y$ such that its projection to X is a proper mapping (i.e., is closed and has compact point-inverses). Thus, the same correspondence may also be denoted as $F \subset X \times Y$; the notation $F : X \rightarrow Y$, however, will most often be used as one that openly displays the nonsymmetric role that the spaces X and Y play here. The image of F under the coordinate-switching map is denoted F^{-1} . It is a correspondence from Y to X provided the projection $F \rightarrow Y$ is proper.

If X_0 is a subset of X , then $F \cap (X_0 \times Y)$ is called the restriction of F to X_0 and denoted $F|X_0$. This notation applies also to single-valued correspondences $X \rightarrow Y$, which we call mappings.

For a correspondence $F : X \rightarrow Y$, the image of the set $F|X_0$ under the projection to Y is denoted by $F(X_0)$ and called the image of X_0 under F . We also write

$$\text{im}(F) := F(X), \quad F(x) := F(\{x\}) \text{ for } x \in X \text{ and } \text{dom}(F) := \{x \in X : F(x) \neq \emptyset\}.$$

If $F : X \rightarrow Y$ is a correspondence, it is not assumed a priori that $F(x) \neq \emptyset$ for all (or even for some) $x \in X$. In fact, the demonstration of the nonemptiness of some of the sets $F(x)$ constitutes the main goal of this section.

In most of our considerations, X is a subset of a Euclidean space E in which a certain simplex Δ is selected along with a retraction $r : E \rightarrow \Delta$. We may then consider an operation on compact subsets A of E defined by the formula

$$(0) \quad \text{co}(A) := \text{conv}(r(A)) \cup \{\lambda a + (1 - \lambda)r(a) : a \in A, \lambda \in [0, 1]\}.$$

(If $A \subset \Delta$, this reduces to taking the convex hull of A .) This operation may be applied levelwise to the correspondence $F : X \rightarrow Y$ to give a new correspondence $cF : E \rightarrow Y$ defined by

$$(1) \quad cF := \bigcup_{y \in Y} \text{co}(F^{-1}(y)) \times \{y\}.$$

Thus,

$$(2) \quad cF(p) = \{y \in Y : p \in \text{co}(X_0) \text{ for some finite set } X_0 \subset E \text{ such that } y \in \bigcap_{x \in X_0} F(x)\}.$$

We say that cF is obtained by **convexifying F level-wise**.

The problem faced in our game-theoretic applications is to establish that, for a given correspondence $F : \Delta \rightarrow Y$, we have $\text{dom}(cF) \supset \Delta$. The importance of such a conclusion is that it allows interpreting the nonemptiness of $(cF)(p)$, for a given point $p \in \Delta$, in terms of the existence of solutions to systems of inequalities encountered in game theory. The approach of handling this problem, developed in [SST], depended on the use of certain homology elements carried by boundaries of open subsets of Euclidean spaces. Below we give an equivalent definition of these elements and prove those of their properties that are needed here.

2.2. Fundamental classes of boundaries of open subsets of \mathbb{R}^n . In what follows, \tilde{H} denotes the reduced Čech homology functor with coefficients in a compact abelian group, cf. [ES, §IX.7]. $\tilde{H}(X, Y)$ is considered as a graded group, i.e., as

a direct sum of the $\tilde{H}_n(X, Y)$'s, and a specific group $\tilde{H}_n(X, Y)$ is isolated only if needed. As is usual, $\tilde{H}(X)$ is identified with $\tilde{H}(X, \emptyset)$.

Let $A \supset B$ and $A' \supset B'$ be compact pairs such that $A \subset A'$ and $B \supset A \cap B' \cup \delta A$. (We consider A' as the ambient space in which δA is taken.) We have canonical homomorphisms

$$\tilde{H}(A', B') \rightarrow \tilde{H}(A', B \cup (A' \setminus \text{int } A)) \rightarrow \tilde{H}(A, B),$$

the first of which is inclusion-induced and the second of which is the composition of the excision of $A' \setminus \text{cl}(\text{int } A)$ and the inclusion into (A, B) . (This excision is legitimate, by the continuity of Čech homology; see [ES, §X.3 and §X.5].) The arising homomorphism $\tilde{H}(A', B') \rightarrow \tilde{H}(A, B)$ will be called **restriction** and denoted $z \mapsto z|(A, B)$.

Remark 1. Let C_0, C_1, C_2 be compact subsets of a space X .

- a) If $C_1 \cap C_2 \subset C_0 \subset C_1 \cup C_2$ then, by the Mayer–Vietoris exact sequence,
(3) $z \mapsto z|(C_1, C_0 \cap C_1) \oplus z|(C_2, C_0 \cap C_2)$ establishes an isomorphism

$$\tilde{H}(C_1 \cup C_2, C_0) \rightarrow \tilde{H}(C_1, C_0 \cap C_1) \oplus \tilde{H}(C_2, C_0 \cap C_2),$$

the inverse of which is the sum of two inclusion-induced homomorphisms.

- b) Assume $C_0 = C_1 \cap C_2$ and $z_i \in \tilde{H}(C_i, C_0)$ for $i = 1, 2$. Then, a $z \in \tilde{H}(C_1 \cup C_2)$ such that $z|(C_1, C_0) = z_1$ and $z|(C_2, C_0) = z_2$ exists iff $\partial z_1 + \partial z_2 = 0$, where ∂ denotes the boundary operator of the corresponding pair (by the same reason).
c) If $z \in \tilde{H}(C_1 \cup C_2)$ and $z|(C_1, C_1 \cap C_2) = 0$, then $z \in \text{im}(\tilde{H}(C_2 \hookrightarrow C_1 \cup C_2))$ (which follows using excision and the exact sequence of the pair $(C_1 \cup C_2, C_2)$).

Now consider \mathbb{R}^n as a subset of the sphere $S^n = \mathbb{R}^n \cup \{\infty\}$, oriented by a certain $\omega \in \tilde{H}_n(S^n)$. Let U be an open bounded set in \mathbb{R}^n . Taking $A' = S^n$ and $B' = \emptyset$, we denote by $[\delta U] \in \tilde{H}(\delta U)$ the image of $\omega|(\overline{U}, \delta U) \in \tilde{H}(\overline{U}, \delta U)$ under the boundary operator.

It needs to be stressed that if δU is the boundary of several bounded open sets in \mathbb{R}^n , then the class $[\delta U]$ depends on which of them is being taken for U . In the most intuitive and still important case, however, when δU is an $(n-1)$ -dimensional compact closed PL-manifold, the choice of U is unique and $[\delta U]$ can be shown to be the homology class of the cycle induced by this manifold (oriented according to ω); this motivates our notation.

If $C \subset \mathbb{R}^n$ is a compact set, then we let $[\delta C] := [\delta(\text{int } C)] \in \tilde{H}(\delta C)$. We note that $[\delta C]$ is the image of $\omega|(C, \delta C) \in \tilde{H}(C, \delta C)$ under the boundary operator, since $\dim(C \setminus \text{int } C) < n$.

Lemma 1. *Let C_1, C_2, C_3 be compact sets in \mathbb{R}^n .*

- a) *Suppose $C_1 \subset C_2 \subset C_3$ and let $z \in \tilde{H}(C_3, \delta C_2)$ be such that its image ∂z under the boundary operator of the pair $(C_3, \delta C_2)$ is equal to $[\delta C_2]$. Then, $z|(C_2, \delta C_2) = \omega|(C_2, \delta C_2)$ and the image of $z|(C_1, \delta C_1)$ under the boundary operator of the pair $(C_1, \delta C_1)$ is $[\delta C_1]$.*
b) *If $\text{int } C_1 \cap C_2 = \emptyset$, then $[\delta(C_1 \cup C_2)]' = [\delta C_1]' + [\delta C_2]'$, where $'$ denotes taking the image in $\tilde{H}(\delta C_1 \cup \delta C_2)$ of the corresponding homology class.*
c) *Suppose either $\text{int } C_1 \cap C_2 = \emptyset$ or $\text{int } C_1 \subset C_2$. With $A_0 := \delta C_1 \cap \delta C_2$ and $B_0 := A_0 \cap \delta(C_1 \cup C_2)$, we have $[\delta C_1]|(A_0, B_0) = \pm[\delta C_2]|(A_0, B_0)$, where the sign is positive if $\text{int } C_1 \subset C_2$ and is negative if $\text{int } C_1 \cap C_2 = \emptyset$.*

- d) If $\delta C_1 \subset C_2$, then the condition $\text{int } C_1 \subset C_2$ is equivalent to $\tilde{H}(\delta C_1 \hookrightarrow C_2)[\delta C_1] = 0$.

Proof. a). Let z_0 be the image of $\omega|(C_2, \delta C_2)$ under the inclusion-induced homomorphism $j : \tilde{H}(C_2, \delta C_2) \rightarrow \tilde{H}(C_3, \delta C_2)$. Then, $\partial z_0 = [\delta C_2] = \partial z$. Thus $z - z_0$ is in the image of $\tilde{H}_n(C_3)$ in $\tilde{H}_n(C_3, \delta C_2)$ and so is equal to 0, since the n -th homology of a compact set in \mathbb{R}^n vanishes. Since j composed with restriction to $(C_2, \delta C_2)$ is the identity, we get $z|(C_2, \delta C_2) = z_0|(C_2, \delta C_2) = \omega|(C_2, \delta C_2)$. The second claim now follows from the fact that $(\omega|(C_2, \delta C_2))|(C_1, \delta C_1) = \omega|(C_1, \delta C_1)$.

b). It is easy to see that indeed $\delta(C_1 \cup C_2) \subset \delta C_1 \cup \delta C_2$. Assuming, as we may, that C_1 and C_2 are closures of open sets, let $y := \omega|(C_1 \cup C_2, \delta(C_1 \cup C_2))$ and $z_i := \omega|(C_i, \delta C_i)$. Then, by (3), the images of z_i in $\tilde{H}(C_1 \cup C_2, \delta C_1 \cup \delta C_2)$ add up to the image of y in that group. Taking boundaries, we get the conclusion.

c). By replacing C_2 by $C'_2 := S^n \setminus \text{int } C_2$ and ∞ by a point of $\text{int } C_2$, we notice that it suffices to consider the case when $\text{int } C_1 \cap C_2 = \emptyset$; we thus assume the latter. By b), $[\delta C_1]' + [\delta C_2]'$ and $[\delta(C_1 \cup C_2)]'$ have the same restrictions to (A_0, B_0) . Moreover, $A_0 \setminus B_0$ is disjoint from $\delta(C_1 \cup C_2)$ whence $[\delta(C_1 \cup C_2)]'|(A_0, B_0) = 0$. So $([\delta C_1]' + [\delta C_2]')|(A_0, B_0) = 0$.

d). We have $\tilde{H}(\delta C_1 \hookrightarrow \text{cl}(\text{int } C_1))[\delta C_1] = 0$ by the exact sequence of a pair. Thus, $\tilde{H}(\delta C_1 \hookrightarrow C_2)[\delta C_1] = 0$ if $C_2 \supset \text{int } C_1$.

Now assume that, conversely, $\tilde{H}(\delta C_1 \hookrightarrow C_2)[\delta C_1] = 0$ and let $Q = [-M, M]^n$ be a cube such that $C_1 \cup C_2 \subset \text{int } Q$. With $C := Q \setminus \text{int } C_1$, we get from b) that $[\delta C] = [\delta Q]' - [\delta C_1]'$, where $'$ denotes taking images in $\tilde{H}(\delta C)$. The image of $[\delta C]$ in $\tilde{H}(C)$ being 0, it follows that $[\delta Q]$ and $[\delta C_1]$ have the same image in $\tilde{H}(C \cup C_2)$. Hence the image of $[\delta Q]$ is 0, which can happen only if $Q \subset C \cup C_2$. Thus $\text{int } C_1 \subset C_2$. \square

2.3. Spanning homological elements.

Definition. A correspondence $F : X \rightarrow Y$ is **acyclic-valued** if each set $F(x), x \in X$, is acyclic, i.e., is nonempty and satisfies $\tilde{H}(F(x)) = \tilde{H}(\{\text{point}\})$; similarly, if $Y = \mathbb{R}^n$, we say that F is **convex-valued** if each set $F(x)$ is nonempty and convex. We call a mapping $f : X \rightarrow Y$ **acyclic** if the correspondence $f^{-1} : \text{im}(f) \rightarrow X$ is acyclic-valued.

We now state a result from [SST] and isolate a property established in it.¹

Theorem 0. Let E be an n -dimensional affine space, U be its open bounded subset, and $F : \overline{U} \rightarrow Y$ be an acyclic-valued correspondence. If $\dim F(U) < n$, then an element of $\tilde{H}_{n-1}(F|\delta U)$ is mapped to $[\delta U]$ by $\tilde{H}(F|\delta U \rightarrow \delta U)$ and to 0 by $\tilde{H}(F|\delta U \hookrightarrow c(F|\delta U))$.

Definition. Let E be as above. A correspondence $F : E \rightarrow Y$ is said to have **property \mathcal{S}** for an open bounded set $U \subset E$ if F is a compact subset of $E \times Y$ and there exists an element $z \in \tilde{H}_{n-1}(F|\delta U)$ whose images under $\tilde{H}(F|\delta U \rightarrow \delta U)$ and under $\tilde{H}(F|\delta U \hookrightarrow F)$ are $[\delta U]$ and 0, respectively. If, moreover, for a given compact set $Z \subset F|\delta U$ one may take z in the image of $\tilde{H}(Z \hookrightarrow F|\delta U)$, then we say that Z **witnesses** the property \mathcal{S} for U . We call z a **spanning element** for

¹Strictly speaking, Theorem 1 of [SST] covers the case where $U = \text{int } \overline{U}$, but its proof establishes the full result.

U because if it exists (that is, if F has property \mathcal{S} for U), then $\text{dom}(F) \supset U$ by Lemma 1(d).

We say that the correspondence F has property \mathcal{S} for a compact set $C \subset E$ if F has property \mathcal{S} for the interior of C .

It turns out that property \mathcal{S} is hereditary and continuous and can under some assumptions be pasted together, as explained below:

Lemma 2. *Let $F : E \rightarrow Y$ be a compact correspondence and $C \subset E$ be a compact set. Then:*

- a) *If F has property \mathcal{S} for a compact set $D \supset C$, then $F|C$ has it for C .*
- b) *F has property \mathcal{S} for C provided that for every neighborhood U of δC in E and for every neighborhood V of F in $E \times Y$, there exists a correspondence $G : E \rightarrow Y$ such that $G \subset V$ and G has property \mathcal{S} for a compact set D satisfying $\delta D \subset U$ and $D \subset C \cup U$.*

In particular, if each of the correspondences $F_i : E \rightarrow Y$ has property \mathcal{S} for C and $F_1 \supset F_2 \supset F_3 \dots$, then $\bigcap_n F_n$ has this property also.

- c) *Let U_1 and U_2 be disjoint open bounded subsets of E . If $G : \overline{U_2} \rightarrow Y$ is an acyclic-valued correspondence and the property \mathcal{S} of F for U_1 is being witnessed by a compact set $Z \subset F|\delta U_1$ such that $Z|\delta U_2 \subset G$, then $F \cup G$ has property \mathcal{S} for $\text{cl}(U_1 \cup U_2)$.*

Proof. a). Let $x \in \tilde{H}(F, F|\delta D)$ satisfy $\tilde{H}(F|\delta D \rightarrow \delta D)(\partial x) = [\delta D]$ and let $y := x|(F|C, F|\delta C)$. By part a) of Lemma 1, applied to $(C_1, C_2, C_3) = (C, D, \text{dom}(F))$ and z equal to the image of x under the projection-induced mapping, $\tilde{H}(F|\delta C \rightarrow \delta C)(\partial y) = [\delta C]$.

b). The first part follows from the fact that the functor \tilde{H} is continuous and takes values in compact groups. The second part follows from the first since the correspondences $F_i|C$ are compact and by a) have property \mathcal{S} for C .

c). Let $z_1 \in \tilde{H}(Z)$ be a spanning element for U_1 and $z_0 := z_1|(Z|A_0, Z|B_0)$, where we let

$$\begin{aligned} A_0 &:= \delta U_1 \cap \delta U_2, \quad D := \overline{U_1} \cup \overline{U_2}, \\ B_0 &:= A_0 \cap \delta D, \quad B_1 := \delta U_1 \cap \delta D, \quad B_2 := \delta U_2 \cap \delta D. \end{aligned}$$

By Lemma 1(c) and Remark 1(b), the image of z_0 in $\tilde{H}(A_0, B_0)$ equals $-[\delta U_2]|(A_0, B_0)$ and $\tilde{H}(Z|B_0 \rightarrow B_0)(\partial z_0) = \partial([\delta U_2]|(B_2, B_0))$.

Claim. There exists a $z_2 \in \tilde{H}(Z|A_0 \cup G|B_2)$ such that $\tilde{H}(Z|A_0 \cup G|B_2 \rightarrow \delta U_2)(z_2) = [\delta U_2]$ and $z_2|(Z|A_0, Z|B_0) = -z_0$.

Proof of the Claim. By Vietoris' theorem [Sp], there exists $\alpha \in \tilde{H}(G|B_2, G|B_0)$ such that $\tilde{H}((G|B_2, G|B_0) \rightarrow (B_2, B_0))(\alpha) = [\delta U_2]|(B_2, B_0)$. Then $\partial \alpha$ is the image of ∂z_0 in $\tilde{H}(G|B_0)$. By Remark 1(b), α and z_0 induce the desired element $z_2 \in \tilde{H}(Z|A_0 \cup G|B_2)$. \square

We return to the proof of part c). Let z'_1 and z'_2 be the images of z_1 and of z_2 , respectively, in $\tilde{H}(Z \cup G|B_2)$. Then $x' := z'_1 + z'_2$ has the property that $x'|(Z|A_0, Z|B_0) = 0$ and thus, by Remark 1(c), there exists an $x \in \tilde{H}(Z|B_1 \cup G|B_2)$ such that the image of x in $\tilde{H}(Z \cup G|B_2)$ is x' . Since z'_1 and z'_2 map to zero in $\tilde{H}(F \cup G)$, so does x .

It remains to show that x maps to $[\delta D]$ under $\tilde{H}(Z|B_1 \cup G|B_2 \rightarrow B_1 \cup B_2 = \delta D)$. However, $[\delta D]$ and x' have, by part (b) of Lemma 1, the same image in $\tilde{H}_{n-1}(\delta U_1 \cup \delta U_2)$, and $\tilde{H}_{n-1}(\delta D \hookrightarrow \delta U_1 \cup \delta U_2)$ is a monomorphism in view of $\dim(\delta U_1 \cup \delta U_2) < n$. Our claim thus follows from the commutativity of a diagram induced by projections and inclusions. \square

Parts a) and b) imply:

Corollary 1. *If a correspondence F has property \mathcal{S} for a compact set $C \subset E$, then so does any compact correspondence $G \subset E \times Y$ containing $F|_{\text{int}(C)}$.*

2.4. Two results about saturated correspondences.

Definition. If $F : \Delta(L) \rightarrow Y$ is a correspondence and $Y \subset \mathbb{R}^L$, then by F^+ we denote the correspondence $\Delta(L) \rightarrow Y$ defined by

$$F^+(p) := \{y \in Y : \exists x \in F(p) \text{ such that } x_l \leq y_l \text{ for all } l \in L \text{ and } x_l = y_l \text{ if } p_l > 0\}.$$

We call F^+ the Y -**saturation** of F . We say that F is **saturated** if $F = F^+$.

In the following we write for brevity Δ for $\Delta(L)$, where L is a fixed finite set. We also often identify vectors in \mathbb{R}^L with the functions they induce on \mathbb{R}^L by scalar multiplication. Thus for $A \subset \mathbb{R}^L$ and $\varphi \in \mathbb{R}^L$, it makes sense to denote by $\varphi|A$ the restricted function $A \rightarrow \mathbb{R}$, and $\varphi|A \geq a$ means for a given function $a : A \rightarrow \mathbb{R}$ that $\varphi \cdot q \geq a(q)$ for all $q \in A$.

Theorem 1. *Let I be a closed segment in \mathbb{R} , let $F : \Delta \rightarrow I^L$ be a convex-valued correspondence, and let $a : \Delta \rightarrow I$ be a lower semi-continuous function such that*

$$(4) \quad a(q) \leq \sup\{\varphi \cdot q : \varphi \in F(p)\} \text{ for all } p, q \in \Delta.$$

With F^+ denoting the I^L -saturation of F , the following correspondence $G : \Delta \rightarrow \mathbb{R}^L$ has property \mathcal{S} for Δ :

$$(5) \quad G := c(F^+) \cap (\Delta \times \{\varphi \in I^L : \varphi| \Delta \geq a\}).$$

Proof. By applying a shift in \mathbb{R} and in \mathbb{R}^L , we may assume that $I = [-M, M]$ for some $M > 0$. We divide the proof into 3 steps:

i) We first assume that a is piecewise linear and the correspondence F is continuous. As in [SST], p. 12, we consider a retraction $r : E \rightarrow \Delta$ and a mapping $u : E \rightarrow [0, \infty)^L$ defined by:

E is the affine hull of $\Delta = \Delta(L)$ in \mathbb{R}^L ,

$$r(v)_k := \max(v_k, 0) / \sum \{v_l : v_l > 0\} \quad \text{and} \quad u(v)_k := \max(-v_k, 0) \quad \text{for all } k \in L.$$

Using r and u we define:

- sets $F(v)$ for $v \in E \setminus \Delta$ by the formula $F(v) := F(r(v)) + u(v)$,
- convex functions $b_v : \Delta \rightarrow \mathbb{R}$ and nonnegative numbers $z(v)$ for $v \in E$ by the formulas:

$$b_v(q) := \max_{\varphi \in F(v)} \varphi(q),$$

$$z(v) := \max_{q \in \Delta} (\text{cav}(a)(q) - b_v(q)) = \max_{q \in \Delta} \min_{\varphi \in F(v)} (\text{cav}(a)(q) - \varphi \cdot q),$$

- correspondences $\Psi_0, \Psi : E \rightarrow \mathbb{R}^L$ by the formulas in which $e = (1, \dots, 1) \in \mathbb{R}^K$:

$$\Psi_0(v) := \{\varphi \in F(v) : \varphi|\Delta \geq a\}, \quad \Psi(v) := \{\varphi \in F(v) + z(v)e : \varphi|\Delta \geq a\},$$

- an open set $U \subset E$ by the formula

$$U := \{v \in E : z(v) > 0 \text{ and } \|v\| < R\},$$

where the value of $R > \sup\{\|v\| : v \in \Delta\}$ is to be specified later.

By the definition of $z(v)$ and by (4), each set $\Psi(v)$ is nonempty and convex and $a \leq \varphi|\Delta \leq b_v + z(v)$ for $\varphi \in \Psi(v)$. This implies as in [SST], p. 10, that $\Psi(U)$ is of dimension lower than that of E . (The needed inequality $a \leq b_v$ follows from (4).) Therefore, $\Psi|\delta U$ witnesses by Theorem 0 the property \mathcal{S} (for U) of the correspondence $c(\Psi|\delta U)$.

Let us now consider the correspondence

$$F' := c(\Psi|\delta U) \cup \Psi_0|(E \setminus U).$$

Clearly, $\Psi(p) = \Psi_0(p)$ if $p \in \delta U \cap \Delta$, while for $p \in E \setminus U$, the set $\Psi_0(p)$ is nonempty, compact and convex. Thus it follows from part (c) of Lemma 2 (applied with $U_1 := U$, $U_2 := \text{int}(\Delta \setminus U)$ and $Z := \Psi|\delta U$) that F' has property \mathcal{S} for $\text{cl}(U \cup U') \supset \Delta$, and from part (a) that $F'|\Delta$ has property \mathcal{S} for Δ . By Corollary 1, it therefore remains to show that

(*) with an appropriate choice of R , we have $F'(p) \subset G(p)$ for each $p \in \text{int } \Delta$.

To this end, we fix $p \in \text{int } \Delta$ and $\varphi \in F'(p)$. If $\varphi \in \Psi_0(p)$, then $\varphi \in F(p)$ and $\varphi|\Delta \geq a$, and so $\varphi \in G(p)$; thus we assume that $\varphi \in c(\Psi|\delta U)(p)$. By definition, $\varphi|\Delta \geq a$ and there exists a finite set $V \subset \delta U$ such that $p \in \text{conv}(r(V))$ and $\varphi \in F(r(v)) + u(v) + z(v)e$ for all $v \in V$.

We use a trick from [SST], which is as follows. Since $p \in \text{int } \Delta \cap \text{conv}(r(V))$, for every $l \in L$ there exists a $w \in V$ such that $r(w)_l > 0$, forcing $u(w)_l = 0$ and $\varphi_l \in [-M, M] + 0 + [-M, M] = [-2M, 2M]$. Now, if v is an arbitrary element of V , then $u(v) \in \varphi - z(v)e - F(r(v))$, whence $u(v)_l \leq 2M + M + M = 4M$ and $v_l \geq -4M$ for all $l \in L$, yielding $\|v\| \leq R_0 := \max\{\|y\| : y \in E \cap [-4M, \infty)^L\} < \infty$. Since for $v \in \delta U$ we have $\|v\| = R$ or $z(v) = 0$, it follows that taking $R > R_0$ forces $z(v) = 0$ for all $v \in V$ and, hence, $\varphi \in F(r(v)) + u(v) \subset F^+(r(v))$ for all $v \in V$. Also, the information that $z(w) = 0$ improves the earlier estimate on φ_l to $\varphi_l \in [-M, M]$ for all $l \in L$. Since $p \in \text{conv}(r(V))$, this establishes that $\varphi \in c(F^+)(p)$ and completes the proof in the special case i) above.

ii) We now continue to assume that F is continuous but drop the assumption that a be PL. Then, a' defined by $a'(q) := \inf_{p \in \Delta(L)} \sup_{y \in F(p)} y \cdot q$ is continuous and thus is a uniform limit of PL-functions $a_n : \Delta(L) \rightarrow \mathbb{R}$ such that $a_1 < a_2 < \dots$. If we define correspondences G_n using formula (5) with a_n in place of a and a correspondence G' using a' in place of a , then we easily see that $\bigcap_n G_n = G' \subset G$. Since, as shown above, each of the correspondences G_n has property \mathcal{S} for $\Delta(L)$, we infer from part (b) of Lemma 2 that G also does have this property.

iii) In the general case, we apply Lemma 2 of Simon [Si] for each $n \in \mathbb{N}$ to get a continuous correspondence $F_n : \Delta(L) \rightarrow \mathbb{R}^L$ such that F_n contains F and is a subset of the $\frac{1}{n}$ -neighborhood of F in $\Delta \times \mathbb{R}^L$. By taking intersections, we may request also that $F_1 \supset F_2 \supset \dots$. The result again follows by a limit argument

as above applied to correspondences G_n , which are defined using formula (5), this time with F_n in place of F . \square

The above theorem is one of the two ingredients of our topological analysis. The other is the following result showing how property \mathcal{S} can be improved (i.e., derived for a larger simplex) under certain operations on saturated correspondences.

Theorem 2. *Let $I = [a, b]$ be a nontrivial closed segment in \mathbb{R} , K a finite set and \mathcal{L} a family of its non-void subsets such that $\bigcup \mathcal{L} = K$. Suppose for every $L \in \mathcal{L}$ there are given a saturated correspondence $F_L : \Delta(L) \rightarrow I^L$ with property \mathcal{S} for $\Delta(L)$ and a closed convex subset U_L of I^K containing the point (b, b, \dots, b) . With $\Delta(L)$ considered as a subset of $\Delta(K)$, set for $L \in \mathcal{L}$:*

$$\tilde{F}_L := F_L \times I^{K \setminus L} = \{(p, y) \in \Delta(L) \times I^K : y^L \in F_L(p)\} \subset \Delta(K) \times I^K.$$

If $\text{im}(\tilde{F}_L) \subset U_L$ for every $L \in \mathcal{L}$ and the correspondence $\Gamma : \Delta(K) \rightarrow I^K$ is defined as

$$\Gamma := \bigcup_{L \in \mathcal{L}} \tilde{F}_L \cap (\Delta(K) \times \bigcap_{L \in \mathcal{L}} U_L),$$

then $c\Gamma$ has property \mathcal{S} for $\Delta(K)$.

In particular, it follows that for every $p \in \Delta(K)$ we have $(c\Gamma)(p) \neq \emptyset$. We formulate this consequence explicitly in a form applied in the sequel:

Corollary 2. *Under assumptions of Theorem 2, let $p \in \Delta(K)$ be given. Then, there exist: finitely many sets $L_1, \dots, L_s \in \mathcal{L}$ and points $p_i \in \Delta(L_i) \subset \Delta(K)$, $i = 1, \dots, s$, and a point $\varphi \in \bigcap_{L \in \mathcal{L}} U_L \subset I^K$ such that the following conditions hold:*

$$p \in \text{conv}\{p_1, \dots, p_s\} \text{ and } \varphi^{L_i} \in F_{L_i}(p_i) \text{ for every } i = 1, \dots, s.$$

The proof of Theorem 2 involves a relation between the property \mathcal{S} of a correspondence $\Delta(K) \rightarrow \mathbb{R}^K$ and certain separation properties of its image. We establish this relation below. As is usual, we say that a closed set S in a space X **separates** X between two sets $A, B \subset X$ if $X \setminus S$ is a union of two disjoint open sets, one of which contains A and the other B .

The next lemma is known, but failing to find a reference we include a proof.

Lemma 3. *Let Δ be an n -simplex in the affine n -space E . A correspondence $F : \Delta \rightarrow (0, 1)$ has property \mathcal{S} for Δ if and only if the set F separates $\Delta \times [0, 1]$ between $\Delta \times \{0\}$ and $\Delta \times \{1\}$.*

Proof. a) The “if” implication. Let C be the component of $\Delta \times [0, 1] \setminus F$ containing $\Delta \times \{0\}$ and let $D := \delta C \cap (\delta \Delta \times [0, 1])$, $D_1 := \Delta \times \{0\}$ and $D_2 := \text{cl}(\delta C \setminus (D_1 \cup D))$. With $z_i := [\delta C](D_i, D_i \cap D)$, we note that the image of $\partial z_2 \in \tilde{H}(D_2 \cap D)$ vanishes in $\tilde{H}(D_2)$ and so it does in $\tilde{H}(F)$ since $D_2 \subset F$.

Let $'$ denote taking the image of homology classes in $\tilde{H}(D \cap (D_1 \cup D_2))$. Then $\partial z'_1 + \partial z'_2 = -\partial([\delta C](D, D \cap (D_1 \cup D_2)))$, by Remark 1(b) applied with $X = \delta C$, $C_2 = D$ and $C_1 = D_1 \cup D_2$. Thus the images in $\tilde{H}(\delta \Delta)$ of ∂z_1 and of $-\partial z_2$ under the projection $D \rightarrow \delta \Delta$ coincide and $-\partial z_2$ is a spanning element for Δ .

b) The “only if” implication. Suppose F fails to separate while having property \mathcal{S} . Then, there exists an arc A connecting $\Delta \times \{0\}$ to $\Delta \times \{1\}$ and missing the set $F \cup \delta \Delta \times [0, 1]$. Due to the assumed property \mathcal{S} for Δ , the cycle generated by $\delta \Delta \times \{0\}$ is null-homologous in $F \cup \delta \Delta \times [0, 1]$. This contradicts the fact that this

cycle is not null-homologous in the complement of the arc A in $\Delta \times [0, 1]$, for by the Mayer–Vietoris theorem, $\tilde{H}(\delta\Delta \times \{0\} \hookrightarrow \Delta \times [0, 1] \setminus A)$ is an isomorphism. \square

Lemma 4. *Let a correspondence $F : \Delta(L) \rightarrow \Delta(L) \times (0, 1)$ satisfy the following conditions:*

- i) $F^{-1} : \text{im}(F) \rightarrow \Delta(L)$ is acyclic-valued,
- ii) $F^{-1}(\Delta(L_0) \times (0, 1)) \subset \Delta(L_0)$ for each nonempty subset L_0 of L .

Then, in order for the image $\text{im}(F)$ of F to separate $\Delta(L) \times [0, 1]$ between $\Delta(L) \times \{0\}$ and $\Delta(L) \times \{1\}$, it is necessary and sufficient that $F \cap (\delta\Delta(L) \times \delta\Delta(L) \times [0, 1])$ witnesses the property \mathcal{S} of F for $\Delta(L)$.

Proof. We write Δ for $\Delta(L)$ and, when willing to distinguish between the factors of $\Delta \times \Delta \times [0, 1]$, we denote the first by Δ_1 and the second by Δ_2 . Consider $F' := \text{im}(F)$ as a correspondence $\Delta \rightarrow (0, 1)$ and denote by π the projection of F onto F' along Δ_1 , by q the projection $\Delta \times [0, 1] \rightarrow \Delta$ and by p the projection $\Delta_1 \times \Delta_2 \times [0, 1] \rightarrow \Delta_1$. By assumptions, π is acyclic and for $L_0 \subset L$ we have

for $x \in T := F \cap (\Delta_1 \times \delta\Delta_2 \times [0, 1])$, if $q\pi(x) \in \Delta(L_0)$ then $p(x) \in \Delta(L_0)$.

Thus T coincides with $F \cap (\delta\Delta_1 \times \delta\Delta_2 \times [0, 1])$ and p and $q\pi$ are on T homotopic as mappings into $\delta\Delta$ (use a straight line homotopy); they hence induce the same homomorphisms $\tilde{H}(T) \rightarrow \tilde{H}(\delta\Delta)$. Also, the acyclic map π induces by Vietoris' theorem isomorphisms $\tilde{H}(F) \rightarrow \tilde{H}(F')$ and $\tilde{H}(T) \rightarrow \tilde{H}(F'|\delta\Delta)$. From these facts and Lemma 3, the conclusion readily follows. \square

Notation. Let $I = [a, b]$ be a nondegenerate closed segment. Given a finite set K , we denote by v_+ and v_- the points (b, \dots, b) and (a, \dots, a) of the cube I^K , and for $L \subset K$ we let

$$B(L) := \{y \in I^K \mid y_k = b \ \forall k \in K \setminus L\}$$

and

$$A(L) := B(L) \cap \{y \in I^K \mid y_k = a \text{ for some } k \in L\}.$$

Remark 2. Each set $B(L)$ is a geometric cone on $A(L)$ with vertex v_+ (in particular, I^K is a cone on $A(K)$); also, $A(K)$ is a cone on $\bigcup_{|L| < |K|} A(L)$ with vertex v_- . Consequently, there exists a homeomorphism h of the cube I^K onto the cone $\text{Cone}(\Delta(K))$ on $\Delta(K)$ that i) sends v_+ to the vertex of the cone, ii) for each set $L \subset K$ sends the set $A(L)$ to the simplex $\Delta(L)$, and iii) is linear on any cone segment. In fact, to construct h , it suffices to use induction with respect to the number of elements of K and to proceed as follows:

- a) first, for every set $K' \subset K$ with $|K'| = |K| - 1$, construct a homeomorphism $B(K') \rightarrow \text{Cone}(\Delta(K'))$ having the desired properties and denote by h_0 the resulting homeomorphism $B \rightarrow \text{Cone}(\delta\Delta(K))$, where $B := \bigcup_{|L| < |K|} B(L)$;
- b) get a homeomorphism $h_1 : A(K) \rightarrow \Delta(K)$ by mapping each segment $[x, v_-]$, $x \in A(K) \cap B$, linearly to the segment from $h(x)$ to the center of $\Delta(K)$;
- c) extend h_1 linearly over any cone segment by considering I^K as a cone over $A(K)$ and using the natural cone structure of $\text{Cone}(\Delta(K))$.

Corollary 3. *Let $F : \Delta(L) \rightarrow I^L$ be a correspondence having the following properties:*

- a) $F^{-1} : \text{im}(F) \rightarrow \Delta(L)$ is acyclic-valued,

- b) $\text{im}(F)$ misses the set $A(L) \cup \{v_+\}$, and
- c) we have $F^{-1}(B(L_0)) \subset \Delta(L_0)$ for every set $L_0 \subset L$.

Then in order for $\text{im}(F)$ to separate I^L between $A(L)$ and $\{v_+\}$, it is necessary and sufficient that $F \cap (\delta\Delta(L) \times \delta I^L)$ witnessed the property \mathcal{S} for F .

Proof. Let $h : I^L \rightarrow \text{Cone}(\Delta(L))$ be the homeomorphism defined in the Remark and let $g : \text{Cone}(\Delta(L)) \rightarrow \Delta(L) \times [0, 1]$ be a homeomorphism that “flattens” to $\Delta(L) \times \{1\}$ a small neighborhood U , in the boundary of the cone, of the vertex point. We request that this neighborhood be so small that it misses the set $\text{im}(hF)$; also, we request that $g^{-1}(\Delta(L_0) \times [0, 1]) \subset B(L_0)$ for each $L_0 \subset L$. Clearly, $\text{im}(F)$ separates I^L between v_+ and $A(L)$ if and only if it separates it between U and $A(L)$ and thus if and only if $\text{im}(ghF)$ separates $\Delta(L) \times [0, 1]$ between $\Delta(L) \times \{0\}$ and $\Delta(L) \times \{1\}$. The result now follows from Lemma 4 applied to the correspondence ghF . \square

Remark 3. Taking levelwise convexification produces a correspondence with property a), while preserving properties b) and c). In turn, the saturation F^+ has properties b) and c) whenever $\text{im}(F) \subset \delta I^L$.

Lemma 5. *Let $F : \Delta(L) \rightarrow I^L$ be a saturated correspondence having property \mathcal{S} for $\Delta(L)$. Then, this property is witnessed by $F \cap (\delta\Delta(L) \times \delta I^L)$.*

Proof. Let us write Δ for $\Delta(L)$. It suffices to prove that the homomorphism $\tilde{H}(j)$ induced by the inclusion $j : F \cap (\delta\Delta \times \delta I^L) \hookrightarrow F|_{\delta\Delta}$ is surjective. To this end write

$$\begin{aligned} Z &:= \{(p, x_1, x_2) : p \in \delta\Delta, x_1 \in F(p), x_2 \in x_1^+(p)\}, \\ Z_0 &:= \{(p, x_1, x_2) \in Z : x_2 \in \delta I^L\} \end{aligned}$$

where by $x_1^+(p)$ we denote the value at p of the saturation of the constant correspondence taking value $\{x_1\}$. Also, let $\pi_i : Z_0 \rightarrow F|_{\delta\Delta}$ be defined by $\pi_i(p, x_1, x_2) = (p, x_i)$ for $i = 1, 2$. Since $x_1^+(p) \subset F(p)$ for $x_1 \in F(p)$ and the set $x_1^+(p)$ is convex, the map π_2 does indeed take values in $F|_{\delta\Delta}$, as does the homotopy between π_1 and π_2 defined by $h_t(p, x_1, x_2) = (p, tx_1 + (1-t)x_2)$.

Moreover, the map π_1 is acyclic and surjective, since each cone $x_1^+(p)$ intersects δI^L along an acyclic set. By Vietoris’ theorem, $\tilde{H}(\pi_1)$ is an isomorphism. Since $\text{im}(\tilde{H}(\pi_1)) = \text{im}(\tilde{H}(\pi_2)) \subset \text{im}(\tilde{H}(j))$, the conclusion follows. \square

The following property of separators enables one to apply the established results:

Lemma 6. *For $i = 1, \dots, n$, let B_i, S_i, U_i be closed subsets of a cube X such that S_i separates X between B_i and $X \setminus U_i$. Then $\bigcup_{i=1}^n S_i \cap \bigcap_{i=1}^n U_i$ separates X between $\bigcap_{i=1}^n B_i$ and $X \setminus \bigcap_{i=1}^n U_i$.*

Proof. By induction on n , it suffices to prove the result for $n = 2$. Let V_i denote the union of those components of $X \setminus S_i$ that intersect B_i . Then $B_i \subset V_i \subset U_i$ and $\delta V_i \subset S_i$. Hence the neighborhood $V := V_1 \cap V_2$ of $B_1 \cap B_2$ misses $X \setminus U_1 \cap U_2$ and $\delta V \subset (\delta V_1 \cap \overline{V_2}) \cup (\overline{V_1} \cap \delta V_2) \subset (S_1 \cap U_2) \cup (U_1 \cap S_2) = (S_1 \cup S_2) \cap U_1 \cap U_2$. \square

Proof of Theorem 2. We temporarily fix an integer n and denote by J the segment $[a - \frac{1}{n}, b + \frac{1}{n}]$. For $L \in \mathcal{L}$ and $p \in \Delta(L) \subset \Delta(K)$, let $F'_L(p) = \{y \in J^K : y^L \in c(F_L^+)(p)\}$, where $+$ denotes saturating a correspondence with respect to J^L . The correspondence $c(F_L^+) : \Delta(L) \rightarrow J^L$ satisfies all the conditions a), b) and c) from Corollary 3 and $c(F_L^+) \cap (\delta\Delta(L) \times \delta J^L)$ witnesses the property \mathcal{S} of $c(F_L^+)$ for

$\Delta(L)$ (since by Lemma 5 this is so with $c(F_L^+)$ replaced by its subset F_L^+). With $v_+ = (b + \frac{1}{n}, \dots, b + \frac{1}{n})$ it therefore follows easily from Corollary 3 that

(6) the set $\text{im}(F'_L)$ separates J^K between v_+

$$\text{and } A'(L) := \{y \in J^K : y^L \in A(L)\}.$$

Next, for $L \in \mathcal{L}$ we denote by U'_L the convex hull of $\text{im}(F'_L) \cup U_L \cup \{v_+\}$. Then the set $I^K \setminus U'_L$ is connected and contains $A'(L)$, while being disjoint from $\{v_+\} \cup \text{im}(F'_L)$. It follows thus from (6) that $\text{im}(F'_L)$ separates v_+ from $J^K \setminus U'_L$, and from Lemma 6 that the set

$$S := \bigcup_{L \in \mathcal{L}} \text{im}(F'_L) \cap \bigcap_{L \in \mathcal{L}} U'_L$$

separates v_+ from $J^K \setminus \bigcap_{L \in \mathcal{L}} U'_L$ and hence also from $A := \bigcup_{L \in \mathcal{L}} A'(L)$ (all separations in J^K).

Since $\bigcup \mathcal{L} = K$ we have $A = A(K)$ and consequently $A(K)$ is separated from v_+ by S .

Let G be the correspondence $\bigcup_{L \in \mathcal{L}} F'_L \cap \bigcap_{L \in \mathcal{L}} \Delta(K) \times U'_L$. Then $\text{im}(cG) \supset \text{im}(G) = S$ and cG satisfies the assumptions of Corollary 3, with K in place of L . Thus, applying this corollary again, we infer that cG has property \mathcal{S} for $\Delta(K)$.

Finally, we allow n to vary and thus for each n produce a correspondence $G_n : \Delta(K) \rightarrow \mathbb{R}^K$ such that $c(G_n)$ has property \mathcal{S} for $\Delta(K)$ and $\Gamma = \bigcap_n G_n$. Then, $c\Gamma = \bigcap_n cG_n$ and it remains to apply Lemma 2(b). \square

The above reasoning also establishes the following (we skip the details):

Proposition 1. *Let a closed segment I be contained in the interior of another closed segment $J = [a, b]$, and let $F : \Delta(K) \rightarrow I^K$ be a saturated correspondence such that the set $F^{-1}(y)$ is acyclic for every $y \in \text{im}(F)$. Then, the correspondence F has property \mathcal{S} for $\Delta(K)$ iff the image of its J^K -saturation separates the cube J^K between the vertex $v_+ = (b, \dots, b)$ and the union of all the faces of J^K disjoint from v_+ .*

3. GAMES OF INCOMPLETE INFORMATION ON ONE SIDE

3.1. The model. A brief description of the games follows. There are two players, Player One and Player Two. There is a finite set K of states of nature. Nature chooses a state $k \in K$ according to a commonly known probability $p_0 \in \Delta(K)$, and the first player, but not the second player, is informed concerning nature's choice. The first player and second player have sets of moves I and J , respectively. The play is repeated indefinitely, with the chosen state remaining constant.

Two independent aspects of the play need explanation, the payoffs and the information structure.

Let A^k and B^k for all $k \in K$ be the corresponding $|I| \times |J|$ payoff matrices for Players One and Two, respectively. At each stage of play, $A^k(i, j)$ and $B^k(i, j)$ are the payoffs to Player One and Player Two, respectively, when Player One chooses $i \in I$, Player Two chooses $j \in J$ and the state of nature is $k \in K$. We assume, without loss of generality, that all payoffs for Player One are positive, meaning that every entry $A^k(i, j)$ will be a positive number.

There are two sets of finite signals R and S , received by the first and second players, respectively. There is a stochastic function, $\Lambda : K \times I \times J \rightarrow \Delta(R \times S)$.

After each stage in which $i \in I$ and $j \in J$ were played and k is the state of nature, a member (r, s) of $R \times S$ is determined by $\Lambda(k, i, j)$; r is communicated to Player One, and s is communicated to Player Two.

A **behavior strategy** of Player One is an infinite sequence $\alpha = (\alpha_1, \alpha_2, \dots)$ such that for each l , α_l is a mapping from $K \times (I \times R)^{l-1}$ to $\Delta(I)$. A behavior strategy of Player Two is an infinite sequence $\beta = (\beta_1, \beta_2, \dots)$ such that for each l , β_l is a mapping from $(J \times S)^{l-1}$ to $\Delta(I)$. (The inclusion of I^{l-1} and J^{l-1} in the definition of behavior strategies for players One and Two, respectively, reflects the important assumption, used later, of perfect recall of one's own past moves.) Let \mathcal{I} and \mathcal{J} be the set of behavior strategies of Players One and Two, respectively.

Define the set of finite play histories of length l to be $\mathcal{H}_l := K \times (I \times J)^l$, and define \mathcal{H}_l^k to be the subset $\{k\} \times (I \times J)^l$.

For every $k \in K$ and $h = (k, i_1, j_1, \dots, i_l, j_l) \in \mathcal{H}_l^k \subset \mathcal{H}_l$, define

$$f_l(h) = \frac{1}{l} \sum_{m=1}^l A^k(i_m, j_m) \quad \text{and} \quad g_l(h) = \frac{1}{l} \sum_{m=1}^l B^k(i_m, j_m).$$

For any fixed $k \in K$, every pair of behavior strategies $\alpha \in \mathcal{I}$ and $\beta \in \mathcal{J}$ induces a probability measure $\mu_{\alpha, \beta}^{l, k}$ on \mathcal{H}_l^k . With the initial probability p_0 , such a pair induces also a probability measure $\mu_{\alpha, \beta}^l$ on \mathcal{H}_l . A **uniform (Nash) equilibrium** (see Renault [Re], also [AM], [So] and [MSZ]) is a pair of behavior strategies $\alpha \in \mathcal{I}$ and $\beta \in \mathcal{J}$ such that for every $k \in K$, $a^k = \lim_{m \rightarrow \infty} \int_{\mathcal{H}_m^k} f_m(h) d\mu_{\alpha, \beta}^{m, k}$ and $b^k = \lim_{m \rightarrow \infty} \int_{\mathcal{H}_m^k} g_m(h) d\mu_{\alpha, \beta}^{m, k}$ exist and, for every pair $\alpha^* \in \mathcal{I}$ and $\beta^* \in \mathcal{J}$,

$$\lim_{m \rightarrow \infty} \sup \int_{\mathcal{H}_m} f_m(h) d\mu_{\alpha^*, \beta}^m \leq \sum_k p_0^k a^k, \quad \lim_{m \rightarrow \infty} \sup \int_{\mathcal{H}_m} g_m(h) d\mu_{\alpha, \beta^*}^m \leq \sum_k p_0^k b^k.$$

3.2. Nonrevealing strategies. For every $j \in J$, let $\Lambda^j : (K \times I) \rightarrow \Delta(S)$ be defined so that $\Lambda^j(k, i)$ is the marginal probability on the signals S determined by the moves $i \in I$, $j \in J$ and the state $k \in K$. Define $\bar{\Lambda}^j : \Delta(I)^K \rightarrow \Delta(S)^K$ so that for every $\sigma \in \Delta(I)^K$ with $\sigma = (\sigma^k \mid k \in K)$, $\bar{\Lambda}^j(\sigma)$ is the $|K|$ -tuple of probabilities on S determined by $\bar{\Lambda}^j(\sigma)^k := \sum_{i \in I} \sigma_i^k \Lambda^j(k, i)$. For every $L \subset K$, define

$$NR(L) := \{\sigma \in \Delta(I)^K \mid \forall j, \text{ for every pair } k, k' \in L, \bar{\Lambda}^j(\sigma)^k = \bar{\Lambda}^j(\sigma)^{k'}\}.$$

By the linearity of the functions $\bar{\Lambda}^j$, for every L , as long as it is nonempty, $NR(L)$ is a convex polytope.

For every $p \in \Delta(K)$, define $\text{supp}(p) := \{k \in K \mid p^k > 0\} \subset K$.

For every $\sigma \in \Delta(I)^K$ and $\tau \in \Delta(J)$, define $\sigma A \tau \in \mathbb{R}^K$ (respectively $\sigma B \tau \in \mathbb{R}^K$) as $(\sigma A \tau)^k := \sigma^k A^k \tau$ (respectively $(\sigma B \tau)^k := \sigma^k B^k \tau$).

Define the function a^* on $\Delta(K)$ by

$$a^*(p) = \max_{\sigma \in NR(\text{supp}(p))} \min_{\tau \in \Delta(J)} p \cdot \sigma A \tau = \min_{\tau \in \Delta(J)} \max_{\sigma \in NR(\text{supp}(p))} p \cdot \sigma A \tau,$$

where we set $a(p) = -\infty$ if $NR(\text{supp}(p)) = \emptyset$.

The function a^* is important because the value of the function $\text{cav}(a^*)$ at $p \in \Delta(K)$ is the value of an infinitely repeated zero-sum game where the payoff matrices of the second player satisfy $B^k(i, j) = -A^k(i, j)$ for every i, j , and k ; see Aumann and Maschler [AM], Theorem C, p. 191 and Mertens, Sorin, and Zamir [MSZ], Theorem 3.5, p. 231. More particularly, for any $x \in \mathbb{R}^K$ such that $x \cdot q \geq a^*(q)$ for

all $q \in \Delta(K)$, there is a behavior strategy $\beta \in \mathcal{J}$ for the second player such that for every behavior strategy $\alpha \in \mathcal{I}$ for the first player and for every $k \in K$,

$$\lim_{m \rightarrow \infty} \sup \int_{\mathcal{H}_m^k} f_m^k(h) d\mu_{\alpha, \beta}^{m, k} \leq x^k.$$

This result, used critically by Renault in the construction of equilibrium behavior strategies, was proven first by E. Kohlberg [Ko] in the context of a deterministic but not necessarily state-independent Λ^j , and in our context by Mertens, Sorin, and Zamir, [MSZ], Section 3d, pp. 234–240.

For every $L \subset K$, define $a_L^* : \Delta(L) \rightarrow \mathbb{R}$ by

$$a_L^*(p) = \max_{\sigma \in NR(L)} \min_{\tau \in \Delta(J)} p \cdot \sigma A \tau = \min_{\tau \in \Delta(J)} \max_{\sigma \in NR(L)} p \cdot \sigma A \tau.$$

Clearly, $a^* = \max_{L \subset K} a_L^*$ and whenever $NR(L)$ is not empty, then a_L^* is Lipschitz.

Define a subset $L \subset K$ to be **extendable** to $N \supset L$ if $NR(L) \neq \emptyset$ and every $\sigma \in NR(L)$ can be replaced by a $\bar{\sigma} \in NR(N)$ such that $\sigma^k = \bar{\sigma}^k$ for every $k \in L$. Define a nonempty subset $L \subset K$ to be **nonextendable** if there exists no properly containing $N \supset L$ with L extendable to N . Define \mathcal{F} to be the family of non-extendable subsets of K . Since $NR(\{k\})$ is always all of $\Delta(I)^K$ for any $k \in K$ and an $L \subset K$ maximal with the property $NR(L) \neq \emptyset$ will be in \mathcal{F} , we know that $\bigcup_{L \in \mathcal{F}} L = K$.

We define an $L \in \mathcal{F}$ to be **singular** if for all $j \in J$, $\bar{\Lambda}^j(NR(L))$ is a singleton. We say that Player One **can send nonrevealing signals** if $K \in \mathcal{F}$ and K is not singular. We will prove the existence of equilibria with a weaker assumption, namely, that every member of $\max \mathcal{F}$ is not singular, which is equivalent to the nonsingularity of all members of \mathcal{F} , since only maximal members of \mathcal{F} can be singular.

For any nonempty $N \subset K$, define $F(N) := \bigcap_{N \subset L \in \mathcal{F}} L$ for all $N \subset K$, with $F(N) = K$ in the case of an empty intersection.

Lemma 7. *\mathcal{F} is closed under nonempty intersection and, for all nonempty $N \subset K$ with $NR(N) \neq \emptyset$, $F(N) = \{k \mid \forall \sigma \in NR(N) \text{ there exists a } \hat{\sigma} \in NR(N \cup \{k\}) \text{ such that for every } n \in N, \sigma^n = \hat{\sigma}^n\}$.*

Proof. Let us assume that L_1 and L_2 are in \mathcal{F} . Suppose for the sake of contradiction that $L_1 \cap L_2$ can be extended to $N \supset L_1 \cap L_2$. By symmetry, assume $N \not\subset L_1$. Since L_1 is not extendable, there is a $j \in J$ and a strategy $\sigma \in NR(L_1)$ such that there is no $\bar{\sigma} \in NR(N \cup L_1)$ with $\bar{\sigma}^k = \sigma^k$ for every $k \in L_1$. Notice that $NR(L_1) \subset NR(L_1 \cap L_2)$. If there were a $\hat{\sigma} \in NR(N)$ with $\hat{\sigma}^k = \sigma^k$ for every $k \in L_1 \cap L_2$, consider $\bar{\sigma}$ defined by $\bar{\sigma}^k := \sigma^k$ if $k \in L_1$ and $\bar{\sigma}^k := \hat{\sigma}^k$ if $k \notin L_1$. For any $j \in J$, consider $r^j \in \Delta(S)$ to be $\bar{\Lambda}^j(\hat{\sigma})^k$ for all $k \in N$. Since $\emptyset \neq L_1 \cap L_2 \subset N$, r^j is $\bar{\Lambda}^j(\sigma)^k$ or $\bar{\Lambda}^j(\hat{\sigma})^k$ for all $k \in L_1 \cap L_2$, r^j is $\bar{\Lambda}^j(\hat{\sigma})^k$ for $k \in N$, and r^j is $\bar{\Lambda}^j(\sigma)^k$ for $k \in L_1$. Therefore r^j is also $\bar{\Lambda}^j(\bar{\sigma})^k$ for all $k \in L_1 \cup N$, a contradiction.

The set $Q := \{k \mid \forall \sigma \in NR(N) \text{ there exists a } \hat{\sigma} \in NR(N \cup \{k\}) \text{ such that for every } n \in N, \sigma^n = \hat{\sigma}^n\}$ is nonextendable, since if Q were extendable to some Q' strictly containing Q , then choosing any $k \in Q' \setminus Q$ we would have to conclude that k was already in Q , a contradiction. Furthermore, if P is a set between N and Q , $NR(P) \subset NR(N)$ would imply that P is extendable to Q . So from the above we conclude that $Q = F(N)$. \square

Lemma 8. $a^* = \max_{L \in \mathcal{F}} (a_L^*)$ holds.

Proof. If $NR(L) \neq \emptyset$, then Lemma 7 shows that L is extendable to $F(L)$, meaning that any strategy in $NR(L)$ can be replaced by an equivalent one in $NR(F(L))$. This means that a_L^* is not greater than $a_{F(L)}^*$ on $\Delta(L)$. \square

3.3. Joint plans. Usually, a joint plan is defined to be a kind of contract between the players concerning their behavior. Due to the complexity of the contractual behavior in this paper needed to establish an equilibrium, we will define a joint plan to be only the underlying probability and payoff structure.

The main component of a joint plan is what we define to be a “proto-joint-plan” for some $p \in \Delta(K)$. A proto-joint-plan is a pair of strategies in $\Delta(I)^K$ and $\Delta(J)$ defining the normal behavior of the players at some subset of the stages of play. A “nonrevealing joint plan” for some $p \in \Delta(K)$ is a mixture of proto-joint-plans for the same $p \in \Delta(K)$, and defines the average payoffs of the players in the limit of play. A “joint plan” for some $p \in \Delta(K)$ is a set $V \subset \Delta(K)$ whose convex hull contains p along with a nonrevealing joint plan for every $p \in V$.

For every $p \in \Delta(K)$, define $F(p) := F(\text{supp}(p))$.

For every $L \subset K$, define the equivalence relation \sim_L on $NR(L)$ by $\sigma \sim_L \hat{\sigma}$ if and only if $\bar{\Lambda}^j(\sigma)^k = \bar{\Lambda}^j(\hat{\sigma})^k$ for all $j \in J$ and $k \in L$. A **proto-joint-plan** for some $p \in \Delta(K)$ is a pair $\sigma \in \Delta(I)^K$, $\tau \in \Delta(J)$ such that $\sigma \in NR(F(p))$ and for every $k \in F(p)$,

$$\sigma^k A^k \tau = \max_{\bar{\sigma} \sim_{F(p)} \sigma} \bar{\sigma}^k A^k \tau.$$

The \mathbb{R}^K **Player One payoff** corresponding to the proto-joint-plan is the vector x in \mathbb{R}^K defined by $x^k = \sigma^k A^k \tau$. A proto-joint-plan is **two-best-reply** (see [Re]) if $p \cdot \sigma B \tau = \max_{\bar{\tau} \in \Delta(J)} p \cdot \sigma B \bar{\tau}$. For any $L \supset F(p)$, an **L -pure proto-joint-plan** for p is a pair $\sigma \in NR(L)$ and $\tau \in \Delta(J)$ such that $\sigma^k A^k \tau = \max_{\bar{\sigma} \sim_L \sigma} \bar{\sigma}^k A^k \tau$ for every $k \in L$.

Remark 4. An L -pure proto-joint-plan is also a proto-joint-plan. The fact that $NR(L) \subset NR(F(p))$ follows from $F(p) \subset L$. Now we need to show that the payoff maximality condition in the definition of L -purity implies the payoff maximality condition in the definition of a proto-joint-plan. Assuming that $\sigma \in NR(L) \subset NR(F(p))$ and σ satisfies $\sigma^k A^k \tau = \max_{\bar{\sigma} \sim_L \sigma} \bar{\sigma}^k A^k \tau$ for every $k \in L$, let $\hat{\sigma}$ be any other member of $NR(F(p))$ with $\hat{\sigma} \sim_{F(p)} \sigma$. We can define, as in the proof of Lemma 7, a $\bar{\sigma} \in NR(L)$ by $\bar{\sigma}^k = \sigma^k$ if $k \notin F(p)$ and $\bar{\sigma}^k = \hat{\sigma}^k$ if $k \in F(p)$, with $\bar{\sigma}^k A^k \tau = \hat{\sigma}^k A^k \tau$ if $k \in F(p)$ and $\bar{\sigma} \sim_L \sigma$.

A **nonrevealing joint plan** for $p \in \Delta(K)$ is a finite convex combination of proto-joint-plans for the same p . It is **two-best-reply** if every proto-joint-plan in the convex combination is two-best-reply. Given that $y \in \mathbb{R}^K$ is the appropriate convex combination of the Player One payoffs from proto-joint-plans corresponding to the nonrevealing joint plan, a **Player One payoff** of the nonrevealing joint plan is any $\bar{y} \in [0, \mathbb{R}]^K$ such that $\bar{y}^k = y^k$ if $p^k > 0$ and $\bar{y}^k \geq y^k$ if $k \in F(p)$. Notice that the payoff of a proto-joint-plan for $p \in \Delta(K)$ is defined with respect to all of K , while the payoffs of a nonrevealing joint plan are defined essentially with respect to $F(p)$ and allow for increasing the values for coordinates outside of $\text{supp}(p)$. The reason for the more restrictive definition for proto-joint-plans is to accommodate the property of purity when dealing with an $L \in \mathcal{F}$ properly containing $F(p)$.

A **joint plan** for $p \in \Delta(K)$ is

- 1) a finite set $V \subset \Delta(K)$ such that $p \in \text{conv}(V)$,
- 2) for every singular $L \in \max \mathcal{F}$ at most one member v of V with $F(v) = L$,
- 3) for every $v \in V$ an associated nonrevealing joint plan for v .

A joint plan is **two-best-reply** if every nonrevealing joint plan associated with the joint plan is two-best-reply. The joint plan for p is **balanced** if there exists an $x \in \mathbb{R}^K$ such that for every $v \in V$, x is a Player One payoff of the nonrevealing joint plan associated with v , and such an $x \in \mathbb{R}^K$ is called a **Player One payoff** for the balanced joint plan. Furthermore, if $N \supset F(p)$ and an $x \in \mathbb{R}^K$ is a Player One payoff for the balanced joint plan, then we will also call x^N a Player One payoff for the balanced joint plan.

Remark 5. Originally for the case of standard information, a “joint plan” was defined using members of $\Delta(I \times J)$, [AM]. Because every member of $\Delta(I \times J)$ can be represented as a convex combination of members of $\Delta(I) \times \Delta(J)$, our definition in the context of the standard information case does not differ essentially from the original definition.

We fix a number M such that $M > \max_{i,j,k} A^k(i, j)$ and, for every $L \in \mathcal{F}$ and $p \in \Delta(L)$, we let the correspondence $\Psi_L : \Delta(L) \rightarrow [0, M]^L$ be defined by $\Psi_L(p)$ being the \mathbb{R}^L coordinates of the Player One payoffs of two-best-reply proto-joint-plans that are L -pure. Renault [Re] proved the following lemma, although only explicitly in the case of state-independent signalling. For the sake of completeness, we repeat his proof.

Lemma 9 (J. Renault [Re]). *For every $L \in \mathcal{F}$, $\Psi_L : \Delta(L) \rightarrow [0, M]^L$ is an upper-semi-continuous correspondence with the property that, for every $p, q \in \Delta(L)$, there is an $x \in \Psi_L(p)$ such that $x \cdot q \geq a_L^*(q)$.*

Proof. First we show that Ψ_L satisfies the latter property. Given $p, q \in \Delta(L)$, let $\sigma \in NR(L)$ and $\tau \in \Delta(J)$ be a Nash equilibrium in the one-shot game defined by the convex polytopal strategy sets $NR(L)$ and $\Delta(J)$ where σ and τ give a payoff of $q \cdot \sigma A \tau$ to the first player and $p \cdot \sigma B \tau$ to the second player. Because a_L^* is defined using $NR(L)$ and $\Delta(J)$, such a Nash equilibrium attains a payoff of at least $a_L^*(q)$ for Player One. By the Nash equilibrium property, for any $k \in \text{supp}(q)$, $\sigma^k A^k \tau$ cannot be improved upon by another $\hat{\sigma}$ with $\hat{\sigma} \sim_L \sigma$. If $k \notin \text{supp}(q)$, then we choose σ^k so as to attain the maximum of $\{\bar{\sigma}^k A^k \tau \mid \bar{\sigma} \sim_L \sigma\}$. We can make this choice freely without upsetting the Nash equilibrium property because $k \notin \text{supp}(q)$. Likewise, the Nash equilibrium property gives the two-best-reply property.

We assume there are sequences $\sigma_l \rightarrow \sigma$, $\tau_l \rightarrow \tau$, and $p_l \rightarrow p$ such that for every l , σ_l and τ_l are two-best-reply proto-joint-plans for p_l that are L -pure. It suffices to show that σ and τ is a two-best-reply proto-joint-plan for p that is L -pure. Showing the two-best-reply property and that σ is in $NR(L)$ are straightforward. It is left to show that if there were a $\hat{\sigma} \in NR(L)$ with $\sigma \sim_L \hat{\sigma}$ for some $k \in L$ and $\hat{\sigma}^k A^k \tau > \sigma^k A^k \tau$, then the same would occur for some σ_l^k and τ_l close enough to σ^k and τ . We consider the directional vector $w := \hat{\sigma}^k - \sigma^k$. Any fraction $0 < t < 1$ of w added to the σ_l^k would generate for large enough l the same distribution on signals for every $j \in J$ and also yield an increase in payoff to the first player at the state k . (A fraction $0 < p < 1$ must be used to avoid having $\sigma_l^k + tw$ outside of $\Delta(I)$ for all l .) \square

For $N \supset L$, a vector $x \in \mathbb{R}^N$ **dominates** a function $f : \Delta(L) \rightarrow \mathbb{R}$ if $x|_{\Delta(L)} \geq f$ or, equivalently, if $x \cdot q \geq f(q)$ for all $q \in \Delta(L)$.

From Lemma 9, Renault [Re] proved essentially something more, namely that for every p in the interior of $\Delta(L)$, there exists a balanced two-best-reply joint plan with a payoff vector for Player One that dominates a_L^* .

Theorem 3. *Given that every member of $\max \mathcal{F}$ is nonsingular, for every $p_0 \in \Delta(K)$ there exists a balanced two-best-reply joint plan such that a payoff vector of Player One dominates a^* .*

Proof. For every $L \in \mathcal{F}$, define $\tilde{\Psi}_L : \Delta(L) \rightarrow [0, M]^L$ by $\tilde{\Psi}_L(p) := \text{conv}(\Psi_L(p))$. From Carathéodory's theorem and Lemma 9, we know that $\tilde{\Psi}_L$ is upper-semi-continuous and, also by Lemma 9, it has the property that for every $p, q \in \Delta(L)$, there is an $x \in \tilde{\Psi}_L(p)$ such that $x \cdot q \geq a_L^*(q)$. Let $\tilde{\Psi}_L^+$ be the $[0, M]^L$ saturation of $\tilde{\Psi}_L$; $\tilde{\Psi}_L^+(p)$ is a subset of the two-best-reply nonrevealing joint plan payoffs in $[0, M]^L$. For every $L \in \mathcal{F}$ define $F_L : \Delta(L) \rightarrow [0, M]^L$ by $F_L := c(\tilde{\Psi}_L^+) \cap (\Delta(L) \times \{y \in [0, M]^L : y|_{\Delta(L)} \geq a_L^*\})$. From Theorem 1 it follows that F_L has property \mathcal{S} . By the fact that all members of \mathcal{F} are nonsingular, and therefore there is no restriction on the forming of convex combinations in $\Delta(L)$ to generate joint plans, we have for every $p \in \Delta(L)$ that $F_L(p)$ is a subset of the \mathbb{R}^L vectors that dominate a_L^* and are Player One payoffs of balanced two-best-reply joint plans for p .

Set $U_L := \{y \in [0, M]^K : y^L|_{\Delta(L)} \geq a_L^*\}$. We know from Corollary 2 that for every $p_0 \in \Delta(K)$, we have a finite number of sets $L_1, \dots, L_s \in \mathcal{F}$, points $p_i \in \Delta(L_i) \subset \Delta(K)$ for $1 \leq i \leq s$, and positive numbers t_1, \dots, t_s with $\sum_{i=1}^s t_i = 1$, $\sum_i t_i p_i = p_0$ and a point $x \in [0, M]^K$ dominating a_L^* for all $L \in \mathcal{F}$ such that $x^{L_i} \in F_{L_i}(p_i)$ for every $i = 1, \dots, s$.

For the conclusion of Theorem 3, it suffices to show that such an $x \in \mathbb{R}^K$ dominates a^* and is a Player One payoff of a balanced two-best-reply joint plan for p_0 .

By Lemma 8, $x \in \mathbb{R}^K$ dominates a^* . For every $1 \leq i \leq s$, there is a finite set $V_i \subset \Delta(L_i)$ such that p_i is in the convex hull of V_i and for every $p \in V_i$, there is a two-best-reply nonrevealing joint plan for p with a payoff of x for the first player. Consider the union $\bigcup_i V_i$; its convex hull contains p_0 . The fact that no member of \mathcal{F} is singular allows the freedom to take any convex combination in $\Delta(K)$ to form a joint plan, and therefore these two-best-reply nonrevealing joint plans giving the same payoff of $x \in \mathbb{R}^K$ to the first player satisfy all the conditions to define a two-best-reply balanced joint plan for p_0 . \square

3.4. From joint plan to equilibrium. In this section we show that a two-best-reply balanced joint plan with a payoff dominating a^* generates a uniform equilibrium of the game. We present a summary of the construction of strategies from such joint plans in the case of a state-independent information structure, done by Renault [Re], and then show that a minimal revision of Renault's construction generates uniform equilibrium strategies in our context of state-dependent information.

Let us first index the joint plan by the finite set $V \subset \Delta(K)$ of probabilities corresponding to the nonrevealing joint plans, such that the initial probability p_0 is in the convex hull of V . For every $v \in V$, we have another index set T_v for the proto-joint-plans. Let $q^{v,t}$ be the proportion given to the proto-joint-plan indexed by $t \in T_v$, with $\sigma_{v,t} \in NR(F(v))$ and $\tau_{v,t} \in \Delta(J)$ the corresponding one-shot mixed strategies. For both index sets, by Carathéodory's theorem, we can assume a finite number of members, no more than $|K|$ for V and no more than $|I||K| + |J|$ for every T_v . We assume that $x \in \mathbb{R}^K$ is the payoff of the joint plan. The pair of

behavior strategies that Renault constructs we will call the **suggested** behavior strategies. Renault's suggested behavior strategies will incorporate four aspects:

- 1) a process by which Player One chooses by lottery a nonrevealing joint plan $v \in V$ and sends to Player Two a noisy signal with the intention of communicating the chosen v , and an interpretation of a $v' \in V$ by Player Two;
- 2) a punishment strategy by the second player against the first player, holding him down in payoff to x^k or less for every k ;
- 3) playing phases, where play should be determined by the chosen nonrevealing joint plan, with proto-joint-plans assigned to subsequences of the playing stages;
- 4) a mechanism for the second player to monitor the actions of the first player in the playing phases and to determine when to replace his adherence to the interpreted nonrevealing joint plan by punishment in some subsequent playing phases.

For our purposes, we need only to add a fifth aspect to Renault's scheme:

- 5) a mechanism for every $L \in \mathcal{F}$, given that the chosen state of nature is in L , for the first player to confirm (in a way nonrevealing with respect to L) to the second player that indeed the state of nature is in L .

The two-best-reply property makes unnecessary a process by which Player One monitors and punishes Player Two for deviation.

Now we describe Renault's approach in more detail. The game is broken into blocks of stages for signaling and blocks of stages for playing, with alternation between signaling and playing.

In the signaling blocks, C_1, C_2, \dots , Player One attempts to send a message that means that they should play a nonrevealing joint plan associated to one of the members of V . In Renault's context of state-independence, there is only one member in \mathcal{F} , namely K , so that $a^* = a_K^*$. If K is singular, then a^* is concave, and equilibrium existence follows easily. Therefore, Renault could assume that K was nonsingular and that there is at least a line segment $\overline{a, b}$ in $\overline{\Lambda}^j(NR(K))$ for some $j \in J$. Let m be large enough such that 2^m is at least as large as the cardinality of V , so that there is a surjective map from $\{a, b\}^m$ to V . We will refer to a member of $\{a, b\}^m$ as a **noisy signal**, because it is a signal that must be interpreted by Player Two; due to stochastic noise, the signal could be interpreted incorrectly. Player One chooses a member of V according to a state-dependent lottery whose outcome of $v \in V$ means exactly that the signal in $\{a, b\}^m$ corresponding to v implies that the posterior probability of $k \in K$ is v^k . (We can generate the lottery through mixtures of behavior strategies, but with the property of perfect recall concerning one's own moves, by Kuhn's theorem it can also be done with behavior strategies; see [Re] and [SST].) The signaling blocks incorporate m sub-blocks of approximately equal length. The purpose of the signaling blocks is to give Player One the opportunity to communicate a member of $\{a, b\}^m$, hence also any member of V . Player Two interprets that a was intended in the l -th coordinate of $\{a, b\}^m$ if the average of the signals in S received in the subsequences corresponding to l is closer to a than to b , and otherwise he interprets that b was intended. Because the length of the blocks C_l grows faster than l^2 , given that Player One is consistent concerning which member in V is to be communicated, with probability one Player Two will receive this message unambiguously. Let y_v be the nonrevealing joint payoff associated with v . From the equality $y_v^k = x^k$ if $v^k > 0$ and the

inequality $y_v^k \leq x^k$ otherwise, we see that the first player is indifferent to which member of V is communicated; (in the state-independent context, K is the only nonextendable set).

The playing blocks alternate with the signaling blocks, but the ratio in length of signaling blocks to playing blocks will approach zero, so that in the limit the signaling blocks are not significant to the payoff. We have for each $v \in V$ a system of mutually disjoint sub-blocks $\{B_l^{v,t} \mid t \in T_v\}$ corresponding to the proto-joint-plans, such that $\bigcup B_l^{v,t} = B_l$ and the ratio of $|B_l^{v,t}|/B_l$ converges to $q^{v,t}$. With the suggested behavior strategies, if v was chosen by the first player, the first player should play the mixed strategy $\sigma_{v,t} \in \Delta(I)^K$ in the blocks $B_l^{v,t}$. If $v' \in V$ is the message interpreted by the second player and the second player is not currently involved in the process of punishing the first player, he should play $(1-1/l)\tau_{v',t} + e/l$ in these same blocks, where $e \in \Delta(J)$ is the even distribution on all moves. The even distribution on all moves of Player Two at a rate of $1/l$ in the l -th playing block is a mechanism to randomly check the behavior of the first player, and to do so in a way that does not influence directly the limit average payoff of the game. For each sub-block $B_l^{v,t}$ and each $j \in J$ played, the second player compares the observed average distribution of signals in S with the expected distribution $\bar{\Lambda}^j(\sigma^{v,t})$. If this difference exceeds $1/l$, then Player Two punishes Player One in sufficiently many following playing blocks so that the length of the punishment dominates the past playing blocks. Player Two does not punish in the signaling blocks, since it is possible that Player Two simply heard the wrong signal and in this case he must give Player One a chance to correct this misinterpretation.

Given that there is a finite stage such that beyond this stage the signals in $\{a, b\}^m$ sent and interpreted are the same and all punishment has terminated, we expect with probability one that the suggested behavior strategies will generate a limit average payoff of x to the first player. Given adherence to the suggested behavior strategies, we can expect successful communication of the chosen $v \in V$ with probability one. Furthermore, if Player One adheres to his suggested behavior strategy, Player Two deviations are not profitable due to the two-best-reply condition. To show that the suggested behavior strategies are in equilibrium, Renault showed that

- 1) with probability one the suggested behavior strategies result in termination of punishment by the second player, and
- 2) deviations by the first player do not yield to him any average payoff advantage.

This second part is the most involved part of Renault's proof. Because of the maximality condition on Player One's payoff in the definition of a proto-joint-plan, a deviant behavior strategy that would give Player One a higher average payoff must fail to yield the appropriate distribution on the signals in S for those random stages where Player Two is checking. Renault showed that the punishments are sufficiently long and the observational powers of the second player are sufficiently sharp that, in the limit, the first player cannot take any advantage from deviations in the playing blocks that he does not subsequently lose by provoking repetitive punishment.

Our addition. The only addition to Renault's definition of the suggested behavior strategies is in the signaling blocks. We need to add a process of **confirmation** in

which the first player guarantees to the second player with probability one that the state of nature is indeed a member of $F(v)$ for the chosen $v \in V$.

Define the set $A_j^k := \{(\bar{\Lambda}^j(\sigma))^k \mid \sigma \in \Delta(I)^K\} \subset \Delta(S)$. We recall from the definition of a nonextendable set $L \subset K$ that for every $k \notin L$ there is a $j_L^k \in J$ and a strategy $\sigma_{L,k} \in NR(L)$ such that the distribution $\chi_L^k \in \Delta(S)$ defined by $\chi_L^k := \bar{\Lambda}^{j_L^k}(\sigma_{L,k})^{k'} \in \Delta(S)$ for all $k' \in L$ is outside of the set $A_{j_L^k}^k$. Let $\xi^k := \min_{k \notin L \in \mathcal{F}}(\text{distance } \chi_L^k, A_{j_L^k}^k)$. We adjoin to the signaling block, C_l , $|J||K|$ sub-blocks, $\tilde{C}_l^{j,k}$, of approximately equal length such that the combined length of the $\tilde{C}_l^{j,k}$ is approximately equal to the length of C_l .

Before attempting to communicate the chosen $v \in V$, the first player must first communicate the appropriate member in \mathcal{F} , namely $F(v)$. In the j, k sub-blocks $\tilde{C}_l^{j,k}$, if $k \notin F(v)$ and $j = j_{F(v)}^k$, the first player, with his suggested behavior strategy, should play $\sigma_{F(v),k}$ and otherwise any member of $NR(F(v))$. The second player, with his suggested behavior strategy, should play j in $\tilde{C}_l^{j,k}$. Given that Player Two has observed an average of $\alpha^{j,k} \in \Delta(S)$ in the blocks $\tilde{C}_l^{j,k}$, we say that the signals received by the second player **confirm** $L \in \mathcal{F}$ if $L = F(N)$ where $N = \{k \in K \mid \forall j \in J, \text{ distance } (\alpha^{j,k}, A_j^k) \leq \xi^k/2\}$.

For every $L \in \mathcal{F}$, define $V_L := \{v \in V \mid F(v) = L\}$.

Next, in the rest of the C_l block, the first player communicates a $v \in V_L$ for the appropriate $L \in \mathcal{F}$ exactly as with Renault, given that L is nonsingular with some distinct a_L and b_L in $\bar{\Lambda}^j(NR(L))$ for some j . If $L \in \max \mathcal{F}$ is singular, by the definition of a joint plan, there will be only one member v of V satisfying $F(v) = L$, so that the confirmation of L is sufficient for the communication of the appropriate nonrevealing joint plan. Exactly as with Renault, the second player must interpret a member of $v \in V_L$ from interpreting a noisy signal in $\{a_L, b_L\}^m$ for some large enough m , where L is the confirmed member of \mathcal{F} . If the signals received by the second player in the sub-blocks $\tilde{C}_l^{j,k}$ do not confirm any member of \mathcal{F} (because $F(N) = \emptyset$ or $F(N) = K$ and $K \notin \mathcal{F}$) or if $V_L = \emptyset$ for the confirmed $L \in \mathcal{F}$, then the second player punishes the first player in the following playing blocks exactly as he would if he were punishing what seemed to be a deviation of play in the B^{l-1} playing block.

With the suggested behavior strategies, since with probability one the intended member of \mathcal{F} along with the intended member of V equals eventually that confirmed and interpreted by Player Two, respectively, with probability one the second player will receive eventually a stable communication of the correct $v \in V$. Due to the definition of A_j^k , Player One has no ability with any positive probability at a state k outside of $F(v)$ to maintain an average signal more than a distance of $\xi^k/2$ outside of $A_{j_{F(v)}^k}^k$. Therefore, no matter how good for Player One is the convex combination of proto-joint-plan payoffs associated with a $v \in V$ at a state $k \notin F(v)$, Player One has no opportunity to enjoy it at the state k – except in a set of measure zero he cannot get any more than x^k by attempting to communicate a $v \in V$ such that $k \notin F(v)$.

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