LOCATION OF THE FERMAT-TORRICELLI MEDIANS
OF THREE POINTS

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ABSTRACT. We prove that a real normed space X with $\dim X \geq 3$ is an inner product space if and only if, for every three points $u, v, w \in X$, the set of points at which the function $x \in X \to \|u - x\| + \|v - x\| + \|w - x\|$ attains its minimum (called the set of Fermat-Torricelli medians of the three points) intersects the convex hull of these three points.

1. INTRODUCTION

The classical Fermat problem [11, p. 153] for three points u, v, w of the Euclidean plane X deals with a fourth point such that the sum of its distances to the three given points is a minimum, i.e., a point $z \in X$ such that

$$\|u - z\| + \|v - z\| + \|w - z\| \leq \|u - x\| + \|v - x\| + \|w - x\|, \text{ for every } x \in X.$$

The point z, called the Fermat-Torricelli median of the three points, is unique and belongs to their convex hull, $\operatorname{co}(u, v, w)$. Moreover, a historical theorem of Torricelli and Cavalieri says that either z is one of the three points, or pairwise the segments $[z, u]$, $[z, v]$, and $[z, w]$ form angles of $2\pi/3$ (see, e.g., [14]).

Suppose now that we are not in the Euclidean plane, but in a real normed space X and that $Z^1(u, v, w)$ denotes the set of Fermat-Torricelli medians of the three points $u, v, w \in X$. It is easy to verify that if X is an inner product space, then $Z^1(u, v, w)$ has, as in the classical problem, a unique point that belongs to $\operatorname{co}(u, v, w)$. In the general case, it is also easy to see that $Z^1(u, v, w)$ is closed and convex, but there arise, amongst others (effective calculation, ...), three basic problems:

Existence. It is possible that $Z^1(u, v, w) = \emptyset$. An example due to Baronti, Casini and Papini [2] is the space

$$X = \left\{ x \in c_0 : \sum_{n=1}^{4} x_n + \sum_{n=5}^{\infty} \frac{1}{2^{n-4}} x_n = 0 \right\},$$

endowed with the sup norm and u, v, w the points $(-1, 0, 0, 1, 0, \dots)$, $(0, -1, 0, \frac{1}{2}, 1, 0, \dots)$, $(0, 0, -1, \frac{1}{2}, 1, 0, \dots)$. (In fact, Veselý [19] proved that either X is reflexive or there exist $u, v, w \in X$ and an equivalent norm in X such that $Z^1(u, v, w) = \emptyset$.)

Uniqueness. The set $Z^1(u, v, w)$ may have more than one element. A simple example, for X the space $\mathbb{R}^2$ endowed with the norm

$$\|(x_1, x_2)\| = \begin{cases} \sup(|x_1|, |x_2|), & \text{if } x_1 x_2 \geq 0, \\ |x_1| + |x_2|, & \text{if } x_1 x_2 < 0, \end{cases}$$

is $Z^1((1,0), (-1,0), (1,2)) = \text{co}((1,0), (-1,0), (1,2))$.

Location. It may be that

$$Z^1(u, v, w) \neq \emptyset, \quad Z^1(u, v, w) \cap \text{co}(u, v, w) = \emptyset.$$

For instance, when $X = \ell_1^3$, $Z^1((1,0,0), (0,1,0), (0,0,1)) = \{(0,0,0)\}$. (It is known [20] that if $Z^1(u, v, w) \cap \text{aff}(u, v, w) \neq \emptyset$, then $Z^1(u, v, w) \cap \text{co}(u, v, w) \neq \emptyset$. Therefore, since $Z^1(u, v, w) \neq \emptyset$ when X is finite-dimensional, it is not possible that $Z^1(u, v, w) \cap \text{co}(u, v, w) = \emptyset$ when $\dim X < 3$.)

In this paper we give an affirmative answer to a plausible conjecture relative to location. Namely, we shall prove that if a real normed space X of dimension ≥ 3 satisfies

(GK1) $Z^1(u, v, w) \cap \text{co}(u, v, w) \neq \emptyset,$ for every $u, v, w \in X,$

then it is an inner product space.

We denote the above hypothesis GK1 in honour of Garkavi and Klee who, forty years ago [12, 13], [1, p. 119], obtained an analogous result for the hypothesis

(GK∞) $Z^\infty(u, v, w) \cap \text{co}(u, v, w) \neq \emptyset,$ for every $u, v, w \in X,$

where $Z^\infty(u, v, w)$ is the set of Chebyshev centers of u, v, w, i.e., the set of points $z \in X$ such that, for every $x \in X$,

$$\sup(\|u - z\|, \|v - z\|, \|w - z\|) \leq \sup(\|u - x\|, \|v - x\|, \|w - x\|).$$

Also, a characteristic property of real inner product spaces of dimension ≥ 3 [3, 4] is the hypothesis GKp, with $p > 1$, relative to the set $Z^p(u, v, w)$ of points $z \in X$ such that, for every $x \in X$,

$$\|u - z\|^p + \|v - z\|^p + \|w - z\|^p \leq \|u - x\|^p + \|v - x\|^p + \|w - x\|^p.$$

Actually it is not difficult to reduce the hypothesis GKp, with $1 < p \leq +\infty$, to the classical Brunn-Blaschke-Kakutani theorem, which states that a real normed space of dimension ≥ 3 is an inner product space if and only if there are norm-1 linear projections of X onto any of its 2-dimensional subspaces (see, e.g., [1, p. 99]).

However, there seems to be no straightforward analogous reduction for the case GK1, i.e., relative to the oldest kind of center, which is the most interesting in a variety of fields (operations research, economics,...).

Some partial answers on the subject of this paper have been obtained by Durier [8], who proved that a characteristic of real inner product spaces of dimension ≥ 3 is the hypothesis

$$Z^1_{\alpha_1, \alpha_2, \alpha_3, \alpha_4}(u_1, u_2, u_3, u_4) \cap \text{co}(u_1, u_2, u_3, u_4) \neq \emptyset,$$

for every four nonnegative weights $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and every four points $u_1, u_2, u_3, u_4 \in X$, where $z \in Z^1_{\alpha_1, \alpha_2, \alpha_3, \alpha_4}(u_1, u_2, u_3, u_4)$ means that

$$\sum_{i=1}^{4} \alpha_i \|u_i - z\| \leq \sum_{i=1}^{4} \alpha_i \|u_i - x\|, \quad \text{for every } x \in X,$$

and Lewicki [16], who proved the same for three, instead of four, weights and points. But whether GK1 characterizes real inner product spaces of dimension ≥ 3 is expressly mentioned as an open question in [9], [20], [7, p. 98], and [6, p. 274], among others.

2. Preliminary results

In this section we take X to be any real normed space of dimension ≥ 3.

Let S and S^* be the unit spheres of X and its topological dual X^*, respectively. For $u \in X$ and $f \in X^*$, write $Ju = \{f \in S^* : f(u) = \|u\|\}$ and $J^*f = \{u \in S : f(u) = \|f\|\}$. It is well-known that:

(i) For $u \neq 0$, Ju is a nonempty, closed and convex subset of S^*, and, for $f \neq 0$, J^*f is a closed and convex subset of S, which is nonempty (for every $f \neq 0$) when X is reflexive.

(ii) X is smooth if and only if Ju has a unique element for every $u \neq 0$ or, equivalently, $J^*f \cap J^*g = \emptyset$ when f and g are linearly independent.

(iii) X is rotund if and only if $Ju \cap Jv = \emptyset$ when u and v are linearly independent or, equivalently, J^*f has at most one element for every $f \neq 0$.

(In the case of uniqueness we shall write $Ju = f$ or $J^*f = u$.)

Let $u, v, w \in X$. It is obvious that $z \in Z^1(u, v, w)$ if and only if

$$(z, z, z) \in P^1_{\mathrm{diag}X^3}(u, v, w),$$

i.e., (z, z, z) is a best approximation to (u, v, w) from the linear subspace $\mathrm{diag}X^3 = \{(x, x, x) : x \in X\}$, when X^3 is endowed with the norm $\|(u, v, w)\|_1 = \|u\| + \|v\| + \|w\|$.

According to [15, p. 8] the following duality result was first discovered by Vecten in 1811 and Fasbender [10] in 1846 for the classical Fermat-Torricelli problem. Since then, it has been frequently rediscovered in different contexts.

Proposition 1. $0 \in Z^1(u, v, w)$ *if and only if there exist* $f \in Ju$, $g \in Jv$, $h \in Jw$ *such that* $f + g + h = 0$.

Proof. It easily follows from the Hahn-Banach theorem (see, e.g., [18, p. 18]) that

$$(0, 0, 0) \in P^1_{\mathrm{diag}X^3}(u, v, w)$$

is equivalent to the existence of $(f, g, h) \in J(u, v, w)$, which is null on $\mathrm{diag}X^3$. That is,

$$\|(f, g, h)\| = \sup(\|f\|, \|g\|, \|h\|) = 1,$$
$$(f, g, h)(u, v, w) = f(u) + g(v) + h(w) = \|u\| + \|v\| + \|w\|,$$
$$(f, g, h)(x, x, x) = f(x) + g(x) + h(x) = 0, \qquad \text{for every } x \in X.$$

Then the inequality $f(u) + g(v) + h(w) \leq \|f\| \|u\| + \|g\| \|v\| + \|h\| \|w\|$ shows that the first and second equalities above are equivalent to $f \in Ju$, $g \in Jv$, $h \in Jw$. $\square$

Remark 2. Analogous arguments show that $0 \in Z^p(u, v, w)$, with $1 < p < +\infty$, if and only if there are $f \in Ju$, $g \in Jv$, $h \in Jw$ such that $\|u\|^{p-1}f + \|v\|^{p-1}g + \|w\|^{p-1}h = 0$. Then, the next corollary is not true for $p > 1$ and, curiously, this is the reason for the greater difficulty of the proof that GKp implies i.p.s. for the case $p = 1$.

Corollary 3. *If* $0 \in Z^1(u, v, w)$ *and* λ, μ, ν *are nonnegative real numbers, then* $0 \in Z^1(\lambda u, \mu v, \nu w)$.

Corollary 4. $z \in Z^1(u, v, w)$ *if and only if there exist* $f \in J(u - z)$, $g \in J(v - z)$, $h \in J(w - z)$ *such that* $f + g + h = 0$.

Proof. $z \in Z^1(u, v, w)$ is equivalent to $0 \in Z^1(u - z, v - z, w - z)$. □

Corollary 5. *Let* $f, g, h \in S^*$ *be such that* $f + g + h = 0$, *let* $u \in J^*f$, $v \in J^*g$, $w \in J^*h$, *(i.e.,* $0 \in Z^1(u, v, w)$*), and let* z *be distinct from* u, v, w. *Then* $z \in Z^1(u, v, w)$ *if and only if*

$$\frac{u - z}{\|u - z\|} \in J^*f, \quad \frac{v - z}{\|v - z\|} \in J^*g, \quad \frac{w - z}{\|w - z\|} \in J^*h.$$

Lemma 6. *Let* P *be a 2-dimensional subspace of* X^* *and* $f \in S^* \cap P$. *Then:*

  (i) *There exist* $g, h \in S^* \cap P$ *such that* $f + g + h = 0$.
 (ii) *If* f *is non-parallel to a segment of* $S^* \cap P$ *of length* > 1, *then the* g *and* h *above are unique.*
(iii) *If* f *is non-parallel to a segment of* $S^* \cap P$ *of length* ≥ 1, *and* g *and* h *are as above, then any* $v \in J^*g$ *and* $w \in J^*h$ *are linearly independent.*

Proof. (i) A simple continuity argument shows the existence of $g \in S^* \cap P$ such that $\|f + g\| = 1$.

(ii) If $g, h, g', h' \in S^* \cap P$ are such that $f + g + h = f + g' + h' = 0$, with $g \neq g'$ and both in the same half plane of P relative to f, then the segments parallel to f, $[g, -h']$ and $[g', -h]$, lie on $S^* \cap P$ and either $\|g + h'\| > 1$ or $\|g' + h\| > 1$.

(iii) If $v \in J^*g$ and $-v \in J^*h$, then $[g, -h] \subset S^*$, $g + h = -f$ and $\|g + h\| = 1$. □

Lemma 7. *If* $f, g, h \in S^*$ *are such that* $f + g + h = 0$ *and* $u \in J^*f$, $v \in J^*g$, $w \in J^*h$ *are such that* $\dim[\mathrm{span}(u, v, w)] = 2$, *then* $0 \in \mathrm{co}(u, v, w)$.

Proof. Let $r, s, t \in \mathbb{R}$ be such that $r + s + t = 1$ and $ru + sv + tw = 0$. Since $g(v) = h(w) = 1$ and $g(u), g(w), h(u), h(v) \in [-1, 0]$, it easily follows from

$$r + s + t = 1,$$
$$rg(u) + sg(v) + tg(w) = 0,$$
$$rh(u) + sh(v) + th(w) = 0$$

that r, s, t are nonnegative. □

3. Main Results

From now on we shall suppose that X satisfies GK1 and, as a consequence of the next lemma, that $\dim X = 3$.

Lemma 8. *A real normed space* X *of dimension* ≥ 3 *satisfies GK1 if and only if all of its 3-dimensional subspaces satisfy GK1.*

Proof. The necessity is obvious. Conversely, suppose that X does not satisfy GK1, i.e., that there are $u, v, w \in X$ such that $Z^1(u, v, w) \cap \mathrm{co}(u, v, w) = \emptyset$. Then $Z^1(u - w, v - w, 0) \cap \mathrm{co}(u - w, v - w, 0) = \emptyset$ and, since $\mathrm{co}(u - w, v - w, 0)$ is compact, there is $y \notin \mathrm{co}(u - w, v - w, 0)$ such that, for every $x \in \mathrm{co}(u - w, v - w, 0)$,

$$\|u - w - y\| + \|v - w - y\| + \|y\| < \|u - w - x\| + \|v - w - x\| + \|x\|.$$

That is,

$$Z^1_M(u - w, v - w, 0) \cap \mathrm{co}(u - w, v - w, 0) = \emptyset,$$

where $M = \operatorname{span}(u - w, v - w, y)$ and $Z_M^1(u - w, v - w, 0)$ denotes the set of points $z \in M$ such that

$$\|u - z\| + \|v - z\| + \|w - z\| \leq \|u - x\| + \|v - x\| + \|w - x\|,$$

for every $x \in M$. □

As a consequence of this lemma and of the well-known fact that X is an inner product space if and only if so are all of its 3-dimensional subspaces, to prove that GK1 characterizes real inner product spaces of dimension ≥ 3, it suffices to consider the case in which $\dim X = 3$.

With this convention ($\dim X = 3$) we can freely use the nice properties that S and S^* are compact, $Z^1(u, v, w) \neq \emptyset$ for every $u, v, w \in X$, and $J^* f \neq \emptyset$ for every $f \in X^*$.

Although the next proposition has a boring repetitive proof, it is one of the cornerstones of our argument. It will be easy to see that the last three of the four different cases that we shall consider in its proof correspond to highly non-regular spheres in S^* that contain some segment of length ≥ 1.

Proposition 9. *Suppose that X satisfies GK1 and $f, g, h \in S^*$ are such that $f + g + h = 0$. Then, for any $u \in J^* f$ and $v \in J^* g$, there exists $w \in J^* h$ such that $\dim[\operatorname{span}(u, v, w)] = 2$.*

Proof. It is obvious when $u = -v$. (This is the case, for example, when $X = \ell_\infty^3$ and $f = (1/2, 1/2, 0), g = (-1, 0, 0), h = (1/2, -1/2, 0), u = (1, 1, 0), v = (-1, -1, 0)$.)

In general, i.e., when $\dim[\operatorname{span}(u, v)] = 2$, let $p \in S^*$ be such that $\ker p = \operatorname{span}(u, v)$. In three of the four cases that follow we shall prove that there is a $w \in J^* h \cap \ker p$.

Suppose, on the contrary, that $p(x) \neq 0$ for every $x \in J^* h$. Since $J^* h$ is convex, the sign of p may be chosen so that $\{p(x) : x \in J^* h\} \subset \mathbb{R}_+$ and, since $J^* h$ is a non-void and compact set, there is a $\bar{w} \in J^* h$ such that

$$p(\bar{w}) = \inf\{p(x) : x \in J^* h\} > 0.$$

An immediate consequence of $f + g + h = 0$ is that $h(u) \leq 0$ and $h(v) \leq 0$, and we shall consider four possible cases with different (but similar) lines of argument.

Case 1 ($h(u) < 0$ and $h(v) < 0$). It follows from Proposition 1 and Corollary 3 that $0 \in Z^1(u, v, 2\bar{w})$. But $0 \notin \operatorname{co}(u, v, 2\bar{w})$, and it follows from

$$\|u - 2\bar{w}\| + \|v - 2\bar{w}\| \geq h(2\bar{w} - u) + h(2\bar{w} - v)$$
$$= 4 - h(u) - h(v) > 4 = \|u\| + \|v\| + \|2\bar{w}\|$$

that $2\bar{w} \notin Z^1(u, v, 2\bar{w})$.

Then, by GK1, there exist $r \geq 0$, $s \geq 0$, $1 > t \geq 0$ such that $r + s + t = 1$ and

$$z = ru + sv + t2\bar{w} \in Z^1(u, v, 2\bar{w}).$$

Therefore, by Corollary 5, $\dfrac{2\bar{w} - z}{\|2\bar{w} - z\|} \in J^* h$. But

$$p\left(\frac{2\bar{w} - z}{\|2\bar{w} - z\|}\right) = \frac{2(1 - t)p(\bar{w})}{2(1 - t) - rh(u) - sh(v)} < p(\bar{w}),$$

in contradiction with the definition of $\bar{w}$.

(Note that in the next three cases the common hypothesis $h(u) = 0$ implies that $(f - g)(u) = 2$ and, hence, that the segment $[f, -g]$ of length 1 lies on S^*.)

Case 2 ($h(u) = 0$, $h(v) < 0$ and $f(\bar{w}) < 0$). As in Case 1, $0 \in Z^1(u, \lambda v, \bar{w})$ for every $\lambda > 0$, but $0 \notin \text{co}(u, \lambda v, \bar{w})$.

For $y = ru + (1 - r)\bar{w}$, with $0 \leq r \leq 1$, and $\lambda > 0$ such that $\lambda f(v) - f(\bar{w}) > 0$, we have that

$$
\begin{aligned}
\|u - y\| + \|\lambda v - y\| + \|\bar{w} - y\| &= \|u - \bar{w}\| + \|\lambda v - y\| \\
&\geq f(u - \bar{w}) + \sup(g(\lambda v - y), h(y - \lambda v)) \\
&= 1 - f(\bar{w}) + \sup(1 + \lambda + (1 - r)f(\bar{w}), 1 - r - \lambda h(v)) \\
&= \sup(2 + \lambda - rf(\bar{w}), 2 + \lambda - r - f(\bar{w}) + \lambda f(v)) \\
&> 2 + \lambda = \|u\| + \|\lambda v\| + \|\bar{w}\|.
\end{aligned}
$$

In other words, $y \notin Z^1(u, \lambda v, \bar{w})$ and, hence, there exist $r \geq 0$, $s > 0$, and $t \geq 0$ such that $r + s + t = 1$ and

$$
z = ru + s\lambda v + t\bar{w} \in Z^1(u, \lambda v, \bar{w}),
$$

from which follows the contradiction that $\dfrac{\bar{w} - z}{\|\bar{w} - z\|} \in J^* h$ and

$$
p\left(\frac{\bar{w} - z}{\|\bar{w} - z\|}\right) = \frac{(1 - t)p(\bar{w})}{1 - t - s\lambda h(v)} < p(\bar{w}).
$$

Case 3 ($h(u) = 0$, $h(v) < 0$, and $f(\bar{w}) = 0$). If $\|\bar{w} - u\| > 1$, then, for every $y = ru + (1 - r)\bar{w}$ with $0 \leq r \leq 1$, we have that

$$
\begin{aligned}
\|u - y\| + \|v - y\| + \|\bar{w} - y\| &= \|\bar{w} - u\| + \|v - y\| \\
&> 1 + g(v - y) = 3 = \|u\| + \|v\| + \|\bar{w}\|,
\end{aligned}
$$

i.e., $y \notin Z^1(u, v, \bar{w})$, and the proof follows as in Case 2.

If $\|\bar{w} - u\| = 1$, then $\bar{w} - u \in J^* h$ and $0 \in Z^1(u, v, \bar{w} - u)$, but $0 \notin \text{co}(u, v, \bar{w} - u)$.

For $0 \leq r \leq 1$ and $y = ru + (1 - r)(\bar{w} - u)$ the inequalities

$$
\begin{aligned}
\|u - y\| + \|v - y\| + \|\bar{w} - u - y\| &= \|\bar{w} - 2u\| + \|v - y\| \\
&\geq f(2u - \bar{w}) + \sup(g(v - y), h(y - v)) \\
&= 2 + \sup(1 + r, 1 - r - h(v)) > 3 = \|u\| + \|v\| + \|\bar{w} - u\|
\end{aligned}
$$

show that $y \notin Z^1(u, v, \bar{w} - u)$. Then, there exist $r \geq 0$, $s > 0$, $t \geq 0$ such that

$$
z = ru + sv + t(\bar{w} - u) \in Z^1(u, v, \bar{w} - u)
$$

and, hence, $\dfrac{\bar{w} - u - z}{\|\bar{w} - u - z\|} \in J^* h$, which is in contradiction with

$$
p\left(\frac{\bar{w} - u - z}{\|\bar{w} - u - z\|}\right) = \frac{(1 - t)p(\bar{w})}{1 - t - sh(v)} < p(\bar{w}).
$$

Case 4 ($h(u) = 0$ and $h(v) = 0$). We do not use the same p and $\bar{w}$ as in the above cases, but a $q \in S^*$ such that $q(u) > 0$ and $q(v) > 0$, whose existence is guaranteed by the linear independence of u and v, and a point $\tilde{w} \in J^* h$ such that

$$
q(\tilde{w}) = \inf\{q(x) : x \in J^* h\}.
$$

We shall prove that $\tilde{w} \in \text{span}(u, v)$. Suppose, on the contrary, that $\tilde{w} \notin \text{span}(u, v)$. Then by Proposition 1 and Corollary 3, $0 \in Z^1(u, v, 3\tilde{w})$, but $0 \notin$

$co(u, v, 3\tilde{w})$. Then, by GK1, there exist $r \geq 0$, $s \geq 0$, and $t \geq 0$ such that $r+s+t = 1$ and

$$z = ru + sv + t3\tilde{w} \in Z^1(u, v, 3\tilde{w}).$$

The inequality

$$5 = \|u\| + \|v\| + \|3\tilde{w}\| = \|u - z\| + \|v - z\| + \|3\tilde{w} - z\|$$
$$\geq h(z - u) + h(z - v) + h(3\tilde{w} - z) = 3 + 3t$$

shows that $t \leq 2/3$ and, hence, that either $r > 0$ or $s > 0$.

Therefore, we have the contradiction

$$\frac{3\tilde{w} - z}{\|3\tilde{w} - z\|} \in J^*h$$

and

$$q\left(\frac{3\tilde{w} - z}{\|3\tilde{w} - z\|}\right) = \frac{(3 - 3t)q(\tilde{w}) - rq(u) - sq(v)}{3 - 3t} < q(\tilde{w}).$$

$\square$

Proposition 10. *Suppose that X satisfies GK1 and that $f, g, h \in S^*$ are such that $f + g + h = 0$. If f is non-parallel to any segment of S^* of length ≥ 1 and $u \in J^*f$, then $g(u) = h(u) = -1/2$.*

Proof. Let $u', u'' \in S \cap \ker f$ be linearly independent. Then, for any $\alpha > 0$, $P_\alpha = \{k \in X^* : k(u' - \alpha u'') = 0\}$ is a 2-dimensional subspace of X^* that contains f, and, by Lemma 6, there is a unique pair $g_\alpha, h_\alpha \in S^* \cap P_\alpha$ such that $f + g_\alpha + h_\alpha = 0$.

Since f and g_α are linearly independent, $u', u'' \in \ker f$, and $g_\alpha(u' - \alpha u'') = 0$, we have that $g_\alpha(u'') \neq 0$. We may assume that $g_\alpha(u'') > 0$ (and $h_\alpha(u'') < 0$) for every $\alpha > 0$. Then it is easy to see that the map $\alpha \in \mathbb{R}_+ \to g_\alpha \in S^*$ is continuous, i.e., it defines a simple arc over S^*, and we shall see that it is contained in $S^* \cap \{k \in X^* : k(u) = -1/2\}$.

Let $(\alpha_n) \searrow \alpha$. Since the sequence

$$(p_{\alpha_n}, -p_{\alpha_n}) = \left(\frac{g_{\alpha_n} - g_\alpha}{\|g_{\alpha_n} - g_\alpha\|}, \frac{h_{\alpha_n} - h_\alpha}{\|h_{\alpha_n} - h_\alpha\|}\right)$$

is in the compact set $S^* \times S^*$, it has a subsequence that converges to $(p_\alpha, -p_\alpha) \in S^* \times S^*$.

By the continuity of the maps $\alpha \to g_\alpha$ and $\alpha \to h_\alpha$, the straight lines $\{g_\alpha + \lambda p_\alpha : \lambda \in \mathbb{R}\}$ and $\{h_\alpha + \lambda p_\alpha : \lambda \in \mathbb{R}\}$ support S^* at g_α and h_α, respectively. Thus, there exist $v_\alpha \in J^*g_\alpha$ and $w_\alpha \in J^*h_\alpha$ such that $p_\alpha(v_\alpha) = p_\alpha(w_\alpha) = 0$. Moreover, for any other $v'_\alpha \in J^*g_\alpha$ and $w'_\alpha \in J^*h_\alpha$ we have that $(g_{\alpha_n} - g_\alpha)(v'_\alpha) \leq 0$, $(h_{\alpha_n} - h_\alpha)(w'_\alpha) \leq 0$. Hence $p_\alpha(v'_\alpha) \leq 0$ and $p_\alpha(w'_\alpha) \geq 0$.

We shall prove that $\dim[\text{span}(u, v_\alpha, w_\alpha)] = 2$, which, taking into account that v_α and w_α are linearly independent (Lemma 6), is equivalent to proving that $p_\alpha(u) = 0$.

Suppose that $p_\alpha(u) \neq 0$, i.e., that $\dim[\text{span}(u, v_\alpha, w_\alpha)] = 3$. Then, by Proposition 9 there exist $v'_\alpha \in J^*g_\alpha$ and $w'_\alpha \in J^*h_\alpha$ such that $\dim[\text{span}(u, v'_\alpha, w_\alpha)] = \dim[\text{span}(u, v_\alpha, w'_\alpha)] = 2$.

By Lemma 7, there exist nonnegative real numbers r, s, t, r', s', t' such that $r + s + t = r' + s' + t' = 1$ and $ru + sv'_\alpha + tw_\alpha = r'u + s'v_\alpha + t'w'_\alpha = 0$. Since both v'_α, w_α and v_α, w'_α are linearly independent pairs, r and r' must be positive, and

we have that $rp_\alpha(u) + sp_\alpha(v'_\alpha) = r'p_\alpha(u) + t'p_\alpha(w'_\alpha) = 0$, which contradicts the aforementioned fact that $p_\alpha(v'_\alpha) \le 0$ and $p_\alpha(w'_\alpha) \ge 0$.

Therefore, every convergent subsequence of (p_{α_n}) converges to the unique $p_\alpha \in S^*$ such that $p_\alpha(u) = 0$, $p_\alpha(v_\alpha) = 0$ for some $v_\alpha \in J^* g_\alpha$. Moreover, $p_\alpha(v'_\alpha) \le 0$ for every $v'_\alpha \in J^* g_\alpha$, and hence,

$$\lim_{\beta \searrow \alpha} \frac{g_\beta - g_\alpha}{\|g_\beta - g_\alpha\|} = p_\alpha.$$

Thus,

$$\lim_{\beta \searrow \alpha} \frac{g_\beta(u) - g_\alpha(u)}{\|g_\beta - g_\alpha\|} = p_\alpha(u) = 0,$$

$$\lim_{\beta \searrow \alpha} \frac{g_\beta(u' - \alpha u'') - g_\alpha(u' - \alpha u'')}{\|g_\beta - g_\alpha\|} = \lim_{\beta \searrow \alpha} \frac{(\beta - \alpha) g_\beta(u'')}{\|g_\beta - g_\alpha\|}$$

$$= g_\alpha(u'') \lim_{\beta \searrow \alpha} \frac{\beta - \alpha}{\|g_\beta - g_\alpha\|} = p_\alpha(u' - \alpha u'').$$

Since $g_\alpha(u'') \ne 0$, it suffices to see that $p_\alpha(u' - \alpha u'') \ne 0$ to obtain that

$$\lim_{\beta \searrow \alpha} \frac{g_\beta(u) - g_\alpha(u)}{\beta - \alpha} = p_\alpha(u) = 0,$$

and, by the mean value theorem, that the continuous function $\alpha \in \mathbb{R}_+ \to g_\alpha(u)$ is constant.

Suppose, on the contrary, that $p_\alpha(u' - \alpha u'') = 0$. Since v_α and w_α are linearly independent and $p_\alpha(v_\alpha) = p_\alpha(w_\alpha) = 0$, there exist nonzero real numbers λ and μ such that $\lambda v_\alpha + \mu w_\alpha = u' - \alpha u''$, from which it follows that

$$\lambda g_\alpha(v_\alpha) + \mu g_\alpha(w_\alpha) = g_\alpha(u' - \alpha u'') = 0,$$

$$\lambda h_\alpha(v_\alpha) + \mu h_\alpha(w_\alpha) = h_\alpha(u' - \alpha u'') = 0.$$

Thus $g_\alpha(w_\alpha) h_\alpha(v_\alpha) = 1$ and $(g_\alpha - h_\alpha)(v_\alpha) = 2 = \|g_\alpha - h_\alpha\|$. So the segment $[g_\alpha, -h_\alpha]$ is parallel to f, of length 1, and contained in S^*, thereby contradicting the hypothesis.

Since we can do the same for $\alpha < 0$ and for $u'' - \alpha u'$ instead of $u' - \alpha u''$, we have obtained that for every $g, h \in S^*$ such that $f + g + h = 0$, $g(u)$ is constant. Obviously the constant must be $-1/2$, as we wished to show. $\qquad \square$

Since all the possible points g cannot lie on two planes simultaneously, we have the following immediate consequence.

Corollary 11. *If X satisfies GK1 and if $f \in S^*$ is non-parallel to a segment of S^* of length ≥ 1, then $J^* f$ has a unique element.*

Proposition 12. *If X satisfies GK1, then X^* is rotund (i.e., X is smooth).*

Proof. Assume by contradiction that there is a non-void open segment $I \subset S^*$, and denote by P the 2-dimensional subspace of X^* that contains I.

Simple 2-dimensional arguments show that there is at most one segment of $S^* \cap P$, of length > 1, that is parallel to some point of I, and that if f and f' are two different points of I and $g, h, g', h' \in S^* \cap P$ are such that $f + g + h = f' + g' + h' = 0$, then $g - h$ is non-proportional to $g' - h'$.

Therefore, we can take $f \in I$ for which the pair $g, h \in S^* \cap P$ satisfying $f + g + h = 0$ is unique and $g - h$ is non-parallel to I. Furthermore, since I is an open segment,

the condition that $g - h$ is non-parallel to I is equivalent to the existence of $c \neq 0$ such that $(g - h)(u) = c$, for every $u \in J^*f$.

Since the set of points of S^* that are non-parallel to any segment contained in S^* is dense in S^* [17], there is a sequence (f_n) that converges to f and is formed by points of S^* that are non-parallel to any segment of S^*. By Corollary 11, each J^*f_n has a unique element u_n.

Let $P_n = \operatorname{span}(f_n, g)$, and let g_n, h_n be the unique pair of $S^* \cap P_n$ such that $f + g_n + h_n = 0$. If we suppose that $\|g_n - g\| \leq \|h_n - g\|$, standard continuity arguments show that the sequences (g_n) and (h_n) converge to g and h, respectively, and that (u_n) has a subsequence that converges to some $u \in J^*f$. Hence, there exists $n \in \mathbb{N}$ such that f_n is non-parallel to any segment of S^*, but $|(g_n - h_n)(u_n)| > \frac{|c|}{2}$, in contradiction with Proposition 10. $\qquad\square$

Corollary 13. *If X satisfies GK1, then X^* is smooth (i.e., X is rotund).*

Proof. Take into account that X^* is rotund, and apply Corollary 11. $\qquad\square$

Corollary 14. *Assume that X satisfies GK1.*

  (i) *If $f, g, h \in S^*$ are such that $f + g + h = 0$ and $u = J^*f$, $v = J^*g$, $w = J^*h$, then $u + v + w = 0$.*

  (ii) *If $u, v, w \in S$ are such that $u + v + w = 0$ and $f = Ju$, $g = Jv$, $h = Jw$, then $f + g + h = 0$.*

Proof. (i) This follows from $f + g + h = 0$, $f(u) = g(v) = h(w) = 1$, and $f(v) = f(w) = g(u) = g(w) = h(u) = h(v) = -\frac{1}{2}$.

(ii) A simple continuity argument shows that there exists $g' \in S^*$ such that $\|f + g'\| = 1$ and $v' = Jg'$ belongs to $\operatorname{span}(u, v)$. Let $h' = -f - g'$ and $w' = Jh'$. It follows from (i) that $u + v' + w' = 0$, which, together with $u + v + w = 0$, implies that $v = v'$, $w = w'$. Hence, $g = g'$, $h = h'$. $\qquad\square$

As in Proposition 1, certain arguments concerning areas which we shall use in the next, final theorem are connected with results of Fasbender [10] for the classical Fermat problem (see, e.g., [14, p. 58]).

Theorem 15. *A real normed space X of dimension ≥ 3 is an inner product space if and only if*

(GK1) $\qquad Z^1(u, v, w) \cap \operatorname{co}(u, v, w) \neq \emptyset, \qquad$ *for every $u, v, w \in X$.*

Proof. On the one hand, it is well known and easy to see that an inner product space satisfies GK1.

On the other hand, we have previously proved that if X satisfies GK1, then it is regular (smooth and rotund) and such that

$$u, v, w \in S, \quad u + v + w = 0, \quad Ju = f, \quad Jv = g, \quad Jw = h$$

imply

$$f(v) = f(w) = g(u) = g(w) = h(u) = h(v) = -\tfrac{1}{2}.$$

Making use of this, we shall see that every two-dimensional subspace Y of X is an inner product space, i.e. (identifying Y with $\mathbb{R}^2$), that $S \cap Y$ is an ellipse.

Denote by $\prec$ the positive orientation of the plane Y. For $u_0 \in S \cap Y$, let $u_0 \prec u_\theta$ be the point of $S \cap Y$ that forms an angle $0 \leq \theta \leq \pi$ with u_0 and let v_θ, w_θ be the unique points of $S \cap Y$ that satisfy $w_\theta \prec u_\theta \prec v_\theta$ and $u_\theta + v_\theta + w_\theta = 0$.

Since $S \cap Y$ is a convex regular curve, $u : \theta \in [0, \pi] \rightarrow u_\theta$ is continuously differentiable and it is easy to see that $v : \theta \in [0, \pi] \rightarrow v_\theta$ and $w : \theta \in [0, \pi] \rightarrow w_\theta$ are continuous and of bounded variation (note that $S \cap Y$ is a convex curve and v, w are increasing, i.e., such that $0 \leq \theta_1 < \theta_2 \leq \pi$ implies $v_{\theta_1} \prec v_{\theta_2}$ and $w_{\theta_1} \prec w_{\theta_2}$). Hence, all the Riemann-Stieltjes integrals that will appear are meaningful.

As usual, for a pair $x = (x_1, x_2)$ and $y = (y_1, y_2)$ of points of Y such that $x \prec y$, we denote by $x \wedge y = x_1 y_2 - x_2 y_1$ the area of the parallelogram with sides x and y. Then $y \wedge x = -x \wedge y$, $x \wedge (y + z) = x \wedge y + x \wedge z$, etc.

Note that $u_\theta + v_\theta + w_\theta = 0$ implies that $u_\theta \wedge (u_\theta + v_\theta + w_\theta) = v_\theta \wedge (u_\theta + v_\theta + w_\theta) = 0$ and, hence,

$$u_\theta \wedge v_\theta = v_\theta \wedge w_\theta = w_\theta \wedge u_\theta.$$

Write $Ju_\theta = f_\theta$. It follows from $f_\theta(v_\theta - w_\theta) = 0$ that, for every partition $0 = \alpha_0 < \alpha_1 < ... < \alpha_n = \alpha$ of the interval $[0, \alpha]$, with $0 < \alpha \leq \pi$, there exist $\alpha_0 < \beta_1 < \alpha_1 < \beta_2 < \alpha_2 < ... < \alpha_{n-1} < \beta_n < \alpha_n$ such that

$$f_{\beta_k}(v_{\beta_k} - w_{\beta_k}) = f_{\beta_k}(u_{\alpha_k} - u_{\alpha_{k-1}}) = 0 \quad (k = 1, ..., n).$$

Then,

$$(v_{\beta_k} - w_{\beta_k}) \wedge (u_{\alpha_k} - u_{\alpha_{k-1}}) = 0 \quad (k = 1, ..., n)$$

and, hence,

$$\int_0^\alpha (v - w) \wedge du = 0,$$

which, taking into account that $u_\theta + v_\theta + w_\theta = 0$, can be written

$$\int_0^\alpha (u + 2v) \wedge du = 0.$$

Analogously,

$$\int_0^\alpha (w - u) \wedge dv = \int_0^\alpha (-2u - v) \wedge dv = \int_0^\alpha (u - v) \wedge dw = 0,$$

and, therefore,

$$\int_0^\alpha (v - w) \wedge du + \int_0^\alpha (w - u) \wedge dv + \int_0^\alpha (u - v) \wedge dw$$

$$= \int_0^\alpha d[(v - u) \wedge (v - w)] = (v_\alpha - u_\alpha) \wedge (v_\alpha - w_\alpha) - (v_0 - u_0) \wedge (v_0 - w_0) = 0,$$

i.e., the function

$$\theta \in [0, \pi] \rightarrow (u_\theta - v_\theta) \wedge (v_\theta - w_\theta) = u_\theta \wedge v_\theta + v_\theta \wedge w_\theta + w_\theta \wedge u_\theta$$

is constant and, since $u_\theta \wedge v_\theta = v_\theta \wedge w_\theta = w_\theta \wedge u_\theta$, the functions

$$\theta \rightarrow u_\theta \wedge v_\theta, \quad \theta \rightarrow v_\theta \wedge w_\theta, \quad \theta \rightarrow w_\theta \wedge u_\theta$$

are also constant.

Therefore,

$$\int_0^\alpha d(u \wedge v) = \int_0^\alpha u \wedge dv - \int_0^\alpha v \wedge du = 0$$

and, hence,

$$\int_0^\alpha u \wedge du = -2 \int_0^\alpha v \wedge du = -2 \int_0^\alpha u \wedge dv = \int_0^\alpha v \wedge dv.$$

Furthermore, since $\theta \in [0, \pi] \to u_\theta$ is continuously differentiable, $\theta \in [0, \pi] \to u_\theta \wedge v_\theta$ is constant, and $\theta \in [0, \pi] \to v_\theta$ is continuous and increasing, this last function is also continuously differentiable.

Then, the above Riemann-Stieltjes integrals can be reduced to ordinary Riemann integrals, and it follows from

$$\int_0^\alpha (u + 2v) \wedge du = \int_0^\alpha (u + 2v) \wedge u' = 0,$$

$$\int_0^\alpha (-2u - v) \wedge dv = \int_0^\alpha (-2u - v) \wedge v' = 0$$

that there exist positive continuous functions $p : [0, \pi] \to \mathbb{R}$ and $q : [0, \pi] \to \mathbb{R}$ such that

$$u' = p(u + 2v), \quad v' = q(-2u - v).$$

Hence

$$\int_0^\alpha u \wedge du = \int_0^\alpha u \wedge u' = \int_0^\alpha u \wedge p(u + 2v)$$

$$= \int_0^\alpha v \wedge dv = \int_0^\alpha v \wedge v' = \int_0^\alpha v \wedge q(-2u - v),$$

and, since $u \wedge v$ is constant, $p = q$.

Finally, solving the simple system of differential equations

$$u_1' = p(u_1 + 2v_1),$$

$$u_2' = p(u_2 + 2v_2),$$

$$v_1' = -p(2u_1 + v_1),$$

$$v_2' = -p(2u_2 + v_2),$$

with the non-restrictive initial conditions $u_0 = (1, 0)$, $v_0 = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$, we obtain

$$u_1^2 + v_1^2 + u_1 v_1 = u_2^2 + v_2^2 + u_2 v_2 = \frac{3}{4}$$

and this, together with

$$u_1 v_2 - u_2 v_1 = \frac{\sqrt{3}}{2},$$

gives

$$S \cap Y = \{(u_1, u_2) : u_1^2 + u_2^2 = 1\},$$

as we wished to show. $\qquad\qquad\square$

Remark 16. It is worth mentioning that much of the work in this paper (cases 2, 3, 4 of Proposition 9, many details in Proposition 11, Proposition 12, ...) is due to the, a priori, non-regularity of X. Our task would have been considerably easier with the additional hypothesis that X is rotund and smooth.

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