HYPERBOLIC MEAN GROWTH
OF BOUNDED HOLOMORPHIC FUNCTIONS
IN THE BALL

E. G. KWON

ABSTRACT. We consider the hyperbolic Hardy class $\varrho H^p(B)$, $0 < p < \infty$. It consists of ϕ holomorphic in the unit complex ball B for which $|\phi| < 1$ and

$$\sup_{0 < r < 1} \int_{\partial B} \{\varrho(\phi(r\zeta), 0)\}^p \, d\sigma(\zeta) \; < \; \infty,$$

where ϱ denotes the hyperbolic distance of the unit disc. The hyperbolic version of the Littlewood-Paley type g-function and the area function are defined in terms of the invariant gradient of B, and membership of $\varrho H^p(B)$ is expressed by the L^p property of the functions. As an application, we can characterize the boundedness and the compactness of the composition operator $\mathcal{C}_\phi$, defined by $\mathcal{C}_\phi f = f \circ \phi$, from the Bloch space into the Hardy space $H^p(B)$.

1. INTRODUCTION

1.1. Let ϕ be a holomorphic map from the unit ball $B = B_n$ of $\mathbb{C}^n$ into the unit disc of $\mathbb{C}$ and let $\mathcal{C}_\phi$ be the composition operator defined by $\mathcal{C}_\phi f = f \circ \phi$. Then properties of ϕ can be observed either in terms of the hyperbolic growth of the image $\phi(rB)$ in B_1 or by the operator-theoretic behavior of $\mathcal{C}_\phi$. Moreover, there should be a relationship between the hyperbolic growth of ϕ and certain properties of $\mathcal{C}_\phi$. For example, when $n = 1$, ϕ is hyperbolically $\mathcal{B}_0$ in the sense of $\lim_{|z| \to 1}(1 - |z|^2)\frac{|\phi'(z)|}{1 - |\phi(z)|^2} = 0$ if and only if $\mathcal{C}_\phi$ is compact on the little Bloch space $\mathcal{B}_0$ ([CM, p. 194]).

From the viewpoint of the boundary behavior, there was a branch of the study of $\mathcal{C}_\phi$ from the Bloch space $\mathcal{B}$ into a nice function space. P. Ahern ([A]) observed that $C_\phi g \in BMOA(B)$ for all $g \in \mathcal{B}$ and for a monomial ϕ. Extensive research followed. Several examples of homogeneous polynomials and conditions for ϕ were found to have the same property and, finally, the relation with hyperbolic mean growth occurred. See [A], [AR1], [AR2], [C1], [C2], [CC], [CK], [CRU], [K5], [Ma], [RU], [Rus] and [Sta]. The boundedness of $\mathcal{C}_\phi : \mathcal{B} \to BMOA$ can be characterized by $\phi \in \varrho BMOA$, where $\varrho BMOA$ is the hyperbolic $BMOA$ class ([K5], see [Y3] for $\varrho BMOA(B_1)$). Also, in the same vein, the boundedness of $\mathcal{C}_\phi : \mathcal{B} \to H^{2p}(B_1)$ was characterized by $\phi \in \varrho H^p(B_1)$ when $1 \le p < \infty$ ([K1]). Note that functions of the

Hardy space H^p and of $BMOA$ should have radial limits almost everywhere but Bloch functions need not.

1.2. In this paper we define and consider the hyperbolic Hardy class $\varrho H^p(B)$, $0 < p < \infty$. It is defined to consist of holomorphic ϕ in B for which $|\phi| < 1$ and

$$\sup_{0<r<1} \int_{\partial B} \varrho\left(\phi(r\zeta), 0\right)^p d\sigma(\zeta) \ < \ \infty,$$

where ϱ denotes the non-Euclidean hyperbolic distance of B_1:

$$\varrho(z, w) \ = \ \frac{1}{2} \ \log \frac{|1 - \bar{z}w| + |z - w|}{|1 - \bar{z}w| - |z - w|}, \quad z, w \in B_1.$$

Our objective is to find a hyperbolic version of the area function and the g-function (of Littlewood-Paley type) which characterize the membership of $\varrho H^p(B)$. This leads us to characterize the boundedness and the compactness of the composition operator $\mathcal{C}_\phi \colon \mathcal{B} \to H^p(B)$ in terms of the growth of ϕ, extending variable results to $n \geq 1$ and to $0 < p < \infty$. The main result shows that the following are equivalent:

$$\phi \ \in \ \varrho H^p(B);$$

$$\int_B (1 - |z|^2)^n \left(\log \frac{1}{1 - |\phi(z)|^2}\right)^{p-1} \frac{|\widetilde{\nabla}\phi(z)|^2}{(1 - |\phi(z)|^2)^2} \ d\tau(z) \ < \ \infty;$$

$$\int_0^1 \frac{|\widetilde{\nabla}\phi(r\zeta)|^2}{(1 - |\phi(r\zeta)|^2)^2} \frac{dr}{1 - r} \ \in L^p(\partial B);$$

$$\int_{D_\alpha(\zeta)} \frac{|\widetilde{\nabla}\phi(z)|^2}{(1 - |\phi(z)|^2)^2} \ d\tau(z) \ \in L^p(\partial B);$$

$$\sup\left\{\log \frac{1}{1 - |\phi(z)|^2} : z \in D_\alpha(\zeta)\right\} \ \in \ L^p(\partial B);$$

$$\mathcal{C}_\phi : \mathcal{B} \to H^{2p}(B) \quad \text{is bounded};$$

$$\mathcal{C}_\phi : \mathcal{B} \to H^{2p}(B) \quad \text{is compact}.$$

Here $\widetilde{\nabla}$ is the $\mathcal{M}$-invariant gradient of B, τ and σ are the $\mathcal{M}$-invariant volume measure on B and the surface measure on ∂B respectively, and $D_\alpha(\zeta)$ is the Koranyi approach region. These concepts will be explained in Section 2. See Theorem 3.6, Theorem 4.2, Theorem 5.1, Theorem 5.10, Theorem 6.1, and Theorem 6.2 for precise results of this paper.

1.3. After introducing some terminology in Section 2, we present and prove our main results in Section 5. Section 3 is the preparatory section dealing with properties of the invariant Laplacian, invariant gradient and Green's theorem. In Section 4, a characterization of $\mathcal{M}$-harmonic Hardy space by an invariant g-function will be given. In terms of tangential and radial gradients, more of the conditions equivalent to the membership of $\varrho H^p(B)$ will be given in Section 6. Section 7 will close this paper by posing a general principle on hyperbolic function classes related to composition operators (defined on the Bloch space) and by giving an example illustrating our result.

2. PRELIMINARIES

2.1. Let $B = B_n$ be the open unit ball of $\mathbb{C}^n$ and S the boundary of B. The Hermitian inner product and the associated Euclidean norm of $\mathbb{C}^n$ are denoted by $\langle z, w \rangle = \sum_{j=1}^n z_j \bar{w}_j$ and $|z| = \sqrt{\langle z, z \rangle}$. The dot product on $\mathbb{C}^n = \mathbb{R}^{2n}$ is denoted by $z \cdot w = \sum_{j=1}^n \{ \text{Re}\, z_j \, \text{Re}\, w_j + \text{Im}\, z_j \, \text{Im}\, w_j \}$, $z, w \in \mathbb{C}^n$. The Möbius group, i.e., the group of biholomorphic self-maps of B, will be denoted by $\mathcal{M}$. Let φ_z, $z \in B$, denote the self-map of B defined by

$$\varphi_z(w) = \frac{z - \frac{\langle w, z \rangle}{\langle z, z \rangle} z - \sqrt{1 - |z|^2} \left(w - \frac{\langle w, z \rangle}{\langle z, z \rangle} z \right)}{1 - \langle w, z \rangle}, \quad w \in \bar{B},$$

if $z \neq 0$ and $\varphi_0(w) = -w$, $w \in B$. It is known that $\mathcal{M}$ consists of functions of the form $U\varphi_z$, where U denotes a unitary operator on $\mathbb{C}^n$.

2.2. We let $\mathcal{D}$, ∇, and Δ denote respectively the complex gradient of $\mathbb{C}^n$, the real gradient of $\mathbb{C}^n = \mathbb{R}^{2n}$ and the Laplacian of $\mathbb{C}^n$:

$$\mathcal{D} = (D_1, D_2, \ldots, D_n), \;\; D_j = \frac{\partial}{\partial z_j}, \;\; j = 1, 2, \ldots, n,$$

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n} \right), \;\; z_j = x_j + iy_j, \;\; j = 1, 2, \ldots, n,$$

$$\Delta = 4 \sum_{j=1}^n D_j \bar{D}_j.$$

The Laplace-Beltrami operator associated with the Bergman kernel of B will be denoted by $\widetilde{\Delta}$:

$$\widetilde{\Delta} f(a) = 4(1 - |a|^2) \sum_{j,k=1}^n (\delta_{j,k} - a_j \bar{a}_k) (D_j \bar{D}_k f)(a), \quad a \in B, \; f \in C^2(B).$$

It is known ([H] or [Sto1, Proposition 3.1]) that $\widetilde{\Delta}$ is $\mathcal{M}$-invariant: $\widetilde{\Delta}(f \circ \psi) = (\widetilde{\Delta} f) \circ \psi$, for all $\psi \in \mathcal{M}$. In particular,

$$\widetilde{\Delta} f(a) = \Delta(f \circ \varphi_a)(0), \quad a \in B,$$

for $f \in C^2(B)$. In a similar way, we will make use of the notation

$$\widetilde{\mathcal{D}} f(a) = \mathcal{D}(f \circ \varphi_a)(0) \text{ and } \widetilde{\nabla} f(z) = \nabla(f \circ \varphi_a)(0), \quad a \in B,$$

for $f \in C^1(B)$. It is easy to see that $|\widetilde{\nabla} f|^2 = 2 \left| \widetilde{\mathcal{D}} f \right|^2 + 2 \left| \widetilde{\mathcal{D}} \bar{f} \right|^2$, which equals $4 \left| \widetilde{\mathcal{D}} f \right|^2$ when f is real-valued and of $C^1(B)$. It is $\mathcal{M}$-invariant in the sense that $\left| \widetilde{\nabla}(f \circ \psi) \right| = \left| (\widetilde{\nabla} f) \circ \psi \right|$ for all $\psi \in \mathcal{M}$ and $f \in C^1(B)$. We will also make use of the tangential gradient ∇_T, which satisfies

$$|\nabla_T f|^2 = |\nabla f|^2 - 2 |Rf|^2 - 2 \left| R\bar{f} \right|^2$$

for $f \in C^1(B)$, where Rf denotes the radial derivative of f:

$$Rf(z) = \sum_{j=1}^n z_j D_j f(z), \quad z \in B.$$

2.3. The surface area measure on S normalized to have total mass one will be denoted by σ. Complex-valued continuous functions f on B satisfying the invariant mean value property:

$$\int_S f \circ \varphi_a(r\zeta) \, d\sigma(\zeta) \; = \; f(a)$$

for all $a \in B$ and all r sufficiently small are called $\mathcal{M}$-harmonic (on B). $\mathcal{M}$-harmonic functions are real-analytic ([R, Theorem 4.2.5]), and are equivalently characterized by $C^2(B)$ functions f satisfying $\widetilde{\Delta} f = 0$ ([R, Theorem 4.2.4]).

Let ν be the Lebesgue volume measure of $\mathbb{C}^n = \mathbb{R}^{2n}$ normalized to be $\nu(B) = 1$ and let τ be the $\mathcal{M}$-invariant volume measure of B:

$$d\tau(z) = \frac{d\nu(z)}{(1 - |z|^2)^{n+1}}.$$

For $\zeta \in S$ and $1 < \alpha < \infty$, $D_\alpha(\zeta)$ denotes the Koranyi approach region defined by

$$D_\alpha(\zeta) = \left\{ z \in B : |1 - \langle z, \zeta \rangle| < \frac{\alpha}{2}(1 - |z|^2) \right\}.$$

Note that $D_\alpha(U\zeta) = U D_\alpha(\zeta)$ for all unitary operators U of $\mathbb{C}^n$.

2.4. For a subharmonic function f on B, we let $\|f\|_p = \lim_{r \to 1} M_p(r, f)$, where

$$M_p(r, f) \; = \; \left(\int_S |f(r\zeta)|^p d\sigma(\zeta) \right)^{1/p}.$$

The $\mathcal{M}$-harmonic Hardy space $\mathcal{H}^p = \mathcal{H}^p(B)$, $0 < p < \infty$, is defined to consist of those f $\mathcal{M}$-harmonic in B for which $\|f\|_{\mathcal{H}^p} = \|f\|_p < \infty$. The Hardy space $H^p(B)$, $0 < p < \infty$, is a subspace of $\mathcal{H}^p$ consisting of holomorphic functions. The hyperbolic Hardy class $\varrho H^p = \varrho H^p(B)$ is defined to consist of $\phi : B \to B_1$ for which $\|\phi\|_{\varrho H^p} = \|\varrho(\phi)\|_p < \infty$, where

$$\varrho(\phi) = \varrho(\phi, 0) = \frac{1}{2} \log \frac{1 + |\phi|}{1 - |\phi|}.$$

The Bloch space $\mathcal{B}$ consists of holomorphic functions f in B_1 for which

$$\sup_{z \in B_1} |f'(z)| \, (1 - |z|^2) < \infty.$$

This is a Banach space, if the norm $\|f\|_{\mathcal{B}}$ of $f \in \mathcal{B}$ is defined to be the sum of $|f(0)|$ and the left side of the above inequality. See [AB], [ABC], [BBG], [CRW], [D], [G], [R], [Sto1], [FS], [Zy], [Zhu] for H^p theory, [K2], [K4], [Y1], [Y2] for ϱH^p and [ACP], [Zhu] for Bloch space.

2.5. Throughout this paper, ϕ always denotes a holomorphic function from B into B_1 and we set

$$\lambda(\phi) \; = \; \lambda \circ \phi \; = \; \log \frac{1}{1 - |\phi|^2}, \quad |\widetilde{\phi}| \; = \; \frac{|\widetilde{\nabla}\phi|}{1 - |\phi|^2}.$$

Other undefined notation and terminology in this paper will follow the books by W. Rudin [R] and M. Stoll [Sto1].

3. The invariant Laplacian and the invariant gradient

Elementary properties involving $\widetilde{\Delta}$ and $\widetilde{\nabla}$ are prepared in this section.

The following representation of $\left|\widetilde{\nabla} f\right|$ is almost direct and so we omit its proof:

3.1. Proposition. *Let Ω be an open subset of B. Then for $f \in C^2(\Omega)$ and $a \in \Omega$ the following equalities hold:*

$$\left|\widetilde{\nabla} f(a)\right|^2 = \lim_{r \to 0} \frac{2n}{r^2} \int_S |f \circ \varphi_a(r\zeta) - f(a)|^2 \, d\sigma(\zeta);$$

$$\left|\widetilde{\nabla} f(a)\right|^2 = \lim_{r \to 0} \frac{2(n+1)}{r^{2(n+1)}} \int_{rB} |f \circ \varphi_a(w) - f(a)|^2 \, d\tau(w);$$

$$\left|\widetilde{\nabla} f(a)\right|^2 = 2(1 - |a|^2) \sum_{j,k=1}^{n} (\delta_{j,k} - a_j \bar{a}_k) \left\{ D_j f \bar{D}_k \bar{f} + D_j \bar{f} \bar{D}_k f \right\}(a);$$

$$(3.1) \qquad \left|\widetilde{\nabla} f(a)\right|^2 = \left(1 - |a|^2\right) |\nabla_T f(a)|^2;$$

$$(3.2) \qquad \left|\widetilde{\nabla} f(a)\right|^2 = \frac{1}{2} \widetilde{\Delta} |f(a)|^2 - \mathrm{Re}\left(\bar{f}(a) \widetilde{\Delta} f(a)\right).$$

3.2. Lemma. *If u and v are real-valued and of $C^2(B)$, then*

$$(3.3) \qquad \widetilde{\Delta}(uv) = u\widetilde{\Delta}v + v\widetilde{\Delta}u + 2\widetilde{\nabla}u \cdot \widetilde{\nabla}v.$$

Proof. For $u, v \in C^2(B)$, elementary calculation gives

$$\widetilde{\Delta}(uv) = u\widetilde{\Delta}v + v\widetilde{\Delta}u + 4\langle \widetilde{\mathcal{D}}u, \widetilde{\mathcal{D}}\bar{v}\rangle + 4\langle \widetilde{\mathcal{D}}v, \widetilde{\mathcal{D}}\bar{u}\rangle.$$

Identifying $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ with $(\mathrm{Re}\, z_1, \mathrm{Im}\, z_1, \ldots, \mathrm{Re}\, z_n, \mathrm{Im}\, z_n) \in \mathbb{R}^{2n}$, it is easy to see that $\nabla f \cdot \nabla g = 4\mathcal{D}f \cdot \mathcal{D}g$ for all real-valued f and g, where $\cdot$ denotes the dot product of $\mathbb{R}^{2n}$. Thus, for $a \in B$,

$$\mathrm{Re}\langle \mathcal{D}(u \circ \varphi_a)(0), \mathcal{D}(v \circ \varphi_a)(0)\rangle = \mathcal{D}(u \circ \varphi_a)(0) \cdot \mathcal{D}(v \circ \varphi_a)(0)$$

$$= \frac{1}{4}\, \widetilde{\nabla}u(a) \cdot \widetilde{\nabla}v(a)$$

provided u and v are real-valued. $\qquad \square$

3.3. Lemma. *Let Ω be an open subset of B and $0 < p < \infty$. Then for $f \in C^2(\Omega)$ and at the points of Ω where f is nonzero, we have*

$$(3.4) \qquad \widetilde{\Delta}|f|^p = p(p-2)|f|^{p-4}\left|\widetilde{\mathcal{D}}|f|^2\right|^2 + \frac{p}{2}|f|^{p-2}\widetilde{\Delta}|f|^2,$$

which equals

$$(3.5) \qquad \widetilde{\Delta}|f|^p = p(p-1)|f|^{p-2}|\widetilde{\nabla}f|^2 + p|f|^{p-2}f\widetilde{\Delta}f$$

if f is real-valued. Moreover, if $\widetilde{\Delta}f = 0$ and $1 < p < \infty$, then $\widetilde{\Delta}|f|^p \geq 0$ and

$$(3.6) \qquad \widetilde{\Delta}|f|^p \sim C_p|f|^{p-2}|\widetilde{\nabla}f|^2.$$

Proof. (3.4) and (3.5) follow by direct calculation. Note that $\left|\widetilde{\mathcal{D}}|f(a)|^2\right| \leq |f|\left|\widetilde{\nabla}f\right|$. Thus, if $\widetilde{\Delta}f = 0$, then $\widetilde{\Delta}|f|^2 = 2|\widetilde{\nabla}f|^2$ by (3.2); so

$$p(p-1)|f|^{p-2}|\widetilde{\nabla}f|^2 \leq \widetilde{\Delta}|f|^p \leq p\,|f|^{p-2}|\widetilde{\nabla}f|^2, \quad \text{if } 1 < p \leq 2,$$

and

$$p\,|f|^{p-2}|\widetilde{\nabla}f|^2 \leq \widetilde{\Delta}|f|^p \leq p(p-1)|f|^{p-2}|\widetilde{\nabla}f|^2, \quad \text{if } 2 \leq p < \infty,$$

off zeros of f. This gives (3.6). $\square$

3.4. Let $0 < r \leq 1$ and let

$$G_r(z) = G(r,z) = \frac{1}{2n} \int_{|z|}^r \frac{(1-t^2)^{n-1}}{t^{2n-1}}\,dt, \quad z \in rB,$$

and $G(z) = G_1(z)$. Then $\widetilde{\Delta}G(z) = 0$ for all $z \in B$, $z \neq 0$. Thus G is $\mathcal{M}$-superharmonic on $B\backslash\{0\}$, $G(0) = \infty$ and $\lim_{|z|\to 1} G(z) = 0$. The function

$$\mathbb{G}(z,w) = G\left(\varphi_w(z)\right), \quad z, w \in B,$$

is called the (invariant) Green's function for $\widetilde{\Delta}$. It satisfies $\mathbb{G}(z,w) = \mathbb{G}(w,z)$ and $\widetilde{\Delta}_z\mathbb{G}(z,w) = 0$ on $B\backslash\{0\}$. We recall the following Green's Theorem ([P, Lemma 2.5]):

$$(3.7) \qquad \int_S f(r\zeta)\,d\sigma(\zeta) - f(0) = \int_{rB} G(r,z)\,\widetilde{\Delta}f(z)\,d\tau(z)$$

for $f \in C^2(B)$ and $0 < r < 1$.

3.5. Lemma. *If $1 < p < \infty$ and f is (complex-valued) $\mathcal{M}$-harmonic on B, then*

$$(3.8) \qquad \int_S |f(r\zeta)|^p\,d\sigma(\zeta) - |f(0)|^p = \int_{rB} G(r,z)\widetilde{\Delta}|f(z)|^p\,d\tau(z).$$

If $0 < p < \infty$ and $\phi : B \to B_1$ is holomorphic, then

$$(3.9) \qquad \int_S (\lambda\circ\phi)^p(r\zeta)\,d\sigma(\zeta) - (\lambda\circ\phi)^p(0) = \int_{rB} G(r,z)\widetilde{\Delta}(\lambda\circ\phi)^p(z)\,d\tau(z).$$

Proof. In [Sto1], (3.8) is proved when f is real-valued $\mathcal{M}$-harmonic on B. By applying a $C^2(B)$ function $f_\epsilon = f_{p,\epsilon} = \left(|f|^2 + \epsilon\right)^{p/2}$, $0 < \epsilon < 1$, which is bounded if $2 \leq p < \infty$ and dominated by $C_p\left(\left|\widetilde{\Delta}\,\mathrm{Re}\,f\right|^2 + \left|\widetilde{\Delta}\,\mathrm{Im}\,f\right|^2\right)$ if $1 < p < 2$, and using the limiting argument $\epsilon \to 1$ as in [Sto1, Proof of Theorem 6.18], we have (3.8). The same idea is applicable in proving (3.9) as follows.

Fix p and r: $0 < p < \infty$, $0 < r < 1$. Let

$$\lambda_\epsilon(z) = \log\frac{1+\epsilon}{1-|z|^2}, \quad z \in B, 0 < \epsilon < 1.$$

Then $(\lambda_\epsilon\circ\phi)^p \in C^2(B)$, $(\lambda_\epsilon\circ\phi)^p \downarrow (\lambda\circ\phi)^p$ as $\epsilon \downarrow 0$ monotonically, and

$$\widetilde{\Delta}(\lambda_\epsilon\circ\phi)^p = p(\lambda_\epsilon\circ\phi)^{p-2}|\widetilde{\phi}|^2\{(p-1)|\phi|^2 + \lambda_\epsilon\circ\phi\}.$$

Also,

$$\widetilde{\Delta}(\lambda_\epsilon\circ\phi)^p \geq 0, \quad \widetilde{\Delta}(\lambda\circ\phi)^p \geq 0$$

and

$$\widetilde{\Delta}(\lambda_\epsilon\circ\phi)^p \to (\lambda\circ\phi)^p$$

pointwise almost everywhere in B.

If $1 \le p < \infty$, then

$$\widetilde{\Delta}(\lambda_\epsilon \circ \phi)^p \le p^2 |\widetilde{\phi}|^2 (\lambda_\epsilon \circ \phi)^{p-1}$$

because $|\phi|^2 \le \lambda_\epsilon \circ \phi$, so that $(\lambda_\epsilon \circ \phi)^p$ is uniformly bounded on rB. Since $G(r, z)$ is τ integrable on rB and $(\lambda_\epsilon \circ \phi)^p \in C^2(B)$, we have

$$
\begin{aligned}
& \int_S (\lambda \circ \phi)^p(r\zeta)\, d\sigma(\zeta) \; - \; (\lambda \circ \phi)^p(0) \\
&= \lim_{\epsilon \to 0} \left\{ \int_S (\lambda_\epsilon \circ \phi)^p(r\zeta)\, d\sigma(\zeta) - (\lambda_\epsilon \circ \phi)^p(0) \right\} \\
&= \lim_{\epsilon \to 0} \int_{rB} G(r, z)\widetilde{\Delta}(\lambda_\epsilon \circ \phi)^p(z)\, d\tau(z) \\
&= \int_{rB} \lim_{\epsilon \to 0} G(r, z)\widetilde{\Delta}(\lambda_\epsilon \circ \phi)^p(z)\, d\tau(z) \\
&= \int_{rB} G(r, z)\widetilde{\Delta}(\lambda \circ \phi)^p(z)\, d\tau(z).
\end{aligned}
$$

(3.10)

This gives (3.9) for $1 \le p < \infty$.

Next, if $0 < p \le 1$, then

(3.11) $$\widetilde{\Delta}(\lambda_\epsilon \circ \phi)^p \le p\,|\widetilde{\phi}|^2(\lambda_\epsilon \circ \phi)^{p-1} \le p\,|\widetilde{\phi}|^2(\lambda \circ \phi)^{p-1} \le \frac{1}{p}\widetilde{\Delta}(\lambda \circ \phi)^p.$$

By Fatou's lemma and Green's theorem,

$$
\begin{aligned}
& \int_{rB} G(r, z)\, \widetilde{\Delta}(\lambda \circ \phi)^p(z)\; d\tau(z) \\
&= \int_{rB} G(r, z)\liminf_{\epsilon \to 0} \widetilde{\Delta}(\lambda_\epsilon \circ \phi)^p(z)\; d\tau(z) \\
&\le \liminf_{\epsilon \to 0} \int_{rB} G(r, z)\widetilde{\Delta}(\lambda_\epsilon \circ \phi)^p(z)\; d\tau(z) \\
&= \lim_{\epsilon \to 0} \left\{ \int_S (\lambda_\epsilon \circ \phi)^p(r\zeta)\; d\sigma(\zeta) - (\lambda_\epsilon \circ \phi)^p(0) \right\} \\
&= \int_S (\lambda \circ \phi)^p(r\zeta)\; d\sigma(\zeta) \; - \; (\lambda \circ \phi)^p(0).
\end{aligned}
$$

(3.12)

(3.11) and (3.12) show that $\widetilde{\Delta}(\lambda_\epsilon \circ \phi)^p$ is dominated by an $L^1(rB, G(r, \cdot)\, d\tau)$ function $p^{-1}\widetilde{\Delta}(\lambda \circ \phi)^p$. Thus, by the dominated convergence theorem, process (3.10) is valid in this case also. This gives (3.9) for $0 < p < 1$. $\qquad \square$

For the case of real $\mathcal{H}^p$, the first half, (3.13) and (3.14) of the following theorem appeared in [Sto2].

3.6. Theorem. *If $1 < p < \infty$, then $f \in \mathcal{H}^p$ if and only if*

(3.13) $$\int_B (1 - |z|^2)^n |f(z)|^{p-2} \left| \widetilde{\nabla} f(z) \right|^2\, d\tau(z) \; < \; \infty.$$

In this case,

(3.14) $$\|f\|_p^p \; = \; |f(0)|^p \; + \; \int_B G(z)\, \widetilde{\Delta}|f(z)|^p\, d\tau(z).$$

If $0 < p < \infty$ and $\phi : B \to B_1$ is holomorphic, then $\phi \in \varrho H^p(B)$ if and only if

(3.15)
$$\int_B (1 - |z|^2)^n (\lambda \circ \phi(z))^{p-1} |\widetilde{\phi}(z)|^2 \, d\tau(z) < \infty.$$

In this case,

(3.16)
$$\|\lambda \circ \phi\|_p^p = |\lambda \circ \phi(0)|^p + \int_B G(z) \widetilde{\Delta}(\lambda \circ \phi)^p(z) \, d\tau(z).$$

Proof. Letting $r \to 1$, the monotone convergence theorem gives (3.14) and (3.16) from (3.8) and (3.9) respectively. Noting that

(3.17)
$$G(z) = \log \frac{1}{|z|}, \quad \text{if } n = 1, \quad G(z) \sim \frac{(1 - |z|^2)^n}{|z|^{2n-2}}, \quad \text{if } n \geq 2$$

(which follows easily via L'Hospital's rule), (3.13) follows from (3.6) and (3.14). By (3.6), (3.19) follows. Since

$$p \, (\lambda \circ \phi)^{p-1} |\widetilde{\phi}|^2 \leq \widetilde{\Delta}(\lambda \circ \phi)^p \leq p^2 \, (\lambda \circ \phi)^{p-1} |\widetilde{\phi}|^2 \quad \text{if} \quad 1 \leq p < \infty,$$
$$p^2 \, (\lambda \circ \phi)^{p-1} |\widetilde{\phi}|^2 \leq \widetilde{\Delta}(\lambda \circ \phi)^p \leq p \, (\lambda \circ \phi)^{p-1} |\widetilde{\phi}|^2 \quad \text{if} \quad 0 < p \leq 1,$$

(3.15) follows from (3.16) and (3.17). $\square$

4. Area function, g-function and the maximal function

In this section, we introduce a well-known characterization of $\mathcal{M}$-harmonic Hardy space in terms of the maximal function, the invariant area function, and the invariant g-function.

For $u \in C^1(B)$ and $1 < \alpha < \infty$, we let

$$M_\alpha u(\zeta) := \sup \{|u(z)|; z \in D_\alpha(\zeta)\}, \quad \zeta \in S;$$

$$\widetilde{g}_u(\zeta) := \left(\int_0^1 \frac{|\widetilde{\nabla} u(r\zeta)|^2}{1 - r} \, dr \right)^{1/2}, \quad \zeta \in S;$$

$$\widetilde{A}_\alpha u(\zeta) := \left(\int_{D_\alpha(\zeta)} |\widetilde{\nabla} u(z)|^2 \, d\tau(z) \right)^{1/2}, \quad \zeta \in S.$$

$\widetilde{A}_\alpha$ and $\widetilde{g}$ are $\mathcal{M}$-invariant in the sense that $\widetilde{g}_u \circ \psi = \widetilde{g}_{u \circ \psi}$ and $\widetilde{A}_\alpha u \circ \psi = \left(\widetilde{A}_\alpha u \right) \circ \psi$ for all $\psi \in \mathcal{M}$.

4.1. Theorem. *For $\mathcal{M}$-harmonic u and $1 \leq p < \infty$, the following four statements are equivalent:*

$$u = P[f] \text{ for some } f \in L^p(S); \quad M_\alpha u \in L^p(S); \quad \widetilde{A}_\alpha u \in L^p(S); \quad \widetilde{g}_u \in L^p(S),$$

where $P[f]$ is the invariant Poisson integral of f:

$$P[f](z) = \int_S \frac{(1 - |z|^2)^n}{|1 - \langle z, \zeta \rangle|^{2n}} f(\zeta) \, d\sigma(\zeta), \quad z \in B.$$

In [ABC, Theorem 4.13], the first three statements of Theorem 4.1 are proved to be equivalent. So we prove the following to fulfill the proof of Theorem 4.1.

4.2. Theorem. *For each $p : 1 < p < \infty$,*

$$\|u\|_{\mathcal{H}^p} \sim \|\widetilde{g}_u\|_{L^p(S)}$$

for all $u \in \mathcal{H}^p$ with $u(0) = 0$.

See [Zy] for a one-variable version of this result. See [S] also. We will prove Theorem 4.2 by a sequence of lemmas, which will also be helpful in proving our main result in Section 5.

For $z \in B$ and $0 < \delta \leq 1$, let us denote, as in [Sto1], $E(z, \delta) = \varphi_z(\delta B)$ in the sequel.

4.3. Lemma. *For each $\alpha : 2 < \alpha < \infty$, there is $\delta_\alpha : 0 < \delta_\alpha < 1$ such that*

$$(4.1) \qquad E(r\zeta, \delta_\alpha) \subset D_\alpha(\zeta), \quad \zeta \in S,$$

for all $r : 0 < r < 1$. For general $\alpha : 1 < \alpha < \infty$, there are $\delta_\alpha : 0 < \delta_\alpha < 1$ and $r_\alpha : 0 < r_\alpha < 1$ such that (4.1) holds for all $r : r_\alpha < r < 1$.

Proof. Let $w \in \delta B$ and let $z = \varphi_{r\zeta}(w)$. Then by [R, Theorem 2.2.5 (2)] and by the fact that $\varphi_{r\zeta}(\zeta) = -\zeta$,

$$
\begin{aligned}
1 - \langle z, \zeta \rangle &= 1 - \langle \varphi_{r\zeta}(w), \zeta \rangle \\
&= 1 - \langle \varphi_{r\zeta}(w), \varphi_{r\zeta} \circ \varphi_{r\zeta}(w) \rangle \\
&= \frac{(1 - r^2)(1 - \langle w, \varphi_{r\zeta}(\zeta) \rangle)}{(1 - \langle w, r\zeta \rangle)(1 - \langle r\zeta, \varphi_{r\zeta}(\zeta) \rangle)} \\
&= \frac{(1 - r)(1 + \langle w, \zeta \rangle)}{1 - r\langle w, \zeta \rangle}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|1 - \langle z, \zeta \rangle| &= \left| \frac{(1 - r)(1 + \langle w, \zeta \rangle)}{1 - r\langle w, \zeta \rangle} \right| \\
&= \frac{|(1 + \langle w, \zeta \rangle)(1 - r\langle w, \zeta \rangle)|}{(1 + r)(1 - |w|^2)} (1 - |z|^2),
\end{aligned}
$$

which is at most

$$
\frac{1 + r\delta}{(1 - \delta)(1 + r)} (1 - |z|^2).
$$

Hence

$$
|1 - \langle z, \zeta \rangle| \leq \frac{\alpha}{2} (1 - |z|^2)
$$

when $\alpha > 2$ by taking $\delta = \delta_\alpha = \frac{(\alpha - 2)}{\alpha}$. This proves (4.1).

For general $\alpha : 1 < \alpha < \infty$, by taking $\delta : \frac{\alpha - 2}{\alpha} < \delta_\alpha < \frac{\alpha - 1}{\alpha + 1}$ and $r = r_\alpha$ that satisfy the equation $\frac{1 + r\delta}{(1 - \delta)(1 + r)} = \frac{\alpha}{2}$, (4.1) also holds for all $r : r_\alpha < r < 1$. $\square$

4.4. Lemma. *For $0 < p < \infty$ and for $\alpha : 1 < \alpha < \infty$, $\widetilde{A}_\alpha u \in L^p(S)$ implies $\widetilde{g}_u \in L^p(S)$. In particular, for each $\alpha : 2 < \alpha < \infty$, there is a positive constant C_α such that*

$$(4.2) \qquad \widetilde{g}_u(\zeta) \leq C_\alpha \widetilde{A}_\alpha u(\zeta), \quad \zeta \in S,$$

for all $\mathcal{M}$ harmonic u.

Proof. Fix $\zeta \in S$. Note that if $w \in E(r\zeta, \delta)$, then $w = \varphi_{r\zeta}(z)$ for some $z : |z| < \delta$. So,

$$
1 - |w|^2 = \frac{(1 - |z|^2)(1 - r^2)}{|1 - \langle z, r\zeta \rangle|^2} \leq (1 - r^2)\left(\frac{1 + |z|}{1 - |z|} \right) \leq \frac{1 + \delta}{1 - \delta}(1 - r^2).
$$

By the symmetry $w \in E(r\zeta, \delta) \iff r\zeta \in E(w, \delta)$,

$$\frac{1-\delta}{1+\delta}(1-r^2) \le (1-|w|^2).$$

Thus,

$$\delta_1(w) := \frac{-2\delta + (1+\delta)|w|}{1-\delta} < r < \frac{2\delta + (1-\delta)|w|}{1+\delta} := \delta_2(w),$$

whence it is not difficult to see that

(4.3)
$$\int_0^1 (1-r)^{-1} \mathcal{X}_{E(r\zeta,\delta)}(w) \, dr$$
$$\le \int_{\max\{\delta_1(w),0\}}^{\delta_2(w)} (1-r)^{-1} \, dr \le C_\delta,$$

where in the sequel $\mathcal{X}_E(w) = 1$ if $w \in E$ and is 0 otherwise.

On the other hand, if u is $\mathcal{M}$-harmonic, then

(4.4)
$$|\widetilde{\nabla} u(r\zeta)|^2 \le C_\delta \int_{E(r\zeta,\delta)} |\widetilde{\nabla} u(w)|^2 \, d\tau(w)$$

(see [Sto1, Proposition 10.6]). Now, let $\delta = \delta_\alpha$ be that of (4.1). Then by (4.1) and (4.3),

(4.5)
$$\widetilde{g}_u(\zeta)^2 \le C \int_0^1 (1-r)^{-1} \int_{E(r\zeta,\delta)} |\widetilde{\nabla} u(w)|^2 \, d\tau(w) \, dr$$
$$\le C \int_{D_\alpha(\zeta)} |\widetilde{\nabla} u(w)|^2 \left(\int_0^1 (1-r)^{-1} \mathcal{X}_{E(r\zeta,\delta)}(w) \, dr \right) d\tau(w)$$
$$\le C \int_{D_\alpha(\zeta)} |\widetilde{\nabla} u(w)|^2 d\tau(w) = C \, \widetilde{A}_\alpha u(\zeta)^2,$$

where $C = C_\delta = C_\alpha$. This proves (4.2).

If $1 < \alpha \le 2$, then by the second part of Lemma 4.3,

$$\widetilde{g}_u(\zeta)^2 \le \int_0^{r_\alpha} (1-r)^{-1} |\widetilde{\nabla} u(r\zeta)|^2 \, dr + C_\alpha \, \widetilde{A}_\alpha u(\zeta)^2,$$

so that $\widetilde{A}_\alpha u \in L^p(S)$ implies $\widetilde{g}_u \in L^p(S)$. $\qquad \square$

4.5. Lemma. *Let $1 < p < \infty$ and $1 < \alpha < \infty$. If $u \in \mathcal{H}^p$, then*

(4.6)
$$\int_S \int_{D_\alpha(\zeta)} \widetilde{\Delta} |u|^p \, d\tau \, d\sigma(\zeta) \le C_\alpha \left(\int_S |u|^p \, d\sigma - |u|^p(0) \right).$$

Proof. Note that

$$r\eta \in D_\alpha(\zeta) \iff |1 - \langle r\eta, \zeta \rangle| < \frac{\alpha}{2}(1-r^2)$$
$$\implies |1 - \langle \eta, \zeta \rangle| \le |1 - \langle r\eta, \zeta \rangle| + (1-r) \le (\alpha+1)(1-r)$$
$$\implies \zeta \in Q(\eta, \sqrt{(\alpha+1)(1-r)}),$$

and since

$$\sigma\left(Q(\eta, \sqrt{(\alpha+1)(1-r)})\right) \sim (1-r)^n$$

([R, Proposition 5.1.4]), by (3.17) we have

$$\int_S \mathcal{X}_{D_\alpha(\zeta)}(z) \, d\sigma(\zeta) \le C_\alpha (1-|z|)^n \le C_\alpha \, G(z).$$

Therefore it follows by (3.14) that

$$\int_S \int_{D_\alpha(\zeta)} \widetilde{\Delta}|u|^p \, d\tau d\sigma(\zeta)$$

$$= \int_B \widetilde{\Delta}|u(z)|^p \left\{ \int_S \mathcal{X}_{D_\alpha(\zeta)}(z) \, d\sigma(\zeta) \right\} d\tau(z)$$

$$\leq C_\alpha \int_B G(z) \widetilde{\Delta}|u(z)|^p \, d\tau(z)$$

$$\leq C_\alpha \left(\int_S |u|^p \, d\sigma - |u|^p(0) \right).$$

$\square$

4.6. Lemma. *For each $p : 1 < p < \infty$ and $\alpha : 1 < \alpha < \infty$, there is a constant $C = C_{p,\alpha}$ such that*

(4.7) $$\|\widetilde{A}_\alpha u\|_{L^p(S)} \leq C\|u\|_{\mathcal{H}^p}$$

for all $u \in \mathcal{H}^p$.

Proof. Let $u \in \mathcal{H}^p$ and fix $\alpha : 1 < \alpha < \infty$. Since

$$\widetilde{\Delta}u = 0 \iff \widetilde{\Delta}\,\mathrm{Re}\,u = 0 = \widetilde{\Delta}\,\mathrm{Im}\,u,$$

with the help of Minkowski's inequality we may assume in proving (4.7) that u is real and nonconstant. Since $D_\alpha \subset D_\beta$ if $\alpha < \beta$, we also may assume that $2 < \alpha < \infty$. We divide the proof into three cases with regard to the magnitude of p and abbreviate $C_{p,\alpha}$ by C.

   Case 1. First, suppose $1 < p \leq 2$. Then, by Lemma 2, on the set where u is not zero,

$$\widetilde{\Delta}|u|^p = p(p-1)|u|^{p-2}\left|\widetilde{\nabla}u\right|^2.$$

Since u is real-analytic ([R, Theorem 4.2.5]), the zero set of u is of volume measure zero. So,

$$\left(\widetilde{A}_\alpha u(\zeta)\right)^p \leq C \left(\int_{D_\alpha(\zeta)} |u|^{2-p}\widetilde{\Delta}|u|^p \, d\tau \right)^{p/2}$$

$$\leq C \left(M_\alpha u(\zeta)\right)^{p(2-p)/2} \left(\int_{D_\alpha(\zeta)} \widetilde{\Delta}|u|^p \, d\tau \right)^{p/2}.$$

Thus, Hölder's inequality, the complex maximal theorem ([R, Theorem 5.4.10]), and Lemma 4.5 give

$$\int_S \left(\widetilde{A}_\alpha u\right)^p d\sigma \leq C \int_S \left(M_\alpha u(\zeta)\right)^{p(2-p)/2} \left(\int_{D_\alpha(\zeta)} \widetilde{\Delta}|u|^p \, d\tau \right)^{p/2} d\sigma(\zeta)$$

$$\leq C \left(\int_S (M_\alpha u)^p \, d\sigma \right)^{(2-p)/2} \left(\int_S \int_{D_\alpha(\zeta)} \widetilde{\Delta}|u|^p d\tau \, d\sigma(\zeta) \right)^{p/2}$$

$$\leq C \|u\|_{\mathcal{H}^p}^{p-(p^2/2)} \left(\|u\|_{\mathcal{H}^p}^p - |u(0)|^p\right)^{p/2}$$

$$\leq C \|u\|_{\mathcal{H}^p}^p,$$

whence follows the desired inequality (4.7).

Case 2. Next, we suppose $4 \le p < \infty$. Let h be a nonnegative polynomial with

$$\|h\|_{L^q(S)} \le 1, \quad \frac{2}{p} + \frac{1}{q} = 1, \quad 1 < q \le 2,$$

and let $H = P[h]$: $H(z) = \int_S P(z, \zeta) h(\zeta) \, d\sigma(\zeta)$.

Since

$$z \in D_\alpha(\zeta) \iff |1 - \langle z, \zeta \rangle| < \frac{\alpha}{2}(1 - |z|^2)$$

(4.8)

$$\implies \frac{1}{(1 - |z|^2)^{n+1}} < \frac{1}{1 - |z|^2} \left(\frac{\alpha}{2}\right)^{2n} P(z, \zeta),$$

it follows that

(4.9)

$$\widetilde{A}_\alpha u(\zeta) = \left\{ \int_{D_\alpha(\zeta)} |\widetilde{\nabla} u(z)|^2 \, d\tau(z) \right\}^{1/2}$$

$$\le \left(\frac{\alpha}{2}\right)^n \left\{ \int_B (1 - |z|)^{-1} P(z, \zeta) |\widetilde{\nabla} u(z)|^2 \, d\nu(z) \right\}^{1/2},$$

whence by (3.17),

(4.10)

$$\int_S h \left(\widetilde{A}_\alpha u\right)^2 d\sigma$$

$$\le C \int_S h(\zeta) \left\{ \int_B (1 - |z|)^{-1} P(z, \zeta) |\widetilde{\nabla} u(z)|^2 \, d\nu(z) \right\} d\sigma(\zeta)$$

$$= C \int_B (1 - |z|)^{-1} H(z) |\widetilde{\nabla} u(z)|^2 \, d\nu(z)$$

$$\le C \int_B G(z) \left(H \widetilde{\Delta} u^2\right)(z) \, d\tau(z).$$

Since the last integrand is nonnegative,

$$\int_B G(z) \left(H \widetilde{\Delta} u^2\right)(z) \, d\tau(z) = \lim_{r \to 1} \int_{rB} G(r, z) \left(H \widetilde{\Delta} u^2\right)(z) \, d\tau(z)$$

by the monotonicity, and by (3.3),

$$H \widetilde{\Delta} u^2 = \widetilde{\Delta}(Hu^2) - 2 \widetilde{\nabla} H \cdot \widetilde{\nabla} u^2,$$

so that

(4.11)

$$\int_B G(z) \left(H \widetilde{\Delta} u^2\right)(z) \, d\tau(z)$$

$$\le \left| \lim_{r \to 1} \int_{rB} G(r, \cdot) \widetilde{\Delta}(Hu^2) \, d\tau \right| + 2 \int_B G \left|\widetilde{\nabla} H\right| \left|\widetilde{\nabla} u^2\right| \, d\tau$$

$$= (I) + (II).$$

Now since H is a bounded function and u is dominated by the radial maximal function of u, an $L^p(S)$ function ([R, Theorem 5.4.10]), (3.7) and the dominated convergence theorem give that

$$\lim_{r \to 1} \int_{rB} G(r, \cdot) \widetilde{\Delta}(Hu^2) \, d\tau = \lim_{r \to 1} \int_S Hu^2(r\zeta) \, d\sigma - Hu^2(0)$$

$$= \int_S hu^2(\zeta) \, d\sigma - Hu^2(0).$$

Hence, by Hölder's inequality with pairing $(\frac{p}{2}, q)$,

$$(I) = \left| \int_S h u^2(\zeta) \, d\sigma - H u^2(0) \right|$$

$$(4.12) \qquad \leq \left(\int_S h^q d\sigma \right)^{1/q} \left(\int_S |u|^p d\sigma \right)^{2/p}$$

$$\leq \|u\|_{\mathcal{H}^p}^2.$$

On the other hand, by a simple calculation, $\left| \widetilde{\nabla} u^2(z) \right| = 2|u(z)| \left| \widetilde{\nabla} u(z) \right|$. So by (4.2) and by Case 1 we just verified

$$\|\widetilde{g}_H\|_{L^q} \leq C \|\widetilde{A}_\alpha H\|_{L^q} \leq C \|H\|_{L^q} \leq C \|h\|_{L^q} \leq C, \quad 1 < q \leq 2,$$

whence it follows by (3.17) that

$$(II) \leq 4 \int_B G \left| \widetilde{\nabla} H \right| |u| \left| \widetilde{\nabla} u \right| d\tau$$

$$= 4 \int_0^1 \mathcal{G}(r) \int_S |u(r\zeta)| \left| \widetilde{\nabla} H(r\zeta) \right| \left| \widetilde{\nabla} u(r\zeta) \right| d\sigma(\zeta) \, dr$$

$$\leq 4 \int_S (M_{rad}u)(\zeta) \left(\int_0^1 \mathcal{G}(r) \left| \widetilde{\nabla} H(r\zeta) \right|^2 dr \right)^{1/2}$$

$$(4.13) \qquad \times \left(\int_0^1 \mathcal{G}(r) \left| \widetilde{\nabla} u(r\zeta) \right|^2 dr \right)^{1/2} d\sigma(\zeta)$$

$$\leq C \left(\int_S (M_{rad}u)^p d\sigma \right)^{1/p} \left(\int_S \widetilde{g}_H^q \, d\sigma \right)^{1/q} \left(\int_S \widetilde{g}_u^p \, d\sigma \right)^{1/p}$$

$$\leq C \|u\|_{\mathcal{H}^p} \|H\|_{L^q} \|\widetilde{g}_u\|_{L^p}$$

$$\leq C \|u\|_{\mathcal{H}^p} \|\widetilde{g}_u\|_{L^p},$$

where $M_{rad}u$ denotes the radial maximal function of u and

$$\mathcal{G}(r) = 2n \frac{r^{2n-1}}{(1-r^2)^{n+1}} G(r).$$

Gathering up from (4.10) to (4.13),

$$\int_S h(\widetilde{A}_\alpha u)^2 \, d\sigma \leq C \left(\|u\|_{\mathcal{H}^p}^2 + \|u\|_{\mathcal{H}^p} \|\widetilde{g}_u\|_{L^p} \right)$$

for all nonnegative polynomials h with

$$\|h\|_{L^q(S)} \leq 1, \quad \frac{2}{p} + \frac{1}{q} = 1, \ 1 < q \leq 2.$$

Since polynomials (of z_j and $\bar{z}_j$, $1 \leq j \leq n$) are dense in $C(S)$ and $C(S)$ is dense in $L^q(S)$ in this case, we obtain

$$(4.14) \qquad \|\widetilde{A}_\alpha u\|_{L^p}^2 \leq C \left(\|u\|_{\mathcal{H}^p}^2 + \|u\|_{\mathcal{H}^p} \|\widetilde{g}_u\|_{L^p} \right)$$

for all nonconstant real $u \in \mathcal{H}^p$. Now by using the arithmetic-geometric mean inequality we conclude (4.7) from (4.14).

Case 3. It now remains to verify the lemma for $2 < p < 4$. Given α, we define the operator T_α on $L^2(S)$ into the set of measurable functions as

$$T_\alpha f(\zeta) = \left(\int_{D_\alpha(\zeta)} \widetilde{\Delta} |P[f]|^2 \, d\tau \right)^{1/2}.$$

Then T_α is subadditive by a simple application of Minkowski's inequality. If $u \in \mathcal{H}^p(B)$, $2 \leq p \leq 4$, then $u = P[f]$ for some $f \in L^p(S)$ ([R, Theorem 4.3.3]), so that $\widetilde{A}_\alpha u = T_\alpha f$, whence (4.7) becomes

$$(4.15) \qquad\qquad\qquad \|T_\alpha f\|_{L^p} \leq C\|P[f]\|_{\mathcal{H}^p} = C\|f\|_{L^p},$$

where the last equality comes from [R, Theorem 5.4.8 or Theorem 5.4.9], or [Sto1, Proposition 5.6]. But we already proved that (4.15) holds for $p = 2$ and $p = 4$ respectively in Case 1 and Case 2. This means that the operator T_α is simultaneously of type $(2,2)$ and of type $(4,4)$, whence, by the interpolation theorem of Marcinkiewicz ([Zy]), the operator is of type (p,p), $2 < p < 4$. This in turn means that (4.15) holds for $2 < p < 4$, and gives that, with $P[f] = u$,

$$\|\widetilde{A}_\alpha u\|_{L^p} = \|T_\alpha f\|_{L^p} \leq C\|f\|_{L^p} = C\|u\|_{L^p} = C\|u\|_{\mathcal{H}^p}.$$

$\square$

Now, by Lemma 4.4 and Lemma 4.6 we have

$$(4.16) \qquad\qquad\qquad \|\widetilde{g}_u\|_{L^p} \leq C_p \|u\|_{\mathcal{H}^p}.$$

So to prove Theorem 4.2 it remains to show

4.7. Lemma. *For each $p : 1 < p < \infty$, there is a constant $C = C_p$ such that*

$$(4.17) \qquad\qquad\qquad \|u\|_{\mathcal{H}^p} \leq C\|\widetilde{g}_u\|_{L^p}$$

for all $u \in \mathcal{H}^p$ with $u(0) = 0$.

Proof. With the help of Minkowski's inequality and a simple inequality: $A^t + B^t \leq C(A + B)^t, 0 < t < \infty$, we may assume u to be real. If f is a polynomial and $F = P[f]$, then by (3.7),

$$\int_S Fu(r\zeta) \, d\sigma(\zeta) = \int_{rB} G(r, \cdot) \, \widetilde{\Delta}(Fu) \, d\tau$$

for all $r : 0 < r < 1$. Thus, by (3.3) and Hölder's inequality,

$$(4.18) \qquad
\begin{aligned}
\left| \int_S Fu(r\zeta) \, d\sigma(\zeta) \right| &\leq C \int_{rB} G(r, \cdot) \left| \widetilde{\nabla} F \right| \left| \widetilde{\nabla} u \right| \, d\tau \\
&\leq C \int_S \int_0^1 (1 - r)^{-1} \left| \widetilde{\nabla} F(r\zeta) \right| \left| \widetilde{\nabla} u(r\zeta) \right| \, dr \, d\sigma(\zeta) \\
&\leq C \|\widetilde{g}_F\|_{L^p} \|\widetilde{g}_u\|_{L^q},
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. But

$$f(\zeta)u(\zeta) = \lim_{r \to 1} F(r\zeta)u(r\zeta)$$

almost everywhere ([R, Theorem 5.4.8]), F is bounded and u is dominated by the radial maximal function of u ([R, Theorem 5.4.10]), so that the dominated convergence theorem guarantees that

$$\lim_{r \to 1} \int_S Fu(r\zeta) \, d\sigma(\zeta) = \int_S fu \, d\sigma.$$

By (4.16)

$$\|\widetilde{g}_F\|_{L^p} \leq C \|F\|_{\mathcal{H}^p} \leq C \|f\|_{L^p}.$$

Hence (4.18) gives

$$\left| \int_S f u \, d\sigma \right| \leq C \|f\|_{L^p} \|\widetilde{g}_u\|_{L^q}$$

for every polynomial f. Therefore we have

$$\|u\|_{L^q} = \|u\|_{\mathcal{H}^q} \leq C \|\widetilde{g}_u\|_{L^q}.$$

$\square$

5. Hyperbolic Hardy classes and Bloch to H^p pullbacks

As was expected from Section 1.2, we prove our main theorem, Theorem 5.1, in this section. It characterizes the membership of $\varrho H^p(B)$ in terms of the hyperbolic area function and the g-function. It also shows that the hyperbolic Hardy class characterizes the Bloch-$H^p(B)$ pullback problem: For which $\phi : B \to B_1$ does $g \circ \phi$ belong to $H^p(B)$ for all $g \in \mathcal{B}$? The problem was naturally called for as explained in Section 1.1.

For $\phi : B \to B_1$ holomorphic and $1 < \alpha < \infty$, let us denote the hyperbolic version of the g-function, the area function and the maximal function as follows:

$$\varrho\widetilde{g}_\phi(\zeta) = \int_0^1 \frac{|\widetilde{\phi}(r\zeta)|^2}{1 - r} \, dr, \quad \zeta \in S;$$

$$\varrho\widetilde{A}_\alpha\phi(\zeta) = \int_{D_\alpha(\zeta)} |\widetilde{\phi}(z)|^2 \, d\tau(z), \quad \zeta \in S;$$

$$\varrho M_\alpha\phi(\zeta) = \sup \left\{ \log \frac{1}{1 - |\phi(z)|^2} : z \in D_\alpha(\zeta) \right\}, \quad \zeta \in S.$$

Recall that $\widetilde{\phi} = \frac{|\widetilde{\nabla}\phi|}{1-|\phi|^2}$. We pay attention to the difference of a half-power in the above definition when compared to the case of the ($\mathcal{M}$-harmonic) invariant g-function $\widetilde{g}_f$ and the area function $\widetilde{A}_\alpha f$ of Section 4. The following is the main theorem of this paper.

5.1. Theorem. *Let $0 < p < \infty$ and $1 < \alpha, \beta < \infty$. If $\phi : B \to B_1$ is holomorphic, then the following are equivalent:*

$$(5.1) \qquad\qquad\qquad \phi \in \varrho H^p(B);$$

$$(5.2) \qquad\qquad\qquad \varrho M_\beta\,\phi(\zeta) \in L^p(S);$$

$$(5.3) \qquad\qquad\qquad \varrho\widetilde{A}_\alpha\phi \in L^p(S);$$

$$(5.4) \qquad\qquad\qquad \varrho\widetilde{g}_\phi \in L^p(S);$$

$$(5.5) \qquad\qquad f \circ \phi \in H^{2p}(B) \quad \text{for all} \quad f \in \mathcal{B}(B_1).$$

In the case $n = 1$, the equivalences between (5.3), (5.4) and (5.5) have been obtained in [PX, Theorem 1.1].

The proof of (5.1) $\implies$ (5.2) follows from the following complex hyperbolic maximal theorem:

5.2. Lemma. *Let $0 < p < \infty$ and $1 < \alpha < \infty$. If $\phi : B \to B_1$ is holomorphic, then $\varrho M_\alpha \phi \in L^p(S)$ if and only if $\phi \in \varrho H^p(B)$. In fact,*

$$(5.6) \qquad \int_S (\varrho M_\alpha \phi)^p (\zeta) \, d\sigma(\zeta) \;\leq\; C_{\alpha,p} \lim_{r \to 1} \int_S \left(\log \frac{1}{1 - |\phi(r\zeta)|^2} \right)^p \, d\sigma(\zeta).$$

Proof. Let $u = \lambda \circ \phi = -\log(1 - |\phi|^2)$. Note that $\varrho M_\alpha \phi(\zeta) = \sup\{u(z) : z \in D_\alpha(\zeta)\}$ $= M_\alpha u(\zeta)$. Fix $r : 0 < r < 1$ for a moment. Since $u_r = \lambda \circ \phi_r$ is $\mathcal{M}$-subharmonic in B and of $C(\bar{B})$,

$$u_r(z) \;\leq\; P[u_r](z), \quad z \in B.$$

Thus,

$$M_\alpha u_r(z) \;\leq\; M_\alpha(P[u_r])(z), \quad z \in B.$$

Denoting Mu_r the (Hardy-Littlewood) maximal function of u_r, by [R, Theorem 5.4.5] and [R, Theorem 5.2.6] we therefore have

$$\int_S (M_\alpha u_r)^p \, d\sigma \;\leq\; \int_S (M_\alpha P[u_r])^p \, d\sigma \;\leq\; C_{\alpha,p} \int_S (Mu_r)^p \, d\sigma$$

$$\leq C_{\alpha,p} \int_S (u_r)^p \, d\sigma \;\leq\; C_{\alpha,p} \sup_r \int_S u^p(r\zeta) \, d\sigma(\zeta)$$

for $1 < p < \infty$. Letting $r \to 1$, we have the conclusion for $p > 1$. For $0 < p \leq 1$, applying the above process with an $\mathcal{M}$-subharmonic function $(\lambda \circ \phi_r)^{1/n} \in L^{np}$, $np > 1$, instead of $u_r = \lambda \circ \phi_r$, we have the same result. $\qquad \square$

In the process of our proof, we need some lemmas as in Section 4. First we will make use of a subharmonicity involved with $\widetilde{\phi}$:

5.3. Lemma. *Let $0 < \delta < 1$ and $E(a,\delta) = \varphi_a(\delta B)$, $a \in B$. Let $\phi : B \to B_1$ be holomorphic. Then*

$$(5.7) \qquad |\widetilde{\phi}(a)|^2 \;\leq\; C_\delta \int_{E(a,\delta)} |\widetilde{\phi}(w)|^2 \, d\tau(w).$$

Proof. Fix $a \in B$. For $f : B \to B_1$ holomorphic let $F = f \circ \varphi_a$ for a moment. For $r : 0 < r < 1$,

$$D_j (F \circ \varphi_{r\zeta}) (0) = \sum_{k=1}^{m} D_k F(r\zeta) \frac{\partial (\varphi_{r\zeta})_k}{\partial z_j}(0)$$

and

$$\frac{\partial (\varphi_{r\zeta})_k}{\partial z_j}(0) = -s\delta_{jk} + \frac{s}{1+s} r^2 \bar{\zeta}_j \zeta_k,$$

where $s = \sqrt{1 - r^2}$ ([R, p. 48]). Since $H_k(z) := (1 - \langle z, a \rangle)^2 D_k F(z)$ is holomorphic for each k, by considering the power series expansion of holomorphic functions and orthogonality of $\{\zeta^\alpha\}$ ([R, Proposition 1.4.8, Proposition 1.4.9])

$$\int_S (1 - \langle r\zeta, a \rangle)^2 \, D_j (F \circ \varphi_{r\zeta}) (0) \, d\sigma(\zeta)$$

$$= \sum_{k=1}^{n} \int_S H_k(r\zeta) \left\{ -s\delta_{jk} + \frac{s}{1+s} r^2 \bar{\zeta}_j \zeta_k \right\} \, d\sigma(\zeta)$$

$$= H_j(0) \left\{ -s + \frac{s}{1+s} r^2 \int_S \bar{\zeta}_j \zeta_j \, d\sigma(\zeta) \right\}$$

$$= D_j F(0) \left\{ -s + \frac{s}{1+s} \frac{r^2}{n} \right\}.$$

Since

$$1 - r^2 = s^2 = s - \frac{s}{1+s}r^2 \leq s - \frac{s}{1+s}\frac{r^2}{n},$$

it follows that

$$(1 - r^2)\,|D_j F(0)| \;\leq\; \left\{-s + \frac{s}{1+s}\frac{r^2}{n}\right\}|D_j F(0)|$$

$$= \left|\int_S (1 - \langle r\zeta, a\rangle)^m\, D_j\,(F \circ \varphi_{r\zeta})\,(0)\,d\sigma(\zeta)\right|,$$

from which it follows that

$$(1 - r^2)^2|D_j F(0)|^2 \;\leq\; \int_S |D_j\,(F \circ \varphi_{r\zeta})\,(0)|^2|1 - \langle a, r\zeta\rangle|^4\,d\sigma(\zeta)$$

for $j = 1, 2, \ldots, n$, whence

$$(1 - r^2)^2|\mathcal{D}F(0)|^2 \;\leq\; \int_S \left|\widetilde{\mathcal{D}}F(r\zeta)\right|^2\,|1 - \langle a, r\zeta\rangle|^4\,d\sigma(\zeta).$$

Since $\widetilde{\Delta}|\phi|^{2l+2} = \frac{(l+1)^2}{4}|\phi|^{2l}\widetilde{\Delta}|\phi|^2$, by letting $f = \phi^{l+1}$, we have

$$(1 - r^2)^2\widetilde{\Delta}|\phi(a)|^2\,|\phi(a)|^{2l} \;\leq\; \int_S \widetilde{\Delta}|\phi \circ \varphi_a(r\zeta)|^2\,|\phi \circ \varphi_a(r\zeta)|^{2l}|1 - \langle a, r\zeta\rangle|^4\,d\sigma(\zeta)$$

for $l = 0, 1, 2, \ldots$. Since $|f(a)| < 1$, by using the geometric series expansion we have

$$(1 - r^2)^2|\widetilde{\phi}(a)|^2 \;\leq\; \int_S |\widetilde{\phi} \circ \varphi_a(r\zeta)|^2\,|1 - \langle a, r\zeta\rangle|^4\,d\sigma(\zeta),$$

which is equivalent to

$$(5.8) \qquad \frac{|\widetilde{\phi}(a)|^2}{(1 - |a|^2)^2} \;\leq\; \int_S \frac{|\widetilde{\phi} \circ \varphi_a(r\zeta)|^2}{(1 - |\varphi_a\,(r\zeta)|^2)^2}\,d\sigma(\zeta).$$

Now, integrating (5.8) with respect to $2nr^{2n-1}(1 - r^2)^{-n-1}\,dr$ over the interval $(0, \delta)$ and noting that $1 - |\phi_a(w)|^2 \geq \frac{1-\delta^2}{4}(1 - |a|^2)$ for all $w : |w| < \delta < 1$, we obtain

$$\frac{|\widetilde{\phi}(a)|^2}{(1 - |a|^2)^2} \;\leq\; C_\delta \int_{\delta B} \frac{|\widetilde{\phi} \circ \varphi_a(w)|^2}{(1 - |\varphi_a(w)|^2)^2}\,d\tau(w)$$

$$\leq\; C_\delta \frac{1}{(1 - |a|^2)^2} \int_{E(a,\delta)} |\widetilde{\phi}(w)|^2\,d\tau(w).$$

This completes the proof. □

If we replace (4.4) by (5.7), then the proof of Lemma 4.4 still holds with $\varrho\widetilde{g}_\phi$ and $\varrho\widetilde{A}_\alpha\phi$ in place of $\widetilde{g}_u$ and $\widetilde{A}_\alpha u$ respectively. Hence, we have

5.4. Lemma. *Let $\phi : B \to B_1$ be holomorphic. For $0 < p < \infty$ and for $\alpha : 1 < \alpha < \infty$, $\varrho\widetilde{A}_\alpha\phi \in L^p(S)$ implies $\varrho\widetilde{g}_\phi \in L^p(S)$. In particular, for each $\alpha : 2 < \alpha < \infty$, there is a positive constant C_α such that*

$$(5.9) \qquad \varrho\widetilde{g}_\phi(\zeta) \;\leq\; C_\alpha\,\varrho\widetilde{A}_\alpha\phi(\zeta), \quad \zeta \in S.$$

It is easy to see that the proof of Lemma 4.5 holds with $u = -\log(1 - |\phi|^2)$ with $\phi \in \varrho H^p(B)$:

5.5. Lemma. *Let* $0 < p < \infty$ *and* $1 < \alpha < \infty$. *If* $u = \lambda \circ \phi = -\log(1 - |\phi|^2)$ *with* $\phi \in \varrho H^p(B)$, *then*

$$(5.10) \qquad \int_S \int_{D_\alpha(\zeta)} \widetilde{\Delta} |u|^p \, d\tau \, d\sigma(\zeta) \ \leq \ C_\alpha \left(\int_S |u|^p \, d\sigma - |u|^p(0) \right).$$

5.6. We now proceed with the proof that (5.2) implies (5.3). The proof below runs in a parallel line with the proof of Lemma 4.6, and here we make use of Theorem 4.2, the main result of Section 4.

Proof of (5.2) $\implies$ (5.3). We will show that for $0 < p < \infty$ and $1 < \alpha < \infty$, there is a constant $C = C_{\alpha,p}$ such that

$$(5.11) \qquad\qquad\qquad \|\varrho \widetilde{A}_\alpha \phi\|_{L^p} \leq C \|\varrho(\phi)\|_p$$

for all $f \in \varrho H^p(B)$. It is sufficient to assume that $2 < \alpha < \infty$. We divide into two cases.

Case 1. $0 < p \leq 1$. Let $\phi \in \varrho H^p(B)$. Since $\lambda(\phi) \geq |\phi|^2$, it follows that

$$\lambda(\phi)^{1-p} \widetilde{\Delta} \lambda(\phi)^p = \ p \, \lambda(\phi)^{1-p} \lambda(\phi)^{p-2} |\widetilde{\phi}|^2 \left\{ \lambda(\phi) + (p-1)|\phi|^2 \right\} \geq \ p^2 |\widetilde{\phi}|^2$$

almost everywhere (off the zeros of ϕ), where $\lambda(\phi) = -\log(1 - |\phi|^2)$. So

$$\int_S \left(\varrho \widetilde{A}_\alpha \phi \right)^p \, d\sigma \ = \ \int_S \left(\int_{D_\alpha(\zeta)} |\widetilde{\phi}(z)|^2 \, d\tau(z) \right)^p \, d\sigma(\zeta)$$

$$\leq \frac{1}{p^2} \int_S \left(\int_{D_\alpha(\zeta)} (\lambda \circ \phi)^{1-p}(z) \, \widetilde{\Delta}(\lambda \circ \phi)^p(z) \, d\tau(z) \right)^{p/2} \, d\sigma(\zeta)$$

$$\leq \frac{1}{p^2} \int_S (\varrho M_\alpha \phi)^{1-p} (\zeta) \left(\int_{D_\alpha(\zeta)} \widetilde{\Delta}(\lambda \circ \phi)^p(z) \, d\tau(z) \right)^{p/2} \, d\sigma(\zeta).$$

Hölder's inequality gives

$$\int_S \left(\varrho \widetilde{A}_\alpha \phi \right)^p \, d\sigma$$

$$\leq \frac{1}{p^2} \left\{ \int_S (\varrho M_\alpha \phi)^p \, d\sigma \right\}^{1-p} \left\{ \int_S \int_{D_\alpha(\zeta)} \widetilde{\Delta}(\lambda \circ \phi)^p(z) \, d\tau(z) \, d\sigma(\zeta) \right\}^p.$$

Hence by Lemma 5.2 and Lemma 5.5,

$$\int_S \left(\varrho \widetilde{A}_\alpha \phi \right)^p \, d\sigma \ \leq C_p \, \|\lambda \circ \phi\|_p^p \leq C_p \, \|\varrho \circ \phi\|_p^p.$$

This gives (5.11).

Case 2. $1 < p < \infty$. To prove (5.11) we consider $\int_S h \left(\varrho \widetilde{A}_\alpha \phi \right) \, d\sigma$ for all nonnegative polynomials h with $\|h\|_{L^q} \leq 1, \frac{1}{p} + \frac{1}{q} = 1$. Let h be such a polynomial and $H = P[h]$.

By (4.8) it follows that

$$\varrho \widetilde{A}_\alpha \phi(\zeta) = \int_{D_\alpha(\zeta)} |\widetilde{\phi}(z)|^2 \, d\tau(z)$$

$$\leq \left(\frac{\alpha}{2} \right)^{2n} \int_B (1 - |z|)^{-1} P(z, \zeta) \, |\widetilde{\phi}(z)|^2 \, d\nu(z),$$

whence by (3.17),

$$\int_S h\left(\varrho\widetilde{A}_\alpha\phi\right)d\sigma$$

$$\le C\int_S h(\zeta)\left\{\int_B (1-|z|)^{-1}P(z,\zeta)\left|\widetilde{\phi}(z)\right|^2 d\nu(z)\right\}d\sigma(\zeta)$$

$$\le C\int_B (1-|z|)^{-1}H(z)\widetilde{\Delta}u(z)\,d\nu(z)$$

$$\le C\int_B G(z)\left(H\widetilde{\Delta}u\right)(z)\,d\tau(z),$$

where we let $\lambda\circ\phi = u$ for simplicity.

Since the last integrand is nonnegative,

$$\int_B G(z)\left(H\widetilde{\Delta}u\right)(z)\,d\tau(z) = \lim_{r\to 1}\int_{rB} G(r,z)\left(H\widetilde{\Delta}u\right)(z)\,d\tau(z)$$

by the monotonicity. Since $\widetilde{\Delta}H \equiv 0$, by (3,3), $H\widetilde{\Delta}u = \widetilde{\Delta}(Hu) - 2\widetilde{\nabla}H\cdot\widetilde{\nabla}u$. Hence,

$$\int_B G(z)\left(H\widetilde{\Delta}u\right)(z)\,d\tau(z)$$

$$\le \left|\lim_{r\to 1}\int_{rB} G(r,\cdot)\,\widetilde{\Delta}(H\,u)\,d\tau\right| + 2\int_B G\left|\widetilde{\nabla}H\right|\left|\widetilde{\nabla}u\right|d\tau$$

$$= (I) + (II).$$

Now since $H\,u \in C^2(B)$, we can apply (3.7), and since H is a bounded function and u is dominated by its radial maximal function, an $L^p(S)$ function by the hypothesis, the dominated convergence theorem gives

$$\lim_{r\to 1}\int_{rB} G(r,\cdot)\,\widetilde{\Delta}(H\,u)\,d\tau = \lim_{r\to 1}\int_S H(r\zeta)\,u(r\zeta)\,d\sigma(\zeta) - H(0)u(0)$$

$$= \int_S h(\zeta)\,u(\zeta)\,d\sigma - H(0)u(0).$$

Hence by Hölder's inequality,

$$(I) = \left|\int_S h(\zeta)u(\zeta)\,d\sigma(\zeta) - H(0)u(0)\right|$$

$$\le \|h\|_{L^q}\|u\|_p \le \|u\|_p.$$

On the other hand, simple calculations give

$$\left|\widetilde{\nabla}u(z)\right| = \left|\widetilde{\nabla}(\lambda\circ\phi)(z)\right| = 2|\phi(z)|\left|\widetilde{\phi}(z)\right|$$

and $|\phi| \leq \lambda(\phi)^{1/2}$, whence it follows that

$$(II) \leq 4 \int_B G\,|\phi||\widetilde{\nabla} H|\,|\widetilde{\phi}|\,d\tau$$

$$\leq C \int_0^1 \frac{dr}{1-r} \int_S u^{1/2}(r\zeta)\,|\widetilde{\nabla} H(r\zeta)|\,|\widetilde{\phi}(r\zeta)|\,d\sigma(\zeta)$$

$$\leq C \int_S M_{rad}u^{1/2} \left(\int_0^1 \frac{|\widetilde{\nabla} H|^2}{1-r}\,dr \right)^{1/2} \left(\int_0^1 \frac{|\widetilde{\phi}|^2}{1-r}\,dr \right)^{1/2}\,d\sigma$$

$$\leq C \left(\int_S M_{rad}u^p\,d\sigma \right)^{1/2p} \left(\int_S g_H^q\,d\sigma \right)^{1/q} \left(\int_S (\varrho\widetilde{g}_\phi)^p\,d\sigma \right)^{1/2p}$$

by an application of Hölder's inequality with indices $(2p, 2p, q)$. Therefore by (5.6) and Theorem 4.2, we obtain

$$(II) \leq C\|u\|_p^{1/2}\|H\|_q\|\varrho\widetilde{g}_\phi\|_{L^p}^{1/2} \leq C\|u\|_p^{1/2}\|\varrho\widetilde{g}_\phi\|_{L^p}^{1/2}.$$

Gathering up,

$$\int_S h\left(\varrho\widetilde{A}_\alpha\phi \right)\,d\sigma \leq C\left(\|u\|_p + \|u\|_p^{1/2}\|\varrho\widetilde{g}_\phi\|_{L^p}^{1/2} \right)$$

for all nonnegative polynomials h with $\|h\|_{L^q(S)} \leq 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 < q < \infty$. Therefore, by (5.9) and the density argument, we obtain

$$\|\varrho\widetilde{A}_\alpha\phi\|_{L^p}^2 \leq C\left(\|u\|_p + \|u\|_p^{1/2}\|\varrho\widetilde{A}_\alpha\phi\|_{L^p}^{1/2} \right).$$

Now by use of the arithmetic-geometric mean inequality we conclude that

$$\|\varrho\widetilde{A}_\alpha\phi\|_{L^p}^2 \leq C\|u\|_p = C\|\lambda \circ \phi\|_p$$

provided $\phi \in \varrho H^p(B)$, where $C = C_{p,\alpha}$. This gives (5.11). $\qquad\square$

5.7. Proof of (5.3) $\implies$ (5.4) follows from Lemma 5.4.

5.8. Proof of (5.4) $\implies$ (5.5): Suppose (5.4). If $1 < p < \infty$, then by Theorem 4.1 it suffices to show that $\widetilde{g}_{f\circ\phi} \in L^{2p}(S)$ for all $f \in \mathcal{B}(B_1)$. But, $\|\varrho\widetilde{g}_{f\circ\phi}\|_{L^{2p}}$ is easily seen to be bounded by $\|f\|_{\mathcal{B}}\|\varrho\widetilde{g}_\phi\|_{L^p}^{1/2}$. If $0 < p \leq 1$, then by [AB] (or [KL]) it suffices to show that

$$\int_S \left(\int_0^1 (1-r)|\nabla(f\circ\phi)(r\zeta)|^2\,dr \right)^p\,d\sigma(\zeta) < \infty$$

for all $f \in \mathcal{B}(B_1)$. Using the inequality $|\widetilde{\nabla}\phi(r\zeta)| \geq 2(1-r^2)|\nabla\phi(r\zeta)|$ (which is valid when ϕ is holomorphic), the left-hand side integral is bounded by a constant times $\|f\|_{\mathcal{B}}^{2p}\|\varrho\widetilde{g}_\phi\|_{L^p}^p$.

5.9. Proof of (5.5) $\implies$ (5.1): Suppose (5.5). Then by the closed graph theorem, the composition operator $\mathcal{C}_\phi$ maps $\mathcal{B}(B_1)$ boundedly into $H^{2p}(B)$. For each nonnegative integer k, let γ_k be the Rademacher functions (see [D] or [Zy]):

$$\gamma_k(t) = \operatorname{sign}\sin(2^{k+1}\pi t), \quad t \in [0,1].$$

Consider for each non-dyadic $t \in [0, 1]$ the function

$$f_t(z) = \sum_0^\infty \gamma_k(t) \, z^{2^k}, \quad z \in B_1.$$

Noting that

$$(1 - |z|)|f'_t(z)| \; < \; (1 - |z|) \sum_0^\infty 2^k |z|^{2^k - 1} \; \leq \; 2(1 - |z|) \sum_0^\infty |z|^k \; \leq \; 2,$$

it follows that $f_t \in \mathcal{B}$, $\|f_t\|_\mathcal{B} \leq 4$. So by the assumption,

$$\|f_t \circ \phi\|_{H^{2p}} \; = \; \|\mathcal{C}_\phi f_t\|_{H^{2p}} \; \leq \; C\|f_t\|_\mathcal{B} \; \leq C$$

for all non-dyadic $t \in [0, 1]$ and so for almost every $t \in [0, 1]$.

On integrating the $2p$ power of each side of the above inequality with respect to t, we have by the monotone convergence theorem,

$$C \; > \; \lim_{r \to 1} \int_0^1 \int_S |f_t \circ \phi(r\zeta)|^{2p} \, d\sigma(\zeta) \, dt.$$

By changing the order of the integration, the last integral equals

$$\int_S \int_0^1 \left| \sum_0^\infty \gamma_k(t) \phi(r\zeta)^{2^k} \right|^{2p} dt \, d\sigma(\zeta).$$

Thus, by [Zy, Theorem V-8.4]), we have

$$C \; > \; \lim_{r \to 1} \int_S \left(\sum_0^\infty |\phi(r\zeta)^2|^{2^k} \right)^p d\sigma(\zeta).$$

On the other hand, it is quite elementary to see that

$$\sum_{k=0}^\infty |\phi(r\zeta)^2|^{2^k} \; \geq \; \sum_{k=1}^\infty \frac{\left| \phi(r\zeta)^2 \right|^k}{k} \; = \; \log \frac{1}{1 - |\phi(r\zeta)|^2}.$$

Therefore, we have

$$\infty \; > \; C \; > \; \lim_{r \to 1} \int_S \log \left(\frac{1}{1 - |\phi(r\zeta)|^2} \right)^{p/2} d\sigma(\zeta).$$

This gives (5.1).

We close this section by showing that the boundedness of our $\mathcal{C}_\phi$ is equivalent to its compactness. See [Zha] for the result when $n = 1$ and $1 \leq p < \infty$. Also, recently in [PX, Theorem 2.3] the result has been established for the case $n = 1$ and $0 < p < \infty$.

5.10. Theorem. *The composition operator $\mathcal{C}_\phi : \mathcal{B} \to H^p$ is bounded if and only if $\mathcal{C}_\phi$ is compact.*

Proof. Suppose that $\mathcal{C}_\phi : \mathcal{B} \to H^p$ is bounded. Then $f \circ \phi \in H^p$ for all $f \in \mathcal{B}$ and $|\phi(\zeta)| := |\lim_{n \to \infty} \phi(r\zeta)|$ exists and is less than 1 for almost every $\zeta \in S$.

Let $\{f_n\} \subset \mathcal{B}$ be such that $\|f_n\|_\mathcal{B} \leq 1$. We are going to show that $\{\mathcal{C}_\phi f_n\}$ has a convergent subsequence in $H^p(B)$. Since

$$|f_n(z)| \; \leq \; |f_n(0)| + \|f_n\|_\mathcal{B} \log \frac{1}{1 - |z|}, \quad z \in B_1,$$

$\{f_n\}$ forms a normal family, so that there is a subsequence of $\{f_n\}$ that converges uniformly on compact subsets of B_1 to a holomorphic function f. Passing to the subsequence, we may assume that the sequence $\{f_n\}$ itself converges to f. f belongs to $\mathcal{B}$ with $\|f\|_\mathcal{B} \le 1$ because

$$|f'(z)|(1 - |z|^2) \ = \ \lim_{n \to \infty} |f_n'(z)|(1 - |z|^2) \ \le \ 1.$$

On the other hand, by (5.5) $\implies$ (5.4) of Theorem 5.1, $\varrho \widetilde{g}_\phi \in L^{p/2}(S)$. In particular, $\varrho \widetilde{g}_\phi(\zeta) < \infty$ a.e. $\zeta \in S$. Since

$$(5.12) \qquad \qquad \left| \widetilde{\nabla}\left((f_n - f) \circ \phi\right) \right| \le \|f_n - f\|_\mathcal{B} \left| \widetilde{\phi} \right| \le 2 \left| \widetilde{\phi} \right|,$$

the dominated convergence theorem gives $\lim \widetilde{g}_{(f_n - f) \circ \phi}(\zeta) = 0$ a.e. $\zeta \in S$. Also, by (5.12), $\widetilde{g}_{(f_n - f) \circ \phi}$ is dominated by $2 \left\{ \varrho \widetilde{g}_\phi \right\}^{1/2}$, which is an $L^p(S)$ function. So, the dominated convergence theorem again gives

$$(5.13) \qquad \qquad \lim_{n \to \infty} \int_S \left\{ \widetilde{g}_{(f_n - f) \circ \phi}(\zeta) \right\}^p \, d\sigma(\zeta) = 0.$$

Using [KL, Theorem 5.1] when $0 < p \le 1$ and using Theorem 4.1 when $1 < p < \infty$,

$$\int_S |(f_n - f) \circ \phi(\zeta) - (f_n - f) \circ \phi(0)|^p \, d\sigma(\zeta) \le C_p \int_S \left\{ \widetilde{g}_{(f_n - f) \circ \phi}(\zeta) \right\}^p \, d\sigma(\zeta),$$

whence

$$\lim_{n \to \infty} \int_S |(f_n - f) \circ \phi(\zeta)|^p d\sigma(\zeta) \ = \ 0$$

by (5.13). Therefore, $\mathcal{C}_\phi f_n \to \mathcal{C}_\phi f$ in $H^p(B)$. $\qquad \square$

6. More equivalent results

As shown in (3.1), the magnitude of the invariant gradient may be estimated only by the tangential gradient ∇_T. It is a general principle that the behavior along the tangential direction is twice better than along the radial direction. So our case is as follows:

6.1. Theorem. *For $\phi : B \to B_1$ holomorphic and $0 < p < \infty$, $1 < \alpha < \infty$, each of the following are equivalent:*

$$(6.1) \qquad \qquad \phi \ \in \ \varrho H^p(B);$$

$$(6.2) \qquad \qquad \int_0^1 \frac{|\nabla_T \phi(r\zeta)|^2}{(1 - |\phi(r\zeta)|^2)^2} \, dr \ \in L^p(S);$$

$$(6.3) \qquad \qquad \int_0^1 (1 - r) \frac{|\nabla \phi(r\zeta)|^2}{(1 - |\phi(r\zeta)|^2)^2} \, dr \ \in L^p(S);$$

$$(6.4) \qquad \qquad \int_0^1 (1 - r) \frac{|R\phi(r\zeta)|^2}{(1 - |\phi(r\zeta)|^2)^2} \, dr \ \in L^p(S).$$

Proof. By (3.1), (6.2) is the same as (5.4). So, by Theorem 5.1, (6.1) $\iff$ (6.2) follows.

To prove (6.1) $\iff$ (6.3), we make use of the following well-known fact for holomorphic f in B:

$$(6.5) \qquad f \in H^p(B) \iff \left(\int_0^1 (1-r)|\nabla f(r\zeta)|^2 \, dr \right)^{1/2} \in L^p(S).$$

See [AB] or [KL] for example.

Suppose (6.1). Then by Theorem 5.1, $g \circ \phi \in H^{2p}$ for all $g \in \mathcal{B}$. On the other hand, there are Bloch functions g_1 and g_2 such that

$$|g_1'(z)| + |g_2'(z)| \geq \frac{1}{1-|z|^2}, \quad z \in B_1$$

([RU, Proposition 5.4]). Thus,

$$|g_1' \circ \phi(z)| + |g_2' \circ \phi(z)| \geq \frac{1}{1-|\phi(z)|^2}, \quad z \in B.$$

But by (6.5),

$$\int_0^1 (1-r)|\nabla(g_j \circ \phi)(r\zeta)|^2 \, dr \in L^p(S), \quad j = 1, 2.$$

Therefore, it follows that

$$\int_S \left(\int_0^1 (1-r) \frac{|\nabla\phi(r\zeta)|^2}{(1-|\phi(r\zeta)|^2)^2} \, dr \right)^p d\sigma(\zeta)$$

$$\leq \int_S \left\{ \int_0^1 (1-r) \left(\sum_{j=1}^2 |g_j' \circ \phi| \, |\nabla\phi|(r\zeta) \right)^2 dr \right\}^p d\sigma(\zeta)$$

$$\leq C_p \sum_{j=1}^2 \int_S \left(\int_0^1 (1-r)|\nabla(g_j \circ \phi)(r\zeta)|^2 \, dr \right)^p d\sigma(\zeta)$$

$$< \infty$$

because $|\nabla(g_j \circ \phi)| = |g_j' \circ \phi| \, |\nabla\phi|$.

Conversely, suppose (6.3). Since

$$|\nabla(g \circ \phi)| \leq |\nabla\phi| \, |g' \circ \phi| \leq \|g\|_{\mathcal{B}} \frac{|\nabla\phi|}{1-|\phi|^2}$$

for $g \in \mathcal{B}$, it follows that

$$\int_0^1 (1-r)|\nabla(g \circ \phi)(r\zeta)|^2 \, dr \in L^p(S).$$

Therefore, by (6.5), $g \circ \phi \in H^{2p}(B)$ for all $g \in \mathcal{B}$. This implies (6.1) by Theorem 5.1. (6.1) $\iff$ (6.4) follows exactly in the same way as (6.1) $\iff$ (6.3) just proven. $\square$

In the same way we directly get

6.2. Theorem. *For $\phi : B \to B_1$ holomorphic and $0 < p < \infty$, $1 < \alpha < \infty$, each of the following are equivalent to $\phi \in \varrho H^p(B)$:*

$$(6.6) \qquad \int_{D_\alpha(\zeta)} (1-r) \frac{|\nabla_T \phi(z)|^2}{(1-|\phi(z)|^2)^2} \, d\tau(z) \in L^p(S);$$

(6.7) $$\int_{D_\alpha(\zeta)} (1-r)^2 \frac{|\nabla \phi(z)|^2}{(1-|\phi(z)|^2)^2}\, d\tau(z) \in L^p(S);$$

(6.8) $$\int_{D_\alpha(\zeta)} (1-r)^2 \frac{|R\phi(z)|^2}{(1-|\phi(z)|^2)^2}\, d\tau(z) \in L^p(S).$$

7. Characterizing the boundedness of the composition operators on Bloch space

7.1. Let Y be a complete holomorphic function space defined in terms of $|f|$ or $|\widetilde{\nabla} f|$. Let $\varrho(Y)$ be the hyperbolic counterpart of Y in the sense that it consists of those bounded functions ϕ, $|\phi| < 1$, whose membership is defined via $\varrho(\phi, 0)$ (or $\lambda(\phi)$) and $\widetilde{\phi}$ in place of Euclidean distance $|f(z)|$ and $|\widetilde{\nabla} f|$ respectively that is used in the definition of the membership "$f \in Y$". Then there is a general phenomenon stating

(7.1) $$\mathcal{C}_\phi \text{ maps } \mathcal{B} \text{ boundedly into } Y \quad \Longleftrightarrow \quad \phi \in \varrho(Y).$$

Examples of classes $\varrho(Y)$ are $\varrho H^p(B_1)$, $1 \leq p < \infty$, $\varrho BMOA(B)$, $\varrho \operatorname{Lip}_\alpha(B_1)$, and $\varrho \operatorname{Lip}_\alpha^p(B_1)$. By [K1], [K5], [Y4], and [KC], (7.1) is known to be true for these classes. Theorem 5.1 states that (7.1) is true with $Y = H^p(B), 0 < p < \infty$, also.

7.2. An Example. Consider the function

$$F(z) = \frac{z_1^2}{1 - z_2^2}, \quad z = (z_1, z_2) \in B_2.$$

The composition operator induced by F was first considered in [CK], and the authors there proved quite complexly that F takes Bloch functions to H^p for all $p : 0 < p < \infty$. In view of Theorem 5.1, this fact can be verified by showing that $F \in \bigcap \varrho H^p(B_2)$. Noting that $|F(\zeta)| = |\lim_{r \to 1} F(r\zeta)| < 1$ almost every $\zeta \in S$, it is sufficient to prove that $\int_S \left(\log \frac{1}{1-|F(\zeta)|^2} \right)^p d\sigma(\zeta) < \infty$ for all p. But the following process is almost direct:

$$\left\{ \int_S \left(\log \frac{1}{1-|F(\zeta)|^2} \right)^p d\sigma(\zeta) \right\}^{1/p} \leq \left\{ \int_S \left(\sum_{k=1}^\infty \frac{|F|^{2k}}{k} \right)^p d\sigma \right\}^{1/p}$$

$$\leq \sum_{k=1}^\infty \left\{ \int_S \left(\frac{|F|^{2k}}{k} \right)^p d\sigma \right\}^{1/p} = \sum_1^\infty \frac{1}{k} \left\{ \int_{B_1} \left(\frac{1-|w|^2}{|1-w^2|} \right)^{2kp} d\nu_1 \right\}^{1/p}$$

$$\leq C \sum_1^\infty \frac{1}{k} \left\{ \int_0^1 (1-r^2)^{2kp}(1-r^2)^{1-kp}\, r dr \right\}^{1/p}$$

$$\leq C \sum_1^\infty \frac{1}{k} \left(\frac{1}{kp+2} \right)^{1/p} \leq C \sum_1^\infty \frac{1}{k^{1+1/p}} < \infty$$

for all $p : p \geq 1$.

It was not known whether F had the Bloch-BMO pullback property ([CK]). Concerning this problem, it was mentioned in [CK] that previously known methods (used by P. Ahern and W. Rudin) do not work for this F. See Remark (a) and (b) of [CK]. We resolve this problem in the sequel to this paper, [K3].

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DEPARTMENT OF MATHEMATICS EDUCATION, ANDONG NATIONAL UNIVERSITY, ANDONG 760-749, S. KOREA

E-mail address: `egkwon@andong.ac.kr`