THE GEOMETRY OF 1-BASED MINIMAL TYPES

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ABSTRACT. In this paper, we study the geometry of a (nontrivial) 1-based SU rank-1 complete type. We show that if the (localized, resp.) geometry of the type is modular, then the (localized, resp.) geometry is projective over a division ring. However, unlike the stable case, we construct a locally modular type that is not affine. For the general 1-based case, we prove that even if the geometry of the type itself is not projective over a division ring, it is when we consider a 2-fold or 3-fold of the geometry altogether. In particular, it follows that in any ω-categorical, nontrivial, 1-based theory, a vector space over a finite field is interpretable.

Geometric stability theory is one of the most important themes in model theory. Originally developing as a pure subject, it has turned out to be the major technical bridge connecting pure model theory and its applications to algebraic geometry and number theory. It mainly focuses on rank-1 types/structures or regular types where a canonical combinatorial geometry can be assigned. In other words, it is primarily concerned with geometric aspects of Shelah's stability theory [19], the study of stable structures. Arguably, the first major achievements of geometric stability theory are Zilber's results from the early 1980s (the translated version is [23]) on a strongly minimal ω-categorical structure. He showed that the geometry assigned to the structure is locally modular, hence, if nontrivial, must either be affine or projective over a finite field. A different proof was discovered independently by Cherlin, Harrington and Lachlan [6]. The results imply both the finiteness in rank of ω-categorical, ω-stable structures [6] and the non-finite axiomatizability of totally categorical theories [23], [6]. Since then, a number of leading researchers have obtained new enlightening results such as Buechler's dichotomy theorem [3], Hrushovski's group configuration theorems in several contexts [10], [11], his classification of a locally modular type [11], and other generalizations in the context of superstable theories [10], [11], [14]. Refined notions such as 1-basedness, regular types and p-weight have also been introduced. Pillay, in his book [17], makes a complete exposition of the subject. Hrushovski has now shifted the direction of his research towards applications in algebraic geometry and number theory, which has deepened and broadened the subject. It is well known that, using geometric stability theory and in particular Zilber's principle on "Zariski structures", he solved the Mordell-Lang conjecture [12] and other problems in number theory.

From the mid 1990s, after the initial papers [15], [16] of Kim and Pillay, simplicity theory, introduced by Shelah [18], has developed rapidly and extensively. Simplicity

theory is the study of a class of structures, called simple, properly containing that of stable structures. The central organizing principle has been the Independence Theorem (type amalgamation), leading to the introduction of Lascar strong types, canonical bases and hyperimaginaries. Definability and the interplay with local stability have also played an important developing role. An account on the subject has been written by Wagner [22], one of the major contributors. However, there are a number of obstacles to initiating an analogous geometric simplicity theory. This is mainly due to the lack of stationarity. Hrushovski's counterexample shows that Zilber's result on ω-categorical strongly minimal structures *cannot* be generalized in the context of ω-categorical rank-1 structures. Namely, there is a *non*-1-based, rank-1 ω-categorical structure [13].

In this paper we attempt to develop initial geometric simplicity theory using classical results in geometry as essential tools. We succeed in obtaining fruitful positive results and examples. We mainly study the solution set D of a nontrivial 1-based (simple) SU rank-1 Lascar strong type. Any finite tuple (not in the algebraic closure of the empty set) in a simple structure realizes a rank-1 Lascar strong type over some parameters. In particular, in any 1-based or supersimple theory, we can find a rank-1 type over a *finite* parameter. Moreover, if the 1-based theory is ω-categorical, then the theory must have finite SU-rank (and is therefore supersimple). So every type is coordinatized by rank-1 types [9]. Hence studying such a 1-based D is important in understanding ω-categorical or even general 1-based theories. (As is well known, any ω-categorical, ω-stable theory is 1-based. The class of ω-categorical 1-based structures *properly* contains that of smoothly approximable structures, too.) Furthermore, in the case of a supersimple rank-1 theory, since any model is the disjoint union of $acl(\emptyset)$ and the solution sets of rank-1 Lascar strong types (over $\emptyset$), we can restrict our attention to the solution set of a Lascar strong type. In [21], using his notion of "generic pair", Vassiliev investigated the rank-1 theory. Here we study rank-1 types using a straightforward argument so that we can provide, for example, a direct proof of the fact that linearity implies 1-basedness. We also obtain the following more general results.

Local modularity of D implies 1-basedness (however, the converse is not true in the general simple theory context, unlike the stable case). We first prove that

$(*)$
> the (local, resp.) modularity of the pregeometry of D
> implies that the (localized, resp.) geometry is isomorphic to a
> projective geometry over a division ring.

These facts give the impression that, as in the stable case, the geometry of locally modular D should be affine over a division ring. However, we construct an example *refuting* this surmise.

On the other hand, it follows from $(*)$ that if D is (locally) modular and ω-categorical, then, after possibly naming a point, a sort of D^{eq} representing the geometry of D has a (definitional) stable reduct preserving independence. The reduct must be strongly minimal, having a projective geometry over a finite field. Then, by the classical result on strongly minimal structures, a definable vector space over a finite field can be recovered inside the reduct. In particular, an infinite group is interpretable. For the more general case of a 1-based ω-categorical D, we also show that a sort representing the geometry on some finite union of sorts in D^{eq} has a generic (i.e., independence-preserving) strongly minimal stable reduct whose

geometry is projective over a finite field. Again, a projective space over a finite field is interpretable. In fact, we obtain a more general result. Namely, for *any* 1-based D, we prove that the canonical geometry, extending (or completing) the geometry of D, on the set of canonical bases of surfaces in D^3, is projective over a division ring (Theorem 3.9) (assuming nontriviality). Here, by a surface in D^3, we mean a rank-2 Lascar strong type of a tuple in D^3. It follows that in any ω-categorical, nontrivial, 1-based theory, a vector space over a finite field is interpretable over a finite parameter.

In this paper we assume that the underlying theory T is simple, having, for convenience, elimination of hyperimaginaries. This assumption can be removed if T is supersimple [4], or by working in $\bar{M}^{heq}$ and replacing notions in $\bar{M}^{eq}$ with corresponding notions in $\bar{M}^{heq}$ (e.g., *acl* by *bdd*, and so on).

1. Locally modular type

We start this section by recalling the definitions of (pre)geometry, modularity and local modularity ([17, 2.1]).

Definition 1.1. Let S be a set. If an operation $cl : \mathcal{P}(S) \to \mathcal{P}(S)$ satisfies the following properties, then we say that (S, cl) is a *pregeometry*.

(1) For $A \subseteq S$, $A \subseteq cl(A) = cl(cl(A))$.
(2) For $A \subseteq B \subseteq S$, $cl(A) \subseteq cl(B)$.
(3) For $A \subseteq S$ and $a, b \in S$, if $a \in cl(Ab) \smallsetminus cl(A)$, then $b \in cl(Aa)$.
(4) If $a \in cl(A)$, then $a \in cl(A_0)$, for some finite $A_0 \subseteq A$.

The pregeometry (S, cl) is said to be *homogeneous* if for any closure X of a finite set and $a, b \in S \smallsetminus X$, there is an automorphism of (S, cl) that fixes X pointwise and moves a to b. We say that the pregeometry (S, cl) is a *geometry* if $cl(\emptyset) = \emptyset$ and $cl(a) = \{a\}$ for all $a \in S$. If (S, cl) is a pregeometry, then we can associate a canonical geometry $(\hat{S}, \hat{cl})$, where $\hat{S} = \{cl(a) | a \in S \smallsetminus cl(\emptyset)\}$ and for $A \subseteq S$ we put $\hat{A} = \{cl(a) | a \in A\}$ and $\hat{cl}(\hat{A}) = \{cl(b) | b \in cl(A)\}$.

Let (S, cl) be a pregeometry. We say that A ($\subseteq S$) is *independent* if $a \notin cl((A \smallsetminus \{a\})$ for all $a \in A$. Given B, a subset $B_0 \subseteq B$ is said to be a *basis* for B if $B \subseteq cl(B_0)$ and B_0 is independent. It follows that any two bases for B have the same cardinality, denoted by $\dim(B)$. Any $A \subseteq S$ gives a *localized* pregeometry on S defined by $cl_A(B) = cl(A \cup B)$, and a notion of dimension *over A* ($\dim(-/A)$). For $A, B, C \subseteq S$, if $\dim(A'/C) = \dim(A'/B \cup C)$ for any finite $A' \subseteq A$, then we say that A is *independent from B over C*.

Definition 1.2. Let (S, cl) be a pregeometry.

(1) (S, cl) is *trivial* if $cl(A) = \bigcup\{cl(\{a\}) : a \in A\}$ for all $A \subseteq S$.
(2) (S, cl) is *modular* if X is independent from Y over $X \cap Y$ for all closed X, Y, or equivalently, if $\dim(X) + \dim(Y) = \dim(X \cup Y) + \dim(X \cap Y)$ for finite dimensional closed X and Y.
(3) (S, cl) is *locally modular* if it is modular over some point in S.
(4) (S, cl) is *locally finite* if the closure of a finite set is finite.
(5) A 3-tuple $(ca_1 a_2) \in S$ is a *non-modular triangle* if $\dim(ca_1 a_2) = 3$, $cl(a_1 a_2) = cl(a_1) \cup cl(a_2)$ and there are $u_i \in cl(ca_i) \smallsetminus (cl(c) \cup cl(a_i))$ for $i = 1, 2$.

Let S' be the geometry of a pregeometry S; S is trivial (modular, or locally modular, resp.) if and only if S' is (resp.). Moreover, each of the properties is preserved under localization. Thus modularity implies local modularity.

Fact 1.3. (S, cl) is modular iff for any subset A and $b \in S$, whenever $c \in cl(Ab)$, then $c \in cl(ab)$ for some $a \in cl(A)$.

Remark 1.4. (1) Suppose that (S, cl) has a non-modular triangle (ca_1a_2). Then (S, cl) is not modular: Let u_i be as in 1.2.5. First, suppose $cl(u_1) = cl(u_2)$. Since $a_1, a_2 \in cl(cu_1) \cup cl(cu_2) = cl(cu_1)$, we get $2 = \dim(cu_1) = \dim(ca_1a_2u_1) = 3$, a contradiction. Hence $\dim(u_1u_2) = 2$. Now $a_1 \notin cl(u_1u_2)$, as otherwise, since $c \in cl(a_1u_1u_2) = cl(u_1u_2)$ and $a_2 \in cl(cu_2)$, we would have $\dim(u_1u_2) = \dim(ca_1a_2u_1u_2) = 3$, a contradiction. Similarly, $a_2 \notin cl(u_1u_2)$. Recall that $cl(a_1a_2) = cl(a_1) \cup cl(a_2)$. Hence $cl(a_1a_2) \cap cl(u_1u_2) = cl(\emptyset)$. Thus

$$4 = \dim(a_1a_2) + \dim(u_1u_2) \neq \dim(a_1a_2u_1u_2) + \dim(cl(a_1a_2) \cap cl(u_1u_2)) = 3.$$

This shows that S is not modular.

(2) If (S, cl) is a modular geometry and $cl(cd) = \{c, d\}$ for any two distinct points $c, d \in S$, then S is trivial: By (4) of 1.1, it is sufficient to show that the closure of any finite set is itself. Suppose not, say there exists $d \notin A = \{a_1, ..., a_n\}$ such that $d \in cl(A)$. Then $cl(a_nd) = \{a_n, d\}$ and $cl(a_1...a_{n-1}) = \{a_1, ..., a_{n-1}\}$ by the induction hypothesis. By modularity $n = \dim(a_1...a_n) = \dim(a_1...a_{n-1}) + \dim(a_nd) = n+1$, a contradiction.

Let us first state two important classical results on geometries, which will be used later.

Fact 1.5 ([1]). A projective (i.e., nontrivial and modular) geometry of dimension ≥ 4 in which each closed set of dimension 2 contains at least 3 elements is isomorphic to a projective geometry over a division ring.

Fact 1.6 ([7]). A locally projective (i.e., nontrivial and locally modular), locally finite geometry of dimension > 4, in which all closed sets of dimension 2 have the same size, is an affine or projective geometry over a finite field.

(For some reason, in the above Fact 1.6, the dimension condition is erroneously known to be ≥ 4 to model theorists. This may be due to the fact that, in Doyen and Hubaut's original paper, they counted *their dimension* 1 less than what is commonly defined by model theorists. They even constructed a counterexample in dimension 4.)

From now on, we consider a pregeometry given on a type. As usual, the saturated model $\bar{M}$ of the simple theory T is fixed. We work in $\bar{M}^{eq}$. (Tuples and sets are small from $\bar{M}^{eq}$.) We also fix a set D, which is a solution set (in the saturated model) of a Lascar strong type. For notational simplicity, we suppose D is over $\emptyset$. Let us further suppose that (D, cl) forms a pregeometry, where cl is an $Aut(\bar{M})$-invariant closure relation on D (i.e., $a \in cl(B)$ if and only if $f(a) \in cl(f(B))$ for each $f \in Aut(\bar{M})$). For safety, we assume that $\dim(D) = \text{Card}(\bar{M})$. The typical examples of D are, of course, rank-1 types with $cl(-) = acl(-) \cap D$, or regular types. D' denotes the geometry of D. We shall use the same closure notation for the pregeometry and its geometry.

Remark 1.7. The closure of $\emptyset$ is $\emptyset$. The size of $cl(p)$ does not depend on $p \in D$; hence (D', cl) is homogeneous iff (D, cl) is.

Now, we observe that a modular (D, cl) is well behaved.[1]

Lemma 1.8 (D modular)**.** *Suppose there exist $a, b \in D$ such that $\dim\{a, b\} = 2$ and $cl(ab) = cl(a) \cup cl(b)$. Then D is trivial and homogeneous.*

Proof. It suffices to show triviality. Moreover, we can assume that D forms a geometry. Then the condition in the lemma simply says that $cl(ab) = \{a, b\}$.

Now, by Remark 1.4.2, it is enough to show that $cl(cd) = \{c, d\}$ for any two distinct points c, d: Suppose not, say there is a third point $u' \in cl(cd)$. Let $q(x, d) = tp(c/d)$. Since D is the solution set of a Lascar strong type, by the Independence Theorem for Lascar strong types, we can amalgamate $q(x, a)$ and $q(x, b)$. Let c' be a realization of a common nonforking extension of $q(x, a)$ and $q(x, b)$. Then $\dim(a, b, c') = 3$, and there are u, v such that $tp(cdu') = tp(c'bu) = tp(c'av)$. Hence the tuple $(c'ab)$ forms a non-modular triangle. By Remark 1.4.1, this contradicts the modularity of D and finishes the proof. $\square$

In the above lemma, the assumption that D is modular is necessary. We shall see in section 2 that even local modularity of D is insufficient to imply the conclusion. There are other 1-based (3.7) and even a non-1-based example [13], where the closure of some pair of points can be 2 or 3 elements.

By now, we have the following theorems, due to Fact 1.5 and the previous lemma.

Theorem 1.9. *If D is modular, then either D is trivial, or the geometry D' of D is projective over a division ring. In either case, (D, cl) is homogeneous.*

Suppose that D is ω-categorical. Then D' is a sort in D^{eq}. Let R_n be an $(n+1)$-ary relation on D' such that $(a_0, a_1, ..., a_n) \in R_n$ if and only if $a_0 \in cl(a_1, ..., a_n)$. Then R_n is $\emptyset$-definable in D' (so in D^{eq}), and $(D', R_n)_{n \in \omega}$ is a (definitional) reduct of D'. Hence, if (D, cl) is nontrivial and modular, then by the previous theorem, $(D', R_n)_{n \in \omega}$ is strongly minimal and quantifier eliminable, with canonical pregeometry (D', cl) that is projective over a finite field F. Then by the general result on strongly minimal structures, a vector space $(V, +, \lambda)_{\lambda \in F}$ over F is definable in $(D', R_n)_{n \in \omega}^{eq}$, possibly over a finite parameter. We summarize this in the following theorem.

Theorem 1.10. *Let D be modular and ω-categorical. Then the sort S_E in D^{eq}, where $E(x, y) \equiv x \in cl(y)$, has a (definitional) stable reduct-preserving independence. Hence, in a nontrivial D, a vector space over a finite field is interpretable (i.e., definable in D^{eq} over some finite set). In particular, the infinite additive group is definable in D^{eq}.*

If, additionally, D has SU-rank 1 and the usual algebraic closure relation, then D itself has a generic stable reduct.

Now we investigate the locally modular case. We fix some notation. Choose $p \in D$, and let D_0 denote the pregeometry $D \setminus cl(p)$ with closure operation cl_p. D_0' denotes the geometry of D_0. Recall that we use the same closure notation for the pregeometry and its geometry. Note that $x \in cl_p(y)$ is an equivalence relation on D_0. We obtain the following results (1.11,12) analogous to the previous ones (1.8,9). However, (1.11,12) are not automatic consequences of (1.8,9), since in the simple context D_0 need not be the solution set of a unique type over p.

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[1]There may exist a reasonable demand for distinguishing between the terminology "modular type" and "type having a modular geometry". Here, we consider the two terms to be equal.

For the rest of this section, D is locally modular.

Lemma 1.11. *Suppose that there are distinct $a', b' \in D_0'$ such that $cl_p(a'b') = \{a'b'\}$. Then $cl(c'd') = \{c', d'\}$ for any two points c', d' in D_0'. Hence, D_0' is trivial.*

Proof. Triviality will follow from the main assertion by Remark 1.4.2. Fix points a, b in D from a', b', respectively. Due to the limitation on type amalgamation, we need more complicated arguments to prove the lemma. We shall calculate dimension in D, unless stated otherwise.

Suppose that the lemma is not true. Then there is $u' \in D_0'$ such that $u' \in cl_p(c'd') \setminus \{c', d'\}$. Now choose (in D) points c, d, u from c', d', u' respectively. Then clearly $\dim(pcdu) = 3$, and

$$(*) \qquad \dim(pX) = 3 \text{ for each 2-subset } X \text{ of } \{c, d, u\}.$$

Case I. $u \in cl(cd)$. Then

$$(**) \qquad \text{the closure of any two of the } u, c, d \text{ contains the third point.}$$

Now, we claim the following.

Claim. There are $v_i \in D$ with $v_i \in v_i' \in D_0'$ (for $i = 1, 2, 3$) such that $tp(cdu) = tp(v_1 b v_2) = tp(v_1 a v_3)$, and $\dim(pabv_1) = 4$.

Proof of claim. Let $q(x, d) = tp(c/d)$. Then we can amalgamate $q(x, b)$ and $q(x, a)$. Let v_1 be a realization of a common nonforking extension. Then clearly there are v_2, v_3 such that $tp(cdu) = tp(v_1 b v_2) = tp(v_1 a v_3)$. Moreover, we can assume

$$v_1 \underset{ab}{\downarrow} p \ ,$$

so that $\dim(pabv_1) = 4$. In particular, $p \notin cl(v_1 b)$. Now by $(**)$, $v_2 \in cl(v_1 b)$. Hence $cl(p) \neq cl(v_2)$, and $v_2' \in D_0$. Similarly $v_3' \in D_0$. Therefore the claim is proved.

Since $u \in cl(cd) \setminus \{c, d\}$, we have $v_3 \in cl(v, a) \setminus \{v, a\}$ and $v_2 \in cl(v, b) \setminus \{v, b\}$. If $v_3 \in cl(ap)$, then $p \in cl(av_3) \subseteq cl(av_1)$, contradicting $\dim(pabv_1) = 4$. Similarly, $v_3 \notin cl(v_1 p)$ and $v_2 \notin cl(bp) \cup cl(v_1 p)$. Thus $(v_1' a' b')$ forms a non-modular triangle in D_0', contradicting modularity. Hence case I does not happen.

We consider the remaining case.

Case II. $u \notin cl(cd)$. Then by $(*)$,

$$(***) \qquad \text{each 3-element subset of } \{p, c, d, u\} \text{ has dimension 3.}$$

This time, let $q(x, cd) = tp(p/cd)$. There are u_2, u_4 such that $tp(cd) = tp(au_2) = tp(bu_4)$ and $\dim(abu_2 u_4) = 4$. Amalgamate $q(x, au_2), q(x, bu_4)$ and let $u_1 \in D$ be a realization of a common nonforking extension, independent of p_0. Then $\dim(pu_1 abu_2 u_4) = 6$. Now there are $u_3 \in u_3' \in D_0'$ and $u_5 \in u_5' \in D_0'$ such that $tp(pcdu) = tp(u_1 au_2 u_3) = tp(u_1 bu_4 u_5)$. By use of $(***)$, one can see that $\dim(pu_2 u_3 u_4 u_5) = 5$ and $a', b' \notin cl_p(u_2' u_3' u_4' u_5')$. Thus $cl_p(a'b') \cap cl_p(u_2' u_3' u_4' u_5') = \emptyset$. This again violates the modularity of D_0'. Hence Case II does not happen either.

Therefore Lemma 1.11 is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 1.12. D_0' *is either trivial or projective over a division ring.*

Lemma 1.13. D *is trivial if and only if D_0 is trivial.*

Proof. We can safely assume that D forms a geometry. It is enough to show that triviality of D_0 implies that of D. Let D_0 be trivial.

Claim. The closure of any two distinct points $\{p, q\}$ in D is itself.

Proof of claim. Suppose not, say $r \in cl(pq) \setminus \{p, q\}$. Choose

$$(*) \qquad\qquad\qquad s \notin cl(pq) = cl(pqr).$$

Since D_0 is trivial, $cl(pqs) = cl(ps) \cup cl(pq)$ and $\dim(pqs) = 3$. Hence $q, r \notin cl(ps)$. Now, since the localized pregeometry of D at s is equivalent to D_0 and D_0 is trivial, $cl(qs) \cup cl(ps) = cl(pqs)$. In particular, $r \in cl(qs) \cup cl(ps)$. Hence, since $r \notin cl(ps)$, $r \in cl(qs)$. Since $\dim(qs) = 2$, $s \in cl(qr)$, this contradicts (*). Thus the claim is verified.

Now, by the claim, for $n + 1$ points $p, q_1, ..., q_n$, since D_0 is trivial, $cl(pq_1...q_n) = cl(pq_1) \cup ... \cup cl(pq_n) = \{p, q_1, ..., q_n\}$. Hence D is also trivial. $\qquad\square$

By the same argument as for Theorem 1.10 above, if D is locally modular ω-categorical, after naming a point p in D, a sort in $(D_p)^{eq}$ representing D_0' has a generic (independence-preserving) stable reduct (over p). Hence, if D is nontrivial, a vector space over a finite field is again interpretable over a finite parameter. We shall in fact prove the interpretability of a vector space in the more general 1-based context (3.23).

By now, one would naturally expect that, as in stable theories, a (nontrivial, non-modular) locally modular D should be affine. However, the example in the following section shows that this is *not* the case.

2. Non-affine locally modular type

First, we fix an extended language $\mathcal{L}' = \mathcal{L}_{\mathcal{A}} \cup \{P\}$ where P is a unary predicate and $\mathcal{L}_{\mathcal{A}}$ is the language of the theory of affine space over a finite field F of cardinality at least 5. (We shall see below a number of reasons why this condition is imposed. In fact, one can do the same construction with a projective space over a finite field of cardinality ≥ 4.) Now, suppose V_1 is a finite affine space over F generated by independent elements $\{x_0, ..., x_n\}$. Let P colour some of the elements in V_1. We call the colouring *good* if for every line l in V_1, we can find at least 2 points in l coloured by P, and let $gc(n) > 0$ be the finite number of good colourings, for $n \geq 1$, described by $\mathcal{L}'$-formulas $Col_{r,n}(x_0, x_1, \dots, x_n)$. Now suppose $V_2 \supset V_1$ is an affine space with basis $\{x_0, ..., x_{n+1}\}$. Then, given a good colouring $Col_{r,n}(x_0, \dots, x_n)$ of V_1, there exists a good colouring extending it to V_2. (For example, colour every point in $V_2 \setminus V_1$ with P while leaving the colouring on V_1. Since every line in V_2 either is already in V_1 or hits V_1 once, the condition $q \geq 5$ implies that this is a good colouring.) We denote the finite number of all possible such colourings by $gc(r, n)$ and let them be described by $\mathcal{L}'$-formulas $Col_{r,n}^{s,n+1}(x_0, x_1, \dots, x_n, x_{n+1})$, for $1 \leq s \leq gc(r, n)$. For $n = 0$, we interpret $Col_{0,0}^{s,1}(x_0, x)$ and $Col_{1,0}^{s,1}(x_0, x)$ to mean that P gives a good colouring on the line generated by $\{x_0, x\}$ extending the colourings $\neg P$ and P respectively on x_0, and we let $gc(0, 0)$ and $gc(1, 0)$ denote the number of such colourings.

Now let T_A denote the theory of affine space over F. Then the $\mathcal{L}_{\mathcal{A}}$-formulas $x \in cl(y_1, ..., y_n)$ and $\dim(y_1...y_n) = m$ ($\leq n$) make sense. (We note here that the

dimension defined throughout the paper is one more than the usual affine subspace dimension.) The theory T_{LM} is given in the language $\mathcal{L}'$ by the following axioms:

$$T_{LM} \equiv T_A \cup \{LM\} \cup \{F_n : n \geq 0\},$$

where LM(local modularity) is the following sentence in $\mathcal{L}'$:

$$LM \equiv \forall x \forall y (x \neq y \rightarrow \exists z_1 \exists z_2 (z_1 \neq z_2 \wedge P(z_1) \wedge P(z_2) \wedge z_1, z_2 \in cl(x,y))),$$

F_n(colour extension), for $n \geq 1$, is the $\mathcal{L}'$-sentence

$$F_n \equiv \bigwedge_{1 \leq r \leq gc(n)} \forall y_0 \ldots y_n (\dim(y_0 \ldots y_n) = n+1 \wedge Col_{r,n}(y_0, y_1, \ldots, y_n)$$

$$\rightarrow \bigwedge_{1 \leq s \leq gc(r,n)} \exists x (\dim(y_0 \ldots y_n x) = n+2 \wedge Col_{r,n}^{s,n+1}(y_0, y_1, \ldots, y_n, x))),$$

and F_0 is the sentence

$$F_0 \equiv \forall y (\neg P(y) \rightarrow \bigwedge_{1 \leq s \leq gc(0,0)} \exists x (\dim(yx) = 2 \wedge Col_{0,0}^{s,1}(y,x)))$$

$$\wedge \forall y (P(y) \rightarrow \bigwedge_{1 \leq s \leq gc(1,0)} \exists x (\dim\ cl(yx) = 2 \wedge Col_{1,0}^{s,1}(y,x))).$$

We first show that the theory T_{LM} is consistent. This can be done by a union of chains argument inside a model V of T_A. We first point out the following on *any* geometry.

Remark 2.1. Suppose that (S, cl) is a geometry. Let $\{A_i \in S \mid i \in I\}$ be independent over $B \subseteq S$, i.e., for finite $\bar{a}_i \in A_i$, $\dim(\bar{a}_i/B) = \dim(\bar{a}_i/B \cup \bigcup_{j \neq i} A_j)$. Then:

   (1) $cl(B) = cl(BA_{i_1} \ldots A_{i_n}) \cap cl(BA_{j_1} \ldots A_{j_m})$ for $i_1, \ldots, i_n, j_1, \ldots j_m$ distinct.
   (2) Each line in $cl(B \cup \bigcup_i A_i)$ either already lies in one of the $cl(BA_i)$, or hits $\bigcup_i cl(BA_i)$ at most twice.

Proof. The first statement clearly comes from the fact that

$$\dim(\bar{a}_{i_1} \ldots \bar{a}_{i_n}/B) = \dim(\bar{a}_{i_1} \ldots \bar{a}_{i_n}/B \cup \bigcup_{j \neq i_1, \ldots, i_n} A_j).$$

For the second, assume that a line in $cl(B \cup \bigcup_i A_i)$ hits $\bigcup_i cl(BA_i)$ 3 times, but does not lie in any $cl(BA_i)$. Let $\{x, y, z\}$ be the hit points. Then clearly $x \in cl(BA_k)$, $y \in cl(BA_m) \setminus cl(BA_k)$, $z \in cl(BA_n) \setminus (cl(BA_m) \cup cl(BA_k))$ for distinct k, m, n, whereas $z \in cl(xy)$ and $z \in cl(BA_n) \cap cl(BA_k A_m)$. But this contradicts (1). $\quad\square$

Now we proceed in a series of steps.

Step 1. Choose a $y_0 \in V$ and label y_0 by P. Now add independent elements $\{x_1, \ldots, x_{gc(1,0)}\}$ over y_0 and colour each $cl(y_0 x_i)$ to witness F_0 for y_0. Let $W_1 = cl(y_0, x_1, \ldots, x_{gc(1,0)})$. Now colour all points in $W_1 \setminus \bigcup_i cl(y_0 x_i)$ with P. Then W_1 satisfies LM; if a line in W_1 already lies in one of $cl(y_0 x_i)$, then we are done by construction. Otherwise, by the above remark (2), the assumption $\mathrm{Card}(F) \geq 5$ ensures that we can pick up at least 3 points on the line labelled with P.

Step 2. We now want to satisfy F_0 in $W_1 \setminus \{y_0\}$. So we repeat the method of Step 1 for each point in $W_1 \setminus \{y_0\}$. Namely, for $a \in W_1 \setminus \{y_0\}$, find independent $\{z_1, \ldots, z_{gc(k,0)}\}$ that are also independent from W_1 ($k = 0, 1$ depends on whether $a \in \neg P$ or $a \in P$). Then colour all the points in $cl(W_1 \cup \{z_i\}_i)$ with P, except

$\bigcup_i cl(az_i)$ and W_1. W_1 is already coloured. Colour each $cl(az_i)$ to witness F_0 for a. Then, for the same reason as in Step 1, with Remark 2.1, $cl(W_1 \cup \{z_i\}_i)$ satisfies LM. Now for $b \in W_1 \setminus \{y_0, a\}$, do the same thing over $cl(W_1 \cup \{z_i\}_i)$, and so on. We then obtain a subspace $U_0 \supset W_1$ such that U_0 satisfies LM and, for all $x \in W_1$, F_0 is witnessed inside U_0.

Step 3. Now by modifying Steps 1 and 2, find U_1 ($\supset U_0$) to make F_1 hold for W_1 inside U_1. Namely, for given distinct $x, y \in W_1$, choose independent points $\{w_k\}_k$ independent from U_0 to extend the good colouring on $cl(xy)$ to all possible good colourings on distinct affine planes $cl(xyw_k)$. (These colourings are compatible, because $cl(xy) = U_0 \cap cl(xyw_k) = cl(xyw_i) \cap cl(xyw_j)$ (for $i \neq j$ by Remark 2.1.1).) Then colour all the points in $cl(U_0 \cup \{w_k\}_k)$ other than U_0 and $\bigcup_k cl(xyw_k)$ with P. Again, as in Step 1, the fact that $\mathrm{Card}(F) \geq 5$ and Remark 2.1.2 guarantee that $cl(U_0 \cup \{w_k\}_k)$ satisfies LM. For the next line from W_1 other than $cl(xy)$, repeat the process over $cl(U_0 \cup \{w_k\}_k)$. By iterating this argument, we obtain a chain of spaces $W_1 \subset U_0 \subset \ldots \subset U_n$ so that U_n witnesses $\{F_0, \ldots, F_n\}$ for W_1, and U_n satisfies LM. We can choose n large enough (depending on the dimension of W_1) so that in fact U_n witnesses all F_i (for $i \geq 0$) for W_1.

Rename U_n as W_2, and repeat the previous steps for W_2. Continuing in this way, we obtain a chain of spaces $y_0 \subset W_1 \subset W_2 \subset \ldots W_i \subset \ldots$ such that each W_i satisfies LM (by 2.1.2), and all F_i ($i \geq 0$) are witnessed for W_i inside W_{i+1}. Let $W = \bigcup_i W_i$; then by construction $W \models T_A \cup \{LM\} \cup \{F_i : i \geq 0\}$. Hence T_{LM} is consistent.

We now have the following theorem.

Theorem 2.2. T_{LM} *is ω-categorical, complete, and has quantifier elimination. Moreover, T_{LM} is a model companion of $T_A \cup \{LM\}$.*

The axioms $\{F_n : n \geq 0\}$ of colour extensions allow a partial isomorphism to be continued. Hence quantifier elimination follows. In particular, for an independent tuple $(a_0, \ldots, a_n)$, $tp(a_0, \ldots, a_n)$ is determined by the colouring on $cl(a_0, a_1, \ldots, a_n)$. ω-categoricity and hence completeness also follow by a straightforward back and forth partial isomorphism argument. Now, given a model M of $T_A \cup \{LM\}$, by doing the above process starting with M, one can obtain an extension N of M which is a model of T_{LM}. Hence T_{LM} is a model companion of $T_A \cup \{LM\}$.

We now want to show that T_{LM} is a simple theory. As we observed, the condition $\mathrm{Card}(F) \geq 4$ is enough to obtain the previous theorem. However, we shall see that we need $\mathrm{Card}(F) \geq 5$ to ensure T_{LM} is simple, particularly for the Independence Theorem. Let us first define a notion of independence on subsets A, B, C of a saturated model $\bar{V}$ of T_{LM} by

$$A \underset{C}{\downarrow} B$$

iff A is *independent* from B over C in the sense of affine space (i.e., in $\mathcal{L}_A$). It follows easily that this notion is automorphism-invariant and satisfies symmetry, transitivity, finite character and local character. We need to check the extension axiom and the Independence Theorem. Below we keep using cl to denote affine closure.

Nonforking extension. So let $\bar{a}$ and B, C be a tuple and two small subsets of $\bar{V}$. Then clearly there is $\bar{a}'$ such that $\bar{a}'$ is ($\mathcal{L}_A$-)independent from C over B. Hence, as in 2.1.1, $cl(\bar{a}'B) \cap cl(BC) = cl(B)$. Therefore the colouring of $cl(\bar{a}'B)$ according

to $cl(\bar{a}B)$ is compatible with the colouring on $cl(BC)$. Again colour all the points in $cl(\bar{a}'BC)$ with P except $cl(\bar{a}'B)$ and $cl(BC)$. By an argument similar to that in Remark 2.1.2, $cl(\bar{a}'BC)$ with the new colouring satisfies LM. Hence by the axioms $\{F_n\}$ we can embed $\bar{a}'$, while fixing BC with the new colouring, into $\bar{V}$, and then get the ($\mathcal{L}'$-)nonforking extension of $tp(\bar{a}/B)$ over BC.

The Independence Theorem for types. We will verify this over any closed set B. Here types are $\mathcal{L}'$-types. Let $\bar{a}$ and $\bar{b}$ be independent tuples over B, and $\bar{c}, \bar{d}$ tuples such that

$$\bar{c} \underset{B}{\perp} \bar{a} \ , \qquad \bar{d} \underset{B}{\perp} \bar{b}$$

and $tp(\bar{c}/B) = tp(\bar{d}/B)$. By the extension property above, we can choose a tuple $\bar{e}$ independent from $\bar{b}$ over $B\bar{a}$ such that $tp(\bar{c}/B\bar{a}) = tp(\bar{e}/B\bar{a})$. Then $\{\bar{e}, \bar{a}, \bar{b}\}$ is independent over B. Consider $cl(B\bar{e}\bar{a}\bar{b})$. We now newly colour $cl(B\bar{e}\bar{a}\bar{b})$ while leaving the (old) colouring on $cl(B\bar{e}\bar{a}) \cup cl(B\bar{a}\bar{b})$, as follows. First, colour $cl(\bar{e}; B\bar{b})$ according to $cl(\bar{d}; B\bar{b})$. Since $tp(\bar{e}/B) = tp(\bar{d}/B)$, on $cl(B\bar{e}\bar{b}) \cap (cl(B\bar{a}\bar{b}) \cup cl(B\bar{a}\bar{e})) = cl(B\bar{b}) \cup cl(B\bar{e})$ (Remark 2.1.1), the two colourings are the same. Hence the new colouring is compatible with the colouring on $cl(B\bar{e}\bar{a}) \cup cl(B\bar{a}\bar{b})$. Then colour the rest of the points in $cl(B\bar{e}\bar{a}\bar{b}) \setminus (cl(B\bar{a}\bar{b}) \cup cl(B\bar{e}\bar{a}) \cup cl(B\bar{e}\bar{b}))$ with P. Now, since every line in $cl(B\bar{e}\bar{a}\bar{b})$ either already lies in $cl(B\bar{a}\bar{b}) \cup cl(B\bar{e}\bar{a}) \cup cl(B\bar{e}\bar{b})$ or hits the set at most 3 times, the condition $\mathrm{Card}(F) \geq 5$ ensures that this new colouring on $cl(B\bar{e}\bar{a}\bar{b})$ is a good colouring. Since we can embed $\bar{e}$ with the new colouring on $cl(B\bar{e}\bar{a}\bar{b})$ while fixing $B\bar{a}\bar{b}$, the Independence Theorem for types holds. (Hence the notion of type coincides with that of Lascar strong type.)

So T_{LM} is simple. Moreover, in $\bar{V}(\models T_{LM})$, since $\mathcal{L}'$-independence is the same as $\mathcal{L}_A$-independence, T_{LM} is an SU-rank-1 theory.

Theorem 2.3. *T_{LM} is simple with SU-rank 1, having the same forking independence notion (and so algebraic relations) as T_A.*

Now we take the solution set $\bar{V}(P) = \{a \in \bar{V} \mid P(a)\}$. Since we checked the Independence Theorem over any closed set, in particular over $\emptyset$, $\bar{V}(P)$ is a 1-(Lascar strong) type. We also have that $\bar{V}(P)$ is simple, rank-1 and ω-categorical. It remains to show that $\bar{V}(P)$ is locally modular but not affine.

Local modularity. Choose $d \in \bar{V}(P)$. Let ef be a pair and B a set from $\bar{V}(P)$ such that $\dim_d(ef) = 2$ and $\dim_d(ef/B) = 1$ inside $\bar{V}(P)$. Working in $\bar{V}$, we also have $\dim_d(ef) = 2$ and $\dim_d(ef/B) = 1$. Since the geometry of $\bar{V}$ is affine, $acl(def) \cap acl(dB) = acl(dg)$, where $g \neq d$ is in $\bar{V}$. By the axiom LM for $\bar{V}$ we can assume that $g \in \bar{V}(P)$. Hence, $\dim_d(cl_1(def) \cap cl_1(dB)) = 1$, where $cl_1(-) = acl(-) \cap \bar{V}(P)$ is the closure inside $\bar{V}(P)$. So $\bar{V}(P)$ is a locally modular geometry.

Not affine. For this it is sufficient to observe that a good colouring of a line in $\bar{V}$ can label r points where $2 \leq r \leq \mathrm{Card}(F)$.

So we have constructed a nontrivial, locally modular ω-categorical rank-1 structure/type whose geometry is neither affine nor projective over a finite field. Another example of this kind can be constructed, as mentioned previously, by repeating the same process on any projective space over a finite field of cardinality ≥ 4. (This is, in a sense, a necessary requirement. See 3.13.) As is well known, there is no such example in a stable structure, and also by 1.9 there is no example of a modular one.

3. 1-BASED TYPE

Let us recall some basic facts on 1-basedness. In this section, algebraic closure is taken over $\bar{M}^{eq}$. The following fact is folklore:

Fact 3.1 (T simple). The following are equivalent:

    (1) T is 1-based, i.e., any sets A and B are independent over $acl(A) \cap acl(B)$.

    (2) For any tuple $\bar{c}$ and a set A, $cb(\bar{c}/A) \subseteq acl(\bar{c})$.

    (3)

$$A \underset{C}{\;\downarrow\;} B$$

       iff $acl(AC) \cap acl(BC) = acl(C)$.

    (4) For any indiscernible sequence $I = \langle \bar{c}_i | i \in \omega \rangle$, $I \setminus \{\bar{c}_0\}$ is a Morley sequence over $\bar{c}_0$.

Proof. We only sketch the proof. The equivalence of 1 and 2 is well known. In 3, the implication from left to right is true for *any* simple theory, and the reverse implication is clearly equivalent to 1. $4 \Rightarrow 2$ follows from the fact that $cb(\bar{c}/A)$ is in the algebraic (bounded) closure of a Morley sequence from $Lstp(\bar{c}/A)$. We supply a rather detailed proof for $1 \Rightarrow 4$. Note first that 1 implies

$$\bar{c}_0 \underset{A}{\;\downarrow\;} \bar{c}_1 \bar{c}_2 ...$$

where $A = acl(\bar{c}_0) \cap acl(\bar{c}_1 \bar{c}_2 ...)$. Then since

($*$)   any indiscernible sequence over B

                 is indiscernible over $acl(B)$ (even over $bdd(B)$),

it follows that $tp(\bar{c}_1 \bar{c}_2 .../acl(\bar{c}_0)) = tp(\bar{c}_2 \bar{c}_3 .../acl(\bar{c}_0))$. Hence

$$A = acl(\bar{c}_0) \cap acl(\bar{c}_2 \bar{c}_3 ...).$$

Again by ($*$), $tp(\bar{c}_0/acl(\bar{c}_2 \bar{c}_3 ...)) = tp(\bar{c}_1/acl(\bar{c}_2 \bar{c}_3 ...))$, and then $A = acl(\bar{c}_1) \cap acl(\bar{c}_2 c_3 ..)$. Then, using 1 again,

$$\bar{c}_1 \underset{A}{\;\downarrow\;} \bar{c}_2 \bar{c}_3 ... \quad .$$

From transitivity,

$$\bar{c}_1 \underset{\bar{c}_0 A}{\;\downarrow\;} \bar{c}_2 \bar{c}_3 ...$$

and

$$\bar{c}_1 \underset{\bar{c}_0}{\;\downarrow\;} \bar{c}_2 \bar{c}_3 ...$$

since $A \subseteq acl(\bar{c}_0)$. Thus 4 follows. $\qquad \square$

When, for example, a theory T is 1-based of finite SU-rank (e.g., any 1-based ω-categorical T; see the remark before 3.23), then, since every tuple can be "built up" (coordinatized) from rank-1 tuples, understanding the properties of rank-1 types is essential. This is the topic of the current section.

From now on, we further assume that D is the solution set of a *rank-1* Lascar strong type with the usual algebraic closure relation. We shall observe a strong connection between 1-basedness and modularity. We first note the following folklore, and supply a simple proof.

Fact 3.2. If D is locally modular, then D is 1-based.

Proof (Remark). Suppose that $W_0, W_1 \subseteq acl(X) \cap acl(Y)$ and

$$X \underset{W_0}{\perp} Y$$

and

$$X \underset{W_1}{\perp} Y \ .$$

Then $acl(W_0) = acl(W_1)$, since

$$W_1 \underset{W_0}{\perp} Y$$

and

$$W_0 \underset{W_1}{\perp} Y \ .$$

(This remark will be used occasionally.)

Claim 1. Choose $p \in D$. For any $A \subseteq D^{eq}$, there is $B \subseteq D$ containing p such that A and B are interalgebraic over p. Indeed, there is $A_0 \subseteq D$ such that $A \subseteq dcl(A_0)$. Let A_1 ($\subseteq D$) realizing $tp(A_0/Ap)$ be independent from A_0 over Ap. Now, since D is modular over p, if we let $B = acl(A_0p) \cap acl(A_1p) \cap D$, then A_0, A_1 are independent over B ($= Bp$). By the remark above, $acl(Ap) = acl(Bp)$. Hence the claim is proved.

Claim 2. Given $B_0, B_1 \subseteq D^{eq}$, B_0 and B_1 are independent over $B_2 = acl(B_0p) \cap acl(B_1p)$ for given $p \in D$ (i.e., D is 1-based over p).

Suppose not; say B_0 forks with B_1 over B_2. By Claim 1, there are $C_0, C_1, C_2 \subseteq D$ containing p such that C_i is interalgebraic with B_i over p. In particular, C_0 forks with C_1 over C_2. (Note that $p \in B_2, C_2$.) Now since D is modular over p, there must exist $d \in acl(C_0C_2) \cap acl(C_1C_2) \cap D$ such that $d \notin acl(C_2) = acl(B_2)$. However, since $acl(C_0C_2) = acl(acl(C_0) \cup acl(C_2)) = acl(B_0pB_2) = acl(B_0B_2)$ and similarly $acl(C_1C_2) = acl(B_1B_2)$, it follows that

$$d \in acl(C_0C_2) \cap acl(C_1C_2) = acl(B_0B_2) \cap acl(B_1B_2) \subseteq B_2.$$

This is a contradiction. Hence Claim 2 is proved.

Now we can finish the proof of the fact. Let $I = \langle \bar{c}_i \mid i \in \omega \rangle$ be an indiscernible sequence from D^{eq}. If $q \in D$ is independent from I, then after a $\bar{c}_0$ conjugation we may assume that I is indiscernible over q. Now by Claim 2 (with a similar proof to 3.1, $1 \Rightarrow 4$), $I \setminus \{\bar{c}_0\}$ is a Morley sequence over $q\bar{c}_0$. Since q is independent from I, by transitivity $I \setminus \{\bar{c}_0\}$ is Morley over $\bar{c}_0$. Hence, D is 1-based by 3.1. $\qquad \square$

The converse of 3.2 is false in general. As in [21], one can easily construct a counterexample by adding a generic predicate $P(x)$ to a vector space over a finite field. In Remark 3.7, we will deal with this example again.

Definition 3.3. We use $G(D)$ to denote the collection of all rank-1 elements (over $\text{dom}(D)$) in D^{eq}.

Clearly $D \subseteq G(D)$, and for any small set A containing $\text{dom}(D)$, $G(D)_A$ forms a pregeometry with the usual algebraic closure relation over A. We recall the definition of *linearity*.

Definition 3.4. For a set A containing $\text{dom}(D)$, we say that D_A $(G(D)_A$, resp.) is *linear* if for any two points $a, b \in D$ $(G(D)$, resp.) and parameters B with $SU(ab/AB) = 1$ we have $SU(C/A) \leq 1$, where $C = cb(ab/AB)$.

Remark 3.5. (1) When defining linearity above, it is safe to assume that the parameters B form a subset of D (or $G(D)$), since we can preserve the rank by taking a Morley sequence as the parameter.

(2) If a canonical base $Cb(p)$ of p $(= Lstp$ of some tuple from D^{eq} over parameters containing $\text{dom}(D)$) has rank 1, then we can find a singleton c in D^{eq} which can be considered as the canonical base *up to interalgebraicity.* Namely, since T has elimination of hyperimaginaries, any hyperimaginary e of ordinal SU-rank is interdefinable with a sequence $(e_i : i < \omega)$ of imaginaries, and there must be a subsequence such that $e \subset acl(e_1, \ldots e_n)$. In this case, we can take $e = Cb(p)$ and find an element c in the sequence such that $SU(c) = 1$ and $Cb(p) \subseteq acl(c)$. From now on, for this case $(SU(Cb(p)) = 1)$, when we refer to a *canonical case* of p, we actually mean *any* such $c \in D^{eq}$. Since we are mainly interested in the geometry of D, as we shall see, this abuse of notation will not cause any problems. (If T does not have elimination of hyperimaginaries, then $G(D)$ is the set of rank-1 hyperimaginaries living in D^{heq}.)

(3) We now single out an easy but useful argument in the proof of Claim 1 in 3.2. Let $D \subseteq F \subseteq G(D)$, and let $(F, cl(-) \cap F)$ form a modular subpregeometry. Then for any $A \subseteq D^{eq}$, there is $B \subseteq F$ such that A and B are interalgebraic: For each $c \in D^{eq}$, there is a finite tuple $\bar{d} \in D$ such that $c \in dcl(\bar{d})$. Hence, there is $A_0 \subseteq D \subseteq F$ such that $A \subseteq dcl(A_0)$. Let A_1 $(\subseteq F)$ realizing $tp(A_0/A)$ be independent from A_0 over A. Let $B = acl(A_0) \cap acl(A_1) \cap F$. Since F is modular, A_0 and A_1 are independent over B. Then, by the remark in the proof of 3.2, $acl(B) = acl(A)$.

Hence, if $a \in G(D)$, then there is a tuple $\bar{b} = (b_1, ..., b_n) \in F$ such that $a, \bar{b}$ are interalgebraic. In particular, $b_1 \in acl(a)$; so a forks with b_1. Since $rk(a) = 1$, $a \in acl(b_1)$, i.e., a, b_1 are interalgebraic. This says that the canonical geometries of F and G are identical.

Theorem 3.6. *The following are all equivalent.*

(1) D (D^{eq}) *is 1-based.*

(2) D *is linear.*

(3) $G(D)$ *is linear.*

(4) $G(D)$ *is modular.*

(5) $G(D)_A$ *is linear, for any (some) small A.*

(6) $G(D)_A$ *is modular, for any (some) small A.*

Proof. The proof of $1 \Rightarrow 2$ is the same as the stable case. For completeness, we supply a proof. Let D be 1-based. Suppose that $SU(Lstp(ab/B)) = 1$ $(a, b, B$ from D). By 1-basedness, $C = cb(ab/B)$ is in $acl(ab)$. Now,

$$SU(abC) = SU(ab/C) + SU(C) = 1 + SU(C) = SU(C/ab) + SU(ab) \leq 2.$$

Hence $SU(C) \leq 1$. Thus D is linear.

$2 \Rightarrow 3$. Assume D is linear. Now let $x, y \in G(D)$, $B \subseteq G(D)$ and $SU(xy/B) = 1$ while

$$(1) \qquad\qquad\qquad SU(xy) = 2.$$

Let $C = cb(xy/B)$. Then

$$(\star) \qquad C \subseteq acl(B) \text{ and } SU(xy/C) = 1 \quad \left(\begin{array}{c} xy \underset{C}{\overset{}{\downarrow}} B \end{array} \right).$$

We will show that $C \subseteq acl(xy)$. Now $\{x, y\}$ is in the definable closure of a finite independent subset of D. In fact, we can find an independent subset $F \cup \{a, b\}$ of D such that $\{x, y\}$ and $\{a, b\}$ are interalgebraic over F, and $F, \{x, y\}$ are independent. Then there is $F' \subseteq D$ such that F and F' realize the same type over xy, and

$$F' \underset{xy}{\overset{}{\downarrow}} abBF \quad ;$$

hence

$$(2) \qquad F' \underset{\emptyset}{\overset{}{\downarrow}} xyabBF$$

Now, since F and F' realize the same type over xy, there is $\{a'b'\}$ such that abF and $a'b'F'$ realize the same type over xy. Note that

$$(3) \qquad acl(xyF') = acl(a'b'F').$$

Claim 1. $SU(a'b'/F'B) = 1$.
First we note that

$$a'b' \underset{F'}{\overset{}{\downarrow}} B \quad .$$

(Otherwise, by (2) and (3), $\{x, y\}$ is independent from B, which contradicts (1).) Now if $SU(a'b'/F'B) = 0$, then by (3), $x, y \in acl(F'B)$, and then by (2), $x, y \in acl(B)$, again contradicting (1). Hence $SU(a'b'/F'B) = 1$. Claim 1 is proved.

Therefore, by Claim 1 and the linearity of D, $SU(a'b'/acl(a'b') \cap acl(F'B)) = 1$. Now

$$acl(a'b') \cap acl(F'B) \subseteq acl(a'b'F') \cap acl(F'B) \subseteq acl(F'B).$$

Thus, by transitivity, Claim 1, and (3),

$$(4)$$
$$SU(a'b'/W) = 1, \text{ where } W = acl(a'b'F') \cap acl(F'B) = acl(xyF') \cap acl(F'B).$$

Claim 2. $SU(xy/W) = SU(xy/WC) = 1 \ (= SU(xy/C))$.
Since $F' \subseteq W$, by (3) and (4),

$$xy \underset{F'}{\overset{}{\not\downarrow}} W \quad .$$

Hence $SU(xy/W) = 0$ or 1. But if $SU(xy/W) = 0$ then by (2) $xy \in acl(B)$, which is not the case. Thus $SU(xy/W) = 1$. Similarly, since $SU(xy/C) = 1$, if $SU(xy/WC) \neq 1$, then by $(\star)$ and (4), $xy \in acl(WC) \subseteq acl(CF'B) \subseteq acl(F'B)$. Then by (2), $xy \in acl(B)$. This contradicts (1) again. Hence $SU(xy/WC) = 1$, and Claim 2 is proved.

Now since $Lstp(xy/C)$ is a canonical type, by Claim 2, $C \subseteq acl(W)$. Therefore by (4), $C \subseteq acl(xyF')$. Then $C \subseteq acl(xy)$, since (2) with $(\star)$ says that

$$C \underset{xy}{\overset{}{\downarrow}} F' \quad .$$

We have proved that $G(D)$ is linear.

$3 \Rightarrow 6$ and 4. Suppose that $G(D)$ is linear. Let $a, b \in G(D)$, and $SU(ab/A) = 2$, whereas $SU(ab/AB) = 1$ for $B \subseteq G(D)$. Now, there is a Morley sequence I of $tp(ab/AB)$ in $G(D)$ such that $SU(ab/AB) = SU(ab/ABI) = SU(ab/I) = 1$. By linearity of $G(D)$, $c \in acl(ab) \cap acl(I) \cap acl(AB)$ where $c = cb(ab/I) = cb(ab/AB)$, and $SU(c) = 1$. Hence $c \in G(D)$ by the previous Remark 3.5.2. Moreover, by the rank calculation of $SU(cab/A)$ we have $SU(c/A) = 1$, i.e., $c \notin acl(A)$. Thus $G(D)_A$ is modular by 1.3.

$4 \Rightarrow 1$. This proof will be similar to the proof of Fact 3.2. Suppose not, say there are $a, b \in D^{eq}$ that are dependent over $U = acl(a) \cap acl(b)$. Now since $G(D)$ is modular, by Remark 3.5.3, there are $a_0, b_0, U_0 \subseteq G(D)$ that are interalgebraic with a, b, U respectively. In particular, a_0 forks with b_0 over U_0. Then, again by modularity, there must exist $c_0 \in acl(a_0 U_0) \cap acl(b_0 U_0) \cap G(D)$ such that $c_0 \notin acl(U_0) = acl(U)$. However, since $acl(a_0 U_0) = acl(acl(a_0) \cup acl(U_0)) = acl(aU)$ and similarly $acl(b_0 U_0) = acl(bU)$, it follows that $c_0 \in acl(aU) \cap acl(bU) \subseteq U$, a contradiction. Therefore, $4 \Rightarrow 1$ is proved.

$3 \Rightarrow 5$. Suppose that $G(D)$ is linear. Let us show that $G(D)_A$ is linear. Let $a, b \in G(D)$, and $SU(ab/B) = 1$ for B containing A. Now there is a Morley sequence I of $tp(ab/B)$ in $G(D)$ such that $SU(ab/B) = SU(ab/BI) = SU(ab/I) = 1$. Then by linearity of $G(D)$ we have $SU(c/A) \leq SU(c) \leq 1$ where $c = cb(ab/I) = cb(ab/B)$. Thus $G(D)_A$ is linear.

5 or $6 \Rightarrow 3$. The proof will be similar to [17, 2.2.6 (i) $\Rightarrow$ (ii)]. Let $a, b \in G(D)$, $B \subseteq G(D)$ and $SU(ab/B) = 1$. Let $c = cb(ab/B)$. If $SU(ab) = 1$, i.e., $\{a, b\}$ is independent from B, then, since $c \in acl(\emptyset)$, there is nothing to prove. Thus we can assume that $SU(ab) = 2$. Now we can further assume that abB is independent from A. Hence, $SU(ab/BA) = 1$, whereas $SU(ab/A) = 2$. Then by modularity (or linearity) of $G(D)_A$, there is $d \in acl(abA) \cap acl(AB)$ with $SU(d/A) = 1$. Hence d and ab are dependent over A. Thus $SU(ab/Ad) = 1 = SU(ab/AB)$. Therefore $c = cb(ab/AB) \in acl(Ad)$, and since $SU(d/A) = 1$, by a rank calculation $SU(c/A) \leq 1$. Since c and A are independent, $SU(c) \leq 1$. $\qquad \square$

Remark 3.7. (1) Similarly, D is linear iff D_A is linear for any (some) A. We note that the geometry of a nonforking extension of D over A is a mere subgeometry of D_A. However, one can still see the fact that linearity (1-basedness) is invariant via parallelism. Indeed, let q be a nonforking extension of D over A. It suffices to check that D is linear when q is linear. The proof is the same as 3.6 (5) $\Rightarrow$ (3) with the trick that when an independent pair (a_0, a_1) is chosen from D, by amalgamation there is (b_0, b_1) such that $tp(a_0, a_1) = tp(b_0, b_1)$ and each b_i realizes q.

However, as is known, (local) modularity is not invariant via parallelism in general. (Frank O. Wagner was the first person to point this out.) Affine or projective space equipped with a binary random graph relation serves as an example, for a reason similar to that described in (2).

(2) Vassiliev introduced a notion of *generic substructure* $\bar{M}_1$ of $\bar{M}$ for a supersimple SU-rank-1 theory (the submodel $\bar{M}_1$ of $\bar{M}$ does not have to be a *small* submodel). From [21], it follows that when we consider D as the universe of a model $\bar{M}$, D is 1-based iff $D_{\bar{M}_1}$ is modular. (He also proved the equivalence of 1 and 2 in Theorem 3.6 using a generic pair argument.)

However, in general, even if D is 1-based, D_A need not be modular over any (small) set A. Indeed, let V be a (saturated) affine space over a field, and let P be a generic predicate on V, as in [5]. Then there are two 1-types over $\emptyset$

isolated by $P(x)$ and $\neg P(x)$, both of which have SU-rank 1. By the genericity of P, both 1-types are 1-based. V is already non-modular over $\emptyset$. Now let W be a nonempty small subspace of V and consider a line l, spanned by $\{a, b\}$ in V, independent from W; so $\dim(l/W) = 2$. Now let l', spanned by $\{a'b'\}$, be parallel to l such that $\dim(l/Wl') = 1$. Then there is $c \in V \setminus W$ such that $acl(cW) = acl(lW) \cap acl(l'W)$. However, since W is small, by genericity of P, we can find l, l' such that $\{a, b, a', b'\} \subseteq P$, whereas $P \cap acl(cW) = P \cap W$. Then clearly P cannot be modular over W, and similarly for $\neg P$. (We noticed later that a similar example is described in [21].)

To conclude, there are examples of nontrivial, 1-based D that are either not locally modular, or locally modular but neither affine nor projective (section 2) and examples of 1-based D that are affine/projective but not stable (e.g. an affine space with a binary random graph relation).

Since triviality of a (pre)geometry implies modularity, we have the following lemma by Remark 3.5.3.

Lemma 3.8. *D is trivial iff $G(D)$ is trivial.*

Theorem 3.9. *Suppose D is 1-based. Then either D is trivial, or the geometry of $G(D)$ is projective geometry over a division ring.*

Proof. Assume that D is nontrivial. By the above lemma and Fact 1.5, it suffices to show that any independent pair $(a, b) \in G(D)$ (that is, $\dim(ab) = 2$) is nontrivial in $G(D)$ (i.e., there is $c \in cl(ab) \setminus (cl(a) \cup cl(b))$ where cl is the closure of $G(D)$).

Claim 1. Any independent pair $(c, d) \in D$ is nontrivial in $G(D)$. Indeed, since $G(D)$ is modular, if there is an independent pair from D trivial in $G(D)$, then any independent pair from D is trivial in $G(D)$. (Otherwise, by type amalgamation one can find a non-modular triangle that violates modularity of $G(D)$. See the proof of 1.8.) Then any independent triple $(a, b, c) \in D$ is also trivial in $G(D)$ (otherwise, say $d \in cl(abc) \setminus (cl(a) \cup cl(b) \cup cl(c))$, then ab and cd violate modularity of $G(D)$). Iterating this argument shows that D is trivial, which is not the case.

Claim 2. Any independent pair $(a, b) \in G(D)$ with $a \in D$ is nontrivial in $G(D)$.

Suppose not, say the independent pair $(a, b) \in G(D)$ with $a \in D$ is trivial. Now since $b \in G(D)$, there is an independent tuple $\bar{c} = (c_1, ..., c_n) \in D$ such that $b \in cl(c_1, ..., c_n)$. By Claim 1, we can assume $n > 1$. We can further assume that any $n-1$ subtuple of $\bar{c}$ is independent from b, and $\{a, b, c_1, ..., c_{n-1}\}$ is independent. Hence

$$(*) \qquad\qquad\qquad \dim(a, b, \bar{c}) = n + 1.$$

By Claim 1, there is $d \in cl(ac_1) \setminus cl(a) \cup cl(c_1)$. It follows that $d \notin cl(c_2, ..., c_n)$ (otherwise $a \in cl(\bar{c})$, violating (*)). Hence the dimension of the set $B = \{d, c_2, ..., c_n\}$ is n, whereas by (*) and triviality of (a, b), it can easily be seen that $cl(B) \cap cl(ab) = \emptyset$. However, by modularity of $G(D)$, this contradicts (*). Hence Claim 2 follows.

Claim 3. Any independent pair $(d, e) \in G(D)$ is nontrivial.

Suppose not, say the independent pair $(d, e) \in G(D)$ is trivial. There is a point $c \in D$ that is independent from (d, e). By Claim 2, (c, d) and (c, e) are nontrivial. Again this violates modularity of $G(D)$, since (c, d, e) forms a non-modular triangle. This finishes the proof of Claim 3 and the theorem. □

The theorem shows Vassiliev's result cheaply. Namely, the geometry of 1-based D is a subgeometry of projective geometry over a division ring.

From now on D will be 1-based.

The (pre)geometry of $G(D)$ is not definable (it is a union of definable sets). However, with the reducibility condition we will discuss below, $G(D)$ can be dealt with as a definable object, at least for the ω-categorical case.

Definition 3.10. We say $G(D)$ is *reducible into* (or has a reduction in) D^k if for any point $a \in G(D)$ there is a k-tuple $\bar{b} \in D$ such that $a \in acl(\bar{b})$.

When D is locally modular, $G(D)$ has a reduction in D^2. In fact, more is true.

Lemma 3.11 (D locally modular). *For $c \in G(D)$ and any $d \in D$, there is $e \in D$ such that $c \in acl(de)$.*

Proof. By Claim 1 in the proof of Fact 3.2, there is $\bar{a} = (a_1, ..., a_n) \in D$ such that $acl(\bar{a}d) = acl(dc)$. If $\bar{a} \in acl(d)$, then $c \in acl(d)$, and we are done. Hence we may assume that some a_i, say a_1, is not in $acl(d)$. Then, since $rk(a_1) = 1$, a_1 forks with c over d. By transitivity, c forks with a_1d. Since $rk(c) = 1$, $c \in acl(da_1)$. $\square$

Now, as by-products, we have the following lemmas.

Lemma 3.12. *Assume that D is nontrivial locally modular, and $p \in D$. The geometry of D_p and the geometry of $G(D)$ are isomorphic as projective geometries.*

Proof. Since $G(D)'$, the geometry of $G(D)$, is that of infinite-dimensional projective space over some ring, $G(D)'_p \cong G(D)'$. Now since $D \subseteq G(D)$, clearly D_p is a subpregeometry of $G(D)_p$. Also, by the previous lemma, given $x \in G(D) \setminus cl(p)$, there is $e \in D$ such that $cl(px) = cl(ex)$ in $G(D)$. Hence $D'_p \cong G(D)'_p \cong G(D)'$, and the lemma holds. $\square$

Lemma 3.13 (D locally modular). *If $G(D)'$ is projective over a field with $\mathrm{Card}(F) \leq 3$, then D', the geometry of D, is affine or projective.*

Proof. This can be shown by checking the possible cases, using amalgamation and 3.11, working in a projective plane. Let us first prove this when $\mathrm{Card}(F) = 3$.

Claim 1. There is no independent $\{a, b\} \subseteq D'$ such that $cl(ab) \cap D' = \{a, b\}$. Indeed, if there exist such a, b, then by amalgamation we can find $c \in D'$ such that $\dim(abc) = 3$ and any closed set in D' generated by a 2-subset of $\{a, b, c\}$ is itself. Let $b' \in (cl(bc) \cap G(D)') \setminus D'$. Then by 3.11, we can find $d \in D'$ such that $b' \in cl(ad)$. By modularity and the fact that any line in $G(D)'$ has size 4, there are $c' \in cl(ac) \cap cl(bd) \cap G(D)$ and $c'' \in cl(ac) \setminus D'$ such that $c'' \neq a, c, c'$. Then it is easy to see that $a, b, c \notin cl(dc'')$. Hence $\emptyset \neq cl(dc'') \cap cl(bc) \in G(D) \setminus D$, and also $cl(dc'') \cap cl(ab) \in G(D) \setminus D$. Therefore $cl(dc'') \cap D' = \{d\}$, which contradicts 3.11. Hence Claim 1 is proved.

Claim 2. There cannot exist independent pairs $\{a_1, a_2\}, \{b_1, b_2\} \subseteq D'$ such that $\mathrm{Card}(cl(a_1a_2) \cap D') = 3$ and $\mathrm{Card}(cl(b_1b_2) \cap D') = 4$.

Suppose that there are such points a_i and b_i in D'. By amalgamation, there are $\{u, v, w\} \subseteq D'$ such that

$$\mathrm{Card}(cl(uv) \cap D') = \mathrm{Card}(cl(vw) \cap D') = 3, \qquad \mathrm{Card}(cl(uw) \cap D') = 4$$

and

$$\dim(uvw) = 3.$$

Choose $v' \in cl(uv) \setminus D'$ and $w' \in cl(vw) \setminus D'v'$. Then, since $\mathrm{Card}(cl(uw) \cap D') = 4$, by modularity there is $z \in cl(uw) \cap cl(v'w') \cap D'$. Then $1 \leq \mathrm{Card}(cl(v'w') \cap D') \leq 2$, contradicting 3.11 and Claim 1. Hence Claim 2 follows, and the lemma is established for $\mathrm{Card}(F) = 3$ by Fact 1.6.

When $\mathrm{Card}(F) = 2$, by a similar argument, it can be seen that one cannot amalgamate independent (c_1c_2), (c_1c_2) and (d_1d_2) from D' such that

$$\mathrm{Card}(cl(c_1c_2) \cap D') = 2 \text{ and } \mathrm{Card}(cl(d_1d_2) \cap D') = 3.$$

Then 1.6 says that the lemma holds for this case as well. $\qquad\square$

Remark 3.14. If D is a generic subset of locally modular U, then $G(D)$ also has a reduction in D^2. Indeed, let $c \in G(D)$. Then by 3.11, there are $d \in D, e \in U$ such that $c \in acl(de)$. Now there exists (a, b) realizing $tp(de/c)$ such that

$$ab \underset{c}{\;\downarrow\;} de \quad.$$

Then, by the remark in the proof of 3.2, $acl(g) = acl(c)$ where $g = cb(ab/de)$. By the genericity of D, there exists $(a', b') \in D$ such that $tp^-(ab/de) = tp^-(a'b'/de)$ in the language of U without the predicate D. Then $a'b'$ preserves the canonical base of $tp(ab/de)$. Hence $acl(c) = acl(g) \subseteq acl(a'b')$. We are done. (In this case, of course, *strong* reducibility in D^2 as in Lemma 3.11 does not follow.)

In general, the following holds. Hence *any* $G(D)$ is reducible into D^3 for 1-based D.

Theorem 3.15. *Let $u \in G(D)$.*

(1) *Suppose that (x, y) is a fixed independent pair from D. Then $u \in acl(x'y'z)$ where $x', y', z \in D$ and $tp(xy) = tp(x'y')$.*

(2) *Suppose that there are independent $\{x, y\}$ from D such that for any $z \in acl(xy) \cap G(D)$, $z \in acl(z')$ for some $z' \in (acl(xy) \cap D)$, i.e., a full line is already in D. Then $G(D)$ is reducible into D^2.*

(3) *For given $d_1 \in D$, there is $d_2d_3 \in D$ such that $u \in acl(d_1d_2d_3)$.*

Proof. We first prove (1) using induction on n where $u \in G(D)$ has a reduction in D^n. The case $n = 0$ (that is, $u \in acl(\emptyset)$) is trivial. Now suppose that $u \in acl(a_1...a_{n+1})$ $(a_i \in D)$ with the induction hypothesis for n. We will verify (1) for $n + 1$. We can assume that $\{a_1, ..., a_{n+1}\}$ is independent. Now let $g = cb(ua_1/a_2...a_{n+1})$. Thus $SU(g) = 1$ and $g \in G(D)$. Since $a_1 \notin acl(g) \subseteq acl(a_2...a_{n+1})$,

$(*)$ $\qquad\qquad\qquad$ u is in the line generated by $\{g, a_1\}$.

Then, since $g \in acl(a_2...a_{n+1})$, by the induction hypothesis, $g \in acl(bcd)$ for some $c, d \in D$ such that $tp(cd) = tp(xy)$. If g is already in $acl(cd)$, then, by $(*)$, $u \in acl(ga_1) \subseteq acl(cda_1)$. Thus, in this case, the theorem holds. Therefore we only need to consider the case when $g \notin acl(cd)$. We can also clearly assume that $g \notin acl(b)$ (otherwise we are done). Then, since $G(D)$ is modular, we can find $v \in acl(cd) \cap acl(bg) \cap G(D)$ such that

$(\dagger)$ $\qquad\qquad\qquad\qquad$ $g \in acl(bv)$,

and $\dim(vg) = 2$. Also,

$(**)$ $\qquad$ at least one of $\{v, c\}$ or $\{v, d\}$ (say $\{v, d\}$) is independent.

Now, since $d, a_1 \in D$, $Lstp(d) = Lstp(a_1)$. Hence, we can find v' such that

$(\star)$ $$Lstp(vd) = Lstp(v'a_1).$$

In particular, $Lstp(v) = Lstp(v')$. Then by $(*),(**),(\star)$, we can amalgamate types $tp(v'/a_1)$ and $tp(v/g)$, so that we obtain $v'' \models tp(v'/a_1) \cup tp(v/g)$. Hence, there are b', c' such that $tp(vgb) = tp(v''gb')$, $tp(vdc) = tp(v''a_1c')$ (by $(\star)$), and so by $(\dagger)$,

$(\ddagger)$ $$tp(c'a_1) = tp(xy), v'' \in acl(c'a_1) \text{ and } g \in acl(b'v'').$$

Then by $(*),(\ddagger)$,

$$u \in acl(ga_1) \subseteq acl(ga_1v'') \subseteq acl(a_1b'v'') \subseteq acl(c'a_1b').$$

Since $tp(c'a_1) = tp(xy)$, the $(n+1)$-th induction hypothesis for (1) is deduced. Therefore we have proved (1).

(2) comes from (1) and the modularity of $G(D)$.

For (3), use the proof of (1), replacing g with u and a_1 with d_1. $\qquad \square$

Remark 3.16. From the above theorem 3.15, we can identify any element u in $G(D)$ as $e = \bar{d}/E \in G(D)$ (i.e., $acl(u) = acl(e)$) where $\bar{d} \in D^6$ and E is a definable equivalence relation on D^6 (over $\mathrm{dom}(D)$), as follows: As usual we take $\mathrm{dom}(D) = \emptyset$ (by naming $\mathrm{dom}(D)$) for notational simplicity. By 3.15, there are triples $(a, b, c), (a', b', c') \in D^3$ such that $u \in acl(abc)$, $(a'b'c') \models tp(abc/u)$ and

$$abc \underset{u}{\bigcup} a'b'c' \quad .$$

Then, by the remark in 3.2,

$$acl(u) = acl(g)$$

where $g = cb(abc/a'b'c')$. Now, if an automorphism fixes $(abca'b'c')$, then, since the parallelism class of $Lstp(abc/a'b'c')$ is setwise fixed, so is $cb(abc/a'b'c')$, i.e., $g \in dcl(abca'b'c')$. Hence we can identify $g = (abca'b'c')/\bar{F}$ where $\bar{F}$ is a type-definable equivalence relation on D^6. Therefore, since $SU(g) = 1$, as in 3.5.2, there is a *definable* equivalence relation $E \in \bar{F}$ such that $(abca'b'c')/E$ in $G(D)$ and g are interalgebraic. Since u and g are interalgebraic, the result follows.

In particular, if D is ω-categorical, then, since there are only finitely many types and equivalence relations over $\mathrm{dom}(D)$ on D^6, by the above remark, the geometry of $G(D)$ is isomorphic to that of some union of finitely many definable sets in finitely many sorts in $G(D)$. Since the union itself is ω-categorical even as a 1-sorted structure, we have the following results.

Corollary 3.17. *Let D be ω-categorical. Then, D is trivial or the geometry of $G(D)$ is projective over a finite field.*

Definition 3.18. Let $G(D)$ have a reduction in D^k where k is minimal such. (Hence $k \leq 3$.) By a hypersurface of D^k, we mean a $Lstp(a_1...a_k/A)$ of rank $k-1$ ($a_i \in D$). We call a hypersurface of D^k with rank-0 canonical base *rigid*.

The *projective closure of D*, denoted $P(D)$, is defined as

$$P(D) = D \cup \{e \mid e \text{ is the canonical base of a non-rigid hypersurface of } D^k\}.$$

By a rank calculation, the rank of the canonical base of a hypersurface of D^k is ≤ 1. Hence $P(D)$ is a collection of canonical bases $e = cb(a_1...a_k/e)$ where $SU(a_1...a_k/e) = k-1$ for independent $(a_1, ..., a_k) \in D^k$. Since $SU(e) = 1$, by

Remark 3.5.2 we can assume that $P(D) \subseteq G(D)$ by identifying e with some inter-algebraic $g \in G(D)$. If D is ω-categorical, we can moreover assume, as in Remark 3.16, that $P(D)$ is a $\mathrm{dom}(D)$-definable subset of $G(D)$.

Lemma 3.19 (Notation as in 3.16). *For any point $u \in G(D)$, there is $g \in P(D)$ such that g and u are interalgebraic. (Hence the geometries of $P(D)$ and $G(D)$ coincide.)*

Proof. This is immediate from Remark 3.16, since there $SU(abc/a'b'c') = 2$. $\square$

Then we get the following theorem immediately. This answers the main conjecture in [21] positively.

Theorem 3.20. *Suppose D is ω-categorical. Then $P(D)$ is $\mathrm{dom}(D)$-definable (in D^{eq}), as is its geometry $P(D)'$. Hence, if D is nontrivial, $P(D)'$ has a strongly minimal stable reduct (over $\mathrm{dom}(D)$) preserving the projective geometry over a finite field F. Hence, over a finite parameter in D^{eq}, D interprets a vector space (as projective and affine spaces do) over F.*

(Recently Ben-Yaacov, Tomasic and Wagner obtained an almost hyperimaginary group configuration in [2]. Using this, they also proved the interpretability of a vector space in an ω-categorical nontrivial D [20].)

Now we study the relationship between the whole theory and rank-1 types. As in the stable case, the notion of triviality can be imposed on the underlying theory.

Definition 3.21. We say that T is *trivial* if for $a, b, c, A \subseteq \bar{M}^{eq}$, whenever $\{a, b, c\}$ is pairwise independent over A, then $\{a, b, c\}$ is independent over A.

By mimicking the proofs in the stable case found in [17, 4.2.5], with some novel argument, we can show Lemma 3.22 below. The following easy fact will be used. In any simple T, if a finite tuple $\bar{a}$ is not in $acl(\emptyset)$, then there is a complete extension p of $tp(\bar{a})$ such that $SU(p) = 1$. Indeed, let Let $tp(\bar{a}) = p_o \subseteq p_1... \subseteq p_i...$ be a maximal non-algebraic forking chain. (The length is at most $|T|$.) Then $p = \bigcup_i p_i$ is non-algebraic as well, and, by maximality, $SU(p) = 1$.

Lemma 3.22. *Suppose that T is 1-based. Then T is trivial if and only if all rank-1 types are trivial.*

Proof. We only need to prove the right to left direction, since the other direction is obvious. Suppose that there exists a nontrivial triple (a, b, c) over A, i.e., (a, b, c) is pairwise independent over A, whereas $\{a, b, c\}$ is dependent over A. We will find a nontrivial rank-1 type.

Claim 1. We may assume that $a \in acl(bcA)$, $b \in acl(acA)$ and $c \in acl(baA)$ (but $\{a, b, c\}$ is still pairwise independent over A).

Indeed, if we choose

$$a_0 \in cb(a/bcA) \setminus acl(A), \quad b_0 \in cb(b/a_0cA) \setminus acl(A), \quad c_0 \in cb(c/a_0b_0A) \setminus acl(A)$$

such that $cb(a/bcA) \subset acl(a_0)$ and similarly for b_0, c_0, then, by 1-basedness, these elements will suffice.

Now, for notational convenience, let $A = \emptyset$.

Claim 2. There is F such that $SU(a/F) = SU(b/F) = SU(c/F) = 1$ and (a, b, c) is still a nontrivial triple over F.

Indeed, by the remark before 3.22, there is d such that $SU(a/d) = 1$. We assume $d = cb(a/d)$. Then, by 1-basedness, $d \in acl(a)$ and

$$bd \underset{F}{\downarrow} cd$$

where $F = acl(bd) \cap acl(cd)$. Then

$$b \underset{F}{\downarrow} c \; .$$

Now we have $SU(a/F) = 1$, since otherwise $a \in acl(bd)$. So $a \in acl(d)$

$$\left(\text{as} \quad ad \underset{\emptyset}{\downarrow} b \right),$$

contradicting the fact that $SU(a/d) = 1$. Similarly,

$$a \underset{F}{\downarrow} b$$

and

$$a \underset{F}{\downarrow} c \; .$$

Now, by the rank calculation of $SU(ab/cF)$,

$$SU(b/cF) = SU(b/F) = 1,$$

since $SU(a/cF) = 1$. Similarly, by the rank calculation of $SU(ac/bF)$,

$$SU(c/bF) = SU(c/F) = 1.$$

Hence, Claim 2 is established.

Claim 3. $tp(a/F)$ is nontrivial. (This finishes the proof.)

First, there are a_1, a_2, b', c' such that

$$(*) \qquad tp(a_1c'/Fb) = tp(ac/Fb), \quad c' \underset{F}{\downarrow} abc \quad , tp(a_2b'/Fc) = tp(ab/Fc)$$

and

$$b' \underset{F}{\downarrow} abcc' \; .$$

Hence

$$(**) \qquad b'c' \underset{F}{\downarrow} abc \; .$$

Now we subclaim that a, a_1, a_2, all realizing the rank-1 type $tp(a/F)$, form a nontrivial triple over $Fb'c'$. For pairwise independence, we only show that

$$a_1 \underset{Fb'c'}{\downarrow} a_2 \; ;$$

the other cases follow similarly. We have

$$b \underset{F}{\downarrow} c \; ;$$

hence

$$b \underset{Fb'c'}{\downarrow} c$$

by $(**)$. Thus

$$Fbc' \underset{Fb'c'}{\downarrow} Fcb' .$$

Then

$$a_1 \underset{Fb'c'}{\downarrow} a_2$$

by Claim 2 and (*). Now since $a \in acl(bc)$, again by (*) and Claim 2, each of a, a_1, a_2 is algebraic over $Fb'c'$ together with the other two. Hence the subclaim is verified.

Secondly, by 1-basedness, there is a tuple $a_3 a_4 a_5$ realizing $tp(aa_1a_2/Fb'c')$ such that

$$aa_1a_2 \underset{a_3a_4a_5}{\downarrow} Fb'c' \quad \text{and} \quad aa_1a_2 \underset{Fb'c'}{\downarrow} a_3a_4a_5 .$$

Then, it can be easily seen that (a, a_1, a_2) forms a nontrivial triple over $Fa_3a_4a_5$. This means that the localized geometry of $tp(a/F)$ at $\{a_3, a_4, a_5\}$ is nontrivial, since all the a and a_i realize the rank-1 type $tp(a/F)$. Hence, the original type $tp(a/F)$ cannot be trivial either. We have proved Claim 3 and the lemma. $\qquad\square$

Recall that if T is 1-based, then for any A and tuple $\bar{a}$, there is $\bar{a}'$ with $|\bar{a}| = |\bar{a}'|$ such that $D(tp(\bar{a}/A), \varphi, k) = D(tp(\bar{a}/\bar{a}'), \varphi, k)$, for each φ, k (by taking an element in a Morley sequence, for example). Thus, if additionally T is ω-categorical, then T is supersimple, since every forking chain must have finite length. Moreover, since $SU(\bar{a}/A) = SU(\bar{a}/\bar{a}')$, any n-type in T has finite SU-rank ([9, 4.7]). (Evans and Wagner [8] showed that any supersimple, ω-categorical group or any CM-trivial theory also has finite SU-rank.)

By previous results in this section, we obtain the following.

Corollary 3.23. *Let T be nontrivial, 1-based and ω-categorical. Then an infinite-dimensional vector space over a finite field, in particular the infinite additive group, is definable in $\bar{M}^{eq}$ over a finite parameter.*

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