

ON THE EQUATIONS DEFINING TORIC L.C.I.-SINGULARITIES

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ABSTRACT. Based on Nakajima’s Classification Theorem we describe the precise form of the binomial equations which determine toric locally complete intersection (“l.c.i.”) singularities.

1. INTRODUCTION

An affine toric variety $U_\sigma = \text{Spec}(\mathbb{C}[M \cap \sigma^\vee])$ associated to a rational strongly convex polyhedral cone σ (where $\text{rank}(M) = \dim(\sigma) = d$) is Gorenstein if and only if σ supports a $(d-1)$ -dimensional lattice polytope P (w.r.t. $N = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$) lying on a “primitive” affine hyperplane of the form $\{\mathbf{x} \in N_{\mathbb{R}} \mid \langle m_\sigma, \mathbf{x} \rangle = 1\}$ (up to lattice automorphism). Nakajima [N] classified in 1986 all affine toric locally complete intersection (“l.c.i.”) varieties U_σ by providing a suitable parametrization for all the corresponding polytopes P . More recently, the class of toric l.c.i.-singularities turned out to have some properties of particular importance in both algebraic and geometric aspects. For instance,

- (i) the algebras $\mathbb{C}[N \cap \sigma]$ have the Koszul property (cf. [BrGT], [DHaZ]),
- (ii) all set-theoretic complete intersections of binomial hypersurfaces in an affine complex space are affine toric ideal-theoretic complete intersections (also in a more general sense, including the nonnormal ones; cf. [BaMT, Thm. 4]),
- (iii) all toric l.c.i.-singularities admit projective crepant resolutions (see [DHeZ], [DHaZ]), and
- (iv) the i -th jet schemes of the underlying spaces U_σ of toric l.c.i.-singularities are irreducible for all $i \geq 1$ (see [M, Thm. 4.13]).

The main theorem of the present paper (see Theorem 3.1 below) gives a precise description of the binomial-type equations defining singular l.c.i. U_σ ’s in terms of the corresponding admissible free-parameter sequence (or “matrix”) $\mathbf{m}$ of the Nakajima polytope $P \sim P_{\mathbf{m}}^{(d)}$, and generalizes a result of Ishida [Ish, §8] which concerns dimensions 2 and 3.

In this section, we introduce the brief “toric glossary” (a)–(e) and the notation which will be used in the sequel. For further details on the theory of toric geometry the reader is referred to the textbooks of Oda [O], Fulton [Fu] and Ewald [Ew], and to the lecture notes [KKMS].

Received by the editors April 30, 2002 and, in revised form, September 27, 2002.

2000 *Mathematics Subject Classification*. Primary 14B05, 14M10, 14M25, 52B20; Secondary 13H10, 13P10, 20M25, 32S05.

The second author would like to thank the Mathematics Department of the University of Crete for hospitality and support during the spring term 2001, where this work was initiated.

(a) The *linear hull*, the *affine hull*, the *positive hull* and the *convex hull* of a set B of vectors of $\mathbb{R}^r$, $r \geq 1$, will be denoted by $\text{lin}(B)$, $\text{aff}(B)$, $\text{pos}(B)$ (or $\mathbb{R}_{\geq 0} B$) and $\text{conv}(B)$, respectively. The *dimension* $\dim(B)$ of a $B \subset \mathbb{R}^r$ is defined to be the dimension of its affine hull.

(b) Let N be a free $\mathbb{Z}$ -module of rank $r \geq 1$. N can be regarded as a *lattice* within $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^r$. The *lattice determinant* $\det(N)$ of N is the r -volume of the parallelepiped spanned by any $\mathbb{Z}$ -basis of it. An $n \in N$ is called *primitive* if $\text{conv}(\{\mathbf{0}, n\}) \cap N$ contains no other points except $\mathbf{0}$ and n .

Let N be as above, $M := \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ its dual lattice, $N_{\mathbb{R}}, M_{\mathbb{R}}$ their real scalar extensions, and $\langle \cdot, \cdot \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \rightarrow \mathbb{R}$ the natural $\mathbb{R}$ -bilinear pairing. A subset σ of $N_{\mathbb{R}}$ is called a *convex polyhedral cone* (c.p.c., for short) if there exist $n_1, \dots, n_k \in N_{\mathbb{R}}$ such that $\sigma = \text{pos}(\{n_1, \dots, n_k\})$. Its *relative interior* $\text{int}(\sigma)$ is the usual topological interior of it and considered as a subset of $\text{lin}(\sigma) = \sigma + (-\sigma)$. The *dual cone* $\sigma^{\vee}$ of a c.p.c. σ is a c.p. cone defined by

$$\sigma^{\vee} := \{\mathbf{y} \in M_{\mathbb{R}} \mid \langle \mathbf{y}, \mathbf{x} \rangle \geq 0, \forall \mathbf{x} \in \sigma\}.$$

Note that $(\sigma^{\vee})^{\vee} = \sigma$ and

$$\dim(\sigma \cap (-\sigma)) + \dim(\sigma^{\vee}) = \dim(\sigma^{\vee} \cap (-\sigma^{\vee})) + \dim(\sigma) = r.$$

A subset τ of a c.p.c. σ is called a *face* of σ (notation: $\tau \prec \sigma$), if for some $m_0 \in \sigma^{\vee}$ we have $\tau = \{\mathbf{x} \in \sigma \mid \langle m_0, \mathbf{x} \rangle = 0\}$. In particular, 1-dimensional faces are called *rays*.

A c.p.c. $\sigma = \text{pos}(\{n_1, \dots, n_k\})$ is called *simplicial* (resp. *rational*) if $n_1, \dots, n_k$ are $\mathbb{R}$ -linearly independent (resp. if $n_1, \dots, n_k \in N_{\mathbb{Q}}$, where $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$). If ϱ is a ray of a rational c.p.c. σ , then we denote by $n(\varrho) \in N \cap \varrho$ the unique primitive vector with $\varrho = \mathbb{R}_{\geq 0} n(\varrho)$, and we set

$$\text{Gen}(\sigma) := \{n(\varrho) \mid \varrho \text{ rays of } \sigma\}.$$

A *strongly convex polyhedral cone* (s.c.p.c., for short) is a c.p.c. σ for which $\sigma \cap (-\sigma) = \{\mathbf{0}\}$, i.e., for which $\dim(\sigma^{\vee}) = r$. The s.c.p. cones are alternatively called *pointed cones* (having $\mathbf{0}$ as their apex).

(c) If $\sigma \subset N_{\mathbb{R}}$ is a rational s.c.p.c., then the subsemigroup $\sigma \cap N$ of N is a monoid. The following proposition follows from results due to Gordan, Hilbert and van der Corput (cf. [Schr, Thm. 16.4, p. 233]).

Proposition 1.1 (Minimal generating system). *$\sigma \cap N$ is finitely generated as an additive semigroup for every rational c.p.c. $\sigma \subset N_{\mathbb{R}}$. Moreover, if σ is strongly convex, then among all the systems of generators of $\sigma \cap N$, there is a system $\text{Hilb}_N(\sigma)$ of minimal cardinality, which is uniquely determined (up to the ordering of its elements) by the following characterization:*

$$(1.1) \quad \text{Hilb}_N(\sigma) = \left\{ n \in \sigma \cap (N \setminus \{\mathbf{0}\}) \mid \begin{array}{l} n \text{ cannot be expressed} \\ \text{as the sum of two other} \\ \text{vectors belonging} \\ \text{to } \sigma \cap (N \setminus \{\mathbf{0}\}) \end{array} \right\}.$$

$\text{Hilb}_N(\sigma)$ is called the *Hilbert basis* of σ w.r.t. N .

(d) For a lattice N of rank r having M as its dual, we define an r -dimensional *algebraic torus* $T_N \cong (\mathbb{C}^*)^r$ by setting $T_N := \text{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) = N \otimes_{\mathbb{Z}} \mathbb{C}^*$. Every

$m \in M$ assigns a character $\mathbf{e}(m) : T_N \rightarrow \mathbb{C}^*$. Moreover, each $n \in N$ determines a 1-parameter subgroup

$$\vartheta_n : \mathbb{C}^* \rightarrow T_N \quad \text{with} \quad \vartheta_n(\lambda)(m) := \lambda^{\langle m, n \rangle}, \quad \text{for } \lambda \in \mathbb{C}^*, m \in M.$$

We can therefore identify M with the character group of T_N and N with the group of 1-parameter subgroups of T_N . On the other hand, for a rational s.c.p. cone σ with

$$M \cap \sigma^\vee = \mathbb{Z}_{\geq 0} m_1 + \cdots + \mathbb{Z}_{\geq 0} m_\nu,$$

we associate to the finitely generated monoidal subalgebra

$$\mathbb{C}[M \cap \sigma^\vee] = \bigoplus_{m \in M \cap \sigma^\vee} \mathbf{e}(m)$$

of the $\mathbb{C}$ -algebra $\mathbb{C}[M] = \bigoplus_{m \in M} \mathbf{e}(m)$ a *toric affine variety*

$$U_\sigma := U_{\sigma, N} := \text{Spec}(\mathbb{C}[M \cap \sigma^\vee]),$$

which can be identified with the set of semigroup homomorphisms:

$$U_\sigma = \left\{ u : M \cap \sigma^\vee \rightarrow \mathbb{C} \mid \begin{array}{l} u(\mathbf{0}) = 1, \quad u(m + m') = u(m) \cdot u(m'), \\ \text{for all } m, m' \in M \cap \sigma^\vee \end{array} \right\},$$

where $\mathbf{e}(m)(u) := u(m)$, $\forall m, m \in M \cap \sigma^\vee$ and $\forall u, u \in U_\sigma$.

U_σ admits a canonical T_N -action which extends the group multiplication of the algebraic torus $T_N = U_{\{\mathbf{0}\}}$:

$$(1.2) \quad T_N \times U_\sigma \ni (t, u) \mapsto t \cdot u \in U_\sigma$$

where, for $u \in U_\sigma$, $(t \cdot u)(m) := t(m) \cdot u(m)$, $\forall m, m \in M \cap \sigma^\vee$. The orbits w.r.t. the action (1.2) are parametrized by the set of all the faces of σ . For a $\tau \prec \sigma$, we denote by $\text{orb}(\tau)$ the orbit which is associated to τ .

Proposition 1.2 (Embedding by binomials). *In the analytic category, U_σ , identified with its image under the injective map*

$$(\mathbf{e}(m_1), \dots, \mathbf{e}(m_\nu)) : U_\sigma \hookrightarrow \mathbb{C}^\nu,$$

can be regarded as an analytic set determined by a finite number of equations of the form (monomial) = (monomial). This analytic structure induced on U_σ is independent of the semigroup generators $\{m_1, \dots, m_\nu\}$ and each map $\mathbf{e}(m)$ on U_σ is holomorphic w.r.t. it. In particular, for $\tau \prec \sigma$, U_τ is an open subset of U_σ . Moreover, if $\dim(\sigma) = r$ and

$$\#(\text{Hilb}_M(\sigma^\vee)) = k \quad (\leq \nu),$$

then (by Prop. 1.1) k is nothing but the (minimal) embedding dimension of U_σ , i.e., the minimal number of generators of the maximal ideal of the local $\mathbb{C}$ -algebra $\mathcal{O}_{U_\sigma, \mathbf{0}}$.

See [O, Prop. 1.2 and 1.3., pp. 4–7] for a proof, and [BiLSR], [BiR], [ES], [HSh], [St1], [St2] for the general theory of the defining equations of toric varieties.

Remark 1.3. In fact, Proposition 1.2 informs us that

$$\mathbb{C}[M \cap \sigma^\vee] \cong \mathbb{C}[z_1, \dots, z_\nu] / \mathcal{I},$$

where the prime ideal $\mathcal{I}$ is generated by binomials belonging to

$$\left\{ \prod_{i=1}^{\nu} z_i^{\kappa_i} - \prod_{i=1}^{\nu} z_i^{\xi_i} \mid \begin{array}{l} (\kappa_1, \dots, \kappa_{\nu}), (\xi_1, \dots, \xi_{\nu}) \in (\mathbb{Z}_{\geq 0})^{\nu} \\ \text{and } (\kappa_1 - \xi_1, \dots, \kappa_{\nu} - \xi_{\nu}) \in L \end{array} \right\},$$

and

$$L = \left\{ (\ell_1, \dots, \ell_{\nu}) \in \mathbb{Z}^{\nu} \mid \sum_{i=1}^{\nu} \ell_i m_i = 0 \right\}$$

(under an appropriate identification $M \cong \mathbb{Z}^r$). Let us fix a $\mathbb{Z}$ -basis $\{v_1, \dots, v_k\}$ of the integer lattice L and denote by $\mathcal{B}$ the $(\nu \times k)$ -matrix having $v_1, \dots, v_k$ as its column-vectors. Then the ideal

$$\mathcal{J}_{\mathcal{B}} := (f_{v_1}, f_{v_2}, \dots, f_{v_k}) \subset \mathbb{C}[z_1, \dots, z_{\nu}],$$

generated by the binomials

$$f_{v_i} := \prod_{j=1}^{\nu} z_i^{(v_i)_j^+} - \prod_{j=1}^{\nu} z_i^{(v_i)_j^-}, \quad 1 \leq i \leq k,$$

with $v_i = ((v_i)_1, \dots, (v_i)_{\nu})^{\top}$, and

$$(v_i)_j^+ := \max\{0, (v_i)_j\}, \quad (v_i)_j^- := \max\{0, -(v_i)_j\}, \quad 1 \leq i \leq k, \quad 1 \leq j \leq \nu,$$

is called the *lattice ideal* associated to $\mathcal{B}$. In general, we have $\mathcal{J}_{\mathcal{B}} \subseteq \mathcal{I}$, with the inclusion possibly strict.

Definition 1.4 (Dominating matrices). A $(\nu \times k)$ -matrix (with integer entries) is called a *mixed matrix* if every column of it has both a positive and a negative entry. A mixed $(\nu \times k)$ -matrix is said to be *dominating* if it does not contain any mixed $(\rho \times \rho)$ -submatrices for $1 \leq \rho \leq \min\{\nu, k\}$.

Theorem 1.5. (i) $\mathcal{J}_{\mathcal{B}} \mathbb{C}[z_1^{\pm 1}, \dots, z_{\nu}^{\pm 1}] = \mathcal{I} \mathbb{C}[z_1^{\pm 1}, \dots, z_{\nu}^{\pm 1}]$, i.e.,

$$\mathcal{J}_{\mathcal{B}} : \left(\prod_{i=1}^{\nu} z_i \right)^{\infty} = \mathcal{I}.$$

(ii) $\mathcal{J}_{\mathcal{B}} = \mathcal{I}$ if and only if $\mathcal{B}$ is a dominating matrix.

Proof. For (i) we refer to [ES, Cor. 2.5] and [BiR, Thm. 2.10], and for (ii) to [FSh, Lemma 2.2] or [HSh, Thm. 1.1]. $\square$

(e) The well-known hierarchy of Noetherian rings

$$(\text{regular}) \implies (\text{l.c.i.}) \implies (\text{Gorenstein}) \implies (\text{Cohen-Macaulay})$$

(cf. [Ku]) is used to describe the punctual algebraic behaviour of affine toric varieties.

Theorem 1.6 (Normality and CM-property). *The toric varieties U_{σ} are always normal and Cohen-Macaulay.*

Proof. For a proof of the normality property see [O, Thm. 1.4, p. 7]. The CM-property for toric varieties was first shown by Hochster in [Ho]. See also [KKMS, Thm. 14, p. 52], and [O, 3.9, p. 125]. $\square$

Definition 1.7 (Multiplicities and basic cones). Let N be a free $\mathbb{Z}$ -module of rank r and $\sigma \subset N_{\mathbb{R}}$ a simplicial, rational s.c.p.c. of dimension $d \leq r$. The cone σ can be obviously written as $\sigma = \varrho_1 + \cdots + \varrho_d$, for distinct rays $\varrho_1, \dots, \varrho_d$. The *multiplicity* $\text{mult}(\sigma; N)$ of σ with respect to N is defined as

$$\text{mult}(\sigma; N) := \frac{\det(\mathbb{Z}n(\varrho_1) \oplus \cdots \oplus \mathbb{Z}n(\varrho_d))}{\det(N_{\sigma})},$$

where N_{σ} is the sublattice of N generated (as a subgroup) by the set $N \cap \text{lin}(\sigma)$. If $\text{mult}(\sigma; N) = 1$, then σ is called a *basic cone* w.r.t. N .

Theorem 1.8 (Smoothness criterion). *The affine toric variety $U_{\sigma} = U_{\sigma, N}$ is smooth (i.e., the corresponding local rings $\mathcal{O}_{U_{\sigma}, u}$ are regular at all points u of U_{σ}) iff σ is basic w.r.t. N .*

Proof. See [KKMS, Ch. I, Thm. 4, p. 14], and [O, Thm. 1.10, p. 15]. $\square$

The next theorem describes necessary and sufficient conditions under which U_{σ} is Gorenstein (cf. [Ish, §7]).

Theorem 1.9 (Gorenstein property). *Let N be a free $\mathbb{Z}$ -module of rank r and σ a rational s.c.p. cone in $N_{\mathbb{R}}$ of dimension $d \leq r$. Then the following conditions are equivalent:*

- (i) U_{σ} is Gorenstein.
- (ii) *There exists an element m_{σ} of M , such that $M \cap (\text{int}(\sigma^{\vee})) = m_{\sigma} + (M \cap \sigma^{\vee})$. Moreover, if $d = r$, then m_{σ} in (ii) is the unique primitive element of $M \cap (\text{int}(\sigma^{\vee}))$ with this property and the above conditions are equivalent to the following one:*
- (iii) $\text{Gen}(\sigma) \subset \mathbf{H}$, where $\mathbf{H} = \{\mathbf{x} \in N_{\mathbb{R}} \mid \langle m_{\sigma}, \mathbf{x} \rangle = 1\}$.

2. NAKAJIMA'S POLYTOPES AND CLASSIFICATION THEOREM

We shall henceforth focus our attention to Gorenstein toric singularities and, in particular, to those which are locally complete intersections (l.c.i.'s).

(a) Let N a free $\mathbb{Z}$ -module of rank $r \geq 2$ and $\sigma \subset N_{\mathbb{R}}$ a rational s.c.p.c. of dimension $d \leq r$. Since N/N_{σ} is torsion free, there exists a lattice decomposition $N = N_{\sigma} \oplus \check{N}$, inducing a decomposition of its dual $M = M_{\sigma} \oplus \check{M}$, where $M_{\sigma} = \text{Hom}_{\mathbb{Z}}(N_{\sigma}, \mathbb{Z})$ and $\check{M} = \text{Hom}_{\mathbb{Z}}(\check{N}, \mathbb{Z})$. Writing σ as $\sigma = \sigma' \oplus \{\mathbf{0}\}$ with σ' a d -dimensional cone in $(N_{\sigma})_{\mathbb{R}}$, we obtain decompositions

$$T_N \cong T_{N_{\sigma}} \times T_{\check{N}} \quad \text{and} \quad M \cap \sigma^{\vee} = (M_{\sigma} \cap (\sigma')^{\vee}) \oplus \check{M},$$

which give rise to the analytic isomorphisms

$$U_{\sigma} \cong U_{\sigma'} \times T_{\check{N}} \cong U_{\sigma'} \times T_{N/N_{\sigma}} \cong U_{\sigma'} \times (\mathbb{C}^*)^{r-d}.$$

$U_{\sigma} = U_{\sigma, N}$ can be therefore viewed as a product of $U_{\sigma'} = U_{\sigma', N}$ and an $(r-d)$ -dimensional algebraic torus. Obviously, the study of the algebraic properties for U_{σ} can be reduced to that of the corresponding properties of $U_{\sigma'}$. For instance, the *singular locus* $\text{Sing}(U_{\sigma})$ of U_{σ} equals

$$\text{Sing}(U_{\sigma}) = \text{Sing}(U_{\sigma'}) \times (\mathbb{C}^*)^{r-d}.$$

In fact, the main reason for preferring to work with the affine variety $U_{\sigma'}$ (or with the germ $(U_{\sigma'}, \text{orb}(\sigma'))$) instead of U_{σ} , is that since $\text{lin}(\sigma') = (N_{\sigma})_{\mathbb{R}}$, the orbit $\text{orb}(\sigma') \in U_{\sigma'}$ is the unique fixed closed point under the action of $T_{N_{\sigma}}$ on $U_{\sigma'}$.

Definition 2.1. If σ is nonbasic w.r.t. N , then $U_{\sigma'}$ will be called *the singular representative* of U_{σ} and $\text{orb}(\sigma') \in U_{\sigma'}$ the associated *distinguished* singular point within the singular locus $\text{Sing}(U_{\sigma'})$ of $U_{\sigma'}$.

Definition 2.2. If σ is nonbasic w.r.t. N , then it is also useful to introduce the notion of the “*splitting codimension*” of $\text{orb}(\sigma') \in U_{\sigma'}$ as the number

$$\max \left\{ \varkappa \in \{2, \dots, d\} \left| \begin{array}{l} U_{\sigma'} \cong U_{\sigma''} \times \mathbb{C}^{d-\varkappa}, \\ \text{for some } \sigma'' \prec \sigma' \\ \text{with } \dim(\sigma'') = \varkappa \\ \text{and } \text{Sing}(U_{\sigma''}) \neq \emptyset \end{array} \right. \right\}.$$

If this number equals d , then $(U_{\sigma'}, \text{orb}(\sigma'))$ will be called an *msc-singularity*, i.e., a singularity having the maximum splitting codimension.

(b) Gorenstein toric affine varieties are completely determined by suitable lattice polytopes.

Definition 2.3 (Lattice equivalence). If N_1 and N_2 are two free $\mathbb{Z}$ -modules (not necessarily of the same rank) and $P_1 \subset (N_1)_{\mathbb{R}}$, $P_2 \subset (N_2)_{\mathbb{R}}$ two lattice polytopes with respect to them, we shall say that P_1 and P_2 are *lattice equivalent* to each other, and denote this by $P_1 \sim P_2$, if P_1 is affinely equivalent to P_2 via an affine map

$$\varpi : (N_1)_{\mathbb{R}} \rightarrow (N_2)_{\mathbb{R}},$$

such that the restriction

$$\varpi|_{\text{aff}(P)} : \text{aff}(P) \rightarrow \text{aff}(P')$$

is a bijection mapping the polytope P_1 onto the (necessarily equidimensional) polytope P_2 , every i -dimensional face of P_1 onto an i -dimensional face of P_2 , for all i , $0 \leq i \leq \dim(P_1) = \dim(P_2)$, and, in addition, N_{P_1} onto the lattice N_{P_2} , where by N_{P_j} is meant the sublattice of N_j generated (as subgroup) by $\text{aff}(P_j) \cap N_j$, $j = 1, 2$. If

$$N_1 = N_2 =: N \quad \text{and} \quad \text{rk}(N) = \dim(P_1) = \dim(P_2),$$

then these ϖ 's are exactly the *affine integral transformations* which are composed of unimodular transformations and N -translations.

Definition 2.4 (Basic simplices). A lattice simplex is said to be *basic* (or *unimodular*) if its vertices constitute a part of a $\mathbb{Z}$ -basis of the reference lattice (or equivalently, if its relative, normalized volume equals 1).

Now let $U_{\sigma} = U_{\sigma, N}$ be a d -dimensional affine toric variety and $U_{\sigma'} = U_{\sigma', N_{\sigma'}}$, as in (a). Assuming that U_{σ} is Gorenstein, we may pass to another analytically isomorphic “standard” representative as follows: Denote by $\mathbb{Z}^d$ the rectangular (standard) lattice in $\mathbb{R}^d$ and by $(\mathbb{Z}^d)^{\vee}$ its dual lattice within $(\mathbb{R}^d)^{\vee} = \text{Hom}_{\mathbb{R}}(\mathbb{R}^d, \mathbb{R})$. Since $\dim(\sigma') = \text{rank}(N_{\sigma'}) = d$, or equivalently, since $(\sigma')^{\vee}$ is strongly convex in $(M_{\sigma'})_{\mathbb{R}}$, Theorem 1.9 (iii) implies

$$\text{Gen}(\sigma') \subset \mathbf{H}^{(d)} \quad \text{with} \quad \mathbf{H}^{(d)} := \{\mathbf{x} \in (N_{\sigma'})_{\mathbb{R}} \mid \langle m_{\sigma'}, \mathbf{x} \rangle = 1\},$$

for a uniquely determined $m_{\sigma'} \in M_{\sigma'}$. Clearly, $\sigma' \cap \mathbf{H}^{(d)}$ is a $(d-1)$ -dimensional lattice polytope (w.r.t. $N_{\sigma'}$). We choose a specific $\mathbb{Z}$ -module isomorphism

$$\Upsilon : N_{\sigma'} \xrightarrow{\cong} \mathbb{Z}^d$$

inducing an $\mathbb{R}$ -vector space isomorphism

$$\Phi = \Upsilon \otimes_{\mathbb{Z}} \text{id}_{\mathbb{R}} : (N_{\sigma})_{\mathbb{R}} \xrightarrow{\cong} \mathbb{R}^d,$$

such that

$$\Phi(\mathbf{H}^{(d)}) = \{\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d \mid x_1 = 1\} =: \bar{\mathbf{H}}^{(d)}.$$

Obviously, $P := \Phi(\sigma' \cap \mathbf{H}^{(d)}) \subset \bar{\mathbf{H}}^{(d)}$ is a lattice $(d-1)$ -dimensional polytope (w.r.t. $\text{aff}(P) \cap \mathbb{Z}^d$). Defining

$$\tau_P := \text{pos}(P) = \{\kappa \mathbf{x} \in \mathbb{R}^d \mid \kappa \in \mathbb{R}_{\geq 0}, \mathbf{x} \in P\}$$

(cf. Figure 1), we easily obtain the following:

Lemma 2.5. (i) *There exists a torus-equivariant analytic isomorphism*

$$U_{\sigma'} \cong U_{\tau_P} (= \text{Spec}(\mathbb{C}[(\mathbb{Z}^d)^{\vee} \cap \tau_P^{\vee}]))$$

mapping $\text{orb}(\sigma')$ onto $\text{orb}(\tau_P)$. Moreover, U_{τ_P} is singular (and its singular locus contains at least $\text{orb}(\tau_P)$) iff P is not a basic simplex w.r.t. $\text{aff}(P) \cap \mathbb{Z}^d$.

(ii) *If $Q \subset \bar{\mathbf{H}}^{(d)}$ is another lattice $(d-1)$ -dimensional polytope (with respect to $\bar{\mathbf{H}}^{(d)} \cap \mathbb{Z}^d$), then $P \sim Q$ iff there exists a torus-equivariant analytic isomorphism $U_{\tau_P} \cong U_{\tau_Q}$ mapping $\text{orb}(\tau_P)$ onto $\text{orb}(\tau_Q)$.*

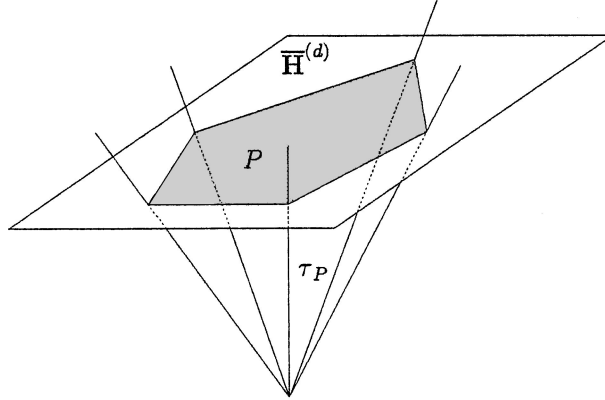


FIGURE 1.

(c) Let $\mathbb{R}^d$ be again the usual d -dimensional euclidean space, $\mathbb{Z}^d$ the usual rectangular lattice in $\mathbb{R}^d$ and $(\mathbb{Z}^d)^{\vee}$ its dual lattice. From now on, we shall denote by $\{e_1, e_2, \dots, e_d\}$ the standard $\mathbb{Z}$ -basis of $\mathbb{Z}^d$, by $\{e_1^{\vee}, e_2^{\vee}, \dots, e_d^{\vee}\}$ its dual basis, and we shall represent the points of $\mathbb{R}^d$ by column vectors and the points of its dual $(\mathbb{R}^d)^{\vee}$ by row vectors.

Definition 2.6. A sequence of free parameters of length ℓ (w.r.t. $\mathbb{Z}^d$) is defined to be a finite sequence

$$\mathbf{m} := (m_1, m_2, \dots, m_{\ell}), \quad 1 \leq \ell \leq d-1,$$

consisting of vectors

$$m_i := (m_{i,1}, m_{i,2}, \dots, m_{i,d}), \quad 1 \leq i \leq \ell, \quad \text{of } (\mathbb{Z}^d)^{\vee} \setminus \{(0, \dots, 0)\}$$

for which $m_{i,j} = 0$ for all i , $1 \leq i \leq \ell$, and all j , $1 \leq j \leq d$, with $i < j$. As $(\ell \times d)$ -matrix such an $\mathbf{m}$ has the form

$$(2.1) \quad \mathbf{m} = \begin{pmatrix} m_{1,1} & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ m_{2,1} & m_{2,2} & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ m_{3,1} & m_{3,2} & m_{3,3} & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots & \cdots & \vdots \\ m_{\ell-1,1} & m_{\ell-1,2} & m_{\ell-1,3} & \cdots & \ddots & 0 & 0 & \cdots & 0 \\ m_{\ell,1} & m_{\ell,2} & m_{\ell,3} & \cdots & \cdots & m_{\ell,\ell} & \underbrace{0 \cdots 0}_{d-\ell \text{ zero-columns}} \end{pmatrix}.$$

Definition 2.7 (Nakajima polytopes). Fixing the dimension d of our reference space, we define the polytopes

$$\left\{ P_{\mathbf{m}}^{(i)} \subset \bar{\mathbf{H}}^{(d)} \mid i \in \mathbb{N}, 1 \leq i \leq d \right\}$$

lying on

$$\bar{\mathbf{H}}^{(d)} = \{ \mathbf{x} = (x_1, \dots, x_d)^{\top} \in \mathbb{R}^d \mid x_1 = 1 \}$$

and being associated to an “admissible” free-parameter sequence (or matrix) $\mathbf{m}$ as in (2.1) w.r.t. $\mathbb{Z}^d$ (with length $\ell = i - 1$, for $2 \leq i \leq d$) by using induction on i ; namely we define

$$P_{\mathbf{m}}^{(1)} := \{e_1\} = \{(1, \underbrace{0, 0, \dots, 0}_{(d-1)\text{-times}}, 0)^{\top}\},$$

and for $2 \leq i \leq d$,

$$(2.2) \quad P_{\mathbf{m}}^{(i)} := \text{conv} \left(\left\{ P_{\mathbf{m}}^{(i-1)} \cup \{(\mathbf{x}', \langle m_{i-1}, \mathbf{x} \rangle, \underbrace{0, \dots, 0}_{(d-i)\text{-times}})^{\top} \mid \mathbf{x} = (\mathbf{x}', \underbrace{0, \dots, 0}_{(d-i+1)\text{-times}})^{\top} \in P_{\mathbf{m}}^{(i-1)} \right\} \right)$$

where $\mathbf{x}' = (x_1, x_2, \dots, x_{i-1})$. $P_{\mathbf{m}}^{(i)}$ is obviously $(i - 1)$ -dimensional. For $\mathbf{m}$ to be “admissible” means that

$$(2.3) \quad \langle m_{i-1}, \mathbf{x} \rangle \geq 0, \quad \forall \mathbf{x}, \quad \mathbf{x} = (x_1, x_2, \dots, x_{i-1}, \underbrace{0, \dots, 0}_{(d-i+1)\text{-times}})^{\top} \in P_{\mathbf{m}}^{(i-1)}.$$

Any lattice $(i - 1)$ -polytope P which is lattice equivalent to a $P_{\mathbf{m}}^{(i)}$ (as defined above) will be called a *Nakajima polytope* (w.r.t. $\mathbb{R}^d$). As explained in [DHaZ], $P_{\mathbf{m}}^{(i)}$ can be written (for $2 \leq i \leq d$) as

$$(2.4) \quad \left\{ \mathbf{x} = (\mathbf{x}', x_i, \underbrace{0, \dots, 0}_{(d-i)\text{-times}})^{\top} \in (P_{\mathbf{m}}^{(i-1)} \times \mathbb{R} \times \{0\}) \hookrightarrow \bar{\mathbf{H}}^{(d)} \mid 0 \leq x_i \leq \langle m_{i-1}, \mathbf{x}' \rangle \right\}$$

i.e., as a polytope determined by means of a suitable cut of a “half-line prism” over $P_{\mathbf{m}}^{(i-1)}$.

Examples 2.8. (i) For $i = d = 1$, we have trivially $P_{\mathbf{m}}^{(1)} = \{1\}$.

(ii) For $i = d = 2$, $\mathbf{m} = (m_{1,1}, 0)$ we have $P_{\mathbf{m}}^{(1)} = \{(1, 0)^{\mathsf{T}}\}$ and

$$\begin{aligned} P_{\mathbf{m}}^{(2)} &= \text{conv}(\{(1, 0)^{\mathsf{T}}\} \cup \{(1, \langle m_1, (1, 0) \rangle)^{\mathsf{T}}\}) \\ &= \text{conv}(\{(1, 0)^{\mathsf{T}}\} \cup \{(1, m_{1,1})^{\mathsf{T}}\}), \quad m_{1,1} > 0. \end{aligned}$$

(iii) For $i = d = 3$, and

$$\mathbf{m} = \begin{pmatrix} m_{1,1} & 0 & 0 \\ m_{2,1} & m_{2,2} & 0 \end{pmatrix}$$

we obtain

$$P_{\mathbf{m}}^{(3)} = \text{conv}(\{(1, 0, 0)^{\mathsf{T}}, (1, m_{1,1}, 0)^{\mathsf{T}}, (1, 0, m_{2,1})^{\mathsf{T}}, (1, m_{1,1}, m_{2,1} + m_{1,1}m_{2,2})^{\mathsf{T}}\})$$

with

$$(2.5) \quad m_{1,1} > 0, \quad m_{2,1} \geq 0, \quad m_{2,1} + m_{1,1}m_{2,2} \geq 0, \quad (m_{2,1}, m_{2,2}) \neq (0, 0).$$

(iv) Finally, for $i = d = 4$, and

$$\mathbf{m} = \begin{pmatrix} m_{1,1} & 0 & 0 & 0 \\ m_{2,1} & m_{2,2} & 0 & 0 \\ m_{3,1} & m_{3,2} & m_{3,3} & 0 \end{pmatrix}$$

we get

$$P_{\mathbf{m}}^{(4)} = \text{conv} \left(\left\{ \begin{pmatrix} (1, 0, 0, 0)^{\mathsf{T}}, (1, m_{1,1}, 0, 0)^{\mathsf{T}}, (1, 0, m_{2,1}, 0)^{\mathsf{T}}, \\ (1, m_{1,1}, m_{2,1} + m_{1,1}m_{2,2}, 0)^{\mathsf{T}}, (1, 0, 0, m_{3,1})^{\mathsf{T}}, \\ (1, m_{1,1}, 0, m_{3,1} + m_{1,1}m_{3,2})^{\mathsf{T}}, \\ (1, 0, m_{2,1}, m_{3,1} + m_{3,3}m_{2,1})^{\mathsf{T}}, \\ (1, m_{1,1}, m_{2,1} + m_{1,1}m_{2,2}, \\ m_{3,1} + m_{3,2}m_{1,1} + m_{2,1}m_{3,3} + m_{1,1}m_{2,2}m_{3,3})^{\mathsf{T}} \end{pmatrix} \right\} \right)$$

with

$$(2.6) \quad \begin{cases} m_{1,1} > 0, \quad m_{2,1} \geq 0, \quad m_{2,1} + m_{1,1}m_{2,2} \geq 0, \\ m_{3,1} \geq 0, \quad m_{3,1} + m_{1,1}m_{3,2} \geq 0, \\ m_{3,1} + m_{3,3}m_{2,1} \geq 0, \quad m_{3,1} + m_{2,1}m_{3,3} + m_{1,1}(m_{3,2} + m_{2,2}m_{3,3}) \geq 0, \\ (m_{2,1}, m_{2,2}) \neq (0, 0), \quad (m_{3,1}, m_{3,2}, m_{3,3}) \neq (0, 0, 0). \end{cases}$$

In Figures 2 and 3 we illustrate the lattice polytopes $P_{\mathbf{m}}^{(3)}$, $P_{\mathbf{m}}^{(4)}$, respectively, for

$$\mathbf{m} = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{m} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & -1 & -1 & 0 \end{pmatrix}.$$

Remark 2.9 (On the set of vertices). A convenient reformulation of (2.2) reads as

$$P_{\mathbf{m}}^{(i)} = \text{conv} \left(P_{\mathbf{m}}^{(i-1)} \cup W_i P_{\mathbf{m}}^{(i-1)} \right),$$

where

$$W_i := (e_1^{\vee}, e_2^{\vee}, \dots, e_{i-1}^{\vee}, m_{i-1}, 0, \dots, 0)^{\mathsf{T}}, \quad i = 2, \dots, d.$$

is the $(d \times d)$ -matrix whose nonzero rows are $e_1^{\vee}, e_2^{\vee}, \dots, e_{i-1}^{\vee}, m_{i-1}$. Setting

$$S_{\mathbf{m}}^{(1)} := \{e_1\}$$

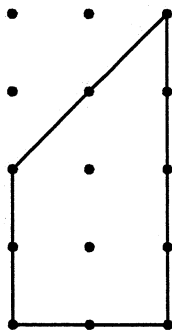


FIGURE 2

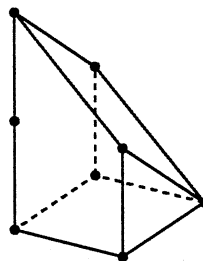


FIGURE 3

and

$$S_{\mathbf{m}}^{(i)} := \left\{ v + (0, \dots, 0, \varepsilon_{i-1} \langle m_{i-1}, v \rangle, \underbrace{0, \dots, 0}_{(d-i)\text{-times}})^{\top} \mid \varepsilon_i \in \{0, 1\}, v \in S_{\mathbf{m}}^{(i-1)} \right\},$$

for $i = 2, \dots, d$, we have

$$P_{\mathbf{m}}^{(i)} = \text{conv} \left(S_{\mathbf{m}}^{(i)} \right),$$

and thus, $\text{vert}(P_{\mathbf{m}}^{(i)}) \subseteq S_{\mathbf{m}}^{(i)}$, where by $\text{vert}(P)$ we denote the set of vertices of a polytope P . Observe, that from the definition of the Nakajima polytopes we know that

$$(2.7) \quad v \geq 0, \quad \text{for all } v \in S_{\mathbf{m}}^{(i)}, \quad i \in \{1, \dots, d\},$$

i.e., in particular, all the coordinates of the vertices are nonnegative.

To provide a more explicit description of the elements of the sets $S_{\mathbf{m}}^{(d)}$ and thus of the possible vertices of $P_{\mathbf{m}}^{(d)}$ also, we define for any choice of

$$\varepsilon = (\varepsilon_1, \dots, \varepsilon_{d-1}) \in \{0, 1\}^{d-1}$$

and any $k \in \{2, \dots, d\}$ the vector

$$(2.8) \quad v_k(\varepsilon) := \varepsilon_{k-1} \left(m_{k-1,1} + \sum_{l=2}^{k-1} m_{k-1,l} v_l(\varepsilon) \right)$$

and we set

$$v(\varepsilon) := (1, v_2(\varepsilon), \dots, v_d(\varepsilon))^{\top}.$$

Because of the definition of the sets $S_{\mathbf{m}}^{(i)}$ we have

$$(2.9) \quad S_{\mathbf{m}}^{(d)} = \{ v(\varepsilon) \mid \varepsilon \in \{0, 1\}^{d-1} \}.$$

The entries $v_2(\varepsilon), \dots, v_d(\varepsilon)$ of $v(\varepsilon)$ can be determined by exploiting the intrinsic recurrence relation which occurs in (2.8). For $\varepsilon \in \{0, 1\}^{d-1}$, $n \in \{2, \dots, d\}$, and $k \in \{0, 1, \dots, n-2\}$ we define the following sum of products:

$$q_{k,n}(\varepsilon) := \sum_{0=i_0 < i_1 < i_2 < \dots < i_k < i_{k+1}=n-1} \prod_{j=1}^{k+1} \varepsilon_{i_j} m_{i_j, i_{j-1}+1}.$$

Observe that $q_{0,n}(\varepsilon) = \varepsilon_{n-1} m_{n-1,1}$.

Proposition 2.10. *For all $n \in \{2, \dots, d\}$ we have*

$$v_n(\varepsilon) = \sum_{k=0}^{n-2} q_{k,n}(\varepsilon).$$

Proof. First we notice that

$$q_{k+1,n+1}(\varepsilon) = \varepsilon_n \sum_{i=k+2}^n m_{n,i} q_{k,i}(\varepsilon),$$

which follows immediately from the definition, because

$$\begin{aligned} & \varepsilon_n \sum_{i=k+2}^n m_{n,i} q_{k,i}(\varepsilon) \\ (2.10) \quad &= \sum_{i=k+2}^n \varepsilon_n m_{n,i} \sum_{0=j_0 < j_1 < j_2 < \dots < j_k < j_{k+1}=i-1} \prod_{l=1}^{k+1} \varepsilon_{j_l} m_{j_l, j_{l-1}+1} \\ &= \sum_{0=j_0 < j_1 < j_2 < \dots < j_k < j_{k+1} < j_{k+2}=n} \prod_{l=1}^{k+2} \varepsilon_{j_l} m_{j_l, j_{l-1}+1} = q_{k+1,n+1}(\varepsilon). \end{aligned}$$

We prove the proposition by using induction w.r.t. n . For $n = 2$ we obtain the identity

$$v_2(\varepsilon) = \varepsilon_1 m_{1,1} = q_{0,2}(\varepsilon).$$

Now let n be > 2 . From the definition (2.8) and our induction hypothesis

$$\begin{aligned} v_{n+1}(\varepsilon) &= \varepsilon_n m_{n,1} + \varepsilon_n \sum_{k=2}^n m_{n,k} v_k(\varepsilon) \\ &= q_{0,n+1}(\varepsilon) + \varepsilon_n \sum_{k=2}^n m_{n,k} \left(\sum_{l=0}^{k-2} q_{l,k}(\varepsilon) \right) \\ &= q_{0,n+1}(\varepsilon) + \sum_{l=0}^{n-2} \varepsilon_n \sum_{k=l+2}^n m_{n,k} q_{l,k}(\varepsilon) \\ &= q_{0,n+1}(\varepsilon) + \sum_{l=0}^{n-2} q_{l+1,n+1}(\varepsilon), \end{aligned}$$

where the last identity follows from (2.10). $\square$

Corollary 2.11. *For any choice of $\varepsilon_\rho \in \{0, 1\}$, $\rho \in \{1, \dots, d-2\}$, we have*

$$\sum_{k=0}^{d-2} \sum_{0=i_0 < i_1 < i_2 < \dots < i_k < i_{k+1}=d-1} m_{d-1, i_k+1} \cdot \prod_{j=1}^k \varepsilon_{i_j} m_{i_j, i_{j-1}+1} \geq 0.$$

Proof. The left-hand side is nothing but another representation of the integer $v_d(\varepsilon)$, where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{d-2}, 1)$, i.e., in the case in which $\varepsilon_{d-1} = 1$ (cf. Proposition 2.10). In view of (2.9) and (2.7) we have $v_d(\varepsilon) \geq 0$ and the corollary is proven. $\square$

Now Nakajima's Classification Theorem [N, Thm. 1.5, p. 86] can be formulated as follows.

Theorem 2.12 (Nakajima’s Classification of Toric L.C.I.’s). *Let N be a free $\mathbb{Z}$ -module of rank $r \geq 2$, and $\sigma \subset N_{\mathbb{R}}$ a s.c.p. cone of dimension d , $2 \leq d \leq r$. Moreover, let U_{σ} denote the affine toric variety associated to σ , and $U_{\sigma'}$ as in (b). Then U_{σ} is a local complete intersection **if and only if** there exists an admissible sequence $\mathbf{m}$ of free parameters of length $d-1$ (w.r.t. $\mathbb{Z}^d$), such that for any standard representative $U_{\tau_P} \cong U_{\sigma'}$ of U_{σ} we have $P \sim P_{\mathbf{m}}^{(d)}$; i.e., P is a Nakajima $(d-1)$ -dimensional polytope (w.r.t. $\mathbb{R}^d$).*

Remark 2.13. (i) Theorem 2.12 was first proved in dimensions 2 and 3 by Ishida [Ish, Thm. 8.1]. Previous classification results, due to Watanabe [W], cover essentially only the class of the $\mathbb{Q}$ -factorial toric l.c.i.’s in all dimensions. In fact, the term “Watanabe simplex” introduced in [DHeZ, 5.13] can be used, *up to lattice equivalence*, as a synonym for a Nakajima polytope (in the sense of 2.7) which happens to be a simplex.

(ii) Obviously, U_{σ} is a l.c.i. $\iff U_{\sigma'} \cong U_{\tau_P}$ is a “g.c.i”, i.e., a *global* complete intersection in the sense of [Ish]. (It is worth mentioning that in the setting of [DHeZ], it was always assumed that $d = r$; therefore, the abelian quotient “g.c.i.”-spaces were abbreviated therein simply as “c.i.’s”).

(iii) For a nonbasic Nakajima polytope P , $(U_{\tau_P}, \text{orb}(\tau_P))$ is a toric g.c.i.-singularity.

(iv) If P is a Nakajima $(d-1)$ -polytope and τ_P nonbasic w.r.t. $\mathbb{Z}^d$, then the orbit $\text{orb}(\tau_P) \in U_{\tau_P}$ has splitting codimension $\varkappa$, with $2 \leq \varkappa \leq d-1$ iff P is lattice-equivalent to the join $\check{P} \star \mathbf{s}$ of a $(\varkappa-1)$ -dimensional (nonbasic) Nakajima polytope $\check{P}$ with a basic $(d-\varkappa-1)$ -simplex $\mathbf{s}$.

(v) It is easy for every $P \subset \check{\mathbf{H}}^{(d)}$, with $P \sim P_{\mathbf{m}}^{(d)}$, to verify that

$$(2.11) \quad d \leq \#(\text{vert}(P)) \leq 2^{d-1} \quad \text{and} \quad d \leq \#(\{\text{facets of } P\}) \leq 2(d-1).$$

(vi) The question:

“what kind of equations define toric l.c.i.-singularities $(U_{\tau_P}, \text{orb}(\tau_P))$?”

will be answered only in dimensions ≥ 3 , because, as it is well-known, in dimension 2 only the classical “Kleinian” hypersurface singularities

$$\{(z, w, t) \in \mathbb{C}^3 \mid z^{\kappa} - wt = 0\}, \quad \kappa \in \mathbb{Z}_{\geq 2},$$

of “type $\mathbf{A}_{\kappa-1}$ ” are present.

3. EQUATIONS DEFINING TORIC L.C.I.-SINGULARITIES

Our main result is the following: Let N be a free $\mathbb{Z}$ -module of rank $r \geq 3$, and $\sigma \subset N_{\mathbb{R}}$ a rational s.c.p.c. of dimension d , $3 \leq d \leq r$, such that $U_{\sigma} = U_{\sigma, N}$ is a local complete intersection. By Theorem 2.12 there exists an admissible sequence

$$(3.1) \quad \mathbf{m} = \begin{pmatrix} m_{1,1} & 0 & 0 & \cdots & \cdots & 0 & 0 \\ m_{2,1} & m_{2,2} & 0 & \cdots & \cdots & 0 & 0 \\ m_{3,1} & m_{3,2} & \ddots & \cdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ m_{d-1,1} & m_{d-1,2} & m_{d-1,3} & \cdots & m_{d-1,d-2} & m_{d-1,d-1} & 0 \end{pmatrix}$$

of free parameters of length $d-1$ (w.r.t. the lattice $\mathbb{Z}^d$), such that for any standard

representative $U_{\tau_P} \cong U_{\sigma'}$ of U_σ we have $P \sim P_{\mathbf{m}}^{(d)}$. We may, without loss of generality, assume that $(U_{\tau_P}, \text{orb}(\tau_P))$ is an msc-singularity.

Theorem 3.1. *If $P \sim P_{\mathbf{m}}^{(d)}$, as above, then*

$$U_{\tau_P} \cong U_{\tau_{P_{\mathbf{m}}^{(d)}}} \cong \text{Spec}(\mathbb{C}[z_1, z_2, \dots, z_d, z_{d+1}, \dots, z_{2d-1}] / \mathcal{I})$$

with defining ideal

$$\mathcal{I} = \left(\left\{ \prod_{1 \leq i \leq j} z_i^{\mu_{i-1,j}} z_{d+i-1}^{-\lambda_{i-1,j}} - z_{j+1} z_{d+j} \mid 1 \leq j \leq d-1 \right\} \right)$$

where the $\lambda_{i,j}$'s, $j \in \{1, \dots, d-1\}$, are nonpositive integers determined recursively by the formula

$$(3.2) \quad \lambda_{i,j} := \begin{cases} \min\{0, m_{j,i+1}\}, & \text{if } i = j-1, \\ \min\left\{0, m_{j,i+1} + \sum_{k=i+1}^{j-1} m_{k,i+1} \lambda_{k,j}\right\}, & \text{if } i \in \{j-2, \dots, 1\}, \\ 0, & \text{if } i = 0, \end{cases}$$

and the $\mu_{i,j}$'s, $j \in \{1, \dots, d-1\}$, are nonnegative integers defined by the formula

$$(3.3) \quad \mu_{i,j} := \begin{cases} \max\{0, m_{j,i+1}\}, & \text{if } i = j-1, \\ \max\left\{0, m_{j,i+1} + \sum_{k=i+1}^{j-1} m_{k,i+1} \lambda_{k,j}\right\}, & \text{if } i \in \{j-2, \dots, 1\}, \\ m_{j,1} + \sum_{k=1}^{j-1} m_{k,1} \lambda_{k,j}, & \text{if } i = 0. \end{cases}$$

Remark 3.2. (i) In the formula (3.2) the $\lambda_{j-1,j}$'s are known from the beginning. For all $\rho \in \{2, \dots, j-1\}$, the $\lambda_{j-\rho,j}$'s are to be found successively by means of integer linear combinations of $\lambda_{j-1,j}$, $\lambda_{j-2,j}$, $\dots$, $\lambda_{j-\rho-1,j}$ (with known coefficients).

(ii) For all $j \in \{1, \dots, d-1\}$ and $i \in \{1, \dots, j\}$, either $\mu_{i-1,j} = 0$ or $\lambda_{i-1,j} = 0$ (by definition). Hence, the first monomial of each of the $d-1$ binomials which generate $\mathcal{I}$ contains only one of the two variables z_i , z_{d+i-1} , $1 \leq i \leq j$.

(iii) If all entries in (3.1) are nonnegative, then

$$(3.4) \quad \mathcal{I} = \left(\left\{ \prod_{1 \leq i \leq j} z_i^{m_{j,i}} - z_{j+1} z_{d+j} \mid 1 \leq j \leq d-1 \right\} \right),$$

because in this case all exponents $\lambda_{i-1,j}$ are 0 and $\mu_{i-1,j} = m_{j,i}$.

Next, we define

$$(3.5) \quad \Omega_{\mathbf{m}}^{(d)} := \left\{ k \in \{1, \dots, d-1\} \mid \begin{array}{l} e_k^\vee = m_{\gamma_k}, \\ \text{for some index} \\ \gamma_k \in \{k, \dots, d-1\} \end{array} \right\}$$

and

$$(3.6) \quad \mathfrak{R}_{\mathbf{m}}^{(d)} := \left\{ l \in \{1, \dots, d-2\} \mid \begin{array}{l} m_l - e_{l+1}^\vee = m_{\delta_l}, \\ \text{for some index} \\ \delta_l \in \{l+1, \dots, d-1\} \end{array} \right\}.$$

Corollary 3.3. *If $P \sim P_{\mathbf{m}}^{(d)}$, as in Theorem 3.1, then $U_{\tau_P} \cong U_{\tau_{P_{\mathbf{m}}^{(d)}}}$ admits the “minimal” embedding*

$$U_{\tau_{P_{\mathbf{m}}^{(d)}}} \hookrightarrow \mathbb{C}^{\#(\mathbf{Hilb}_{(\mathbb{Z}^d)^\vee}(\tau_{P_{\mathbf{m}}^{(d)}}^\vee))}$$

after eliminating the redundant variables of $\mathbb{C}[z_1, z_2, \dots, z_d, z_{d+1}, \dots, z_{2d-1}]$. More precisely,

$$U_{\tau_{P_{\mathbf{m}}^{(d)}}} \cong \text{Spec} \left(\mathbb{C} \left[\left\{ z_k \mid k \in \{1, \dots, d\} \setminus \mathfrak{Q}_{\mathbf{m}}^{(d)} \right\} \cup \left\{ z_{d+l} \mid l \in \{1, \dots, d-1\} \setminus \mathfrak{R}_{\mathbf{m}}^{(d)} \right\} \right] / \mathcal{I} \right),$$

where the ideal $\mathcal{I}$ is generated by $\#(\mathbf{Hilb}_{(\mathbb{Z}^d)^\vee}(\tau_{P_{\mathbf{m}}^{(d)}}^\vee)) - d$ binomials. These binomials are exactly those remaining from

$$\left\{ \prod_{1 \leq i \leq j} z_i^{\mu_{i-1,j}} z_{d+i-1}^{-\lambda_{i-1,j}} - z_{j+1} z_{d+j} \mid 1 \leq j \leq d-1 \right\}$$

after the elimination of the variables

$$\left\{ z_k \mid k \in \mathfrak{Q}_{\mathbf{m}}^{(d)} \right\} \cup \left\{ z_{d+l} \mid l \in \mathfrak{R}_{\mathbf{m}}^{(d)} \right\}$$

by means of the substitutions

$$z_k = z_{\gamma_k+1} z_{d+\gamma_k} \quad \text{and} \quad z_{d+l} = z_{\delta_l+1} z_{d+\delta_l}.$$

Proofs of 3.1 and 3.3 are given in the next section. Let us first apply them to a couple of examples of Nakajima polytopes.

Examples 3.4. (i) For the Nakajima quadrilateral $P_{\mathbf{m}}^{(3)}$ of Figure 2, (3.4) gives

$$U_{\tau_{P_{\mathbf{m}}^{(3)}}} \cong \text{Spec} \left(\mathbb{C}[z_1, z_2, z_3, z_4, z_5] / (z_1^2 - z_2 z_4, z_1^2 z_2 - z_3 z_5) \right).$$

(ii) Let $P_{\mathbf{m}}^{(3)}$ be the Nakajima triangle with

$$\mathbf{m} = \begin{pmatrix} k & 0 & 0 \\ k & -1 & 0 \end{pmatrix}, \quad k \in \mathbb{Z}_{\geq 2}.$$

Eliminating the variable $z_4 (= z_3 z_5)$ as in Corollary 3.3, and setting $w = z_1$, $t_1 = z_2$, $t_2 = z_3$, $t_3 = z_5$, we obtain the hypersurface

$$U_{\tau_{P_{\mathbf{m}}^{(3)}}} \cong \text{Spec} \left(\mathbb{C}[w, t_1, t_2, t_3] / (w^k - t_1 t_2 t_3) \right).$$

(iii) For the Nakajima solid $P_{\mathbf{m}}^{(4)}$ of Figure 3, we have

$$\begin{cases} \mu_{0,1} = 1, & \mu_{0,2} = 1, & \mu_{0,3} = 0, \\ \mu_{1,2} = 0, & \mu_{1,3} = 0, & \mu_{2,3} = 0, \\ \lambda_{0,1} = 0, & \lambda_{0,2} = 0, & \lambda_{0,3} = 0, \\ \lambda_{1,2} = 0, & \lambda_{1,3} = -1, & \lambda_{2,3} = -1. \end{cases}$$

Hence, Theorem 3.1 gives

$$U_{\tau_{P_{\mathbf{m}}^{(4)}}} \cong \text{Spec} \left(\mathbb{C}[z_1, z_2, z_3, z_4, z_5, z_6, z_7] / (z_1 - z_2 z_5, z_1 - z_3 z_6, z_5 z_6 - z_4 z_7) \right).$$

Therefore it is possible (by Corollary 3.3) to erase the redundant variable z_1 (cf. (3.5)) and describe $U_{\tau_{P_{\mathbf{m}}^{(4)}}}$ as a complete intersection of two binomials in $\mathbb{C}_{(t_1, \dots, t_6)}^6$,

where $t_i = z_{i+1}$, $1 \leq i \leq 6$, as follows:

$$U_{\tau_{P_{\mathbf{m}}^{(4)}}} \cong \text{Spec} \left(\mathbb{C}[t_1, t_2, t_3, t_4, t_5, t_6] / (t_1 t_4 - t_2 t_5, t_4 t_5 - t_3 t_6) \right).$$

(iv) Now let us give an example of a Nakajima polytope with the smallest number of vertices (cf. (2.11)), generalizing slightly (ii). For $k \in \mathbb{Z}_{\geq 2}$, $d \in \mathbb{Z}_{\geq 4}$, let

$$\mathbf{s}_k^{(d)} \subset \bar{\mathbf{H}}^{(d)} \hookrightarrow \mathbb{R}^d$$

denote the $(d-1)$ -simplex

$$\mathbf{s}_k^{(d)} := \text{conv}(\{e_1, e_1 + k e_2, e_1 + k(e_2 + e_3), \dots, e_1 + k(e_2 + e_3 + \dots + e_d)\})$$

being constructed by the k -th dilation of a basic $(d-1)$ -simplex. Obviously,

$$\mathbf{s}_k^{(d)} = P_{\mathbf{m}}^{(d)}$$

with $\mathbf{m}$ denoting the $((d-1) \times d)$ -matrix having entries $m_{1,1} = k$, $m_{i,i} = 1$ in its diagonal, $\forall i$, $2 \leq i \leq d-1$, and zero entries otherwise. By Theorem 3.1 we can embed $U_{\tau_{\mathbf{s}_k^{(d)}}}$ into $\mathbb{C}^{2d-1}$ via the $d-1$ equations

$$\begin{cases} z_1^k - z_2 z_{d+1} = 0, \\ z_2 - z_3 z_{d+2} = 0, \\ \vdots \\ z_{d-1} - z_d z_{2d-1} = 0. \end{cases}$$

In fact, replacing successively z_2 by $z_3 z_{d+2}$, z_3 by $z_4 z_{d+3}$, etc., in the first equation (according to the pattern of Corollary 3.3) and setting $w = z_1$, $t_i = z_{d+i-1}$, for all $i \in \{1, \dots, d\}$, we may represent $U_{\tau_{\mathbf{s}_k^{(d)}}}$ as a hypersurface embedded into $\mathbb{C}^{d+1}_{(w, t_1, t_2, t_3, \dots, t_d)}$:

$$U_{\tau_{\mathbf{s}_k^{(d)}}} \cong \text{Spec} \left(\mathbb{C}[w, t_1, t_2, t_3, \dots, t_d] / (w^k - \prod_{j=1}^d t_j) \right).$$

(Notice that $U_{\tau_{\mathbf{s}_k^{(d)}}} \cong \mathbb{C}^d / G(d; k)$ is an abelian quotient space w.r.t. a group $G(d; k) \cong (\mathbb{Z} / k\mathbb{Z})^{d-1}$; cf. [W, Ex. 1.5, p. 90] and [DHeZ, Prop. 5.10, p. 217].)

(v) Finally, let $k_1, k_2, \dots, k_{d-1}$ be a $(d-1)$ -tuple of positive integers ($d \geq 4$), with $k_1 \geq 2$, and let

$$\begin{aligned} \mathbf{RP}(k_1, k_2, \dots, k_{d-1}) &= \left\{ (x_1, \dots, x_d)^\top \in \mathbb{R}^d \mid \begin{array}{l} x_1 = 1, \ 0 \leq x_{j+1} \leq k_j, \\ \forall j, \ 1 \leq j \leq d-1 \end{array} \right\} \\ &= \{1\} \times [0, k_1] \times [0, k_2] \times \dots \times [0, k_{d-1}] \end{aligned}$$

denote the $(d-1)$ -dimensional *rectangular parallelepiped* in $\bar{\mathbf{H}}^{(d)} \hookrightarrow \mathbb{R}^d$ having them as lengths of its edges (cf. [DHaZ]).

$\mathbf{RP}(k_1, k_2, \dots, k_{d-1})$ has 2^{d-1} vertices (i.e., the greatest possible number of vertices; cf. (2.11)), namely

$$\{e_1 + \varepsilon_1 \cdot k_1 \cdot e_2 + \varepsilon_2 \cdot k_2 \cdot e_3 + \dots + \varepsilon_{d-1} \cdot k_{d-1} \cdot e_d \mid \varepsilon_1, \dots, \varepsilon_{d-1} \in \{0, 1\}\}$$

and equals $P_{\mathbf{m}}^{(d)}$ with $\mathbf{m}$ denoting the $((d-1) \times d)$ -matrix with entries $m_{i,1} = k_i$, for all i , $1 \leq i \leq d-1$, in its first column, and all the other entries = 0. By (3.4) we obtain

$$U_{\tau_{\mathbf{RP}(k_1, \dots, k_{d-1})}} \cong \text{Spec}(\mathbb{C}[z_1, \dots, z_{2d-1}] / (\{z_j^{k_j} - z_{j+1} z_{d+j} \mid 1 \leq j \leq d-1\})).$$

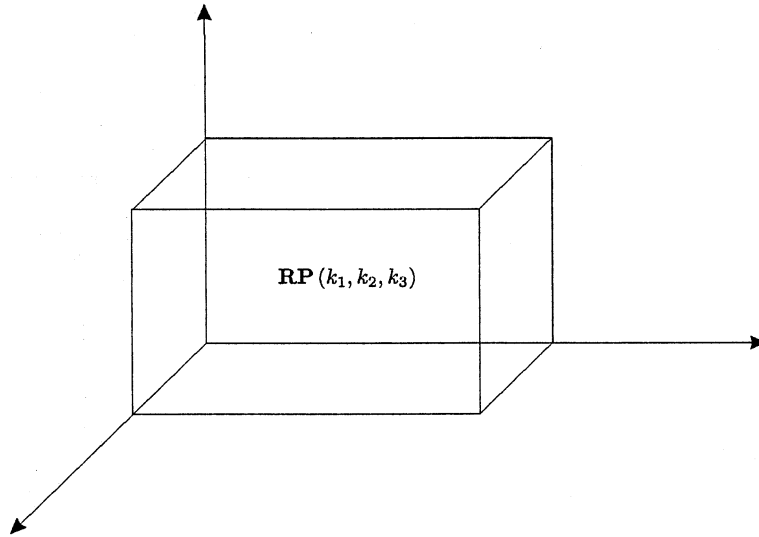


FIGURE 4.

4. PROOF OF THE MAIN THEOREM

To prove Theorem 3.1 we need several auxiliary lemmas. First, starting with an admissible sequence $\mathbf{m}$ as in (3.1), where $d \geq 3$, we define the set

$$\mathcal{L}_{\mathbf{m}}^{(d)} := \{e_1^\vee\} \cup \{e_k^\vee, m_{k-1} - e_k^\vee \mid 2 \leq k \leq d\}.$$

Lemma 4.1. $\tau_{P_{\mathbf{m}}^{(d)}}^\vee = \text{pos}(\mathcal{L}_{\mathbf{m}}^{(d)})$.

Proof. See Nakajima [N, p. 92]. □

For $\varepsilon_i \in \{0, 1\}$, $i \in \{1, \dots, d-1\}$, we define the s.c.p. cones $C_{\varepsilon_1, \dots, \varepsilon_{d-1}}^{(d)} \subset (\mathbb{R}^d)^\vee$ as follows:

$$C_{\varepsilon_1, \dots, \varepsilon_{d-1}}^{(d)} := \text{pos}(\{e_1^\vee\} \cup \{\varepsilon_i e_{i+1}^\vee + (1 - \varepsilon_i)(m_i - e_{i+1}^\vee) \mid 1 \leq i \leq d-1\}).$$

Lemma 4.2. The 2^{d-1} s.c.p. cones $\left\{ C_{\varepsilon_1, \dots, \varepsilon_{d-1}}^{(d)} \mid \varepsilon_i \in \{0, 1\}, 1 \leq i \leq d-1 \right\}$ form a subdivision of $\tau_{P_{\mathbf{m}}^{(d)}}^\vee$ into basic cones w.r.t. $(\mathbb{Z}^d)^\vee$.

Proof. Obviously, we have

$$|\det(e_1^\vee, \varepsilon_1 e_2^\vee + (1 - \varepsilon_1)(m_1 - e_2^\vee), \dots, \varepsilon_{d-1} e_d^\vee + (1 - \varepsilon_{d-1})(m_{d-1} - e_d^\vee))| = 1,$$

which means that all the cones $C_{\varepsilon_1, \dots, \varepsilon_{d-1}}^{(d)}$ are basic w.r.t. $(\mathbb{Z}^d)^\vee$. Next, we show that the intersection of two of these simplicial cones, say of $C_{\varepsilon_1, \dots, \varepsilon_{d-1}}^{(d)}$ and $C_{\varepsilon'_1, \dots, \varepsilon'_{d-1}}^{(d)}$,

is either a face of both or empty. More precisely, we shall prove that

$$(4.1) \quad C_{\varepsilon_1, \dots, \varepsilon_{d-1}}^{(d)} \cap C_{\varepsilon'_1, \dots, \varepsilon'_{d-1}}^{(d)} \\ = \text{pos} \left(\{e_1^\vee\} \cup \left\{ \varepsilon_{i-1} e_i^\vee + (1 - \varepsilon_{i-1}) (m_{i-1} - e_i^\vee) \mid \begin{array}{l} \text{for all} \\ i \in \{2, \dots, d\} \\ \text{for which} \\ \varepsilon_{i-1} = \varepsilon'_{i-1} \end{array} \right\} \right).$$

The inclusion “ $\supseteq$ ” is obvious. For every element $c \in C_{\varepsilon_1, \dots, \varepsilon_{d-1}}^{(d)} \cap C_{\varepsilon'_1, \dots, \varepsilon'_{d-1}}^{(d)}$ there exist

$$(\nu_1, \dots, \nu_d), (\xi_1, \dots, \xi_d) \in (\mathbb{R}_{\geq 0})^d,$$

such that

$$(4.2) \quad c = \nu_1 e_1^\vee + \sum_{i=2}^d \nu_i (\varepsilon_{i-1} e_i^\vee + (1 - \varepsilon_{i-1}) (m_{i-1} - e_i^\vee)) \\ = \xi_1 e_1^\vee + \sum_{i=2}^d \xi_i (\varepsilon'_{i-1} e_i^\vee + (1 - \varepsilon'_{i-1}) (m_{i-1} - e_i^\vee)).$$

For the last coordinate of c we obtain

$$\nu_d (\varepsilon_{d-1} e_d^\vee + (1 - \varepsilon_{d-1}) (-e_d^\vee)) = \xi_d (\varepsilon'_{d-1} e_d^\vee + (1 - \varepsilon'_{d-1}) (-e_d^\vee)),$$

and thus either $\nu_d = \xi_d = 0$ or $\nu_d = \xi_d > 0$ and $\varepsilon_{d-1} = \varepsilon'_{d-1}$. Hence, (4.2) is reduced to

$$\nu_1 e_1^\vee + \sum_{i=2}^{d-1} \nu_i (\varepsilon_{i-1} e_i^\vee + (1 - \varepsilon_{i-1}) (m_{i-1} - e_i^\vee)) \\ = \xi_1 e_1^\vee + \sum_{i=2}^{d-1} \xi_i (\varepsilon'_{i-1} e_i^\vee + (1 - \varepsilon'_{i-1}) (m_{i-1} - e_i^\vee)).$$

Applying the same argumentation to the $(d-1)$ -th coordinate of this vector and repeating (actually by induction w.r.t. d ; cf. (2.4)) the whole procedure, we get

$$(\nu_i = \xi_i = 0) \text{ or } (\nu_i = \xi_i > 0 \text{ and } \varepsilon_{i-1} = \varepsilon'_{i-1}), \text{ for } i \in \{2, \dots, d\},$$

which implies the “ $\subseteq$ ” part of (4.1). Finally, we use induction w.r.t. d to prove the equality

$$(4.3) \quad \bigcup_{\varepsilon_i \in \{0,1\}, 1 \leq i \leq d-1} C_{\varepsilon_1, \dots, \varepsilon_{d-1}}^{(d)} = \tau_{P_m^{(d)}}^\vee.$$

For $d = 3$ this was proved in [Ish, Lemma 8.12, p. 142]. Suppose that d is > 3 . Clearly, the inclusion “ $\subseteq$ ” in (4.3) is always valid (by Lemma 4.1). Now let v be an arbitrary element of $\tau_{P_m^{(d)}}^\vee$ and $\nu_1, \dots, \nu_d, \xi_1, \dots, \xi_d \in \mathbb{R}_{\geq 0}$ such that

$$v = \left(\nu_1 e_1^\vee + \sum_{i=2}^{d-1} \nu_i e_i^\vee + \xi_i (m_{i-1} - e_i^\vee) \right) + \nu_d e_d^\vee + \xi_d (m_{d-1} - e_d^\vee).$$

Define

$$\tilde{v} := \nu_1 e_1^\vee + \sum_{i=2}^{d-1} \nu_i e_i^\vee + \xi_i (m_{i-1} - e_i^\vee).$$

Then $\tilde{v} \in \tau_{P_{\tilde{\mathbf{m}}}^{(d-1)}}^\vee \subset \tau_{P_{\mathbf{m}}^{(d)}}^\vee$, with

$$\mathbf{m} = \begin{pmatrix} \tilde{\mathbf{m}} & 0 & 0 \\ m_{d-1} & 0 & 0 \end{pmatrix}$$

and we may write

$$v = \begin{cases} \tilde{v} + \xi_d m_{d-1} + (\nu_d - \xi_d) e_d^\vee, & \text{if } \nu_d \geq \xi_d, \\ \tilde{v} + \nu_d m_{d-1} + (\xi_d - \nu_d) (m_{d-1} - e_d^\vee), & \text{if } \nu_d \leq \xi_d. \end{cases}$$

Since we know that $\tilde{\mathbf{m}}$ is admissible, we have $\tilde{v} + r m_{d-1} \in \tau_{P_{\tilde{\mathbf{m}}}^{(d-1)}}^\vee$ for any $r \in \mathbb{R}_{\geq 0}$. Thus, in the case in which $\nu_d \geq \xi_d$, there exists, by induction hypothesis, a $(d-1)$ -dimensional cone $C_{\varepsilon_1, \dots, \varepsilon_{d-2}}^{(d-1)} \subset \tau_{P_{\tilde{\mathbf{m}}}^{(d-1)}}^\vee$ containing $\tilde{v} + \xi_d m_{d-1}$. But this shows that $v \in C_{\varepsilon_1, \dots, \varepsilon_{d-2}, 1}^{(d)}$. In the other case, i.e., whenever $\nu_d \leq \xi_d$, we find, in the same way, a cone of type $C_{\varepsilon_1, \dots, \varepsilon_{d-2}, 0}^{(d)}$ containing v . $\square$

Lemma 4.3. *The set $\mathcal{L}_{\mathbf{m}}^{(d)}$ is a system of generators of the additive semigroup $\tau_{P_{\mathbf{m}}^{(d)}}^\vee \cap (\mathbb{Z}^d)^\vee$.*

Proof. For the subdivision of $\tau_{P_{\mathbf{m}}^{(d)}}^\vee$ into *basic* cones (constructed in Lemma 4.2), we have

$$\bigcup_{\varepsilon_i \in \{0,1\}, 1 \leq i \leq d-1} \text{Gen} \left(C_{\varepsilon_1, \dots, \varepsilon_{d-1}}^{(d)} \right) = \mathcal{L}_{\mathbf{m}}^{(d)}.$$

Hence, every element of $\tau_{P_{\mathbf{m}}^{(d)}}^\vee \cap (\mathbb{Z}^d)^\vee$ can be written as a nonnegative integral linear combination of the elements of $\mathcal{L}_{\mathbf{m}}^{(d)}$. $\square$

Now let $\mathcal{A}_{\mathbf{m}}^{(d)} = (\mathbf{I}_d \quad \mathbf{M}^\top)$ be the $d \times (2d-1)$ integral matrix with

$$\mathbf{M} := \begin{pmatrix} m_{1,1} & -1 & 0 & \cdots & \cdots & 0 & 0 \\ m_{2,1} & m_{2,2} & -1 & \cdots & \cdots & 0 & 0 \\ m_{3,1} & m_{3,2} & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & -1 & 0 \\ m_{d-1,1} & m_{d-1,2} & m_{d-1,3} & \cdots & m_{d-1,d-2} & m_{d-1,d-1} & -1 \end{pmatrix},$$

and $\mathbf{I}_d$ the $(d \times d)$ -identity matrix. (Notice that the transposes of the column vectors of $\mathcal{A}_{\mathbf{m}}^{(d)}$ are precisely the elements of the set $\mathcal{L}_{\mathbf{m}}^{(d)}$). Consider the integer lattice $\Lambda_{\mathcal{L}_{\mathbf{m}}^{(d)}}$ w.r.t. $\mathcal{L}_{\mathbf{m}}^{(d)}$, i.e.,

$$\begin{aligned} \Lambda_{\mathcal{L}_{\mathbf{m}}^{(d)}} &= \left\{ \ell = (\ell_1, \dots, \ell_{2d-1}) \in \mathbb{Z}^{2d-1} \mid \sum_{\kappa=1}^d \ell_\kappa e_\kappa^\vee + \sum_{i=1}^{d-1} \ell_{d+i} (m_i - e_{i+1}^\vee) = 0 \right\} \\ &= \left\{ \ell = (\ell_1, \dots, \ell_{2d-1}) \in \mathbb{Z}^{2d-1} \mid \mathcal{A}_{\mathbf{m}}^{(d)} \ell^\top = 0 \right\} = \text{Ker}(\psi), \end{aligned}$$

where ψ is the homomorphism $\mathbb{Z}^{2d-1} \ni \ell \mapsto \psi(\ell) = \mathcal{A}_{\mathbf{m}}^{(d)} \ell \tau \in \mathbb{Z}^d$. By Lemma 4.3, the characters $\mathbf{e}(e_1^\vee), \dots, \mathbf{e}(e_d^\vee), \mathbf{e}(m_1 - e_2^\vee), \mathbf{e}(m_2 - e_3^\vee), \dots, \mathbf{e}(m_{d-1} - e_d^\vee)$ generate $\mathbb{C}[\tau_{P_{\mathbf{m}}^{(d)}}^\vee \cap (\mathbb{Z}^d)^\vee]$. Hence, the affine toric variety $U_{\tau_P} \cong U_{\tau_{P_{\mathbf{m}}^{(d)}}}$ admits an embedding into $\mathbb{C}^{2d-1}$ w.r.t. $\mathcal{L}_{\mathbf{m}}^{(d)}$, and is, in particular, a g.c.i. of $d-1$ binomials (by Theorem 2.12, Remark 2.13(ii), and Theorem 1.2). The map

$$\theta : \mathbb{C}[z_1, z_2, \dots, z_d, z_{d+1}, \dots, z_{2d-1}] \longrightarrow \mathbb{C}[\tau_{P_{\mathbf{m}}^{(d)}}^\vee \cap (\mathbb{Z}^d)^\vee]$$

defined by

$$\theta(z_\kappa) := \mathbf{e}(e_\kappa^\vee), \quad \forall \kappa, \quad \kappa \in \{1, \dots, d\},$$

and

$$\theta(z_{d+i}) := \mathbf{e}(m_i - e_{i+1}^\vee), \quad \forall i, \quad i \in \{1, \dots, d-1\},$$

is a $\mathbb{C}$ -algebra epimorphism. Let $\mathcal{I} := \mathcal{I}_{\mathcal{A}_{\mathbf{m}}^{(d)}} := \text{Ker}(\theta)$ denote its kernel. The column-vectors, say $b_1, b_2, \dots, b_{d-1}$, of the $(2d-1) \times (d-1)$ -matrix

$$\mathcal{B}_{\mathbf{m}}^{(d)} := \begin{pmatrix} \mathbf{M}^\top \\ -\mathbf{I}_{d-1} \end{pmatrix} = \begin{pmatrix} m_{1,1} & m_{2,1} & m_{3,1} & \dots & m_{d-2,1} & m_{d-1,1} \\ -1 & m_{2,2} & m_{3,2} & \dots & m_{d-2,2} & m_{d-1,2} \\ 0 & -1 & m_{3,3} & \dots & m_{d-2,3} & m_{d-1,3} \\ 0 & 0 & -1 & \ddots & m_{d-2,4} & m_{d-1,4} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & -1 & m_{d-1,d-1} \\ 0 & \dots & \dots & 0 & 0 & -1 \\ -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & -1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & -1 & 0 \\ 0 & \dots & \dots & 0 & 0 & -1 \end{pmatrix}$$

build up a $\mathbb{Z}$ -basis of $\Lambda_{\mathcal{L}_{\mathbf{m}}^{(d)}}$. Let $\mathcal{J}_{\mathcal{B}_{\mathbf{m}}^{(d)}}$ be the lattice ideal of $\mathbb{C}[z_1, \dots, z_d, \dots, z_{2d-1}]$ which is associated to $\mathcal{B}_{\mathbf{m}}^{(d)}$ (cf. 1.3). In order to determine a generating system of $\mathcal{I}$ consisting of binomials whose exponents are expressed in terms of the entries of our initial admissible sequence of free parameters (3.1), it seems to be reasonable to specify the saturation of $\mathcal{J}_{\mathcal{B}_{\mathbf{m}}^{(d)}}$ w.r.t. the product $\prod_{j=1}^{2d-1} z_j$ of all available variables (see Theorem 1.5(i)). Nevertheless, this method would be rather laborious from the computational point of view, because it would involve elimination techniques or even primary decompositions of $\mathcal{J}_{\mathcal{B}_{\mathbf{m}}^{(d)}}$, and would necessarily demand to perform a relatively high number of Gröbner basis algorithms (see [BiLSR], [HSh], [St1]). Instead, we shall pass to another $\mathbb{Z}$ -basis of $\Lambda_{\mathcal{L}_{\mathbf{m}}^{(d)}}$ whose matrix $\widehat{\mathcal{B}}_{\mathbf{m}}^{(d)}$ is dominating and we shall apply Theorem 1.5(ii). (For affine semigroup rings which are complete intersections the *existence* of an integral basis of their relation space having a dominating coefficient matrix is guaranteed by a result of Fischer and Shapiro; cf. [FSh, Cor. 2.10, p. 47]. In the case at hand, $\widehat{\mathcal{B}}_{\mathbf{m}}^{(d)}$ will be constructed *explicitly*.)

Lemma 4.4. *All integral $(2d - 1) \times (d - 1)$ -matrices of the form*

$$(4.4) \quad \begin{pmatrix} \alpha_{1,1} & \alpha_{2,1} & \alpha_{3,1} & \cdots & \alpha_{d-2,1} & \alpha_{d-1,1} \\ -1 & \alpha_{2,2} & \alpha_{3,2} & \cdots & \alpha_{d-2,2} & \alpha_{d-1,2} \\ 0 & -1 & \alpha_{3,3} & \cdots & \alpha_{d-2,3} & \alpha_{d-1,3} \\ 0 & 0 & -1 & \ddots & \alpha_{d-2,4} & \alpha_{d-1,4} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & -1 & \alpha_{d-1,d-1} \\ 0 & \cdots & \cdots & 0 & 0 & -1 \\ -1 & \beta_{2,2} & \beta_{3,2} & \cdots & \beta_{d-2,2} & \beta_{d-1,2} \\ 0 & -1 & \beta_{3,3} & \cdots & \beta_{d-2,3} & \beta_{d-1,3} \\ 0 & 0 & -1 & \ddots & \beta_{d-2,4} & \beta_{d-1,4} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & -1 & \beta_{d-1,d-1} \\ 0 & \cdots & \cdots & 0 & 0 & -1 \end{pmatrix}$$

where

$$\alpha_{i,j}, \quad 1 \leq i \leq d-1, \quad 1 \leq j \leq i \leq d-1,$$

and

$$\beta_{i,j}, \quad 2 \leq j \leq i \leq d-1,$$

are nonnegative, and each of their columns contains at least one positive element, are dominating matrices.

Proof. Suppose that such an (obviously mixed) matrix contains a mixed $(\rho \times \rho)$ -submatrix, for $\rho \in \{2, \dots, d-1\}$, with “column indices” $l_1, \dots, l_\rho$, where

$$l_1 < l_2 < \cdots < l_\rho.$$

Then the negative entries of the column having index l_i are to be found in the rows whose indices belong to the set

$$\mathfrak{N}_i := \{l_i + 1, l_i + d\},$$

for every $i \in \{1, \dots, \rho\}$. Since the $(\rho \times \rho)$ -submatrix under consideration is assumed to be mixed, it has to contain in its first column a positive entry which is located in the rows whose indices are within

$$\mathfrak{N}_{\rho+1} := \{1, \dots, l_1, d+1, \dots, l_1 + d-1\}.$$

Since $l_\rho \leq d-1$, the $\rho+1$ sets $\{\mathfrak{N}_i \mid 1 \leq i \leq \rho+1\}$ are pairwise disjoint, but our $(\rho \times \rho)$ -submatrix must contain a “row index” from each of these sets, which is impossible. Consequently, all integral matrices of the form (4.4) are dominating matrices. $\square$

Remark 4.5. Our intention is to prove that after having performed (at most) $d-2$ suitable unimodular transformations to the entries of our initial matrix $\mathcal{B}_{\mathbf{m}}^{(d)}$, we construct a $\mathbb{Z}$ -basis of $\Lambda_{\mathcal{L}_{\mathbf{m}}^{(d)}}$ whose matrix $\widehat{\mathcal{B}}_{\mathbf{m}}^{(d)}$ is of type (4.4). This procedure will be realized in three steps. In the first step, which explains where our motivation comes from, we discuss what happens in the “low” dimensions $d=3$ and $d=4$. In the second step, we present the recursive principle by means of which we modify the last column of $\mathcal{B}_{\mathbf{m}}^{(d)}$. Finally, in the third step we apply unimodular transformations of the same sort to the next coming column vectors. (In the particular case in

which *all* the entries of (3.1) are nonnegative, $\mathcal{B}_{\mathbf{m}}^{(d)}$ is itself dominating and there is no need to proceed; cf. 3.2(iii)).

► **Step 1: Low dimensions.** Let us start with $d = 3$. In this case,

$$\mathcal{B}_{\mathbf{m}}^{(3)} = \begin{pmatrix} m_{1,1} & m_{2,1} \\ -1 & m_{2,2} \\ 0 & -1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If $m_{2,2} \geq 0$, there is nothing to do. So we may assume that $m_{2,2} < 0$. Adding $m_{2,2}$ times the first column to the second one (which corresponds to a unimodular transformation) results in

$$\begin{pmatrix} m_{1,1} & m_{2,1} + m_{1,1}m_{2,2} \\ -1 & 0 \\ 0 & -1 \\ -1 & -m_{2,2} \\ 0 & -1 \end{pmatrix}.$$

In view of the “admissibility conditions” (2.5), the matrix is of type (4.4). (We should mention at this point that, for $d = 3$, Ishida makes similar choices by using some purely geometric arguments; cf. [Ish, proof of Thm. 8.1, in particular pp. 140-141]).

Let us now increase the dimension by one. For $d = 4$,

$$\mathcal{B}_{\mathbf{m}}^{(4)} = \begin{pmatrix} m_{1,1} & m_{2,1} & m_{3,1} \\ -1 & m_{2,2} & m_{3,2} \\ 0 & -1 & m_{3,3} \\ 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We start by looking at the last column. Assume, for instance, that $m_{3,3} < 0$. Adding $m_{3,3}$ times the second column to the last one we obtain

$$\begin{pmatrix} m_{1,1} & m_{2,1} & m_{3,1} + m_{2,1}m_{3,3} \\ -1 & m_{2,2} & m_{3,2} + m_{2,2}m_{3,3} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & -m_{3,3} \\ 0 & 0 & -1 \end{pmatrix}.$$

If

$$m_{3,2} + m_{2,2}m_{3,3} \geq 0,$$

the “admissibility conditions” (2.6) tell us that the entries of the last column of our matrix are like those of the last column of the matrices of type (4.4). If

$$m_{3,2} + m_{2,2}m_{3,3} < 0,$$

then we add $(m_{3,2} + m_{2,2}m_{3,3})$ -times the first row to the last one and we get

$$\begin{pmatrix} m_{1,1} & m_{2,1} & m_{3,1} + m_{2,1}m_{3,3} + m_{1,1}(m_{3,2} + m_{2,2}m_{3,3}) \\ -1 & m_{2,2} & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & -0 & -(m_{3,2} + m_{2,2}m_{3,3}) \\ 0 & -1 & -m_{3,3} \\ 0 & 0 & -1 \end{pmatrix}.$$

Again by (2.6) the last column has the desired property. Now we can start to transform the first two columns as we did before in dimension 3.

Next, let us assume that

$$m_{3,3} \geq 0, \text{ but } m_{3,2} < 0.$$

In this case, we add $m_{3,2}$ times the first column to the last one and we get

$$\begin{pmatrix} m_{1,1} & m_{2,1} & m_{3,1} + m_{1,1}m_{3,2} \\ -1 & m_{2,2} & 0 \\ 0 & -1 & m_{3,3} \\ 0 & 0 & -1 \\ -1 & -0 & -m_{3,2} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Again the entries of the last column of our matrix are like those of the last column of the matrices of type (4.4).

► **Step 2: The recursive principle.** We define appropriate matrix operations, so that the last column looks like (4.4). Since these operations (as we shall see below) do not affect the other columns, we can apply a recursive argument.

More precisely, for $i = d-2, \dots, 1$ we define recursively some nonpositive integers λ_i and some nonnegative integers μ_i as follows:

$$\lambda_i := \begin{cases} \min \{0, m_{d-1, d-1}\}, & \text{if } i = d-2, \\ \min \left\{ 0, m_{d-1, i+1} + \sum_{k=i+1}^{d-2} \lambda_k m_{k, i+1} \right\}, & \text{if } i \leq d-3, \end{cases}$$

$$\mu_i := \begin{cases} \max \{0, m_{d-1, d-1}\}, & \text{if } i = d-2, \\ \max \left\{ 0, m_{d-1, i+1} + \sum_{k=i+1}^{d-2} \lambda_k m_{k, i+1} \right\}, & \text{if } i \leq d-3. \end{cases}$$

Furthermore, we set

$$\mu_0 := m_{d-1, 1} + \sum_{i=1}^{d-2} \lambda_i m_{i, 1}.$$

Since

$$\mu_i = -\lambda_i + \left(m_{d-1, i+1} + \sum_{k=i+1}^{d-2} \lambda_k m_{k, i+1} \right), \quad i = 1, \dots, d-2,$$

the vector $\widehat{b}_{d-1} := b_{d-1} + \sum_{i=1}^{d-2} \lambda_i b_i$ can be written as

$$\widehat{b}_{d-1} = (\mu_0, \mu_1, \dots, \mu_{d-2}, -1, -\lambda_1, \dots, -\lambda_{d-2}, -1)^\top.$$

Lemma 4.6. $\widehat{b}_{d-1}$ contains both negative and positive entries.

Proof. Apparently, $\widehat{b}_{d-1}$ has two negative coordinates. Suppose that $\widehat{b}_{d-1}$ has no positive entries. Then we have $\lambda_i = \mu_i = 0$, $i = 1, \dots, d-2$, which implies $m_{d-1,i+1} = 0$, for $i = d-2, \dots, 1$, and $\mu_0 = m_{d-1,1}$. However, by the definition of the free parameters $\mathbf{m}$ (see Definition 2.6) we know that $m_{d-1} \neq 0$ and from (2.3) we get $m_{d-1,1} = \langle m_{d-1}, e_1 \rangle \geq 0$, which leads to a contradiction. $\square$

Next, we shall show that the first entry of $\widehat{b}_{d-1}$, i.e., μ_0 , is also nonnegative. To this end we define

$$\varepsilon_i := \begin{cases} 0, & \text{if } \lambda_i = 0, \\ 1, & \text{if } \lambda_i < 0, \end{cases} \quad \text{for all } i \in \{1, \dots, d-2\},$$

and write λ_i as follows:

$$\lambda_i = \varepsilon_i \left(m_{d-1,i+1} + \sum_{k=i+1}^{d-2} \lambda_k m_{k,i+1} \right).$$

Now for $n \in \{1, \dots, d-2\}$ and $k \in \{0, 1, \dots, d-2\}$ we set

$$p_{k,n} := \sum_{n=i_0 < i_1 < \dots < i_k < i_{k+1}=d-1} \prod_{j=1}^{k+1} \varepsilon_{i_{j-1}} m_{i_j, i_{j-1}+1}.$$

Note that for all $n \in \{1, \dots, d-2\}$, we obtain

$$p_{0,n} = \varepsilon_n \cdot m_{d-1,n+1}.$$

Lemma 4.7. For all $n \in \{1, 2, \dots, d-2\}$ we have

$$\lambda_n = \sum_{k=0}^{d-2-n} p_{k,n}.$$

Proof. First we check the identity

$$(4.5) \quad p_{l+1,n-1} = \sum_{k=n}^{d-2-l} \varepsilon_{n-1} m_{k,n} p_{l,k}.$$

Again this follows immediately from the definition, since

$$\begin{aligned} & \sum_{k=n}^{d-2-l} \varepsilon_{n-1} m_{k,n} p_{l,k} \\ &= \sum_{k=n}^{d-2-l} \varepsilon_{n-1} m_{k,n} \sum_{k=i_0 < i_1 < \dots < i_l < i_{l+1}=d-1} \prod_{j=1}^{l+1} \varepsilon_{i_{j-1}} m_{i_j, i_{j-1}+1} \\ &= \sum_{k=n}^{d-2-l} \varepsilon_{n-1} m_{k,n} \sum_{k=i_1 < i_2 < \dots < i_{l+1} < i_{l+2}=d-1} \prod_{j=2}^{l+2} \varepsilon_{i_{j-1}} m_{i_j, i_{j-1}+1} \\ &= \sum_{n-1=i_0 < i_1 < i_2 < \dots < i_{l+1} < i_{l+2}=d-1} \prod_{j=1}^{l+2} \varepsilon_{i_{j-1}} m_{i_j, i_{j-1}+1} = p_{l+1,n-1}. \end{aligned}$$

To prove the proposition we apply (backwards) induction with respect to n . If $n = d - 2$, then we have $\lambda_{d-2} = \varepsilon_{d-2} m_{d-1, d-1}$. So let n be $< d - 2$. Then we may write

$$\begin{aligned} \lambda_{n-1} &= \varepsilon_{n-1} \left(m_{d-1, n} + \sum_{k=n}^{d-2} \lambda_k m_{k, n} \right) = p_{0, n-1} + \sum_{k=n}^{d-2} \sum_{l=0}^{d-2-k} p_{l, k} \varepsilon_{n-1} m_{k, n} \\ &= p_{0, n-1} + \sum_{l=0}^{d-2-n} \sum_{k=n}^{d-2-l} p_{l, k} \varepsilon_{n-1} m_{k, n} \\ &= p_{0, n-1} + \sum_{l=0}^{d-2-n} p_{l+1, n-1}, \end{aligned}$$

where the last equation follows from (4.5). $\square$

Lemma 4.8.

$$\mu_0 \geq 0.$$

Proof. From the definition of μ_0 and Lemma 4.7 we get

$$\begin{aligned} (4.6) \quad \mu_0 &= m_{d-1, 1} + \sum_{n=1}^{d-2} \lambda_n m_{n, 1} = m_{d-1, 1} + \sum_{n=1}^{d-2} \sum_{k=0}^{d-2-n} p_{k, n} m_{n, 1} \\ &= m_{d-1, 1} + \sum_{k=0}^{d-3} \sum_{n=1}^{d-2-k} p_{k, n} m_{n, 1} = m_{d-1, 1} + \sum_{k=1}^{d-2} \sum_{n=1}^{d-1-k} p_{k-1, n} m_{n, 1}. \end{aligned}$$

Furthermore, by the definition of $p_{k, n}$ we have

$$\begin{aligned} \sum_{n=1}^{d-1-k} p_{k-1, n} m_{n, 1} &= \sum_{n=1}^{d-1-k} m_{n, 1} \sum_{n=i_0 < i_1 < \dots < i_{k-1} < i_k = d-1} \prod_{j=1}^k \varepsilon_{i_{j-1}} m_{i_j, i_{j-1}+1} \\ &= \sum_{n=1}^{d-1-k} m_{n, 1} \sum_{n=i_1 < i_2 < \dots < i_k < i_{k+1} = d-1} \prod_{j=2}^{k+1} \varepsilon_{i_{j-1}} m_{i_j, i_{j-1}+1} \\ &= \sum_{0=i_0 < i_1 < i_2 < \dots < i_k < i_{k+1} = d-1} m_{i_1, i_0+1} \prod_{j=2}^{k+1} \varepsilon_{i_{j-1}} m_{i_j, i_{j-1}+1} \\ &= \sum_{0=i_0 < i_1 < i_2 < \dots < i_k < i_{k+1} = d-1} m_{d-1, i_k+1} \prod_{j=1}^k \varepsilon_{i_j} m_{i_j, i_{j-1}+1}. \end{aligned}$$

This, combined with (4.6), gives

$$\mu_0 = \sum_{k=0}^{d-2} \sum_{0=i_0 < i_1 < i_2 < \dots < i_k < i_{k+1} = d-1} m_{d-1, i_k+1} \cdot \prod_{j=1}^k \varepsilon_{i_j} m_{i_j, i_{j-1}+1},$$

which is nonnegative by Corollary 2.11. $\square$

What we have done so far can be summarized in the following corollary.

Corollary 4.9. *Let $\mathcal{U}^{(d)} \in \mathrm{GL}(d-1, \mathbb{Z})$ be the unimodular matrix given by*

$$\mathcal{U}^{(d)} = \begin{pmatrix} 1 & 0 & 0 & \cdots & \lambda_1 \\ 0 & 1 & 0 & \cdots & \lambda_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \lambda_{d-2} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Then we have

$$\mathcal{B}_{\mathbf{m}}^{(d)} \mathcal{U}^{(d)} = \begin{pmatrix} m_{1,1} & m_{2,1} & m_{3,1} & \cdots & m_{d-2,1} & \mu_0 \\ -1 & m_{2,2} & m_{3,2} & \cdots & m_{d-2,2} & \mu_1 \\ 0 & -1 & m_{3,3} & \cdots & m_{d-2,3} & \mu_2 \\ 0 & 0 & -1 & \ddots & m_{d-2,4} & \mu_3 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & -1 & \mu_{d-2} \\ 0 & \cdots & \cdots & 0 & 0 & -1 \\ -1 & 0 & 0 & \cdots & 0 & -\lambda_1 \\ 0 & -1 & 0 & \cdots & 0 & -\lambda_2 \\ 0 & 0 & -1 & \ddots & 0 & -\lambda_3 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & -1 & -\lambda_{d-2} \\ 0 & \cdots & \cdots & 0 & 0 & -1 \end{pmatrix}$$

where $\mu_i \geq 0$ for all $i \in \{0, 1, \dots, d-2\}$, $\lambda_i \leq 0$ for all $i \in \{1, \dots, d-2\}$, and the last column contains both positive and negative entries.

We observe that the first $d-2$ columns of the matrix $\mathcal{B}_{\mathbf{m}}^{(d)} \mathcal{U}^{(d)}$ are the same as the columns of the matrix $\mathcal{B}_{\mathbf{m}}^{(d)}$, namely

$$\begin{pmatrix} m_{1,1} \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} m_{2,1} \\ m_{2,2} \\ -1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} m_{3,1} \\ m_{3,2} \\ m_{3,3} \\ -1 \\ \vdots \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} m_{d-2,1} \\ m_{d-2,2} \\ m_{d-2,3} \\ m_{d-2,4} \\ \vdots \\ -1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ -1 \\ 0 \end{pmatrix},$$

and that the d -th and $(2d-1)$ -th row contain nonzero entries only in the last column of $\mathcal{B}_{\mathbf{m}}^{(d)} \mathcal{U}^{(d)}$. This allows us to apply our transformations, which have been carried out so far only for the last column, successively to the other columns also.

► **Step 3: Generalizing the recursion for all column vectors.** It is enough to equip our lambdas and mus with an additional index, just for keeping track of

the next coming columns (viewed backwards). That's why we define recursively nonnegative integers $\lambda_{i,j}$ and nonpositive integers $\mu_{i,j}$ by the formulae (3.2) and (3.3), respectively. ($\lambda_{i,d-1}, \mu_{i,d-1}$ are exactly the numbers which we called before λ_i and μ_i). Moreover, using the column vectors $b_1, \dots, b_{d-1}$ of $\mathcal{B}_{\mathbf{m}}^{(d)}$, we introduce the integer linear combinations

$$\widehat{b}_j := \begin{cases} b_1, & \text{if } j = 1, \\ b_j + \sum_{i=1}^{j-1} \lambda_{i,j} b_i, & \text{if } j \in \{2, \dots, d-1\}. \end{cases}$$

Finally, we define $\mathcal{U}_j^{(d)} \in \mathrm{GL}(d-1, \mathbb{Z})$ for $j = 2, \dots, d-1$ as follows:

$$\mathcal{U}_j^{(d)} := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \lambda_{1,j} & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & \lambda_{2,j} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots & \vdots & 0 & \cdots & 0 \\ \vdots & & 0 & \ddots & \vdots & \vdots & 0 & \cdots & 0 \\ & & & & \ddots & \lambda_{j-1,j} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & & \ddots & & \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}.$$

Each $\mathcal{U}_j^{(d)}$ is obviously an upper triangular matrix, with 1's as diagonal elements. All the other nontrivial elements are contained in the j -th column.

Proposition 4.10. *Using the above matrices, the product*

$$\widehat{\mathcal{B}}_{\mathbf{m}}^{(d)} := (\widehat{b}_1, \dots, \widehat{b}_{d-1}) = \mathcal{B}_{\mathbf{m}}^{(d)} \cdot \mathcal{U}_{d-1}^{(d)} \cdot \mathcal{U}_{d-2}^{(d)} \cdot \cdots \cdot \mathcal{U}_2^{(d)}$$

reads as

$$(4.7) \quad \widehat{\mathcal{B}}_{\mathbf{m}}^{(d)} = \begin{pmatrix} m_{1,1} & \mu_{0,2} & \mu_{0,3} & \cdots & \mu_{0,d-2} & \mu_{0,d-1} \\ -1 & \mu_{1,2} & \mu_{1,3} & \cdots & \mu_{1,d-2} & \mu_{1,d-1} \\ 0 & -1 & \mu_{2,3} & \cdots & \mu_{2,d-2} & \mu_{2,d-1} \\ 0 & 0 & -1 & \ddots & \mu_{3,d-2} & \mu_{3,d-1} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & -1 & \mu_{d-2,d-1} \\ 0 & \cdots & \cdots & 0 & 0 & -1 \\ -1 & -\lambda_{1,2} & -\lambda_{1,3} & \cdots & -\lambda_{1,d-2} & -\lambda_{1,d-1} \\ 0 & -1 & -\lambda_{3,3} & \cdots & -\lambda_{3,d-2} & -\lambda_{2,d-1} \\ 0 & 0 & -1 & \ddots & -\lambda_{4,d-2} & -\lambda_{3,d-1} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & -1 & -\lambda_{d-2,d-1} \\ 0 & \cdots & \cdots & 0 & 0 & -1 \end{pmatrix}$$

which is a dominating matrix with only -1 's as negative entries.

Proof. According to the “admissibility conditions” (2.3) we have always $m_{1,1} > 0$. We observe that the unimodular matrix $\mathcal{U}_j^{(d)}$ affects only the j -th column; and since $\mathcal{B}_m^{(d)}$ is partitioned into two upper triangular matrices, $\mathcal{U}_j^{(d)}$ affects only the nontrivial elements of the j -th column of the matrix $\mathcal{B}_m^{(d)} \cdot \mathcal{U}_{d-1}^{(d)} \cdots \mathcal{U}_{j+1}^{(d)}$. Hence, it suffices to use induction w.r.t. d (by exploiting Corollary 4.9), and to take into account Lemma 4.4 and Lemma 4.8. $\square$

Proof of Theorem 3.1. By Proposition 4.10, $\widehat{\mathcal{B}}_m^{(d)}$ is a dominating matrix and its column vectors $\widehat{b}_1, \dots, \widehat{b}_{d-1}$ constitute a $\mathbb{Z}$ -basis of $\Lambda_{\mathcal{L}_m^{(d)}}$. Applying Theorem 1.5(ii) we obtain

$$\mathcal{I} = \mathcal{J}_{\widehat{\mathcal{B}}_m^{(d)}},$$

and the j -th binomial of the constructed $d-1$ generators of $\mathcal{I}$ is exactly that one containing the nonnegative entries of $\widehat{b}_j$ as exponents of the variables $z_1, \dots, z_{2d-1}$ in its first monomial and the opposites of the nonpositive entries of $\widehat{b}_j$ as exponents of $z_1, \dots, z_{2d-1}$ in its second monomial, $\forall j, 1 \leq j \leq d-1$; cf. 1.3 and (4.7). Since $\widehat{\mathcal{B}}_m^{(d)}$ has only two -1 's as negative entries in each column, the second monomials contain just two variables, namely z_{j+1} and z_{d+j} for all $j \in \{1, \dots, d-1\}$. $\square$

Remark 4.11. By Lemma 4.3, we have $\mathbf{Hilb}_{(\mathbb{Z}^d)^\vee}(\tau_{P_m^{(d)}}^\vee) \subseteq \mathcal{L}_m^{(d)}$, and the inclusion may be strict. The next lemma gives an explicit characterization of the Hilbert basis elements and allows us to realize the “minimal” embedding of $U_{\tau_{P_m^{(d)}}}$ by eliminating redundant variables.

Lemma 4.12. *Let $v \in \mathcal{L}_m^{(d)}$. Then*

$$v \in \mathbf{Hilb}_{(\mathbb{Z}^d)^\vee}(\tau_{P_m^{(d)}}^\vee) \iff [\nexists i \in \{2, \dots, d-1\} \text{ such that } v = m_i].$$

Proof. Suppose that $v = m_i$ for an index $i \in \{2, \dots, d-1\}$. Then v can be written as the sum

$$v = (m_i - e_{i+1}^\vee) + e_{i+1}^\vee$$

which means that $v \notin \mathbf{Hilb}_{(\mathbb{Z}^d)^\vee}(\tau_{P_m^{(d)}}^\vee)$ by (1.1). Conversely, assume that

$$v \notin \mathbf{Hilb}_{(\mathbb{Z}^d)^\vee}(\tau_{P_m^{(d)}}^\vee).$$

It is worth mentioning that $e_d^\vee$ and $m_{d-1} - e_d^\vee$ are elements of $\mathcal{L}_m^{(d)}$ belonging always to the Hilbert basis. We shall use again induction w.r.t. d . For $d = 3$ the assertion can be verified easily (cf. [Ish, Prop. 8.7, p. 138]). Now let d be > 3 . Since $v \in \mathcal{L}_m^{(d)} \setminus \mathbf{Hilb}_{(\mathbb{Z}^d)^\vee}(\tau_{P_m^{(d)}}^\vee)$, by Lemma 4.3 we may express v as a linear combination of the form

$$v = \left(\sum_{\overline{v} \in \mathcal{L}_m^{(d)} \setminus \{v, e_d^\vee, m_{d-1} - e_d^\vee\}} \alpha_{\overline{v}} \overline{v} \right) + \beta e_d^\vee + \beta' (m_{d-1} - e_d^\vee),$$

for some $\alpha_{\overline{v}}, \beta, \beta' \in \mathbb{Z}_{\geq 0}$. Since $v \notin \{e_d^\vee, m_{d-1} - e_d^\vee\}$, its last coordinate has to be zero. Therefore $\beta = \beta'$ with

$$(4.8) \quad v = \left(\sum_{\overline{v} \in \mathcal{L}_m^{(d)} \setminus \{v, e_d^\vee, m_{d-1} - e_d^\vee\}} \alpha_{\overline{v}} \overline{v} \right) + \beta m_{d-1}.$$

The admissibility of $\mathbf{m}$ has as direct consequence that $m_{d-1} \in \tau_{P_{\mathbf{m}}^{(d-1)}}^\vee$. Hence we have $v \in \tau_{P_{\mathbf{m}}^{(d-1)}}^\vee$, where $\tilde{\mathbf{m}}$ is the matrix $\mathbf{m}$ without the last row.

On the other hand, by the induction hypothesis, we may assume that

$$v \in \mathbf{Hilb}_{(\mathbb{Z}^d)^\vee}(\tau_{P_{\mathbf{m}}^{(d-1)}}^\vee)$$

and hence, v cannot be written as a nontrivial sum of elements of $\tau_{P_{\mathbf{m}}^{(d-1)}}^\vee \cap (\mathbb{Z}^d)^\vee$. So (4.8) can be simplified as

$$v = \beta m_{d-1}.$$

Since some coordinate of v necessarily equals either 1 or -1 , and $m_{d-1} \in (\mathbb{Z}^d)^\vee$, we conclude that $\beta = 1$, as required. $\square$

Examples 4.13. Computing (by 4.12) the Hilbert bases of the duals of the cones which support the Nakajima polytopes 3.4 (i)–(v) w.r.t. the rectangular lattice, we obtain:

- For (i):

$$\mathbf{Hilb}_{(\mathbb{Z}^3)^\vee}(\tau_{P_{\mathbf{m}}^{(3)}}^\vee) = \mathcal{L}_{\mathbf{m}}^{(3)} = \{e_1^\vee, e_2^\vee, e_3^\vee, 2e_1^\vee - e_2^\vee, 2e_1^\vee + e_2^\vee - e_3^\vee\}.$$

- For (ii):

$$\mathbf{Hilb}_{(\mathbb{Z}^3)^\vee}(\tau_{P_{\mathbf{m}}^{(3)}}^\vee) = \mathcal{L}_{\mathbf{m}}^{(3)} \setminus \{ke_1^\vee - e_2^{\vee\top}\} = \{e_1^\vee, e_2^\vee, e_3^\vee, ke_1^\vee - e_2^\vee - e_3^\vee\}.$$

- For (iii):

$$\mathbf{Hilb}_{(\mathbb{Z}^4)^\vee}(\tau_{P_{\mathbf{m}}^{(4)}}^\vee) = \mathcal{L}_{\mathbf{m}}^{(4)} \setminus \{e_1^\vee\},$$

where

$$\mathcal{L}_{\mathbf{m}}^{(4)} = \{e_1^\vee, e_2^\vee, e_3^\vee, e_4^\vee, e_1^\vee - e_2^\vee, e_1^\vee - e_3^\vee, 2e_1^\vee - e_2^\vee - e_3^\vee - e_4^\vee\}.$$

- For (iv):

$$\mathcal{L}_{\mathbf{m}}^{(d)} = \{e_i^\vee \mid 1 \leq i \leq d\} \cup \{ke_1^\vee - e_2^\vee\} \cup \{e_i^\vee - e_{i+1}^\vee \mid 2 \leq i \leq d-1\},$$

while

$$\mathbf{Hilb}_{(\mathbb{Z}^d)^\vee}(\tau_{\mathbf{s}_k^{(d)}}^\vee) = \{e_1^\vee, e_d^\vee\} \cup \{ke_1^\vee - e_2^\vee\} \cup \{e_i^\vee - e_{i+1}^\vee \mid 2 \leq i \leq d-1\}.$$

- For (v):

$$\begin{aligned} \mathbf{Hilb}_{(\mathbb{Z}^d)^\vee}(\tau_{\mathbf{RP}(k_1, \dots, k_{d-1})}^\vee) &= \mathcal{L}_{\mathbf{m}}^{(d)} \\ &= \{e_i^\vee \mid 1 \leq i \leq d\} \cup \{k_i e_i^\vee - e_{i+1}^\vee \mid 1 \leq i \leq d\}. \end{aligned}$$

Proof of Corollary 3.3. We define $\mathfrak{Q}_{\mathbf{m}}^{(d)}$ and $\mathfrak{R}_{\mathbf{m}}^{(d)}$ as in (3.5) and (3.6), respectively. Suppose that one of the unit vectors, say $e_k^\vee$, $k \in \{1, \dots, d-1\}$, does not belong to the Hilbert basis. Then by Lemma 4.12 for some index $\gamma = \gamma_k$, which is $\geq k$, we have necessarily $e_k^\vee = m_\gamma$, i.e., $k \in \mathfrak{Q}_{\mathbf{m}}^{(d)}$. Looking at the definition of $\lambda_{i,\gamma}$ and $\mu_{i,\gamma}$ we find

$$\lambda_{i,\gamma} = 0, \quad \text{for all } i \in \{0, \dots, \gamma-1\},$$

and

$$\mu_{i,\gamma} = \begin{cases} 0, & \text{if } i \in \{0, \dots, \gamma-1\} \setminus \{k-1\}, \\ 1, & \text{if } i = k-1, \end{cases}$$

and the binomial corresponding to the γ -th column of $\widehat{\mathcal{B}}_{\mathbf{m}}^{(d)}$ equals $z_k - z_{\gamma+1}z_{d+\gamma}$; and conversely, if one of the $d-1$ initial binomials is of this type, then $e_k^\vee = m_\gamma$ with

$e_k^\vee$ not belonging to the Hilbert basis. Analogously, if a vector of type $m_l - e_{l+1}^\vee$, for some $l \in \{1, \dots, d-2\}$, does not belong to the Hilbert basis, then there exists an index $\delta = \delta_l$, which is $\geq l+1$, so that $m_l - e_{l+1}^\vee = m_\delta$, i.e., $l \in \mathfrak{R}_m^{(d)}$. Again by the definition of $\lambda_{i,\delta}$ and $\mu_{i,\delta}$ we find

$$\lambda_{i,\delta} = \begin{cases} 0, & \text{if } i \in \{0, \dots, \delta-1\} \setminus \{l\}, \\ -1, & \text{if } i = l, \end{cases}$$

and

$$\mu_{i,\delta} = 0, \quad \text{for all } i \in \{0, \dots, \delta-1\},$$

and the binomial corresponding to the δ -th column of $\widehat{\mathcal{B}}_m^{(d)}$ equals $z_{d+l} - z_{\delta+1}z_{d+\delta}$; and conversely, if one of the $d-1$ initial binomials is of this type, then $m_l - e_{l+1}^\vee = m_\delta$ with $m_l - e_{l+1}^\vee$ not belonging to the Hilbert basis. Obviously,

$$\#(\text{Hilb}_{(\mathbb{Z}^d)^\vee}(\tau_{P_m^\vee}^\vee)) = \#(\mathcal{L}_m^{(d)}) - \#(\mathfrak{Q}_m^{(d)}) - \#(\mathfrak{R}_m^{(d)}),$$

and the assertion is true. $\square$

Comment and open problem. The first partial verification of the fact that the affine semigroup rings which are complete intersections admit an “inductive characterization” in *all dimensions* appeared already in the 1980’s, in the works of Watanabe [W] and Nakajima [N] (who classified those which are invariant subrings of finite abelian groups, and affine torus embeddings, respectively). This was completely proved in 1997 by Fischer, Morris and Shapiro [FMSH] via the theory of dominating matrices.

“Watanabe simplices” (introduced in [DHeZ]) and, more general, but in a slightly different context, “Nakajima polytopes” provide a *geometric parametrization* of the classes of semigroup rings treated in [W] and [N], respectively. In the present paper, based on Nakajima’s classification, we ascertained that, besides the above-mentioned inductive characterization, there is also some kind of *recursion principle* governing a natural set of generators of the relation space of the involved semigroups. This gives rise to ask if this property can be generalized for a wider class of affine semigroup rings, probably in connection with a suitable combinatorial parametrization resulting from partitions of minimal generating sets (also called “semigroup gluings”), decomposition theorems of dominating matrices or even from graph-theoretic objects (such as coloured paths, etc.); cf. [RG], [R], [FMSH], [BaMT], and [SSS].

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