UNIQUENESS OF THE DENSITY
IN AN INVERSE PROBLEM
FOR ISOTROPIC ELASTODYNAMICS

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ABSTRACT. We consider the unique determination of the density of a nonhomogeneous, isotropic elastic object from measurements made at the surface. We model the behavior of the bounded, 3-dimensional object by the linear, hyperbolic system of operators for isotropic elastodynamics. The material properties of the object (its density and elastic properties) correspond to the smooth coefficients of these differential operators. The data for this inverse problem, in the form of the correspondence between applied surface tractions and resulting surface displacements, is modeled by the dynamic Dirichlet-to-Neumann map on a finite time interval. In an earlier paper we show that the speeds $c_{p/s}$ of (compressional and sheer) wave propagation through the object are uniquely determined by the Dirichlet-to-Neumann map. Here we extend that result by showing that the density is also determined in the interior by the Dirichlet-to-Neumann map in the case, for example, that $c_p = 2c_s$ at only isolated points in the object. We use techniques from microlocal analysis and integral geometry to solve this fully three-dimensional problem.

1. INTRODUCTION AND MAIN RESULT

One purpose in studying dynamic inverse problems for bounded objects is to describe the material properties of the object using only measurements made at its surface. In the case of anisotropic elastodynamics the material properties of the object are modeled by 22 parameters. We may ask, then, how many of these 22 parameters may be determined by surface measurements. As a first step in addressing this question we show in [R III] that those parameters needed to describe paths of wave propagation (in the setting of rather general hyperbolic systems) are determined, to a certain extent, by surface measurements. In particular, the metrics, whose geodesics model the paths along which wave propagation occurs, are determined by surface measurements, up to pullback by a diffeomorphism. An open question is whether these metrics are determined exactly by surface measurements, and whether the other material parameters (parameters other than those needed to describe the paths of wave propagation) are also determined.

As an indication of what might be expected to be true for the dynamic inverse problem for general hyperbolic systems (using Dirichlet-to-Neumann-type data),

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here we consider the remaining uniqueness question in the case of isotropic elastodynamics. In [R I] we have shown that the (compressional and sheer) speeds $c_p = \sqrt{(\lambda + 2\mu)/\rho}$ and $c_s = \sqrt{\mu/\rho}$ of wave propagation through the object, written in terms of the elasticity parameters λ and μ and the density ρ, are uniquely determined in the interior by surface measurements. In this paper we extend that result by showing that the third parameter, the density ρ, is also determined (in the case, for example, that the compressional wave speed c_p is twice the sheer wave speed c_s at only isolated points in the object). That is, we show that if two isotropic elastic objects differ only in their densities, then the surface measurements of those objects will also differ.

It is interesting that "top-order" boundary data, in the form of the boundary polarization, does not determine the (smoothly varying) density (cf. Section 3). (We have made this observation in [R I], Section 5.3, in the case of constant wave speeds $c_{p/s}$.) Rather, it is necessary to consider "lower-order polarization". We conclude, then, that in more general settings we may expect to determine parameters other than those describing the wave paths, but that in determining other parameters, it will likely be necessary to use "lower-order" data.

A problem which has some of the features of elastodynamics is the problem solved by Rakesh and Symes in [Ra-Sy]. In [Ra-Sy] a disturbance that propagates through an object is modelled by a solution of the wave equation, $\left[\partial_t^2 - \Delta + q(x)\right] u = 0$ in $\Omega \times (0, T)$, where Ω is a bounded region. Rakesh and Symes show that the integral of the potential, $q(x)$, over any path along which wave energy propagates, is determined by surface measurements. Knowing these integrals of the potential is enough to recover the potential itself inside Ω. In this model problem the paths along which energy propagates are straight lines, and the differential equation is scalar.

In elastodynamics, though, the paths along which wave energy propagates are not straight lines, and the elasticity equation is a system of three equations in three unknowns.

Ikehata, Nakamura, and Yamamoto [I-N-Y] use modified Carleman estimates to show that the density $\rho \in C^2(\overline{\Omega})$ of a bounded, isotropic elastic object in n-dimensional space is determined by n pieces of data (f_i, ψ_i, g_i), where f_i is the surface displacement, ψ_i is the initial displacement, and g_i is the corresponding surface traction. (The n initial stresses corresponding to the initial displacements ψ_i must be uniformly linearly independent on $\overline{\Omega}$. No condition is placed on the initial velocity.) Note that ψ_i is the initial displacement of the entire object, and so the data (f_i, ψ_i, g_i) do not correspond to measurements that are made only at the surface of the object. In addition, the time interval for data collection must be longer than the greatest travel time of both compressional and sheer wave propagation through the object.

In comparison, in this paper we use techniques from microlocal analysis and integral geometry to determine the density $\rho \in C^\infty(\overline{\Omega})$, but using only surface data of the form (f, g), where f runs through every (sufficiently regular function modeling the) surface displacement, and $g = \Lambda_{\lambda, \mu, \rho} f$ is the corresponding surface traction. This Dirichlet-to-Neumann-type data is *non-invasive* or *non-destructive* in the sense that it corresponds to measurements that need be made only at the surface of the object. (The initial displacements and velocity are taken to be zero for each measurement.) Also, the time interval $[0, T]$ for data collection is taken

to be the greatest travel time of just the compressional wave propagation through the object. See Uhlmann's review article [U] on inverse problems using Dirichlet-to-Neumann-type data.

In Sections 1.1 and 1.2 we state the main result and give a sketch of the proof. In Section 2 we present preliminaries to the proof of the main result. In Section 3 we show that the polarization at $\partial\Omega$ does not have information about ρ in Ω. In Section 4 we use "lower-order polarization" to show that ρ is determined in Ω.

1.1. **Statement of the Main Result.** Let Ω be a bounded region in $\mathbb{R}^3$ with smooth boundary. Ω represents a linearly elastic, nonhomogeneous, isotropic object if the function $\rho(x)$ representing the density is positive on $\overline{\Omega}$, if the Lamé parameters $\lambda(x)$ and $\mu(x)$, representing the elastic properties of the object, satisfy $\mu > 0$ on $\overline{\Omega}$ and $3\lambda(x) + 2\mu(x) > 0$ on $\overline{\Omega}$ (the strong convexity condition), and if the vector $u(x, t)$, which represents the displacement of the object, solves the initial-boundary-value problem (2) associated with the hyperbolic system of operators P for isotropic elastodynamics. The operator P for elastodynamics is given by

$$(1) \qquad (Pu)_i = \rho\, \partial_t^2 u_i - \sum_{j,k,l=1}^3 \partial_{x_j}(c_{ijkl}\partial_{x_l} u_k) = 0, \qquad i = 1, 2, 3,$$

where $c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \mu\delta_{il}\delta_{jk}$ is the elasticity tensor for an isotropic elastic medium. The initial-boundary-value problem is (for $0 < T < \infty$)

$$(2) \qquad \begin{cases} Pu = 0 & \text{in } \Omega \times (0, T), \\ u|_{\partial\Omega} = f & \text{for } t \in [0, T], \\ u|_{t=0} = 0, \quad (\partial_t u)|_{t=0} = 0 & \text{in } \Omega. \end{cases}$$

The Dirichlet-to-Neumann map $\Lambda_{\lambda,\mu,\rho}$ models surface measurements in that it models the correspondence between a displacement $f(x, t)$ at the surface and the applied surface traction $\Lambda_{\lambda,\mu,\rho}f$ that would generate this displacement. (See [U] for a review of the use of the Dirichlet-to-Neumann map in modeling surface measurements in inverse problems.) The Dirichlet-to-Neumann map $\Lambda_{\lambda,\mu,\rho}$ is defined in terms of solutions u of the initial-boundary-value problem (2) by

$$(\Lambda_{\lambda,\mu,\rho}f)_i = \Big(\sum_{j,k,l=1}^3 \nu_j c_{ijkl}\partial_{x_l} u_k\Big)\big|_{\partial\Omega \times (0,T)}, \qquad i = 1, 2, 3,$$

where $u|_{\partial\Omega \times (0,T)} = f$, and ν is the unit outer normal vector to $\partial\Omega$.

We introduce the following notation and terminology, following [Sh II], Sections 3.1, 3.2, and 3.4. Given a Riemannian manifold M with boundary ∂M and given a point $x \in \partial M$, the *second quadratic form of the boundary* is defined by $\Pi(\xi, \xi) = \langle \nabla_\xi \nu, \xi \rangle$ on $\xi \in T_x(\partial M)$, where $\nu = \nu(x)$ is the unit outer normal vector to the boundary. We say that the boundary is *strictly convex* if this form is positive-definite for all $x \in \partial M$. We say that a compact Riemannian manifold (M, g) with boundary is a *compact, dissipative Riemannian manifold (CDRM)* (1) if the boundary ∂M is strictly convex, and (2) if there is no geodesic of infinite length in M.

By $TM = \{(x, \xi) \mid x \in M, \xi \in T_x M\}$ we denote the tangent bundle of a manifold M, and by $\Omega M = \{(x, \xi) \in TM \mid |\xi| = 1\}$ we denote the unit tangent bundle. Let $C^\infty(\partial_\pm \Omega M)$ be the space of smooth functions on the manifold $\partial_\pm \Omega M = \{(x, \xi) \in \Omega M \mid x \in \partial M \text{ and } \pm\langle \xi, \nu(x) \rangle \geq 0\}$, the outward/inward unit tangent bundle at the boundary.

Given a Riemannian manifold M, any point $x \in M$, and a two-dimensional subspace $\sigma \subset T_x M$, we denote by $K(x, \sigma)$ the sectional curvature of the section σ. For $(x, \xi) \in T^0 M = \{(x, \xi) \in TM \mid \xi \neq 0\}$ we set $K(x, \xi) = \sup_{\{\xi \in \sigma\}} K(x, \sigma)$ and $K^+(x, \xi) = \max\{0, K(x, \xi)\}$. For a CDRM (M, g) we define

$$k^+(M, g) = \sup_{(x, \xi) \in \partial_- \Omega M} \int_0^{\tau_+(x, \xi)} t K^+\Big(\gamma_{x, \xi}(s), \dot{\gamma}_{x, \xi}(s)\Big)\, ds,$$

where $\gamma_{x, \xi} : [0, \tau_+(x, \xi)]$ is the maximal geodesic in $M \cup \partial M$ satisfying the initial (i.e., entry) conditions $\big(\gamma_{x, \xi}(0), \dot{\gamma}_{x, \xi}(0)\big) = (x, \xi) \in \partial_- \Omega M$.

Our main result is the unique determination of the density ρ in Ω by surface measurements:

Theorem 1 (Main Result). *Let Ω be a bounded region in $\mathbb{R}^3$ with smooth boundary. Suppose that the coefficients $\lambda_i, \mu_i, \rho_i \in C^\infty(\overline{\Omega})$ satisfy $\rho_i, \mu_i > 0$, $3\lambda_i + 2\mu_i > 0$ (strong convexity) for $i = 1, 2$. In addition, suppose that (i) the metrics $g_p^i = \big(\rho_i/(\lambda_i + 2\mu_i)\big) e$, $g_s^i = (\rho_i/\mu_i) e$, where e is the Euclidean metric, do not give rise to caustics, (ii) the geodesics of $g_{p/s}^i$ all exit Ω before some time $T < \infty$, and (iii) the geodesics of $g_{p/s}^i$ do not graze the boundary. Finally, suppose that (Ω, g_p^i) is a compact, dissipative Riemannian manifold with $3k^+(M, g_p^i) < 1$. Then $\Lambda_{\lambda_1, \mu_1, \rho_1} = \Lambda_{\lambda_2, \mu_2, \rho_2}$ on $\partial\Omega \times (0, T)$ implies*

$$\rho_1 = \rho_2$$

in the closure of an open set $\mathcal{C}$ in Ω (defined in (56)) where $\lambda_i \neq 2\mu_i$. In particular, if $\lambda_i = 2\mu_i$, $i = 1, 2$, only at isolated points in $\overline{\Omega}$, then $\rho_1 = \rho_2$ on $\overline{\Omega}$.

Remark 1. The condition $3k^+(M, g) < 1$ is satisfied if the curvature of $\big(\Omega, (1/c_p^2)e\big)$ is bounded above and is small enough along each geodesic, relative to the length of the geodesic. For example, in the case that the wave speed c_p is constant, the manifold $\big(\Omega, (1/c_p^2)e\big)$ is flat (zero curvature), and thus satisfies $3k^+(M, g) < 1$. Also, in the case that $\big(\Omega, (1/c_p^2)e\big)$ has negative curvature (e.g., the hyperbolic disk), the condition $3k^+(M, g) < 1$ is easily satisfied due to the fact that $K\big(\gamma_{x, \xi}(s), \dot{\gamma}_{x, \xi}(s)\big) < 0$, and so $K^+\big(\gamma_{x, \xi}(s), \dot{\gamma}_{x, \xi}(s)\big) \equiv 0$.

Other upper bound conditions on $k^+(M, g)$ may be used to describe classes of CDRM. For example (cf. [Sh II], Section 3.4), if a CDRM (M, g) has $k^+(M, g) < 1$, then M is diffeomorphic to the ball, and (M, g) is *simple*; that is, (i) the boundary is strictly convex, and (ii) any two points $x, y \in M$ are joined by a unique geodesic in a way that depends smoothly on x and y (that is, the mapping $\exp_x^{-1} : M \to T_x M$ is smooth).

Remark 2. In this paper we consider (Ω, g_p) with $g_p = (1/c_p^2)e$; so we find

$$K\big(\gamma_{x, \xi}(s), \dot{\gamma}_{x, \xi}(s)\big)$$

$$= \sup_{\vec{n}} \frac{1}{c_p^2}\Bigg[\vec{n}^t (\log c_p)_{xx} \vec{n} + \dot{\gamma}_{x, \xi}(s)^t (\log c_p)_{xx} \dot{\gamma}_{x, \xi}(s)$$

$$- (1/2)\Big([\dot{\gamma}_{x, \xi}(s) \cdot (\log c_p)_x]^2 + [\vec{n} \cdot (\log c_p)_x]^2 + |(\log c_p)_x|^2\Big)\Bigg],$$

where the supremum is taken over $\vec{n}$ with $\vec{n}$ orthogonal to $\dot{\gamma}_{x, \xi}(s)$. It follows that $3k^+(M, g) < 1$ is satisfied, for example, if $\log c_p$ is concave on Ω (that is, if

$\log c_p(\alpha x + [1-\alpha]y) \geq \alpha \log c_p(x) + [1-\alpha]\log c_p(y)$ for any constant $\alpha \in [0,1]$ and $x, y \in \Omega$).

Remark 3. The result of Rakesh and Symes [Ra-Sy] for the wave equation $[\partial_t^2 - \Delta + q(x)]u = 0$ applies to media with a constant wave speed and so curvature $K(x, \xi) \equiv 0$; that is, $k^+(M, g) \equiv 0$. In this paper we consider a generalization to elastodynamics and to the case of media with $k^+(M, g)$ not necessarily zero. (See the introduction of this article for a description of Rakesh and Symes' result.)

Remark 4. Conditions (i)–(iii) of Theorem 1 are imposed in order to apply [R I], Theorem 1, which finds that the wave speeds $c_{p/s}^i$ are determined in Ω. This result is preliminary to proving uniqueness of the density in Ω (cf. Section 2.5). The conditions that (Ω, g_p^i) be a compact, dissipative Riemannian manifold with $3k^+(M, g_i) < 1$ are imposed in Theorem 1 in order to apply Sharafutdinov's inversion of the ray transform for a tensor field (cf. Section 4.2). We restrict the conclusion of Theorem 1 to the region $\mathcal{C}$ due to the need to guarantee uniqueness for the fourth-order, linear, elliptic differential equation (53). The singular points of (53), that is, the zeros of the leading coefficient γ, are the points in $\overline{\Omega}$ where $\lambda_i = 2\mu_i$, i.e., where $c_p = 2c_s$. (Compare $c_p \neq 2c_s$, i.e., $c_p^2 \neq 4c_s^2$, with the strong convexity condition $3\lambda + 2\mu > 0$, i.e., $3c_p^2 > 4c_s^2$.)

1.2. Sketch of the Proof.
In elastodynamics, wave energy from a disturbance propagates through the medium along paths that are determined by the density and elasticity. In fact, these paths are, in the isotropic case, geodesics, curves along which travel time is minimized; so they are given in terms of the wave speeds c_p and c_s of the medium, which, in turn, are given in terms of the density $\rho(x)$ and elastic parameters, $\lambda(x)$ and $\mu(x)$. We derive explicit descriptions of these wave paths (cf. Section 2.3) in terms of λ, μ and ρ.

We then apply the boundary determination (Theorem 2.1): if $\Lambda_{\lambda_1,\mu_1,\rho_1} = \Lambda_{\lambda_2,\mu_2,\rho_2}$, then it follows that $\partial_\nu^j \lambda_1 = \partial_\nu^j \lambda_2$, $\partial_\nu^j \mu_1 = \partial_\nu^j \mu_2$, and $\partial_\nu^j \rho_1 = \partial_\nu^j \rho_2$ on $\partial\Omega$, $j = 0, 1, \ldots$, where ∂_ν is the normal derivative at $\partial\Omega$. We conclude in [R I] that the object may be viewed as being embedded in a medium of infinite extent with smoothly varying density and elasticities at the interface of the surface of the object and the surrounding medium. In particular, we show that the displacement of the medium is uniquely determined outside the object by the Dirichlet-to-Neumann map (cf. Section 2.1). In [R I] we conclude that asymptotic expansions of the displacement of the object and the surrounding medium may be written, in this case, in terms of global Fourier integral operators on $\mathbb{R}^3$. We have applied results on the propagation of singularities and results from integral geometry to conclude in [R I] that the compressional and sheer wave speeds $c_{p/s}$ associated with the medium are determined in Ω by the Dirichlet-to-Neumann map.

We now consider the determination of ρ in the interior. We remark that a consequence of the determination of the density ρ, given the determination of the wave speeds $c_p = \sqrt{(\lambda + 2\mu)/\rho}$ and $c_s = \sqrt{\mu/\rho}$, will be that each of the three parameters λ, μ, and ρ is determined.

In Section 3 we show that polarization boundary data contains no information about the (smooth) density in the interior.

In Section 4 we show that the boundary values of "lower-order polarization" do determine ρ in Ω. In particular, we consider the decomposition of the displacement vector as the sum of terms, graded by the strength of their singularities. In fact,

the displacement vector is represented in terms of Fourier integral operators (FIOs) acting on initial ($t = 0$) data (an FIO for each of the forward/backward p/s-waves). The amplitudes of the FIOs are written as the sum of terms $a_0, a_{-1}, a_{-2}, \ldots$ of homogeneity $0, -1, -2, \ldots$ with respect to the frequency. The top-order terms a_0 describe the most singular behavior of the displacement vector (the polarization). The lower-order terms $a_{-1}, a_{-2}, \ldots$ describe less singular behavior. For the forward p-wave FIO we write $a_0 = \alpha_0 N$, with N the (unit) vector in the direction of the gradient of the phase, i.e., parallel to the direction of displacement of the p-wave, and we write $a_{-1} = h_1 + \alpha_{-1} N$, with h_{-1} orthogonal to N. (Similarly, $a_{-J} = h_{-J} + \alpha_{-J} N$.) We refer to the term $\alpha_{-1} N$ as the lower-order polarization.

The α_{-J} solve ODEs (the transport equations) along the bicharacteristics of P. Therefore, we write the (lower-order) polarization in terms of line integrals along rays in order to derive a ray transform defined on ray segments between boundary points of the object. It follows from the fact that (lower-order) polarization is determined at the boundary (and outside the object), that this ray transform is determined by the Dirichlet-to-Neumann map. We invert the ray transform by applying results of Sharafutdinov. It follows that the tensor field appearing in the integrand of the ray transform is determined (modulo its potential part) at each interior point. We filter the integrand (via use of the Saint-Venant operator) to derive a fourth-order, linear, elliptic partial differential equation $E\beta^- = 0$ which has solution involving the difference between ρ_1 and ρ_2. Then, applying uniqueness results, we conclude that the density is, in fact, determined in a part of the interior described in terms of the topology of the singular points of E.

2. PRELIMINARIES TO THE PROOF OF THE MAIN RESULT

2.1. **Embedding the object in an unbounded elastic domain.** The author shows in [R II] that λ, μ, ρ, and their normal derivatives are determined to infinite order by the Dirichlet-to-Neumann map at the boundary. In fact,

Theorem 2.1 (Rachele [R II], Theorem 1). *Let Ω be a bounded region in $\mathbb{R}^3$ with smooth boundary. Suppose that the coefficients λ_i, μ_i, $\rho_i \in C^\infty(\overline{\Omega})$ satisfy ρ_i, $\mu_i > 0$, and $3\lambda_i + 2\mu_i > 0$ (strong convexity) on $\overline{\Omega}$. Then $\Lambda_{\lambda_1,\mu_1,\rho_1} = \Lambda_{\lambda_2,\mu_2,\rho_2}$ on $\partial\Omega \times (0,T)$ for any $T > 0$ implies*

$$\partial_\nu^j \lambda_1 = \partial_\nu^j \lambda_2, \quad \partial_\nu^j \mu_1 = \partial_\nu^j \mu_2, \quad \partial_\nu^j \rho_1 = \partial_\nu^j \rho_2 \qquad on \ \partial\Omega$$

for $j = 0, 1, \ldots$.

Theorem 2.1 is applied in [R I], Section 3, to extend λ, μ, and ρ smoothly to all of $\mathbb{R}^3$ so that each is determined by $\Lambda_{\lambda,\mu,\rho}$ outside Ω. As in [S-U], we then approach the question of uniqueness in the interior from the point of view of the Cauchy problem for elastodynamics on $\mathbb{R}^3 \times (0,T)$,

$$(3) \qquad \begin{cases} P\,U &=& 0 & \text{on } \mathbb{R}^3 \times (0,T), \\[2mm] U\,|_{t=0} &=& \psi_0 & \text{on } \mathbb{R}^3, \\[2mm] \partial_t U\,|_{t=0} &=& \psi_1 & \text{on } \mathbb{R}^3, \end{cases}$$

rather than the initial-boundary-value problem (2) on $\Omega \times (0,T)$. We may view the elastic object as being embedded in a larger (unbounded) elastic domain, due to [R I], Theorem 3.1, which we state below. In [R I], Theorem 3.1, we show that the

displacement vectors U that solve the Cauchy problem (3) on $\mathbb{R}^3$ for the operator P for isotropic elastodynamics are determined outside Ω by the Dirichlet-to-Neumann map, given fixed boundary data and fixed initial data that is supported outside Ω.

Theorem 2.2 (Rachele [R I], Theorem 3.1). *Suppose that the conditions of Theorem 2.1 hold. Let $P_j = P(\lambda_j, \mu_j, \rho_j)$, $j = 1, 2$, be the hyperbolic system of operators of the form* (1), *but defined on all of $\mathbb{R}^3$, and suppose that $\lambda_1 = \lambda_2$, $\mu_1 = \mu_2$, $\rho_1 = \rho_2$ outside Ω. In addition, suppose that U_j $(j = 1, 2)$ solves the Cauchy problem* (3) *on $\mathbb{R}^3 \times (0, T)$, where $(\operatorname{supp} \psi_k) \cap \overline{\Omega} = \emptyset$ for $k = 0, 1$, and*

$$U_1 \left.\right|_{\partial\Omega\times(0,T)} = U_2 \left.\right|_{\partial\Omega\times(0,T)}.$$

Then $\Lambda_{\lambda_1,\mu_1,\rho_1} = \Lambda_{\lambda_2,\mu_2,\rho_2}$ on $(0,T)$ implies

$$U_1(x,t) = U_2(x,t) \quad outside\ \Omega,\ for\ t \in (0,T).$$

2.2. Constructing Fourier integral operator representations of the elastic displacement wave.

In [R I] we construct asymptotic expansions of certain solutions U_j of the Cauchy problem (3). These solutions U_j have wave front set that is minimal in the sense that it is generated by initial data with wave front set at a single point along a single ray (cf. [R I], Lemma 4.3). The construction of these asymptotic expansions is central to the main result of this paper and is summarized below.

We construct solutions U of the Cauchy problem (3) on $\mathbb{R}^3 \times (0, T)$ by writing

$$(4) \qquad\qquad U = E_0\psi_0 + E_1\psi_1,$$

in terms of the initial data ψ_0, ψ_1 and solution operators E_0, $E_1(t, x, D_x)$. These solution operators solve

$$\left\{ \begin{array}{llll} P_c E_{\underline{k}} & \equiv & 0 & \text{on } \mathbb{R}^3 \times (0,T) \quad \text{for } \underline{k} = 0, 1, \\ E_0 \left.\right|_{t=0} & \equiv & I, & E_1 \left.\right|_{t=0} \equiv 0, \\ \partial_t E_0 \left.\right|_{t=0} & \equiv & 0, & \partial_t E_1 \left.\right|_{t=0} \equiv I \end{array} \right\} \text{mod smoothing,}$$

and, in fact, may be written in terms of the Fourier integral operator Ansatz

$$(5) \qquad (E_{\underline{k}}\mu)_{\underline{l}} = \sum_{p/s,\underline{m},\pm} \int e^{i\varphi_{p/s}^{\pm}} \; e_{p/s,\pm}^{\underline{l},\underline{m}}(\underline{k}) \; \widehat{\mu}_{\underline{m}}(\eta) \; d\eta, \qquad \left(\begin{array}{c} \underline{k} = 0,1 \\ \underline{l} = 1,2,3 \end{array} \right).$$

In [R I], Section 2, we derive conditions on the phase functions $\varphi_{p/s}^{\pm}(t, x, \eta)$ and amplitudes $e_{p/s}^{\underline{l},\underline{m}}(\underline{k})(t, x, \eta)$ that appear in this oscillatory integral representation (5) of the solution operators $E_{\underline{k}}$. We find, for example, that the phase functions $\varphi_{p/s}^{\pm}$ are homogeneous of order 1 in $|\eta|$ and solve the eikonal equations $\det p(t, x, \partial_{t,x}\varphi_{p/s}^{\pm}) = 0$, i.e.,

$$(6) \qquad\qquad \partial_t \varphi_{p/s}^{\pm} = \mp c_{p/s} |\nabla_x \varphi_{p/s}^{\pm}|.$$

The eikonal equations are noncharacteristic, first-order nonlinear equations which can be solved by Hamilton-Jacobi theory (cf., for example, [Gr-Sj]), with initial values

$$(7) \qquad\qquad \varphi_{p/s}^{\pm}|_{t=0} = x \cdot \eta$$

given in terms of the parameter $\eta \in \mathbb{R}^3 \setminus 0$. It follows that the gradient of $\varphi_{p/s}$ takes on the following simple form at $t = 0$:

$$\nabla_x \varphi_{p/s}^{\pm} |_{t=0} = \eta.$$

The amplitudes $e_{p/s}(\underline{k})$ can be written in the form

$$
(8) \qquad \left(e^{\cdot,\frac{m}{p/s}}(\underline{k})\right)_J \;=\; \left(h^{\cdot,\frac{m}{p/s}}\right)_J + \left\{ \begin{array}{c} (\alpha^{\underline{m}})_J N \\ (\alpha^{\underline{m}}_1)_J N_1 + (\alpha^{\underline{m}}_2)_J N_2 \end{array} \right. \quad \left(\begin{array}{c} J = 0, -1, \ldots \\ \underline{k} = 0, 1 \\ p/s, \ \pm \\ \underline{m} = 1, 2, 3 \end{array} \right)
$$

where $\left(e_{p/s}(\underline{k})\right)_J$ is the part of $e_{p/s}(\underline{k})$ that is homogeneous of order J in $|\eta|$. In addition, the $(\alpha^{\underline{m},\pm})_J(k)$ and $(\alpha^{\underline{m},\pm}_j)_J(\underline{k})(t,x,\eta)$ are scalars, $(h^{\cdot,\frac{m}{p/s}})_0 = 0$, $(h^{\cdot,\frac{m}{p/s}})_J$ is in the cokernel of $p(t,x,\partial_{t,x}\varphi_{p/s})$ for $J = -1, -2, \ldots$, and N and N_1, N_2 form an orthonormal basis of the kernel of $p(t,x,\partial_{t,x}\varphi)$ for $\varphi = \varphi_p$ and $\varphi = \varphi_s$, respectively. That is, $N = \dfrac{\nabla_x \varphi_p}{|\nabla_x \varphi_p|}$, and $N_1, N_2, \dfrac{\nabla_x \varphi_s}{|\nabla_x \varphi_s|}$ are mutually orthonormal.

The amplitudes solve algebraic equations:

$$
(9) \qquad \begin{aligned} & p(t,x,\partial_{t,x}\varphi^{\pm}_{p/s}) \left(e_{p/s}(\underline{k})\right)_{J-1} \\ & = B_{p/s}\left(e_{p/s}(\underline{k})\right)_J + C_{p/s}\left(e_{p/s}(\underline{k})\right)_{J+1}, \end{aligned} \quad \left(\begin{array}{c} \underline{k} = 0, 1 \\ p/s, \pm \\ J = 0, -1, \ldots \end{array} \right)
$$

where

$$
(10) \qquad \begin{aligned} p(t,x,\tau,\xi) &= (-\rho\tau^2 + \mu|\xi|^2)I + (\lambda + \mu)(\xi \otimes \xi) \\ &= -\rho\Big[(\tau^2 - c_s^2|\xi|^2)I - (c_p^2 - c_s^2)(\xi \otimes \xi)\Big] \end{aligned}
$$

is the principal symbol of $P(t,x,D)$, $p_1(t,x,\tau,\xi)$ is the sum of the lower-order terms in the symbol of P,

$$
\begin{aligned}
B_{p/s}M &= i\partial_{\tau,\xi}p(t,x,\partial_{t,x}\varphi^{\pm}_{p/s}) \cdot \partial_{t,x}M + i(ip_1)(t,x,\partial_{t,x}\varphi^{\pm}_{p/s})M \\
&\quad + \frac{i}{2}\sum_{|\alpha|=2}\sum_{l=1}^{3}(\partial^{\alpha}_{\tau,\xi}p^{il})(t,x,\partial_{t,x}\varphi^{\pm}_{p/s}) \cdot (\partial^{\alpha}_{t,x}\varphi^{\pm}_{p/s})M^{l\underline{m}},
\end{aligned}
$$

$$
\begin{aligned}
C_{p/s}M &= \partial_{\tau,\xi}(ip_1)(t,x,\partial_{t,x}\varphi^{\pm}_{p/s}) \cdot \partial_{t,x}M \\
&\quad + \frac{1}{2}\sum_{|\alpha|=2}\sum_{l=1}^{3}(\partial^{\alpha}_{\tau,\xi}p^{il})(t,x,\partial_{t,x}\varphi^{\pm}_{p/s}) \cdot \partial^{\alpha}_{t,x}M^{l\underline{m}},
\end{aligned}
$$

and, for ease of notation, we set

$$
\left(e_{p/s}(\underline{k})\right)_1 = 0.
$$

In Section 4.1 we reduce the *compatibility conditions* for (9),

$$
(11) \qquad N_{p/s}\Big[B_{p/s}\left(e_{p/s}(\underline{k})\right)_J + C_{p/s}\left(e_{p/s}(\underline{k})\right)_{J+1}\Big] = 0 \quad \left(\begin{array}{c} J = 0, -1, \ldots \\ p/s, \pm \\ \underline{k} = 0, 1 \end{array} \right)
$$

to the *transport equations*, first-order ordinary differential equations in $(\alpha^{\underline{m}})_J$ along p-wave characteristics of P, and systems of two first-order ordinary differential equations in $(\alpha^{\underline{m}}_1)_J$ and $(\alpha^{\underline{m}}_2)_J$ along s-wave characteristics of P. Here $N_p = N$ spans $\ker p(t,x,\partial_{t,x}\varphi_p)$ and $N_s \in \{N_1, N_2\}$, which span $\ker p(t,x,\partial_{t,x}\varphi_s)$. It follows from (9) that the h and α defining the $e(\underline{k})$ at one level are given in terms of the $e(\underline{k})$ from the two previous levels. We can therefore choose the lower-order $(h^{\cdot,\frac{m},\pm}_{p/s})_{J-1}$ to solve the (algebraic) equations (9), and we can define the $(\alpha^{\underline{m}})_{J-1}$ and $(\alpha^{\underline{m}}_j)_{J-1}$ to be solutions of the transport equations for $J-1$.

Assuming that the initial data ψ_0, ψ_1 are both homogeneous of order zero in $|\eta|$, and taking

$$N_1^+ = N_1^- \qquad \text{and} \qquad N_2^+ = N_2^- \quad \text{at } t = 0,$$

we find that there is a unique choice at $t = 0$ for the scalars $(\alpha^m)_J$ and $(\alpha_i^m)_J$ for each $J = 0, -1, \ldots$ (see [R I], Section 2.4). In the case $J = 0$, for example, we have

$$(\alpha^{m,\pm})_0 \;=\; \tfrac{1}{2} e_{\underline{m}} \delta_{\underline{k}0} \cdot N \;=\; \tfrac{1}{2} \tfrac{\eta_m}{|\eta|} \delta_{\underline{k}0},$$

$$(\alpha_i^{m,\pm})_0 \;=\; \tfrac{1}{2} e_{\underline{m}} \delta_{\underline{k}0} \cdot N_i \;=\; \tfrac{1}{2} (N_i)_{\underline{m}} \delta_{\underline{k}0}, \qquad i = 1, 2.$$

It follows that the principal term $(e_{p,+})_0$ (at $t = 0$) of $e_{p,+} = e_{p,+}(0)(0, x, \eta)$ is given by

$$(12) \qquad \left(e_{p,+} \right)_0 \;=\; \Big((\alpha^1)_0 N, \ (\alpha^2)_0 N, \ (\alpha^3)_0 N \Big) \;=\; \tfrac{1}{2} \frac{\eta}{|\eta|} \otimes \frac{\eta}{|\eta|}$$

since (at $t = 0$) $\nabla_x \varphi_p(0, x, \eta) = \eta$ by (7) and so (at $t = 0$) $N = \dfrac{\eta}{|\eta|}$ by (8). (Recall that $(v \otimes w)_{ij} = v_i w_j$.) That is, $(e_{p,+})_0|_{t=0} = \tfrac{1}{2} \mathrm{Proj}_\eta$ where $\mathrm{Proj}_\eta = \frac{\eta}{|\eta|} \otimes \frac{\eta}{|\eta|}$ is the projection onto the space spanned by η. Similarly, $e_{s,\pm}(0)|_{t=0} = \tfrac{1}{2} \mathrm{Proj}_{\eta^\perp}$, where $\mathrm{Proj}_{\eta^\perp} = I - \frac{\eta}{|\eta|} \otimes \frac{\eta}{|\eta|}$ is the projection onto the space orthogonal to η.

2.3. **Modeling wave paths by bicharacteristics of P.** We apply Dencker's [De] propagation of singularities result ([De], Theorem 4.2; see also [De], Definition 4.1 and equation (3.4)) for systems of real principal type (cf. [R I], Section 4.1) to conclude that singularities propagate along bicharacteristics of P. That is, bicharacteristic curves of P can be used to model the paths along which wave propagation occurs. Bicharacteristic curves, for flow out of $t = 0$, are integral curves $\big(t(s), x(s), \tau(s), \xi(s)\big)$ of Hamilton vector fields $V_{H_{p/s}^\pm}$, with the restriction that the (null) bicharacteristic curves lie in the characteristic set $\{(t, x, \tau, \xi) \mid \det p(t, x, \tau, \xi) = 0\}$ of P. The Hamilton vector field V_H of any (smooth) Hamiltonian function $H(t, x, \tau, \xi)$ is defined by

$$(13) \qquad V_H \;=\; (\partial_\tau H)\partial_t \;+\; (\nabla_\xi H) \cdot \nabla_x \;-\; (\partial_t H)\partial_\tau \;-\; (\nabla_x H) \cdot \nabla_\xi.$$

The Hamiltonians

$$H_{p/s}^\pm(t, x, \tau, \xi) \;=\; \tau \pm c_{p/s}|\xi|$$

are factors of the determinant of the principal symbol $p(t, x, \tau, \xi)$ of P (cf. (10)). (See Hörmander [Hö] for a description of bicharacteristics in terms of the Hamilton flow in the scalar case; replace H_p by $H_{\det \, \sigma_{\mathrm{pr}}(P)}$ in the proof of Hörmander's Theorem 6.4.3 in the case that P is a system.) That is, bicharacteristics have direction

$$(14) \qquad \frac{dt}{ds} \;=\; \frac{1}{c_{p/s}}, \quad \frac{dx}{ds} \;=\; \frac{\pm \xi}{|\xi|}, \quad \frac{d\tau}{ds} \;=\; 0, \quad \frac{d\xi}{ds} \;=\; \mp (\log c_{p/s})_x |\xi|$$

and satisfy $\det p(t, x, \tau, \xi) = -\rho^3 (\tau^2 - c_p^2 |\xi^2|)(\tau^2 - c_s^2 |\xi^2|)^2 = 0$.

The component $x(s)$ of the bicharacteristic curves corresponds to position along the wave path, and is in fact a geodesic with respect to the metric $g_{ij} = \frac{1}{c_{p/s}^2} \delta_{ij}$ (cf. [R I], Proposition 4.1). The component $t(s)$ of the bicharacteristics of P corresponds to the travel time of the wave, i.e., geodesic distance.

Given the eikonal equations $\det p(t, x, \partial_{t,x}\varphi_{p/s}) = 0$ (cf. (6)), we conclude that wave paths are described in terms of the phase functions $\varphi_{p/s}$ by bicharacteristics (t, x, τ, ξ) with $\tau = \partial_t \varphi_{p/s}$ and $\xi = \nabla_x \varphi_{p/s}$.

2.4. The bicharacteristics of P are determined outside Ω by the Dirichlet-to-Neumann map. We denote by

$$\Gamma_{p/s,\pm} = \Gamma_{p/s,\pm}(\underline{x}, \underline{\xi}) \tag{15}$$

the (forward or backward) null bicharacteristic of P that passes through $(t, x, \tau, \xi) = (0, \underline{x}, \mp c_{p/s}|\underline{\xi}|, \underline{\xi})$ where $\underline{s} = 0$, $\underline{x} = x(\underline{s}) \notin \Omega$, $\underline{\xi} = \xi(\underline{s})$.

Theorem 2.3 (Rachele [R I], Theorem 4.2). *Suppose that the conditions of Theorem 2.1 hold. Suppose further that the coefficients λ_i, μ_i, ρ_i do not give rise to caustics for $i = 1, 2$. Then for any $\underline{x} \notin \overline{\Omega}$ and $\underline{\xi} \in \mathbb{R}^3 \setminus 0$ the bicharacteristics $\Gamma_{p/s,\pm}(\underline{x}, \underline{\xi})$ are determined by $\Lambda_{\lambda,\mu,\rho}$ outside Ω for $t \in (0, T)$.*

Proof. For $(\lambda_1, \mu_1, \rho_1)$ and $(\lambda_2, \mu_2, \rho_2)$ with $\Lambda_{\lambda_1,\mu_1,\rho_1} = \Lambda_{\lambda_2,\mu_2,\rho_2}$ we construct solutions U_j of the Cauchy problem (3) as in the proof of [R I], Theorem 3.1, with initial data ψ^0 and ψ^1 (to be chosen later) supported outside Ω. By [R I], Theorem 3.1, these U_j are determined outside Ω, and so their wave front sets agree outside Ω:

$$\mathrm{WF}\, U_1 \;=\; \mathrm{WF}\, U_2 \qquad \text{outside } \overline{\Omega}. \tag{16}$$

We now choose initial data ψ_0 and ψ_1 with "minimal" wave front set, so that the wave front sets of the corresponding solutions U_1 and U_2 will consist of only a few null bicharacteristic strips of P. In fact, we choose

$$\psi_0 = h(x)\xi_0 \quad \text{and} \quad \psi_1 = 0$$

where $\xi_0 \cdot \underline{\xi} \neq 0$, $\xi_0 \times \underline{\xi} \neq 0$, and $h(x)$ is a scalar function on $\mathbb{R}^3$ with $\mathrm{WF}\, h(x) = \{(\underline{x}, \alpha\underline{\xi}) \;:\; \alpha > 0\}$. (See [R I], Remark 4.1 on the existence of such distributions $h(x)$.)

Given these initial values, we write the components of U_j as $U_j^l = \Sigma_{p/s,\pm} E_{p/s,\pm}^l h$, $l = 1, 2, 3$, where $E_{p/s,\pm}^l(t, x, D_x)$ is the scalar Fourier integral operator $\big($cf. (4) and (5)$\big)$ given by

$$E_{p/s,\pm}^l(t, x, D_x)\, h \;=\; \int e^{i\varphi_{p/s,\pm}(t,x,\eta)} \big[e_{p/s,\pm}^{l,\cdot}(0)\xi_0\big]\, \hat{h}(\eta)d\eta.$$

Then, given the initial values (7) for the phase, we describe the wave front set of the pseudodifferential operators $E_{p/s,\pm}^l|_{t=0}$ acting on h. In particular, we observe that the principal symbol of $E_{p/s,\pm}|_{t=0}$ is given by $(E_{p/s,\pm})_0(t, x, \eta)|_{t=0} = (e_{p/s,\pm})_0(j, 0)(t, x, \eta)\xi_0|_{t=0}$, which is $\frac{1}{2}\mathrm{Proj}_\eta \xi_0$ in the case of p-waves, and $\frac{1}{2}\mathrm{Proj}_{\eta^\perp} \xi_0$ in the case of s-waves by (12) and the text that follows. Therefore, by the choice of ξ_0, $(\underline{x}, \underline{\xi}) \in \mathrm{WF}\,(E_{p/s,\pm}^l|_{t=0}h) \neq \emptyset$ for some $l \in \{1, 2, 3\}$. In addition, by [Eg], Corollary 4.1, p. 141, for example,

$$\mathrm{WF}\,(E_{p/s,\pm}^l|_{t=0}\, h) \subseteq \pi_{(x,\xi)}\Big[\mathrm{WF}\,(E_{p/s,\pm}^l h)\Big]\big|_{t=0}, \tag{17}$$

and so

$$\mathrm{WF}\,(E_{p/s,\pm}^l h) \quad \text{is not empty for some } l \in \{1, 2, 3\}. \tag{18}$$

On the other hand, $\mathrm{WF}\,(E^l_{p/s,\pm}h)$ can be written in terms of the canonical transformation associated with the scalar Fourier integral operator $E^l_{p/s,\pm}$:

$$(19) \qquad \begin{aligned} \mathrm{WF}\,(E^l_{p/s,\pm}h) \quad &\subseteq \quad C_{p/s,\pm} \circ \{(\underline{x},\underline{\xi}) : \beta > 0\} \\ &= \quad \text{conic-closure}\big(\Gamma^j_{p/s,\pm}(\underline{x},\underline{\xi})\big) \end{aligned}$$

by [Tr], Theorem 4.1, for example.

We note that the wave front set of U_j is the union of the wave front sets of its components, and that the bicharacteristic strips $\Gamma_{p/s,\pm}$ are disjoint. Also, by Dencker [De], Theorem 4.2, (see also Section 6.1), the wave front set of U_j is invariant under the Hamilton flow for P and thus equals a union of null bicharacteristics for P. Therefore, by (18) and (19),

$$\mathrm{WF}\,U_j = \text{conic-closure}(\Gamma^j_{p+} \cup \Gamma^j_{p-} \cup \Gamma^j_{s+} \cup \Gamma^j_{s-}), \quad \text{for} \ \ j = 1,2.$$

The bicharacteristics $\Gamma_{p/s,\pm}$ are distinguishable (the p-wave is faster than the s-wave), and the forward and backward waves for each j are distinguishable. (The "forward" wave is on the light cone $\{\tau < 0\}$ and the "backward" wave is on the light cone $\{\tau > 0\}$.) It follows from (16) that $\Gamma^1_{p+} = \Gamma^2_{p+}$, $\Gamma^1_{p-} = \Gamma^2_{p-}$, $\Gamma^1_{s+} = \Gamma^2_{s+}$, and $\Gamma^1_{s-} = \Gamma^2_{s-}$ outside Ω for $t \in (0,T)$. $\qquad \square$

2.5. The wave speeds $c_{p/s}$ are determined in the interior by the Dirichlet-to-Neumann map.

Theorem 2.4 (Rachele [R I], Theorem 4.4). *Suppose that the conditions of Theorem 2.3 hold, and suppose that there are unique distance-minimizing geodesics between boundary points with respect to the metrics $(1/c^2_{p/s})dx^i dx^i$ for wave speeds $c_p = \sqrt{(\lambda + 2\mu)/\rho}$ and $c_s = \sqrt{\mu/\rho}$; these geodesics all exit Ω before time $T < \infty$; and they do not graze $\partial\Omega$. Then $\Lambda_{\lambda_1,\mu_1,\rho_1} = \Lambda_{\lambda_2,\mu_2,\rho_2}$ on $\partial\Omega \times (0,T)$ implies*

$$\sqrt{\frac{\lambda_1 + 2\mu_1}{\rho_1}} \quad = \quad \sqrt{\frac{\lambda_2 + 2\mu_2}{\rho_2}} \quad and \quad \sqrt{\frac{\mu_1}{\rho_1}} \quad = \quad \sqrt{\frac{\mu_2}{\rho_2}} \qquad in \ \ \Omega.$$

3. The boundary polarization in isotropic elastodynamics has no information about the density in the interior

3.1. Definition of the polarization set $\mathrm{WF}_{\mathrm{pol}}U$ and polarization vectors w.

To define the polarization set $\mathrm{WF}_{\mathrm{pol}}\,U$ of a distribution $U \in \mathcal{D}'(\mathbb{R}^4,\mathbb{R}^4)$ we consider, following Dencker [De], 1×3 systems F of pseudodifferential operators of order zero. The polarization set of U is the collection of $(t,x,\tau,\xi) \in T^*(\mathbb{R}^4)$ together with the polarization vectors $w(t,x,\tau,\xi)$ that make up the intersection of the kernels (that is, the orthogonal spaces) of the principal symbols $f(t,x,\tau,\xi)$ of such F with $FU \in C^\infty$. That is, we define

$$(20) \qquad \mathrm{WF}_{\mathrm{pol}}\,U \quad = \bigcap_{FU \in C^\infty(\mathbb{R}^3)} \{(t,x,\tau,\xi;w) \, : \, w \in \ker f(t,x,\tau,\xi)\}.$$

The product FU is given by the usual inner product.

Dencker [De], Theorem 4.2, characterizes the polarization set (for solutions U of systems of differential equations of real principal type) as a union of sections $w(t,x,\tau,\xi)$ over bicharacteristics $\Gamma \in \mathrm{WF}\,U$ of P, that solve

$$(21) \qquad\qquad\qquad\qquad V_q w = Aw \qquad \text{on} \ \ \Gamma.$$

Here V_p is the Hamilton vector field of the principal symbol p of P (cf. (10) and (13)); and $A = -\frac{1}{2}\{\tilde{p}, p\} - i\tilde{p}p^s_{m-1}$ is defined in terms of p,

$$(22) \qquad \widetilde{p}(t, x, \tau, \xi) = -\left[(\tau^2 - c_p^2|\xi|^2)I + (c_p^2 - c_s^2)(\xi \otimes \xi)\right],$$

the subprincipal symbol p^s_{m-1} of P, and the Poisson bracket $\{\cdot, \cdot\}$.

We denote by

$$(23) \qquad w_{p/s, \pm}(\alpha) = w_{p/s, \pm}(\alpha)(t, x, \tau, \xi)$$

(with $\alpha > 0$) the smooth section of the cotangent bundle $T^*(\mathbb{R} \times \mathbb{R}^3)$ that solves (21) on $\Gamma_{p/s, \pm}(\underline{x}, \alpha\underline{\xi})$. In particular, $w_{p/s, \pm} = w_{p/s, \pm}(1)$ denotes the polarization vector with $\alpha = 1$. Given $\underline{\xi} \neq 0$ and $\underline{\xi}^\perp$ with $\underline{\xi}^\perp \cdot \underline{\xi} = 0$, we denote by

$$w_{p/s}^\perp(\alpha)(t, x, \tau, \xi)$$

the polarization vector $w_{p/s}(\alpha)(t, x, \tau, \xi)$ with initial values $w_{p/s, \pm}^\perp(\alpha) = \underline{\xi}^\perp$ at $(0, \underline{x}, \mp c_{p/s}\alpha|\underline{\xi}|, \alpha\underline{\xi})$. In Section 3.2, $w_{p/s, \pm}(\alpha)(t, x, \tau, \xi)$ denotes the polarization vector with initial values $w_{p/s, \pm}(\alpha) = \underline{\xi}$ at $(0, \underline{x}, \mp c_{p/s}\alpha|\underline{\xi}|, \alpha\underline{\xi})$. We write $w_{p/s} = w_{p/s, +}$ and $w_{p/s}^\perp = w_{p/s, +}^\perp$ for the polarization vectors over forward bicharacteristics of P.

3.2. Uniqueness of the polarization vectors $w_{p/s, +}$ and $w_{p/s, +}^\perp$ outside Ω.

We show in [R I], Theorem 5.1, that the s-wave polarization vectors $w_{s, +}^\perp$ are determined outside Ω by the Dirichlet-to-Neumann map:

Theorem 3.1 (Rachele [R I], Theorem 5.1). *For any $\underline{x} \notin \Omega$, any initial direction $\underline{\xi} \neq 0$, and any initial polarization $\underline{\xi}^\perp \neq 0$ with $\underline{\xi}^\perp \cdot \underline{\xi} = 0$, let $w_{s, +}^\perp(t, x, \tau, \xi)$ denote the solution of (21) on $\Gamma_{s, +}(\underline{x}, \underline{\xi})$ with initial values $w_{s, +}^\perp = \underline{\xi}^\perp$ at $(0, \underline{x}, -c_s(\underline{x})|\underline{\xi}|, \underline{\xi})$. If $\overline{x} \notin \Omega$ with $(\overline{t}, \overline{x}, -c_s(\overline{x})|\overline{\xi}|, \overline{\xi}) \in \Gamma_{s, +}(\underline{x}, \underline{\xi})$ for some $\overline{t}, \overline{\xi}$, then the polarization vector $w_{s, +}^\perp$ is determined by the Dirichlet-to-Neumann map at $(\overline{t}, \overline{x}, -c_s(\overline{x})|\overline{\xi}|, \overline{\xi})$.*

Below we modify the proof of [R I], Theorem 5.1, to arrive at the analogous result for p-waves:

Theorem 3.2. *For any $\underline{x} \notin \Omega$ and any initial direction $\underline{\xi} \neq 0$, let $w_{p, +}(t, x, \tau, \xi)$ denote the solution of (21) on $\Gamma_{p, +}(\underline{x}, \underline{\xi})$ with initial values $w_{p, +} = \underline{\xi}$ at $(0, \underline{x}, -c_p(\underline{x})|\underline{\xi}|, \underline{\xi})$. If $\overline{x} \notin \Omega$ with $(\overline{t}, \overline{x}, -c_p(\overline{x})|\overline{\xi}|, \overline{\xi}) \in \Gamma_{p, +}(\underline{x}, \underline{\xi})$ for some $\overline{t}, \overline{\xi}$, then the polarization vector $w_{p, +}$ is determined by the Dirichlet-to-Neumann map at $(\overline{t}, \overline{x}, -c_p(\overline{x})|\overline{\xi}|, \overline{\xi})$.*

Proof. For $(\lambda_1, \mu_1, \rho_1)$ and $(\lambda_2, \mu_2, \rho_2)$ with $\Lambda_{\lambda_1, \mu_1, \rho_1} = \Lambda_{\lambda_2, \mu_2, \rho_2}$ we construct solutions U_j of the Cauchy problem (3) as in the proof of [R I], Theorem 3.1, with initial data ψ^0 and ψ^1 (to be chosen later) supported outside Ω. By [R I], Theorem 3.1, these U_j are determined outside Ω, and so their polarization sets agree outside Ω:

$$(24) \qquad \mathrm{WF}_{\mathrm{pol}}\, U_1 = \mathrm{WF}_{\mathrm{pol}}\, U_2 \qquad \text{outside } \Omega.$$

We now choose initial data ψ_0 and ψ_1, as in the proof of [R I], Theorem 5.1, again with "minimal" wave front set so that the wave front sets of the corresponding solutions U_1 and U_2 will consist of only a few null bicharacteristic strips of P. In fact, we choose

$$(25) \qquad \psi_0 = h(x)\underline{\xi} \quad \text{and} \quad \psi_1 = 0$$

where WF $h(x) = \{(\underline{x}, \alpha\underline{\xi}) \ : \ \alpha > 0\}$. (See [R I], Remark 4.1 on the existence of such distributions $h(x)$.)

We compute upper and lower bounds on the polarization set of U_j in terms of the polarization vectors $w_{p/s,\pm}$ (cf. (23)). In fact, we show in Lemma 3.3 that the polarization set of U_j does contain the polarization vector $w_{p,+}$, but contains at most the span of the polarization vectors $w_{p/s,\pm}(\alpha)$, $\alpha > 0$. It follows from the fact that the bicharacteristics $\Gamma^j_{p/s,\pm}$ are disjoint and distinguishable (for each j) and from the fact that the polarization sets $\mathrm{WF}_{\mathrm{pol}}\, U_j$ agree outside Ω (cf. (24)) that the polarization vectors making up the polarization set of U_j are determined outside Ω. In particular, the polarization vector $w_{p,+}$ is determined. This completes the proof of the theorem. $\qquad\square$

Lemma 3.3. *For the U_j constructed in the proof of Theorem 3.2 and for polarization vectors $w^j_{p/s,\pm}(\alpha)$ solving (21) on $\Gamma^j_{p/s,\pm}$ with initial values $w^j_{p/s,\pm}(\alpha) = \underline{\xi}$ at $(0, \underline{x}, \mp c_{p/s}\alpha|\underline{\xi}|, \alpha\underline{\xi})$ we have*

$$\{\ (\gamma, w^j_{p,+}(\gamma))\ :\ \gamma = (t, x, \tau, \xi) \in \Gamma^j_{p,+}(\underline{x}, \underline{\xi})\ \}$$

(26)
$$\subseteq\quad \mathrm{WF}_{\mathrm{pol}}\, U_j$$

$$\subseteq \bigcup_{p/s,\pm} \{\ (\gamma, \beta\, w^j_{p/s,\pm}(\alpha)(\gamma))\ :\ \alpha, \beta > 0,\ \gamma \in \Gamma^j_{p/s,\pm}(\underline{x}, \alpha\,\underline{\xi})\ \}.$$

Proof. To show that the upper bound of (26) holds, we describe the polarization set of U_j by referring to Dencker's characterization (21). In particular, we observe that the polarization vectors in the polarization set $\mathrm{WF}_{\mathrm{pol}}\, U_j$ of U_j

- lie over bicharacteristics in the wave front set of U_j, and
- have initial polarization that agrees with some polarization vector of the initial data $U_j|_{t=0} = \psi_0$.

Therefore, to describe $\mathrm{WF}_{\mathrm{pol}}\, U_j$, we describe $\mathrm{WF}\, U_j$ and $\mathrm{WF}_{\mathrm{pol}}\, (U_j|_{t=0})$.

We have observed in the proof of Theorem 2.3 that the wave front set of U_j is contained in the union of the bicharacteristics $\Gamma^j_{p/s,\pm}(\underline{x}, \alpha\underline{\xi})$ that pass through $(0, \underline{x}, \mp c_{p/s}(\underline{x})\alpha|\underline{\xi}|, \alpha\underline{\xi})$, that is,

$$\mathrm{WF}\ U_j \subseteq \bigcup_{p/s,\pm} \{\ \Gamma^j_{p/s,\pm}(\underline{x}, \alpha\underline{\xi})\ :\ \alpha > 0\ \}.$$

We show below in Lemma 3.4 that the polarization set of the initial data $U_j|_{t=0}$ is restricted to the span of $\underline{\xi}$ over the points $(\underline{x}, \alpha\underline{\xi})$, $\alpha > 0$. It follows that the upper bound in (26) holds.

Lemma 3.4. *For the U_j constructed in the proof of Theorem 3.2,*

(27)
$$\mathrm{WF}_{\mathrm{pol}}\, (U_j|_{t=0})\ =\ \{\ (\underline{x}, \alpha\underline{\xi}; \beta\underline{\xi}) : \alpha > 0, \beta \in \mathbb{R}\ \}.$$

Proof. We recall that $U_j|_{t=0} = \psi_0 = h(x)\underline{\xi}$. We first show that

(28)
$$\mathrm{WF}_{\mathrm{pol}}\, \psi_0 \subseteq \{\ (\underline{x}, \alpha\underline{\xi}; \beta\underline{\xi})\ :\ \alpha > 0,\ \beta \in \mathbb{R}\ \}$$

by considering the pseudodifferential operators F with (principal) symbol $f(x, \xi) = \left[I - \psi(x, \xi)\frac{\xi}{|\xi|} \otimes \frac{\xi}{|\xi|}\right]v(x, \xi)$, where $\psi(x, \xi)$ is any smooth function on $\mathbb{R}^3 \times \mathbb{R}^3$ that is homogeneous of order 0 in ξ and has values $\psi = 1$ on $\{(\underline{x}, \alpha\underline{\xi}) : \alpha > 0\}$, and

$v(x,\xi)$ is any smooth vector field on $T^*\mathbb{R}^3$ that is homogeneous of order 0 in ξ and has values in $\mathbb{R}^3$.

To show that (28) holds, we will apply the definition (20) of the polarization set. We note that the intersection, over all ψ and v, of $\{(x,\xi,w) : w \in \ker f(x,\xi)\}$ is actually equal to the right side of (28). Then, to see that each $F\psi_0$ is smooth, we notice that each of the (scalar) amplitudes $f(x,\xi)\underline{\xi}$ of

$$h(x) \longrightarrow (F(x,D_x)\underline{\xi})(h(x))$$
$$= \int e^{ix\cdot\xi} f(x,\xi)\underline{\xi}\,\hat{h}(\xi)\,d\xi$$
$$= F\big(h(x)\underline{\xi}\big) = F\psi_0$$

vanishes in a conic neighborhood of $\mathrm{WF}\,h = \{(\underline{x},\alpha\underline{\xi}) : \alpha > 0\}$.

To show that $\mathrm{WF}_{\mathrm{pol}}\,\psi_0$ is actually given by the upper bound in (28), we observe that for any zeroth-order pseudodifferential operator F on $\mathbb{R}^3$ with $(F\underline{\xi})h = F\psi_0 \in C^\infty$ it is the case that

$$\mathrm{WF}\,(h) \subseteq \mathrm{WF}\Big((F\underline{\xi})h\Big) \cup \{(x,\xi) : f(x,\xi)\underline{\xi} = 0\},$$

that is,

$$\{(\underline{x},\alpha\underline{\xi}) : \alpha > 0\} \subseteq \{(x,\xi) : f(x,\xi)\underline{\xi} = 0\},$$

where $f(x,\xi)$ is the principal symbol of F. It follows that $f(\underline{x},\alpha\underline{\xi})\underline{\xi} = 0$; that is, $\underline{\xi} \in \ker f(\underline{x},\alpha\underline{\xi})$ for any $\alpha > 0$. $\qquad\square$

To show that the lower bound in (26) also holds, that is, that the initial data ψ_0 does induce compressional-wave polarization of the form $w^j_{p,+}$ in U_j, we show that it is enough to prove that $\Gamma^j_{p,+} \subseteq \mathrm{WF}\,(U_j)$. In fact, by [De], Proposition 2.5, $\pi_{(t,x,\tau,\xi)}(\mathrm{WF}_{\mathrm{pol}}\,U_j) \setminus 0 = \mathrm{WF}\,U_j$. It follows that if $\Gamma^j_{p,+} \subseteq \mathrm{WF}\,(U_j)$ holds, then some part of $(\mathrm{WF}_{\mathrm{pol}}\,U_j) \setminus 0$ lies over $\Gamma^j_{p,+}$. The polarization set over $\Gamma^j_{p,+}$ is a union of solutions of (21) that have initial polarization contained in $\mathrm{WF}_{\mathrm{pol}}\,(U_j|_{t=0})$. By (27) the initial values ψ_0 of U_j have polarization only with direction $\beta\underline{\xi}$, $\beta \in \mathbb{R}$ (at $(\underline{x},\alpha\underline{\xi})$, $\alpha > 0$). This implies that the $(p,+)$-polarization of U_j is, in fact, the span of the polarization vectors $w^j_{p,+}(\alpha)$, $\alpha > 0$. Therefore, the lower bound in (26) holds.

Next, we observe that $\Gamma^j_{p,+} \subseteq \mathrm{WF}\,(U_j)$ holds if $(\underline{x},\underline{\xi}) \in \mathrm{WF}\,(u^+_p|_{t=0})$, where

$$u^{\pm}_{p/s} = \sum_{\underline{m}} \int e^{i\varphi^{\pm}_{p/s}(t,x,\eta)} e^{\cdot,\underline{m}}_{p/s,\pm}(0)(t,x,\eta)(\hat{\psi}_0)_{\underline{m}}(\eta)\,d\eta$$

are terms in the sum (4) making up U_j. In fact, as in the proof of Theorem 2.3 (cf. (19) and (17)), we have $\mathrm{WF}\,(u^+_p|_{t=0}) \subseteq \pi_{(x,\xi)}(\mathrm{WF}\,u^+_p)|_{t=0}$ and $\mathrm{WF}\,u^{\pm}_{p/s} \subseteq \{\Gamma_{p/s,\pm}(\underline{x},\alpha\underline{\xi}) : \alpha > 0\}$. Thus, if $(\underline{x},\underline{\xi}) \in \mathrm{WF}\,(u^+_p|_{t=0})$ holds, then $\emptyset \neq \mathrm{WF}\,(u^+_p) \subseteq \{\Gamma_{p,+}(\underline{x},\alpha\underline{\xi}) : \alpha > 0\}$. Again, referring to the proof of Theorem 2.3 we note that the $\Gamma^j_{p/s,\pm}$ are disjoint (for each $j = 1,2$), and so $\mathrm{WF}\,U_j = \bigcup_{p/s,\pm} \mathrm{WF}\,u^{\pm}_{p/s}$. Also, $\mathrm{WF}\,(U_j)$ is a union of bicharacteristics of P, and so $\mathrm{WF}\,(u^+_p) = \{\Gamma_{p,+}(\underline{x},\alpha\underline{\xi}) : \alpha > 0\}$, and $\{\Gamma_{p,+}(\underline{x},\alpha\underline{\xi}) : \alpha > 0\} \subseteq \mathrm{WF}\,(U_j)$.

Now, to show $(\underline{x}, \underline{\xi}) \in \mathrm{WF}\,(u_p^+|_{t=0})$ we observe that

$$
\begin{aligned}
u_p^+|_{t=0} &= \sum_{\underline{m}} \int e^{i\varphi_p^+(t,x,\eta)}\ e_{p,+}^{\cdot,\underline{m}}(0)(t,x,\eta)(\widehat{\psi_0})_{\underline{m}}(\eta)\ d\eta\ \Big|_{t=0} \\
&= \sum_{\underline{m}} \int e^{i\left[\varphi_p^+(0,x,\eta)-y\cdot\eta\right]}\ e_{p,+}^{\cdot,\underline{m}}(0)(0,x,\eta)(\psi_0)_{\underline{m}}(y)\ dyd\eta \\
&= \int e^{i\left[\varphi_p^+(0,x,\eta)-y\cdot\eta\right]}\ e_{p,+}(0)(0,x,\eta)\underline{\xi}\,h(y)\ dyd\eta \\
&= \int e^{ix\cdot\eta}\ e_{p,+}(0)(0,x,\eta)\underline{\xi}\,\hat{h}(\eta)\ d\eta,
\end{aligned}
$$

with the last equality holding due to the initial values $\varphi_{p/s}^{\pm}(0,x,\eta) = x\cdot\eta$ of φ_p^+ given in (7). By (12) $u_p^+|_{t=0} = \frac{1}{2}\,\mathrm{P}_\eta(x,D_x)\underline{\xi}(h)$, where $\mathrm{P}_\eta(x,D_x)\underline{\xi}(h) = \int e^{ix\cdot\eta}\left[\frac{\eta}{|\eta|}\right.$ $\left.\otimes\frac{\eta}{|\eta|}\right]\underline{\xi}\,\hat{h}(\eta)\ d\eta$ is the 1×3 system of pseudodifferential operators with symbol $\mathrm{P}_\eta(x,\eta)\underline{\xi} = (\frac{\eta}{|\eta|}\otimes\frac{\eta}{|\eta|})\underline{\xi}$ (acting on the scalar h).

For each component $(l=1,2,3)$ of the amplitude $\mathrm{P}_\eta(x,\eta)\underline{\xi}$, we have

$$
\mathrm{WF}\,h \quad \subseteq \quad \mathrm{WF}\left((\mathrm{P}_\eta(x,D_x)\underline{\xi})_l h\right) \ \cup \ \left\{(x,\eta) : \left(\mathrm{P}_\eta(x,\eta)\underline{\xi}\right)_l = 0\right\}.
$$

The wave front set of h is the ray through $(\underline{x}, \underline{\xi})$, and $\left(\mathrm{P}_\eta(\underline{x}, \underline{\xi})\underline{\xi}\right)_l = \underline{\xi}_l \neq 0$ for some l. Therefore, $(\underline{x}, \underline{\xi})$ is, in fact, in $\mathrm{WF}\left(\mathrm{P}_\eta(x,D_x)\underline{\xi}(h)\right) = \mathrm{WF}\,(u_p^+|_{t=0})$.

This completes the proof of the lower bound in (26). $\qquad\square$

3.3. Conclusion: Boundary polarization contains no information about the density in the interior.

We first compute the terms involved in the ODE (21) which describes the propagation of the polarization vectors $w_{p/s}^+$. To shorten the calculation of A we follow Dencker [De], Example 4.3. We define $\widetilde{P}$ to be the pseudodifferential operator with (principal) symbol $\tilde{p}$ given in (22). Then the pseudodifferential operator $Q = \widetilde{P}P$ has principal symbol qI with q scalar and of real principal type. That is, by (10) and (22),

$$
q(t,x,\tau,\xi) \ = \ \tilde{p}p \ = \ \rho(\tau^2 - c_s^2|\xi|^2)(\tau^2 - c_p^2|\xi|^2).
$$

For $w \in \ker p(t,x,\tau,\xi)$ we have $A = A_q$ where $A_q = -iq^s$ with q^s the subprincipal symbol of Q. That is, for $Q_j = Q(\lambda_j, \mu_j, \rho_j)$, the polarization vectors of U_j solve

$$
(29) \qquad\qquad\qquad V_q w = A_q w,
$$

where V_q is the Hamilton vector field of q (cf. (13)). In fact, writing $c_\lambda^2 = \lambda/\rho$, we have

$$
V_q\,|_{\tau=\mp c_{p/s}|\xi|} \ = \ \mp(+/-)2\rho c_{p/s}^2(c_p^2 - c_s^2)|\xi|^3 \cdot \left[\frac{\partial_t}{c_{p/s}} \pm \frac{\xi}{|\xi|}\cdot\nabla_x \mp (\log c_{p/s})_x|\xi|\cdot\nabla_\xi\right],
$$

$p_1(t,x,\tau,\xi)$ consists of the lower-order terms of the symbol of P and is given by

$$
\begin{aligned}
(30) \qquad p_1(t,x,\tau,\xi) \ &= \ -i\left[\lambda_x \otimes \xi \ + \ (\mu_x\cdot\xi)I \ + \ \xi\otimes\mu_x\right] \\
&= \ -i\rho\left[c_\lambda^2(\log\rho c_\lambda^2)_x\otimes\xi \ + \ [c_s^2(\log\rho c_s^2)_x\cdot\xi]I \ + \ c_s^2\xi\otimes(\log\rho c_s^2)_x\right],
\end{aligned}
$$

where

$$u \textcircled{s} v = \frac{1}{2}\big[u \otimes v \; + \; v \otimes u\big].$$

In addition,

$$
\begin{aligned}
A_q \;=\;& -iq^s \;=\; -i\widetilde{p}p_1 \;-\; \sum_j (\partial_{\xi_j}\widetilde{p})(\partial_{x_j}p) \;+\; \tfrac{1}{2}\big(\sum_j \partial_{x_j}\partial_{\xi_j}q\big)I \\
\;=\;& \rho\sum_j \Big[2c_p^2\xi_j I \;-\; 2(c_p^2 - c_s^2)(\xi\textcircled{s}e_j)\Big]\cdot\Big[-(c_p^2 - c_s^2)_{x_j}(\xi\otimes\xi) \\
& + (\log\rho)_{x_j}\Big((\tau^2 - c_s^2|\xi|^2)I - (c_p^2 - c_s^2)(\xi\otimes\xi)\Big) \;-\; (c_s^2)_{x_j}|\xi|^2 I\Big] \\
& + \rho\Big[(\tau^2 - c_p^2|\xi|^2)I \;+\; (c_p^2 - c_s^2)(\xi\otimes\xi)\Big] \\
& \quad\cdot\Big((c_p^2 - 2c_s^2)\big[\log\rho(c_p^2 - 2c_s^2)\big]_x \otimes \xi \\
& \qquad\qquad\qquad + c_s^2\big[(\log\rho c_s^2)_x\cdot\xi\big]I + c_s^2\,\xi\otimes(\log\rho c_s^2)_x\Big) \\
& - \rho\sum_j \Big((\log\rho)_{x_j}\Big[c_p^2(\tau^2 - c_s^2|\xi|^2)\xi_j \;+\; c_s^2(\tau^2 - c_p^2|\xi|^2)\xi_j\Big] \\
& \qquad + (c_p^2)_{x_j}(\tau^2 - c_s^2|\xi|^2)\xi_j \;+\; (c_s^2)_{x_j}(\tau^2 - c_p^2|\xi|^2)\xi_j \\
& \qquad\qquad\qquad - c_p^2|\xi|^2(c_s^2)_{x_j}\xi_j \;-\; c_s^2|\xi|^2(c_p^2)_{x_j}\xi_j\Big)I \\
\;=\;& \rho\Big(\gamma_1 I \;+\; \gamma_2(\xi\otimes\xi) \;+\; \xi\otimes v_1 \;+\; v_2\otimes\xi\Big),
\end{aligned}
$$

for

$$
\begin{aligned}
\gamma_1 \;=\;& \big[\xi\cdot(\log\rho)_x\big]c_p^2(c_p^2 - c_s^2)|\xi|^2\delta_{\tau^2 = c_p^2|\xi|^2} \;-\; \big[\xi\cdot(c_s^2)_x\big]c_p^2|\xi|^2 \\
& \qquad\qquad + \big[\xi\cdot(c_p^2)_x\big]\Big[(c_p^2 - c_s^2)|\xi|^2\delta_{\tau^2 = c_p^2|\xi|^2} \;-\; c_s^2|\xi|^2\Big], \\
\gamma_2 \;=\;& \big[\xi\cdot(\log\rho)_x\big]\big[-2c_s^2(c_p^2 - c_s^2)\big] \;-\; \big[\xi\cdot(c_p^2 - c_s^2)_x\big][2c_s^2], \\
v_1 \;=\;& -(\log\rho)_x(c_p^2 - c_s^2)|\xi|^2\big[(c_p^2 - 2c_s^2)\delta_{\tau^2 = c_p^2|\xi|^2}\big] \\
& \qquad\qquad + (c_s^2)_x(c_p^2 - c_s^2)|\xi|^2\big[2 \;-\; \delta_{\tau^2 = c_s^2|\xi|^2}\big], \\
v_2 \;=\;& (\log\rho)_x(c_p^2 - c_s^2)|\xi|^2\big[c_s^2\delta_{\tau^2 = c_s^2|\xi|^2}\big] \;+\; 2(c_s^2)_x(c_p^2 - c_s^2)|\xi|^2\delta_{\tau^2 = c_s^2|\xi|^2} \\
& \qquad\qquad + (c_p^2)_x(c_p^2 - c_s^2)|\xi|^2\delta_{\tau^2 = c_p^2|\xi|^2}.
\end{aligned}
$$

(31)

We denote by $'$ and by $\frac{d}{ds}$ the derivative with respect to arclength in the direction of the bicharacteristic $\Gamma^{\pm}_{p/s}(\underline{x},\underline{\xi})$ (cf. (14)); that is,

(32)
$$\frac{d}{ds} = \frac{\partial_t}{c_{p/s}} \pm \frac{\xi}{|\xi|}\cdot\nabla_x \mp (\log c_{p/s})_x|\xi|\cdot\nabla_\xi.$$

It follows that the ODE (29) reduces to

$$(w^{\pm}_{p/s})' \;=\; \frac{dw_{p/s}}{ds} \;=\; \frac{(+/-)\Big[\gamma_1 I + \gamma_2(\xi\otimes\xi) + \xi\otimes v_1 + v_2\otimes\xi\Big]}{2c_{p/s}^2(c_p^2 - c_s^2)|\xi|^3}\, w^{\pm}_{p/s}.$$

In the case of a forward p-wave (i.e., $\tau = -c_p|\xi|$) we have $w_p = \alpha N$ where $N = \frac{\xi}{|\xi|} = \frac{dx}{ds}$. Therefore, the ODE in this case reduces to

$$\alpha'N \;+\; \alpha N' \;=\; \frac{dw_p}{ds}\big|_{\tau = -c_p|\xi|} \;=\; \alpha\frac{\Big[\gamma_1 \;+\; \gamma_2|\xi|^2 \;+\; v_1\cdot\xi\Big]N \;+\; v_2|\xi|}{2c_p^2(c_p^2 - c_s^2)|\xi|^3};$$

that is, given (31),

$$\alpha' = \alpha G(c_p, c_s),$$

where $G(c_p, c_s)$ does not depend on ρ. (Note that by (14),

$$N' = \left(\frac{\xi}{|\xi|}\right)' = (\log c_p)_x + (\log c_p)'N,$$

while $v_2|\xi|/(2c_p^2(c_p^2 - c_s^2)|\xi|^3) = (\log c_p)_x)$. It follows that the p-wave boundary polarization does not contain any information about the value of the density ρ in the interior.

In the case of a forward s-wave (i.e., $\tau = -c_s|\xi|$) we have $w_s \perp \xi$; so the ODE reduces to

$$\frac{dw_s(t, x, \tau, \xi)}{ds}\Big|_{\tau=-c_s|\xi|} = \frac{-[\gamma_1 w_s + (v_1 \cdot w_s)\xi]}{2c_s^2(c_p^2 - c_s^2)|\xi|^3}.$$

It follows that

(33) $$\frac{d}{ds}\left(\log |w_s|^2\right) = \frac{2w_s}{|w_s|^2} \cdot \frac{dw_s}{ds} = \frac{-\gamma_1}{c_s^2(c_p^2 - c_s^2)|\xi|^3}$$

and

(34) $$\frac{d}{ds}\left(w_s \cdot \frac{w_s(0)}{|w_s(0)|}\right) = \left[\mathrm{proj}_{w_s^\perp} w_s'\right] \cdot \frac{w_s(0)}{|w_s(0)|}$$

where $w_s(0)$ is the polarization vector at the boundary.

Each of these ODEs, (33) and (34), has right-hand sides that do not depend on $\rho(x)$ with $x \in \Omega$, when $\tau = -c_s|\xi|$. It follows that $|w_s|$ and $\mathrm{Arccos}\left(\frac{w_s}{|w_s|} \cdot \frac{w_s(0)}{|w_s(0)|}\right)$ do not depend on ρ in the interior. Therefore, the s-wave boundary polarization also contains no information about the value of the density ρ in the interior.

4. Uniqueness of the density

4.1. **Using the transport equations to describe the propagation of "lower-order polarization".** Here we return to the representation of the elastic displacement wave in terms of solutions of the Cauchy problem (3), in particular, in terms of the Fourier integral operators described in Section 2.2. The amplitudes of these Fourier integral operators solve algebraic equations (9) and satisfy compatibility conditions (11).

We reduce the compatibility conditions (11) (for top-order and lower-order terms of the p-wave FIO amplitudes) to the transport equations (37) and (39), ODEs along the bicharacteristics of P. Writing α_0 for $(\alpha^{\underline{m}})_0(t, x, \eta)$, we have by (8), (10), (30),

for forward p-waves (i.e., $\tau = -c_p|\xi|$) :

$$\partial_\tau p = -2\rho\tau I,$$

$$\partial_{\xi_j} p = 2\rho\Big[c_s^2\xi_j I + (c_p^2 - c_s^2)(e_j \circledS \xi)\Big],$$

$$\partial_{\xi_i}\partial_{\xi_j} p = 2\rho\Big[c_s^2\delta_{ij} I + (c_p^2 - c_s^2)(e_i \circledS e_j)\Big],$$

$$iN\partial_{\tau,\xi} p \cdot \partial_{t,x}\big[\alpha_0 N\big]$$

(35)
$$= i\rho\Big[2c_p^2|\xi|\Big(\tfrac{\partial_t}{c_p} + \tfrac{\xi}{|\xi|}\cdot\nabla_x\Big)\alpha_0 + (c_p^2 - c_s^2)|\xi|(\nabla_x\cdot N)\alpha_0\Big],$$

$$-Np_1[\alpha_0 N] = i\Big([\xi\cdot\lambda_x] + 2[\xi\cdot\mu_x]\Big)\alpha_0 = i\rho c_p^2[\xi\cdot(\log\rho c_p^2)_x]\alpha_0,$$

$$N\cdot(i/2)\sum_{|\alpha|=2}\sum_{l=1}^{3}(\partial_{\tau,\xi}^\alpha p)\cdot(\partial_{t,x}^\alpha\varphi_p)\big[\alpha_0 N\big]$$

$$= i\rho\Big(-c_p^2(\log c_p)'|\xi| + c_s^2(\nabla_x\cdot\xi) - c_s^2[N(\nabla_x\otimes\xi)N]\Big)\alpha_0$$

due to the fact that

$$2N\cdot N_t = \partial_t(N\cdot N) = \partial_t(1) = 0,$$

$$(\nabla_x\otimes N)N = \frac{1}{2}\big[\nabla_x(N\cdot N)\big] = \frac{1}{2}\big[\nabla_x(1)\big] = 0,$$

(36)
$$\partial_t^2\varphi_p(t,x,\eta) \overset{(32)}{=} \Big[c_p\frac{d}{ds} - c_p N\cdot\nabla_x\Big]\tau = c_p^2\Big[(\log c_p)'|\xi| + N(\nabla_x\otimes\xi)N\Big],$$

$$\nabla_x\cdot N = \frac{\nabla_x\cdot\xi}{|\xi|} - \frac{N(\nabla_x\otimes\xi)N}{|\xi|}.$$

Given (8), it follows from the compatibility condition (11) for $J = 0$, $NB_p\big(e_p^{\cdot,m}(\underline{k})\big)_0 = 0$, that

(37)
$$\alpha_0' = -\Big[(\log\sqrt{\rho c_p})' + \frac{1}{2}(\nabla_x\cdot N)\Big]\alpha_0.$$

Therefore, for $\underline{x} = x(\underline{s}) \notin \Omega$, $\underline{t} = t(\underline{s})$, $\xi(s) = \nabla_x\varphi_p\big(t(s),x(s),\eta\big)$, and $\alpha_0 = \alpha_0\big(t(s),x(s),\eta\big)$

(38)
$$\alpha_0 = \alpha_0(\underline{t},\underline{x},\eta)\frac{\sqrt{(\rho c_p)(\underline{t},\underline{x},\eta)}}{\sqrt{\rho c_p(x)}}\exp\Big(-\frac{1}{2}\int_{\underline{s}}^{s}\nabla_x\cdot N\,ds\Big) = \frac{H}{\sqrt{\rho}},$$

where $H = H\big(t(s),x(s),\eta\big)$ depends on c_p, but not $\rho(x)$ for $x \in \Omega$.

To derive the transport equation for $\alpha_{-1} = (\alpha^{\underline{m}})_{-1}(t,x,\eta)$ from the compatibility condition (11) for $J = -1$, $N[B_p(e_p)_{-1} + C_p(e_p)_0] = 0$, we first observe, as in (35) and the calculations that follow, that

$$NB_p(\alpha_{-1}N) = 2i\rho c_p^2|\xi|\Big(\alpha_{-1}' + \big[(\log\sqrt{\rho c_p})' + \frac{1}{2}(\nabla_x\cdot N)\big]\alpha_{-1}\Big).$$

It follows that this compatibility condition reduces to the transport equation

(39)
$$\alpha_{-1}' + \big[(\log\sqrt{\rho c_p})' + \frac{1}{2}(\nabla_x\cdot N)\big]\alpha_{-1} = G,$$

where
(40)
$$G = \frac{-1}{2i\rho c_p^2|\xi|}\big[NB_p h_{-1} + NC_p(\alpha_0 N)\big].$$

We solve the transport equation (39):

$$(41) \qquad g\alpha_{-1} = \int gG + C,$$

where C is a constant, the integration is over p-wave bicharacteristic segments (cf. Section 2.3), and

$$(42) \qquad g = \frac{\sqrt{\rho}}{H} = \frac{\sqrt{\rho c_p(x)}}{\alpha_0(\underline{t},\underline{x},\eta)\sqrt{(\rho c_p)(\underline{t},\underline{x},\eta)}} \exp\left(\frac{1}{2}\int_{\underline{s}}^{s} \nabla_x \cdot N \, ds\right)$$

has $g' = \left[(\log\sqrt{\rho c_p})' + \frac{1}{2}(\nabla_x \cdot N)\right] g$.

We compute G, given (10), (35), (36), and (38):

$$NC_p(\alpha_0 N)$$
$$= \rho\left(2(c_p^2 - 2c_s^2)\left[\log\sqrt{\rho(c_p^2 - 2c_s^2)}\right]'\left[\nabla_x\cdot\left(\frac{H}{\sqrt{\rho}}N\right)\right]\right.$$
$$+ 2c_s^2(\log\sqrt{\rho}c_s)_x\left[\nabla_x \otimes \left(\frac{H}{\sqrt{\rho}}N\right)\right]N - \partial_t^2\left(\frac{H}{\sqrt{\rho}}N\right)\cdot N$$
$$+ 2c_s^2 N\left[\nabla_x\otimes\left(\frac{H}{\sqrt{\rho}}N\right)\right](\log\sqrt{\rho}c_s)_x + c_s^2\Delta\left(\frac{H}{\sqrt{\rho}}N\right)\cdot N$$
$$\left. + (c_p^2 - c_s^2)N\left[\nabla_x \otimes \nabla_x\right]\cdot\left(\frac{H}{\sqrt{\rho}}N\right)\right)$$

$$(43)$$
$$= -H\sqrt{\rho}\left(c_s^2(\Delta\log\sqrt{\rho}) + (c_p^2 - c_s^2)[N(\log\sqrt{\rho})_{xx}N]\right.$$
$$+ c_s^2|(\log\sqrt{\rho})_x|^2 + (c_p^2 - c_s^2)[(\log\sqrt{\rho})']^2$$
$$+ (\log\sqrt{\rho})'\left[2(c_p^2 - c_s^2)(\log\sqrt{c_p^2 - c_s^2})' - (c_p^2 - 3c_s^2)(\nabla_x \cdot N)\right]$$
$$+ (\log\sqrt{\rho})_x \cdot \left[(c_p^2 - 3c_s^2)(\nabla_x \otimes N)^t N + 2c_s^2(\log c_s)_x\right]$$
$$\left. + \text{ terms that do not depend on } \rho\right)$$

since, by (30) and (38),

$$\partial_{\xi_j}(ip_1) = \rho\left(2(c_p^2 - 2c_s^2)\left[\left(\log\sqrt{\rho(c_p^2 - 2c_s^2)}\right)\right]_x \otimes e_j\right.$$
$$\left. + 2c_s^2\left(\log\sqrt{\rho}c_s\right)_{x_j}I + 2c_s^2\left[e_j \otimes (\log\sqrt{\rho}c_s)_x\right]\right),$$

$$(44) \qquad \sqrt{\rho}(\alpha_0)_x = H[-(\log\sqrt{\rho})_x + (\log H)_x],$$

$$\sqrt{\rho}(\alpha_0)_{xx} = H\left[(\log\sqrt{\rho})_x\otimes(\log\sqrt{\rho})_x - (\log\sqrt{\rho})_{xx} + (\log H)_{xx}\right.$$
$$\left. - 2(\log\sqrt{\rho})_x\circledS(\log H)_x + (\log H)_x \otimes (\log H)_x\right].$$

We note, by (32) and (37), that $\alpha_0 = \alpha_0(t,x,\eta)$ solves

$$\left[\frac{\partial_t}{c_p} + \frac{\xi}{|\xi|}\cdot\nabla_x\right]\alpha_0 = \alpha_0' = -\left[(\log\sqrt{\rho c_p})' + \frac{1}{2}(\nabla_x \cdot N)\right]\alpha_0;$$

so, by the definition of H in (38),

$$(45) \qquad \partial_t \log H \;=\; -c_p\Big[(\log\sqrt{c_p})' \;+\; \frac{1}{2}(\nabla_x \cdot N) \;+\; [N\cdot(\log H)_x]\Big].$$

It follows, given the algebraic equations (9) with $J = 0$, given that N is in $\ker p(t,x,\partial_{t,x}\varphi_p)$ (cf. (8)), and given that h_{-1} is orthogonal to ξ (cf. (8)), that

$$
\begin{aligned}
h_{-1} &= \frac{-1}{\rho(c_p^2 - c_s^2)|\xi|^2} B_p(\alpha_0 N) \\[4pt]
&= \frac{-1}{\rho(c_p^2 - c_s^2)|\xi|^2}\cdot i\rho|\xi|\Bigg(2c_p\partial_t(\alpha_0 N) \;+\; 2c_s^2\big[\nabla_x \otimes (\alpha_0 N)\big]^t N \\
&\qquad +\; (c_p^2 - c_s^2)\Big[\big[\nabla_x \otimes (\alpha_0 N)\big]N \;+\; \big[\nabla_x \cdot (\alpha_0 N)\big]N\Big] \\
&\qquad +\; \Big[2(c_p^2 - 2c_s^2)\Big(\log\sqrt{\rho(c_p^2 - 2c_s^2)}\Big)_x \;+\; 4c_s^2\big(\log\sqrt{\rho}c_s\big)' N \\
&\qquad -\; \frac{\partial_t^2\varphi_p}{|\xi|}N \;+\; c_s^2\frac{(\Delta\varphi_p)}{|\xi|}N \;+\; (c_p^2 - c_s^2)\frac{[\nabla_x \otimes \xi]}{|\xi|}N\Big]\alpha_0\Bigg) \\[4pt]
&= \frac{iH}{(c_p^2 - c_s^2)|\xi|\sqrt{\rho}}(V \;+\; W),
\end{aligned}
$$

especially due to (10), (14), (30), (35), (36), (38), (44), (45), and

$$
\begin{aligned}
\partial_t N(t,x,\eta) &= c_p\Big[\big(\tfrac{\xi}{|\xi|}\big)' - (\nabla_x \otimes N)^t N\Big] \\
&= -c_p\Big[(\nabla_x \otimes N)^t N \;+\; (\log c_p)_x \;-\; (\log c_p)' N\Big], \\
(\nabla_x \otimes N)^t N &= \frac{(\nabla_x \otimes \xi)}{|\xi|}N \;-\; [N^t\frac{(\nabla_x \otimes \xi)}{|\xi|}N]N.
\end{aligned}
$$

Here we use the notation

$$
\begin{aligned}
V &= 2N@\Big[(c_p^2 - 3c_s^2)(\log\sqrt{\rho})_x\Big]N, \\
W &= 2N@\Big[-4c_s^2(\log c_s)_x \;+\; (c_p^2 - c_s^2)(\log H)_x\Big]N \\
&\qquad\qquad\qquad\qquad +\; (c_p^2 - c_s^2)(\nabla_x \otimes N)^t N, \\
[u@v]w &= \frac{1}{2}\Big([u\otimes v]w \;-\; [v\otimes u]w\Big) = \frac{1}{2}\Big(u[v\cdot w] \;-\; v[u\cdot w]\Big).
\end{aligned}
$$

Therefore, given that $N \cdot h_{-1} = 0$, we have

$$
\begin{aligned}
NB_p = i\rho|\xi|\Bigg(&-2c_p(\partial_t N)\cdot h_{-1} \;+\; (c_p^2 + c_s^2)N^t[\nabla_x \otimes h_{-1}]N \\
&+\; (c_p^2 - c_s^2)(\nabla_x \cdot h_{-1}) \;+\; 2c_s^2[(\log\sqrt{\rho}c_s)_x \cdot h_{-1}] \\
&\qquad\qquad +\; (c_p^2 - c_s^2)N^t\frac{[\nabla_x \otimes \xi]}{|\xi|}h_{-1}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
= i\rho|\xi| \cdot \frac{iH}{(c_p^2 - c_s^2)|\xi|\sqrt{\rho}} \Bigg(& -(c_p^2 + c_s^2)(\log\sqrt{\rho})'(W\cdot N) \\
& - (c_p^2 - 3c_s^2)\big[(\log\sqrt{\rho})_x \cdot (V+W)\big] + (c_p^2 - c_s^2)(\nabla_x \cdot V) \\
& + (c_p^2 + c_s^2)\big[N^t(\nabla_x \otimes V)N\big] \; + \; 2c_p^2\big[N^t(\nabla_x \otimes N)V\big] \\
& + (c_p^2 - c_s^2)\big[(\log H)_x \cdot V\big] + 4c_s^2\big[(\log c_s)_x \cdot V\big]
\end{aligned}
$$

(46)
$$
\hspace{3cm} + \text{ terms that do not depend on } \rho \Bigg)
$$

$$
\begin{aligned}
= \frac{-H\sqrt{\rho}}{(c_p^2 - c_s^2)} \cdot \Bigg[& \Big(|(\log\sqrt{\rho})_x|^2 \; - \; [(\log\sqrt{\rho})']^2 \Big)(c_p^2 - 3c_s^2)^2 \\
& + \Big([N^t(\log\sqrt{\rho})_{xx}N] - \Delta(\log\sqrt{\rho}) \Big)(c_p^2 - c_s^2)(c_p^2 - 3c_s^2) \\
& + (\log\sqrt{\rho})'\Big((c_p^2 - c_s^2)(c_p^2 - 3c_s^2)(\nabla_x \cdot N) \\
& \hspace{2cm} + (c_p^2 - c_s^2)(c_p^2 + c_s^2)' \; - \; 8c_s^2(c_s^2)' \Big) \\
& - (\log\sqrt{\rho})_x \cdot \Big((c_p^2 - c_s^2)(c_p^2 - 3c_s^2)(\nabla_x \otimes N)^t N \\
& \hspace{2cm} + (c_p^2 - c_s^2)(c_p^2 + c_s^2)_x \; - \; 8c_s^2(c_s^2)_x \Big) \\
& + \text{ terms that do not depend on } \rho \Bigg].
\end{aligned}
$$

It follows from (40), (41), (42), (43), and (46) that

(47)
$$
\begin{aligned}
g\alpha_{-1} \; &= \; \int gG \; + \; C \\
&= \; \int \frac{\sqrt{\rho}}{H} \cdot \frac{-1}{2i\rho c_p^2 |\xi|} \cdot \left(-H\sqrt{\rho} \right) [N^t \cdot A(x) \cdot N] \; + \; C \\
&= \; \frac{-1}{2i\tau} \int [N^t \cdot \frac{A(x)}{c_p} \cdot N] \; + \; C,
\end{aligned}
$$

where the rank-2 tensor $A(x)$ depends only on x and is given by

$$
\begin{aligned}
A(x) \; = \; & (\log\sqrt{\rho})_{xx}\Big[2(c_p^2 - 2c_s^2) \Big] \\
& + [(\log\sqrt{\rho})_x \otimes (\log\sqrt{\rho})_x]\left[4c_s^2 \frac{(c_p^2 - 2c_s^2)}{(c_p^2 - c_s^2)} \right] \\
& + (\log\sqrt{\rho})_x \otimes \left[2(c_p^2)_x - \frac{8c_s^2}{c_p^2 - c_s^2}(c_s^2)_x \right] \\
& - I \Bigg[\Delta(\log\sqrt{\rho})(c_p^2 - 4c_s^2) \\
& \hspace{1cm} -|(\log\sqrt{\rho})_x|^2 \left((c_p^2 - 4c_s^2) + \frac{4c_s^4}{c_p^2 - c_s^2} \right) \\
& \hspace{1cm} - (\log\sqrt{\rho})_x \cdot \left(-(c_p^2)_x + \frac{8c_s^2}{c_p^2 - c_s^2}(c_s^2)_x \right) \Bigg].
\end{aligned}
$$

It follows from the fact that $N = \nabla_x\varphi_p/|\nabla_x\varphi_p|$ is the unit tangent vector to the spatial component $x(s)$ of the bicharacteristics (cf. Section 2.3), and from the fact

that the spatial components $x(s)$ are geodesics with respect to the metric $(1/c_p^2)e$, that these line integrals (47), when taken over all p-wave bicharacteristics $\big(\text{cf. } (14)\big)$ $\big($that is, for all $\underline{x} \notin \Omega$, $\underline{\xi} \in \mathbb{R}^3 \smallsetminus 0$, cf. (15)$\big)$ make up a ray transform of the symmetric tensor field A (cf. (51) and Theorem 4.2) over the family of geodesics with respect to $(1/c_p^2)e$.

The Dirichlet-to-Neumann map determines the value of the ray transform (47) over geodesic segments between boundary points of Ω. (In fact, the Dirichlet-to-Neumann map determines the wave displacement U_j outside Ω (cf. Theorem 2.2) and thus it determines $g\alpha_{-1}$ at $\partial\Omega$.) We conclude, for any pair of boundary points $\underline{x}, \overline{x} \in \partial\Omega$, that

$$(48) \qquad\qquad \int_\gamma N^t \cdot B(x) \cdot N \, ds \;\; = \;\; 0,$$

where γ is the p-wave geodesic segment between $\underline{x}$ and $\overline{x}$ with respect to $(1/c_p^2)e$, and

$$
\begin{aligned}
B(x) \;\; = \;\;& \frac{A_1}{c_p} - \frac{A_2}{c_p} \\[6pt]
= \;\;& \kappa\Big[(\beta_1)_x \otimes (\beta_1)_x - (\beta_2)_x \otimes (\beta_2)_x\Big] - \alpha I + \left[\Big(\log\sqrt{\tfrac{\rho_1}{\rho_2}}\Big)_x \cdot V\right] I \\[6pt]
& + \Big(\log\sqrt{\tfrac{\rho_1}{\rho_2}}\Big)_x \textcircled{s}\big(V + 2(c_p)_x\big) \;+\; 2(c_p^2 - 2c_s^2)\Big(\log\sqrt{\tfrac{\rho_1}{\rho_2}}\Big)_{xx},
\end{aligned}
$$

(49)

$$
\begin{aligned}
\kappa \;\; = \;\;& \frac{4c_s^2}{c_p}\frac{(c_p^2 - 2c_s^2)}{(c_p^2 - c_s^2)}, \\[4pt]
\beta_i \;\; = \;\;& \log\sqrt{\rho_i}, \quad i = 1, 2, \\[4pt]
\alpha \;\; = \;\;& \frac{(c_p^2 - 4c_s^2)}{c_p}\Delta\Big(\log\sqrt{\tfrac{\rho_1}{\rho_2}}\Big) - \omega\Big[|(\log\sqrt{\rho_1})_x|^2 - |(\log\sqrt{\rho_2})_x|^2\Big], \\[4pt]
\omega \;\; = \;\;& \frac{(c_p^2 - 4c_s^2)}{c_p} \;+\; \frac{4c_s^4}{c_p(c_p^2 - c_s^2)}, \\[4pt]
V \;\; = \;\;& 2(c_p)_x \;-\; \frac{8c_s^2}{c_p(c_p^2 - c_s^2)}(c_s^2)_x.
\end{aligned}
$$

4.2. **Inverting the ray transform.** We begin with some preliminary notation, following [Sh II], Sections 2.1–2.4. Let $C^\infty(\tau_s^r M)$ denote the r-times contravariant and s-times covariant tensor fields on a manifold M. Let $C^\infty(S^m\tau_M')$ denote the *symmetric tensor fields* of degree m on M, i.e., the $C^\infty(M)$-submodule of $C^\infty(\tau_m^0 M)$ that consists of the tensor fields invariant with respect to every transposition of the indices. Let $\sigma : C^\infty(\tau_m^0 M) \to C^\infty(S^m\tau_M')$ denote the *canonical projection (symmetrization)* defined by $\sigma(v) \;=\; \frac{1}{m!}\sum_{\pi\in\Pi_m} v_{j_{\pi(1)}\cdots j_{\pi(m)}}$, where Π_m is the group of permutations of the set with m elements.

We define *inner differentiation* $d : C^\infty(S^m\tau_M') \to C^\infty(S^{m+1}\tau_M')$ as in [Sh II], Section 2.4, by

$$(50) \qquad\qquad\qquad\qquad d = \sigma\nabla,$$

where ∇ denotes the Levi-Civita connection associated with the Riemannian manifold (M, g). The *divergence operator* $\delta : C^\infty(S^{m+1}\tau_M') \to C^\infty(S^m\tau_M')$ is defined

in coordinate form by $(\delta v)_{i_1\ldots i_m} = v_{i_1\ldots i_m j;k} g^{jk}$, where g^{jk} is the inverse matrix of the metric tensor g_{jk} on the Riemannian manifold (M, g).

Let $H^k(S^m \tau'_M)$, for any integer $k \geq 0$, be the completion of $C^\infty(S^m \tau'_M)$ with respect to the Sobolev norm $||\cdot||_k$ corresponding to the scalar product $(\cdot, \cdot)_k$ that is defined inductively in k by $(u, v)_k = (\nabla u, \nabla v)_{k-1} + (u, v)_{L_2}$. In particular, $H^0(S^m \tau'_M) = L_2(S^m \tau'_M)$.

Sharafutdinov shows in [Sh II], Theorem 2.4.2, that any symmetric tensor field has a unique decomposition into the sum of "solenoidal" and "potential" parts. That is,

Theorem 4.1 (Sharafutdinov [Sh II]). *Let M be a compact Riemannian manifold with boundary. Let $k \geq 1$ and $m \geq 0$ be integers. For every field $f \in H^k(S^m \tau'_M)$, there exist uniquely determined $^s f \in H^k(S^m \tau'_M)$ and $v \in H^{k+1}(S^{m-1} \tau'_M)$ such that*

$$f = {}^s f + dv,$$

where

$$\delta \, {}^s f = 0 \quad and \quad v|_{\partial M} = 0.$$

The fields $^s f$ and v are called the *solenoidal* and *potential* parts of the tensor field f.

The ray transform on a CDRM M (cf. notation and terminology introduced in Section 1.1 and in this section) is the linear operator $I : C^\infty(S^m \tau'_M) \rightarrow C^\infty(\partial_+ \Omega M)$ defined by

$$(51) \qquad If(x, \xi) = \int_{\tau_-(x,\xi)}^0 f_{i_1\ldots i_m}(\gamma_{x,\xi}(t)) \, \dot\gamma_{x,\xi}^{i_1}(t) \ldots \dot\gamma_{x,\xi}^{i_m}(t) \, dt,$$

where $\gamma_{x,\xi} : [\tau_-(x, \xi), 0]$ is the maximal geodesic on $M \cup \partial M$ satisfying the initial (i.e., exit) conditions $\gamma_{x,\xi}(0) = x$ and $\dot\gamma_{x,\xi}(0) = \xi$.

The ray transform may be inverted, modulo the potential part:

Theorem 4.2 (Sharafutdinov [Sh II]). *If (M, g) is a compact, dissipative Riemannian manifold with $(m+1)k^+(M, g) < 1$, $m \in \{0, 1, 2, \ldots\}$, then the solenoidal part $^s f$ of any tensor field $f \in H^1(S^m \tau'_M)$ is uniquely determined by its ray transform If.*

We define *symmetrization with respect to two indices* as in [Sh II], in coordinate form, by

$$\sigma(i_1 i_2) M_{i_1\ldots i_4} = \frac{1}{2} \sum_{\pi \in \Pi_2} M_{i_{\pi(1)} i_{\pi(2)} i_3 i_4},$$

where M is a symmetric (here, rank-4) covariant tensor. The Saint-Venant operator W is defined on the space of smooth, symmetric (here, rank-2) tensor fields, mapping into the smooth, symmetric (here, rank-4) tensor fields, by

$$
\begin{aligned}
(WB)_{i_1 i_2 j_1 j_2} &= \sigma(i_1 i_2)\sigma(j_1 j_2) \sum_{p=0}^{2} (-1)^p \binom{2}{p} B_{i_1 i_{2-p} j_1 j_p, j_{p+1} j_2 i_{2-p+1} i_2} \\
&= \sigma(i_1 i_2)\sigma(j_1 j_2) \left[B_{i_1 i_2, j_1 j_2} - 2 B_{i_1 j_1, j_2 i_2} + B_{j_1 j_2, i_1 i_2} \right],
\end{aligned}
$$

where $\binom{2}{p} = \frac{2!}{p!(2-p)!}$ are the binomial coefficients and $B_{,l}$ denotes partial differentiation of B with respect to x_l.

Sharafutdinov [Sh I] has shown that the potential parts of the compactly supported, rank-2 symmetric tensor fields $C_c^\infty(S^2\tau'_{\mathbb{R}^3})$ on Euclidean space $(\mathbb{R}^3, e)$ form the kernel of W:

Theorem 4.3 (Sharafutdinov [Sh I]).   *The Saint-Venant operator*

$$W : C_c^\infty(S^2\tau'_{\mathbb{R}^3}) \to C_c^\infty(S^2\tau'_{\mathbb{R}^3} \otimes S^2\tau'_{\mathbb{R}^3})$$

has kernel

$$\ker W = \{d_{(\mathbb{R}^3, e)}v \mid v \in C^\infty(S^1\tau'_{\mathbb{R}^3}) \text{ is compactly supported}\}.$$

Remark 5.   We note that $d_{(\mathbb{R}^3, e)}v = \nabla_x \textcircled{s} v$ (cf. notation (30)).

4.3. **Proof of Theorem 1.**   In Section 2 we apply results from [R I] and [R II] to show that the wave speeds $c_{p/s}$ are determined in Ω by the Dirichlet-to-Neumann map. In Section 4.1 we compute the "lower-order polarization" of the elastic wave, and deduce that the ray transform (48) is identically zero if $\Lambda_{\lambda_1,\mu_1,\rho_1} = \Lambda_{\lambda_2,\mu_2,\rho_2}$.

By Theorem 4.1 we decompose the tensor B that appears in the integrand of the ray transform as $B = {}^sB + dv$, where v is a rank-1 symmetric tensor field on Ω and d is the symmetrization of the covariant derivative. By Theorem 4.2 we invert the ray transform (48) to conclude that the solenoidal part ${}^sB$ of $B(x)$ vanishes on Ω. We note that v involves derivatives of $\rho_i(x)$, $i = 1, 2$, of order at most 1 since B involves derivatives of ρ_i of order at most 2. Here we have $(M, g) = (\Omega, e/c_p^2)$; so we write $dv = \nabla_x \textcircled{s} v + R(v)$, where $R(v)$ depends on v and derivatives of c_p (cf. notation (30)). It follows from Theorem 4.3 that $W(B) = W(dv) = W(\nabla_x \textcircled{s} v) + WR(v) = WR(v)$ involves derivatives of $\rho_i(x)$ of order at most 3. We write $T_l u$ for the sum of terms of u that involve the lth-order derivatives of ρ_i, $i = 1, 2$. Then

$$(52) \qquad\qquad T_4 W(B) \;=\; 0.$$

In the following we derive from this fact a partial differential equation whose solution involves the difference between ρ_1 and ρ_2.

We compute the value of $T_4 W(B)$ acting on certain input. In fact, we first observe (cf. (49) and Theorem 4.3) that

$$W\big((\log \sqrt{\rho_i})_{xx}\big) = \; W\big(\nabla_x \textcircled{s} (\log \sqrt{\rho_i})_x\big) \;=\; 0, \qquad\qquad i = 1, 2,$$

$$\big[W(\alpha I)\big]_{i_1 i_2 j_1 j_2} \;=\; \sigma(i_1 i_2)\sigma(j_1 j_2)\Big[\delta_{i_1 i_2}\alpha_{,j_1 j_2} - 2\delta_{i_1 j_1}\alpha_{,j_2 i_2} + \delta_{j_1 j_2}\alpha_{,i_1 i_2}\Big]$$

$$\;=\; \delta_{i_1 i_2}\alpha_{x_{j_1} x_{j_2}} \;+\; \delta_{j_1 j_2}\alpha_{x_{i_1} x_{i_2}} \;-\; \frac{1}{2}\Big[\delta_{i_1 j_1}\alpha_{x_{j_2} x_{i_2}}$$

$$+ \; \delta_{i_2 j_1}\alpha_{x_{j_2} x_{i_1}} \;+\; \delta_{i_1 j_2}\alpha_{x_{j_1} x_{i_2}} \;+\; \delta_{i_2 j_2}\alpha_{x_{j_1} x_{i_1}}\Big],$$

$$\big[W(\beta_x \otimes \beta_x)\big]_{i_1 i_2 j_1 j_2} \;=\; \sigma(i_1 i_2)\sigma(j_1 j_2)\Big[\partial_{x_{j_2}}\partial_{x_{j_1}}(\beta_{x_{i_1}}\beta_{x_{i_2}})$$

$$- \; 2\partial_{x_{i_2}}\partial_{x_{j_2}}(\beta_{x_{i_1}}\beta_{x_{j_1}}) \;+\; \partial_{x_{i_2}}\partial_{x_{i_1}}(\beta_{x_{j_1}}\beta_{x_{j_2}})\Big]$$

$$\;=\; \partial_{x_{j_1}}\partial_{x_{j_2}}(\beta_{x_{i_1}}\beta_{x_{i_2}}) \;+\; \partial_{x_{i_1}}\partial_{x_{i_2}}(\beta_{x_{j_1}}\beta_{x_{j_2}})$$

$$- \; 2\sigma(i_1 i_2)\sigma(j_1 j_2)\partial_{x_{i_2}}\partial_{x_{j_2}}(\beta_{x_{i_1}}\beta_{x_{j_1}}),$$

where $\delta_{ij} = \left\{ \begin{array}{ll} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{array} \right\}$ is the Kronecker delta. Writing

$$\varphi' = \varphi_x \cdot X \qquad \text{and} \qquad \dot{\varphi} = \varphi_x \cdot Y$$

for any constant vectors X, Y and any smooth scalar function φ, we find that

$$
\begin{aligned}
W(\alpha I)(X, X, Y, Y) &= |X|^2 \ddot{\alpha} - 2(X \cdot Y)\dot{\alpha}' + |Y|^2 \alpha'', \\
W(\beta_x \otimes \beta_x)(X, X, Y, Y) &= [(\beta')^2]^{\cdot\cdot} + [(\dot{\beta})^2]'' - 2(\dot{\beta}\beta')^{\cdot\prime} \\
&= 2\left[\dot{\beta}'\dot{\beta}' - \ddot{\beta}\beta''\right].
\end{aligned}
$$

We choose $X = e_i$ and $Y = e_j$, where e_i is the unit vector in $\mathbb{R}^3$ with 1 in the ith component and 0 elsewhere. We sum over $i, j = 1, 2, 3$ to conclude, given (49), that

$$
\begin{aligned}
0 &= \sum_{i,j} T_4 W(B)[e_i, e_i, e_j, e_j] \\
&= T_4 \sum_{i,j}\left[\kappa W(\beta_{1,x} \otimes \beta_{1,x}) - \kappa W(\beta_{2,x} \otimes \beta_{2,x}) - W(\alpha I)\right][e_i, e_i, e_j, e_j] \\
&= -4 T_4 \Delta\alpha = -\frac{4(c_p^2 - 4c_s^2)}{c_p}\Delta^2(\log\sqrt{\rho_1} - \log\sqrt{\rho_2}) \\
&\quad + 4\omega\Delta\left[(\log\sqrt{\rho_1} + \log\sqrt{\rho_2})_x \cdot (\log\sqrt{\rho_1} - \log\sqrt{\rho_2})_x\right].
\end{aligned}
$$

Note that

$$
\omega = \frac{c_p^4 - 5c_p^2 c_s^2 + 8c_s^4}{c_p(c_p^2 - c_s^2)}
$$

is never zero on $\overline{\Omega}$, and is bounded on $\overline{\Omega}$, given the strong convexity condition $3\lambda + 2\mu > 0$, i.e., $3c_p^2 > 4c_s^2$, on $\overline{\Omega}$. It follows that the difference $\beta^- = \log\sqrt{\rho_1} - \log\sqrt{\rho_2}$ solves the fourth-order, linear, elliptic partial differential equation

$$
(53) \qquad \gamma\Delta^2(\beta^-) - \Delta[(\beta^+)_x \cdot (\beta^-)_x] = 0
$$

in Ω, where $\beta^+ = \beta_1 + \beta_2 = \log\sqrt{\rho_1} + \log\sqrt{\rho_2}$ and

$$
\gamma = \frac{(c_p^2 - 4c_s^2)/c_p}{\omega} = \frac{(c_p^2 - c_s^2)(c_p^2 - 4c_s^2)}{c_p^4 - 5c_p^2 c_s^2 + 8c_s^4}.
$$

Since $c_{p/s}$ are real, γ is bounded on $\overline{\Omega}$.

We apply strong uniqueness results for (53) to conclude that β^- vanishes on parts of Ω. In particular, we first note that β^- vanishes to infinite order outside $\partial\Omega$ by Theorem 2.1 and the extension of λ_j, μ_j, and ρ_j to $\mathbb{R}^3$ (see Section 2.1). We then assume, without loss of generality, that λ_j, μ_j, and ρ_j were extended to $\mathbb{R}^3$ so that $c_p \neq 2c_s$ in a neighborhood of ∂B_R, where $B_R \subseteq \mathbb{R}^3$ is an open ball containing $\overline{\Omega}$. Writing

$$
(54) \qquad S_R = \{\, x \in B_R \,:\, \gamma = 0 \,\} = \{\, x \in B_R \,:\, c_p = 2c_s \,\}
$$

for the collection of singular points of (53) in B_R, we conclude that S_R does not intersect some neighborhood of ∂B_R. Let U_0 be any connected, open neighborhood in B_R of ∂B_R whose closure has empty intersection with the singular points S_R. Then $1/\gamma$ is bounded on U_0. By (53) and the fact that, outside Ω, $\beta^- \equiv 0$ solves the PDE in (53), we conclude that for any $x_0 \in U_0 \smallsetminus \overline{\Omega}$,

$$
(55) \qquad |\Delta^2\beta^-(x_0 - x)| \leq C \sum_{|\alpha|\leq 3} |D^\alpha\beta^-(x_0 - x)|
$$

on $\{x_0\} - U_0$, where $C > 0$ is constant. By the strong uniqueness result of [Pr], page 90, for example, with $u(x) = \beta^-(x_0 - x)$, it follows that $\beta^- \equiv 0$ on U_0. We vary U_0 and refer to the continuity of ρ_j to conclude that $\beta^- = 0$ on the closure of the open, connected component $\mathcal{C}_0$ of $B_R \smallsetminus S_R$ that contains a neighborhood in B_R of ∂B_R. That is, $\rho_1 = \rho_2$ on the closure of

$$(56) \qquad\qquad \mathcal{C} \; = \; \mathcal{C}_0 \cap \Omega.$$

Aknowledgements

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